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COERCIVITY PROPERTIES AND WELL-POSEDNESS IN VECTOR OPTIMIZATION

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Abstract. This paper studies the issue of well-posedness for vector optimization. It is shown that coercivity implies well-posedness without any convexity assumptions on problem data. For convex vector optimization problems, solution sets of such problems are non-convex in general, but they are highly structured. By exploring such structures carefully via convex analysis, we are able to obtain a number of positive results, including a criterion for well-posedness in terms of that of associated scalar problems. In particular we show that a well-known relative interiority condition can be used as a sufficient condition for well-posedness in convex vector optimization.

Keywords. Vector optimization, weakly efficient solution, well-posedness, level-coercivity, error bounds, relative interior.

1. Introduction

In this paper we are concerned with the following vector optimization problem,

$$(\mathcal{P}) \qquad \begin{array}{ll} \min & F(x) \\ \text{s.t.} & x \in C, \end{array}$$

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where $F(x) = (f_1(x), ..., f_m(x)) : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function, and $C \subset \mathbb{R}^n$ is a nonempty closed set. For the clarity of exposition, we assume throughout that each f_i is a continuous function. When each f_i is a finite convex function, and C is a closed convex set, $(\mathcal{P})$ is a convex vector optimization problem (CVOP for short).

There are several solution concepts associated with $(\mathcal{P})$. Throughout this paper, we will use the notion of weakly efficient solution, which is defined as follows.

Let $W = \mathbb{R}^m \backslash (-\text{int } \mathbb{R}^m_+)$, where int denotes interior. A vector $\bar{x} \in C$ is a weakly efficient solution of problem $(\mathcal{P})$ if and only if

$$F(x) - F(\bar{x}) \in W \quad \forall x \in C.$$

Denote by E_w the set of all weakly efficient solutions to problem $(\mathcal{P})$. In what follows, we assume that, for each $i \in [1, m]$,

$$\arg\min_{x \in C} f_i(x) \neq \emptyset. \tag{1}$$

Assumption (1) implies in particular that $E_w \neq \emptyset$ since the following inclusion holds always

$$E_w \supset \cup_{\lambda \in \Lambda} \arg\min_{x \in C} f_\lambda(x),$$

where $f_\lambda(x) = \lambda^T F(x)$ and $\Lambda = \{(\lambda_1, \dots, \lambda_m)^T \mid \lambda_i \geq 0 \; \forall i \in [1, m], \; \sum_{i=1}^m \lambda_i = 1\}$. This implies also that $F(E_w)$ is nonempty. When $(\mathcal{P})$ is a CVOP, by [18, Th. 11.2] or [21, Cor. 3.4.1], we have

$$E_w = \cup_{\lambda \in \Lambda} \arg\min_{x \in C} f_\lambda(x). \tag{2}$$

Denote the parametric problem: minimize $f_\lambda(x) = \lambda^T F(x)$ subject to the constraint $x \in C$ by $(\mathcal{P}(\lambda))$. Then (2) reveals a basic connection between $(\mathcal{P})$ and the parametric problem $(\mathcal{P}(\lambda))$. This basic observation is a key for much of our studies on CVOPs, which enables us to study CVOPs through a family of parametric scalar convex optimization problems.

For scalar optimization problems, the theory of well-posedness has a long history, and has been extensively studied. See [10] for surveys of the results up to 1993, and [14, 22] and references therein for recent developments on the subject. Well-posedness properties play an important role in optimization theory because of their links to several basic issues in optimization as well as the usefulness in the convergence analysis of many algorithms. For vector optimization problems, "local" properties of well-posedness have been studied in [9, 13, 17], among others. But it has not been studied in a "global" sense, which will be carried out in this paper. We say that problem $(\mathcal{P})$ is *well-posed* if, $F(E_w)$ is closed, and for any sequence $\{x(n)\} \subset C$,

$$[\text{dist}\,(F(x(n)) \mid F(E_w)) \to 0 \;\text{ as } n \to \infty] \;\Rightarrow [\text{dist}\,(x(n) \mid E_w) \to 0 \;\text{ as } n \to \infty], \tag{3}$$

where $\text{dist}\,(x \mid E_w)$ and $\text{dist}\,(y \mid F(E_w))$ denote the Euclidean distances between the vector x and E_w, and the vector y and $F(E_w)$ respectively. This notion of well-posedness is an extension of Tikhonov's well-posedness in scalar optimization. The generalization amounts to removing the uniqueness assumption on solution sets.

In view of (2) and (3), we observe that there are two basic questions to be asked:

(a) Suppose that $(\mathcal{P})$ is a CVOP, that the parametric problem $(\mathcal{P}(\lambda))$ is well-posed over Λ (formal definition to follow), and that $F(E_w)$ is closed. Is $(\mathcal{P})$ well-posed in the sense of (3)?

(b) When is $F(E_w)$ closed?

Resolutions for question (a) will enable us to study well-posedness of a CVOP in terms of that of the parametric problem $(\mathcal{P}(\lambda))$. As noted above, well-posedness of scalar optimization problems have been well studied. Question (b) has an affirmative answer when E_w is compact, and F is continuous. This is a straightforward consequence of the fact that the image of a compact set under a continuous mapping is compact. Apart from the compactness assumption, to our knowledge, this issue has not been very well studied in the vector optimization literature. The closeness of $F(E_w)$, however, is a natural requirement in view of well-posedness defined by (3), and serves as a type of regularity condition for $(\mathcal{P})$. The aim of this paper is to study questions (a) and (b).

The paper is organized as follows. In Section 2, we show that $(\mathcal{P})$ is well-posed under the level-coercivity assumption regardless of whether $(\mathcal{P})$ is a CVOP or not. Moreover, our analysis reveals that the level-coercivity of an associated function for $(\mathcal{P})$ yields certain error bounds, which is potentially useful for the convergence analysis of algorithms to solve $(\mathcal{P})$. When $(\mathcal{P})$ is a CVOP, we do obtain some very useful conditions under which $(\mathcal{P})$ is well-posed. In particular, we give an affirmative answer for question (a) (see Sect. 3). Section 4 is devoted to study question (b). By establishing a result on the closeness of $F(C)$ first, which is interesting in its own right, we are able to obtain some verifiable conditions under which $F(E_w)$ is closed without the boundedness assumption on E_w. As a consequence, we show that a well-known relative interior (or weak coercivity) condition, which is shown to be fundamental in scalar convex optimization [1, 3], is sufficient for well-posedness of $(\mathcal{P})$. We note that it is possible to replace $F(E_w)$ by $cl(F(E_w))$ when $F(E_w)$ is not necessarily closed. But we will not pursue this approach in this work.

The following notation will be used throughout the paper. We denote by $[1, m]$ the set $\{1, 2, \ldots, m\}$. For problem $(\mathcal{P})$, let the level set $L(q) = \{x \in C \mid F(q) - F(x) \in W\}$, where $q \in C$. It is easy to see that

$$E_w = \cap_{q \in C} L(q).$$

The above observation implies in particular that E_w is a closed set since each $L(q)$ is a closed set, and E_w is the intersection of closed sets $L(q)$.

For any given closed nonempty set $X \subset \mathbb{R}^n$, we write I_X for the indicator function of set X, and we define the recession (horizon) cone of X [20, Def. 3.3] as

$$X^\infty = \{v \mid \exists \text{ sequences } \{v(n)\} \text{ in } X \text{ and } \lambda(n) \downarrow 0 \text{ with } \lim_{n\to+\infty} \lambda(n)v(n) = v\} \cdot$$

For any proper lower semi-continuous function $h : I\!R^n \to I\!R \cup \{+\infty\}$, we say that h is level-bounded if $\{x \mid h(x) < \alpha\}$ is either empty or bounded for any real number α, and h is level-coercive if $h^\infty(x) > 0$ for all $x \neq 0$ where the recession (horizon) h^∞ [20, Th. 3.21] is defined as follows

$$h^\infty(v) = \lim_{\delta \downarrow 0} \inf_{x \in I\!B(v,\delta), \lambda \in (0,\delta)} \lambda h(\lambda^{-1}x) \qquad \forall v \in I\!R^n, \tag{4}$$

where $I\!B(v, \delta)$ denotes the closed ball with radius δ and centered at v. The level boundedness of h corresponds to having $h(x) \to \infty$ as $\|x\| \to \infty$, and the level-coercivity of h is equivalent to the following: for some $\alpha > 0$, there exists a $\beta \in (-\infty, \infty)$ such that

$$h(x) \geq \alpha\|x\| + \beta \qquad\qquad \forall x \in R^n.$$

For the parametric problem $(\mathcal{P}(\lambda))$, we denote $\inf_{x \in C} \lambda^T F(x)$ by $p(\lambda)$, and $\arg\min_{x \in C} \lambda^T F(x)$ by S_λ respectively.

2. Coercivity, level boundedness and error bounds

The level-coercivity and level boundedness of a given function are important notions in optimization. They describe the growth behavior of the given function. For a given scalar optimization problem, these growth properties are often used as sufficient conditions for the existence of optimal solutions as well as for the convergence analysis of many iterative algorithms for finding these solutions. The following theorem illustrates the level-coercivity and level boundedness properties also play an important role in vector optimization.

Theorem 2.1. *For problem $(\mathcal{P})$, assume that (1) holds, and that $F(E_w)$ is closed. Let $g(x) = \text{dist}\,(F(x) \mid F(E_w))$. Consider the following statements:*

(1) $f_i + I_C$ is level-coercive for each $i \in [1, m]$.

(2) The set $F(E_w)$ is compact, and $g + I_C$ is level-coercive.

(3) The set $F(E_w)$ is compact, and the function $g(x)$ has the property that $g(x(n)) \to \infty$ whenever $\{x(n)\} \subset C$ and $||x(n)|| \to \infty$, i.e., $g + I_C$ is level-bounded.

(4) The set E_w is nonempty and compact.

Then the following holds:

Statement (1) implies statement (2), statement (2) implies statement (3), and statement (3) implies statement (4). When $(\mathcal{P})$ is a CVOP, statements (1) through (4) are equivalent.

Proof. [(1) ⇒ (2)]: It is easy to see that E_w is nonempty. If E_w were unbounded, then there would exist some unbounded sequence $\{x(n)\} \subset E_w$. By the level-coercivity of $f_i + I_C$ for each $i \in [1, m]$, $f_i(x(n)) \to \infty$ as $n \to \infty$. Let $\bar{x}$ be a given point in C. Then $F(\bar{x}) - F(x(N)) \in -\text{int}\,(R^m_+)$ for some N sufficiently large. This shows that $x(N)$ cannot be in E_w. The contradiction proves that E_w is compact. This in turn implies that $F(E_w)$ is nonempty and compact by the continuity of F.

By the hypothesis, $f_1 + I_C$ is level-coercive in particular. Since $F(E_w)$ is bounded, a simple calculation shows that $g(x) \geq |f_1(x)| + \beta$ for some $\beta \in R$. Hence $g + I_C$ is level-coercive since $|f_1| + I_C$ is.

[(2) $\Rightarrow$ (3)]: This is evident since the level-coercivity implies the level boundedness.

[(3) $\Rightarrow$ (4)]: If E_w were unbounded, there would exist an unbounded sequence $\{x(n)\} \subset E_w \subset C$. By the hypothesis, $F(x(n))$ is bounded since $F(E_w)$ is bounded. This in turn implies that $g(x(n))$ is bounded and $\|x(n)\| \to \infty$ as $n \to \infty$, which contradicts statement (3).

When $(\mathcal{P})$ is a CVOP, we have proved in [6] that (4) and (1) statements are equivalent. Therefore, all statements are equivalent. □

Remark. When $(\mathcal{P})$ is not a CVOP, it is easy to see that (4) does not imply (3). The following example shows that (3) does not imply (2): let $F(x) : R \to R$ be given by $F(x) = \sqrt{|x|}$, and $C = R$. Then (2) fails but (3) holds. To illustrate that statement (2) does not imply statement (1) in general, we consider the following example: let $F(x) : R^2 \to R^2$ be given by $F(x) = (f_1(x_1, x_2), f_2(x_1, x_2)) = (x_1^2 + \sqrt{|x_2|}, \sqrt{|x_1|} + x_2^2)$, and $C = R^2$. Since f_1 and f_2 are level-bounded, but not level-coercive, (1) fails. We now show that (2) holds. To see this, we observe first that the level-boundedness of f_1 and f_2 implies that E_w is nonempty. The boundedness argument for E_w follows the same line as for the boundedness of E_w in the proof of the implication [(1) $\Rightarrow$ (2)]. Secondly, for any $(y_1, y_2) \in F(E_w)$, which is a bounded set since E_w is, we have

$$\begin{aligned} g(x) &= \inf_{(y_1,y_2)\in F(E_w)} \sqrt{(f_1(x_1,x_2) - y_1)^2 + (f_2(x_1,x_2) - y_2)^2} \\ &\geq \frac{1}{2} \inf_{(y_1,y_2)\in F(E_w)} (|f_1(x_1,x_2) - y_1| + |f_2(x_1,x_2) - y_2|) \\ &\geq \frac{1}{2}(x_1^2 + x_2^2) - \beta \end{aligned}$$

for some real number β. This is enough to show that g is level-coercive.

When $(\mathcal{P})$ is a CVOP, various characterizations of the nonemptiness and compactness can be found in [6]. For generalizations of these characterizations and applications, see [7, 11, 12].

The level-coercivity property is closely related to certain error bounds as been illustrated by the author [8] for scalar optimization problems. We now give a consequence of statement (2) in Theorem 2.1 in terms of error bounds and well-posedness for vector optimization $(\mathcal{P})$. In [8] results were also given under conditions weaker than those of level-coercivity in the presence of the convexity of certain residual functions. However, it is very unlikely that g in Theorem 2.1 will be convex for vector optimization problems. The proof follows the same line of the arguments used by the author in [8]. For completeness, we include a proof. We begin with a lemma first.

Lemma 2.2. *Assume that $F(E_w)$ is nonempty and closed, and that the sequence $\{x(n)\} \subset C$ satisfies* $\operatorname{dist}(F(x(n)) \mid F(E_w)) \to 0$ *as $n \to \infty$. If $\{F(x(n))\}$ converges to y, then $y \in F(E_w)$. If it is further assumed that $\{x(n)\}$ is a bounded sequence, then any cluster point of $\{x(n)\}$ is in E_w.*

Proof. Let $y(n) \in F(E_w)$ be such that $\operatorname{dist}(F(x(n)) \mid F(E_w)) = ||F(x(n)) - y(n)||$ for each n. By the triangular inequality for a norm, we have

$$\|y - y(n)\| \leq \|F(x(n)) - y\| + \|F(x(n)) - y(n)\|.$$

This shows that y is the limit of $\{y(n)\}$. Since $F(E_w)$ is closed, $y \in F(E_w)$.

Suppose now that $\{x(n)\}$ is a bounded sequence. Let $\hat{x}$ be a cluster point of the sequence. Then there is a subsequence of $\{x(n)\}$ converging to $\hat{x}$. Without loss of generality, suppose $\{x(n)\}$ is a convergent sequence. By the continuity of F, $F(x(n))$ converges to $F(\hat{x})$. So $F(\hat{x}) \in F(E_w)$, which means

$$F(x) - F(\hat{x}) \in W \qquad \forall x \in C.$$

This shows that $\hat{x} \in E_w$ by definition. □

The main result on error bounds and well-posedness of $(\mathcal{P})$ follows.

Theorem 2.3. *For problem $(\mathcal{P})$, suppose that the basic assumptions in statement (2) of Theorem 2.1 hold. Then the following error bound holds: for any $\epsilon > 0$, there is some $\tau(\epsilon) > 0$ such that*

$$\operatorname{dist}(x \mid E_w) \leq \tau(\epsilon) g(x), \qquad \forall x \in \{y \in C \mid \operatorname{dist}(y \mid E_w) \geq \epsilon\}\cdot \tag{5}$$

As a consequence, the error bound (5) implies $(\mathcal{P})$ is well-posed in the sense of (3).

Proof. Suppose that (5) does not hold for some $\epsilon > 0$. Then there is a sequence $\{x(n)\} \subset C$ with

$$\operatorname{dist}(x(n) \mid E_w) \geq \epsilon \ \forall n, \ \text{ and } \ g(x(n))/\operatorname{dist}(x(n) \mid E_w) \to 0 \ \text{ as } \ n \to \infty. \tag{6}$$

If $\{x(n)\}$ is bounded, then $g(x(n)) \to 0$ by (6). So any cluster point of $\{x(n)\}$ must be in E_w by Lemma 2.2. Again by (6), this is impossible. We conclude that any subsequence of $\{x(n)\}$ must be divergent. Without loss of generality, we may assume that $\|x(n)\| \to \infty$ as $n \to \infty$, and $x(n)/\|x(n)\| \to d \neq 0$. Let $\hat{x}(n) \in E_w$ such that $\operatorname{dist}(\hat{x}(n) \mid E_w) = \|x(n) - \hat{x}(n)\|$. Since E_w is bounded, $x(n)/\|x(n) - \hat{x}(n)\| \to d \in C^\infty$. It follows that $\frac{g(x(n)) + I_C(x(n))}{\|x(n) - \hat{x}(n)\|} \to 0$ as $n \to \infty$, which is equivalent to saying that $\lim_{n\to\infty} \frac{g(x(n)) + I_C(x(n))}{||x(n)||} = 0$ since $\{\hat{x}(n)\}$ is bounded. This implies that $g + I_C$ is not level-coercive. The contradiction establishes (5). For the second part, suppose that $(\mathcal{P})$ is not well-posed. Then for some $\epsilon > 0$, there is a sequence $\{x(n)\} \subset C$ with $g(x(n)) \to 0$ and $\operatorname{dist}(x(n) \mid E_w) \geq \epsilon$ for all $n = 1, 2, \ldots$. In view of (5), there is some $\tau(\epsilon) > 0$ such that $\operatorname{dist}(x(n) \mid E_w) \leq \tau(\epsilon) g(x(n))$. But this would imply that, for $n \to \infty$, $\operatorname{dist}(x(n) \mid E_w) \to 0$ as $g(x(n)) \to 0$. The contradiction completes the proof. □

3. Well-posedness

When $(\mathcal{P})$ is a CVOP, E_w is not convex in general. However, thanks to (2), the structure of E_w can be explored by a family of convex optimization problems $(\mathcal{P}(\lambda))$. To understand $(\mathcal{P})$ better, we need to examine $(\mathcal{P}(\lambda))$ carefully. Generally speaking, the parametric problem $(\mathcal{P}(\lambda))$ can be studied in a parametric optimization setting. The references on parametric optimization are vast. The lower semi-continuity property of the value function has been extensively studied in the literature under the boundedness assumption on the solution set at a reference point as well as certain convexity assumptions on problem data, see [5, 20, Chap. 3] and references therein. Due to the special structures of $(\mathcal{P}(\lambda))$, we are able to sharpen the results by weakening some of the underlying assumptions. The following result gives a sufficient condition for the lower semi-continuity of the value functions. Our contribution is that we are able to establish the lower semi-continuity of value functions without either the boundedness assumption on solution sets or the convexity assumption on problem data.

Theorem 3.1. *For the parametric problem $(\mathcal{P}(\lambda))$, assume that, for each $\lambda \in \Lambda$, S_λ is nonempty. Then $p(\cdot)$ is continuous on Λ.*

Proof. We first show the easy part that p is upper semi-continuous. For any $\bar{\lambda} \in \Lambda$, let $\{\lambda(n)\} \subset \Lambda$ be any sequence converging to $\bar{\lambda}$, then there exists $x(n) \in \arg\min_{x\in C} \lambda(n)^T F(x)$ such that $p(\lambda(n)) = \lambda(n)^T F(x(n)) \leq \lambda(n)^T F(\bar{x})$, where $\bar{x} \in \arg\min_{x\in C} \bar{\lambda}^T F(x)$. From $\lambda(n)^T F(\bar{x}) \to \bar{\lambda}^T F(\bar{x})$ as $n \to \infty$, it follows that $p(\cdot)$ is upper semi-continuous at $\bar{\lambda}$.

We now proceed to show that $p(\cdot)$ is lower semi-continuous at $\bar{\lambda}$. Suppose that $p(\cdot)$ is not lower semi-continuous at $\bar{\lambda}$. Then there exist sequences $\{\lambda(n)\}$ with $\lambda(n) \to \bar{\lambda}$, $\{x(n)\} \subset C$, and some $\epsilon > 0$ such that

$$\lambda(n)^T F(x(n)) = p(\lambda(n)) \leq p(\bar{\lambda}) - \epsilon \quad \forall n. \tag{7}$$

Let $I = \{i \in [1,m] \mid \bar{\lambda}_i \neq 0\}$, and $I' = [1,m] \backslash I$. Since $\arg\min_{x\in C} f_i(x)$ is nonempty for each $i \in [1,m]$, f_i is bounded below on C for each $i \in [1,m]$; that is, for some $\alpha \in R$,

$$f_i(x) \geq \alpha \qquad \forall i \in [1,m], \ \ \forall x \in C. \tag{8}$$

This implies that, for any $x \in C$,

$$\sum_{i\in I'} \lambda_i(n) f_i(x) \geq \alpha \sum_{i\in I'} \lambda_i(n). \tag{9}$$

From (7) and (9), we have, for all n,

$$\alpha \sum_{i\in I'} \lambda_i(n) + \sum_{i\in I} \lambda_i(n) f_i(x(n)) \leq \sum_{i\in[1,m]} \lambda_i(n) f_i(x(n)) \leq p(\bar{\lambda}) - \epsilon. \tag{10}$$

Since $\lambda_i(n) \to \bar{\lambda}_i > 0$ for each $i \in I$, and $\alpha \sum_{i \in I'} \lambda_i(n) \to 0$ as $n \to \infty$, (10) implies there is some $\beta > 0$ such that $f_i(x(n)) \leq \beta$ for all $i \in I$. This observation along with (8) implies that

$$\{f_i(x(n))\} \text{ is a bounded sequence for each } i \in I \text{ and all } n. \tag{11}$$

Choose N such that

$$\left| \alpha \sum_{i \in I'} \lambda_i(N) \right| + \left| \sum_{i \in I} (\lambda_i(N) - \bar{\lambda}_i) f_i(x(N)) \right| \leq \epsilon/2.$$

Again, it follows from (10) that

$$\begin{aligned} \sum_{i \in I} \bar{\lambda}_i f_i(x(N)) - \epsilon/2 &\leq \sum_{i \in I} \bar{\lambda}_i f_i(x(N)) + \alpha \sum_{i \in I'} \lambda_i(N) + \sum_{i \in I} (\lambda_i(N) - \bar{\lambda}_i) f_i(x(N)) \\ &\leq \lambda(N)^T F(x(N)) \leq p(\bar{\lambda}) - \epsilon. \end{aligned} \tag{12}$$

Since $\sum_{i \in I} \bar{\lambda}_i f_i(x(N)) \geq p(\bar{\lambda})$, the inequality (12) implies that $p(\bar{\lambda}) \leq p(\bar{\lambda}) - \epsilon/2$, which is impossible. This completes the proof. $\square$

Definition 3.2. For a given $\bar{\lambda} \in \Lambda$, $(\mathcal{P}\ (\bar{\lambda}))$ is *well-posed* if, $S_{\bar{\lambda}} \neq \emptyset$, and for any sequence $\{x(n)\} \subset C$ with

$$\left[\bar{\lambda}^T F(x(n)) \to p(\bar{\lambda}) \ \text{ as } n \to \infty\right] \Rightarrow [\operatorname{dist}(x(n) \mid S_{\bar{\lambda}}) \to 0 \ \text{ as } n \to \infty].$$

We say that the parametric problem $(\mathcal{P}(\lambda))$ is *well-posed over* Λ if $(\mathcal{P}(\bar{\lambda}))$ is well-posed for every $\bar{\lambda} \in \Lambda$.

The main result of this section follows.

Theorem 3.3. *Assume that (*$\mathcal{P}$*) is a CVOP, that (1) holds, and that* $F(E_w)$ *is closed. Then (*$\mathcal{P}$*) is well-posed in the sense of (3) if*

$$(\mathcal{P}(\lambda)) \text{ is well-posed over } \Lambda. \tag{13}$$

Proof. Let $\{x(n)\} \subset C$ be a sequence with $\operatorname{dist}(F(x(n)) \mid F(E_w)) \to 0$. We note that the following holds without (13). Since $F(E_w)$ is closed, for each $x(n)$, there is an $\hat{x}(n) \in E_w$ such that

$$\operatorname{dist}(F(x(n)) \mid F(E_w)) = ||F(x(n)) - F(\hat{x}(n))|| \to 0 \ \text{ as } n \to \infty. \tag{14}$$

Since $(\mathcal{P})$ is a CVOP, there is some $\lambda(n) \in \Lambda$ such that $\hat{x}(n) \in \arg\min_{x \in C} \lambda(n)^T F(x)$. By the compactness of Λ, $\{\lambda(n)\}$ is a bounded sequence. Without loss of generality, suppose that $\lambda(n) \to \bar{\lambda} \in \Lambda$. By (14), and the boundedness of $\lambda(n)$,

$$\lambda(n)^T F(x(n)) - p(\lambda(n)) = \lambda(n)^T F(x(n)) - \lambda(n)^T F(\hat{x}(n)) \to 0 \ \text{ as } n \to \infty. \tag{15}$$

If (13) holds, by Theorem 3.1,

$$\lambda(n)^T F(\hat{x}(n)) = p(\lambda(n)) \to p(\bar{\lambda}) \text{ as } \lambda(n) \to \bar{\lambda}. \tag{16}$$

Relations (15) and (16) imply that $\lambda(n)^T F(x(n)) \to p(\bar{\lambda})$ as $n \to \infty$. We claim that

$$\bar{\lambda}^T F(x(n)) \to p(\bar{\lambda}).$$

To see this, let $I = \{i \in [1,m] \mid \bar{\lambda}_i \neq 0\}$, and $I' = [1,m]\backslash I$ as in the proof of Theorem 3.1. Since f_i is bounded below on C for $i \in [1,m]$, the sequence $\{f_i(x(n))\}$ is bounded below in particular. For each $i \in I$, since $p(\lambda(n)) \to p(\bar{\lambda})$, we have the sequence $\{f_i(\hat{x}(n))\}$ is bounded by a similar argument used in the proof of Theorem 3.1 (see (11)). This in turn implies that, for each $i \in I$, $\{f_i(x(n))\}$ is bounded by (14). So for any $\epsilon > 0$, there is an N such that, for all $n \geq N$,

$$\left| \sum_{i\in I} (\bar{\lambda}_i - \lambda_i(n)) f_i(x(n)) \right| < \epsilon/2,$$

and

$$\sum_{i \in I'} \lambda_i(n) f_i(x(n)) \geq -\epsilon/2$$

since $\lambda(n) \to \bar{\lambda}$ and $\lambda_i(n) \downarrow 0$ for $i \in I'$. It follows that, for all $n \geq N$,

$$\begin{aligned}
0 &\leq \bar{\lambda}^T F(x(n)) - p(\bar{\lambda}) \\
&= \sum_{i\in I} \bar{\lambda}_i f_i(x(n)) - p(\bar{\lambda}) \\
&= \sum_{i\in I} \lambda_i(n) f_i(x(n)) - p(\bar{\lambda}) + \sum_{i\in I} (\bar{\lambda}_i - \lambda_i(n)) f_i(x(n)) \\
&\leq \sum_{i=1}^{m} \lambda_i(n) f_i(x(n)) - p(\bar{\lambda}) + \epsilon/2 - \sum_{i\in I'} \lambda_i(n) f_i(x(n)) \\
&\leq \lambda(n)^T F(x(n)) - p(\bar{\lambda}) + \epsilon.
\end{aligned}$$

Since $\lambda(n)^T F(x(n)) \to p(\bar{\lambda})$ and ϵ is arbitrary, we proved the claim that $\bar{\lambda}^T F(x(n)) \to p(\bar{\lambda})$. It follows that $\operatorname{dist}(x(n) \mid S_{\bar{\lambda}}) \to 0$ since $(\mathcal{P}(\bar{\lambda}))$ is well-posed. By $\cup_{\lambda \in \Lambda} S_\lambda = E_w$,

$$\operatorname{dist}(x(n) \mid E_w) \leq \operatorname{dist}(x(n) \mid S_{\bar{\lambda}}) \to 0. \qquad \square$$

4. Closeness criteria for $F(E_w)$

As noted before, $F(E_w)$ is closed whenever E_w is compact, and F is continuous. For the convex case, this is equivalent to the level-coercivity of each $f_i + I_C$ (see Th. 2.1). When E_w is unbounded, we can easily construct examples such that $F(E_w)$ is not closed. This is a *distinct feature* for vector optimization problems. The following lemma gives a sufficient condition under which $F(E_w)$ is closed. This result was given in [16, Cor. 1.2, p. 136]. For completeness, we include an elementary proof.

Lemma 4.1. *For problem $(\mathcal{P})$, assume that $F(E_w)$ is nonempty, and $F(C)$ is closed. Then $F(E_w)$ is closed.*

Proof. Let $\{y(n)\} \subset F(E_w)$ such that $y(n) \to \bar{y}$ as $n \to \infty$. Since $F(C)$ is closed, $\bar{y} \in F(C)$. If $\bar{y} \notin F(E_w)$, then there is some $x \in C$ such $f_i(x) < \bar{y}_i$ for all $i \in [1, m]$. Let $\epsilon = \min_{i \in [1,m]}(\bar{y}_i - f(x))$. Since $y(n) \to \bar{y}$, there is some natural number N such that $y_i(N) > \bar{y}_i - \frac{\epsilon}{2} > f_i(x)$ for $i \in [1, m]$. But $y(N) = F(x(N))$ for some $x(N) \in E_w$. This shows that $x(N)$ cannot be in E_w. The contradiction completes the proof. □

Lemma 4.1 indicates that we can prove the closeness of $F(C)$ instead. In the presence of the convexity on C and on each f_i, the closeness of $F(C)$ can be proved under weaker conditions. The closeness of images of sets under operations is an important issue in optimization. It has been mainly studied for the image of a closed convex set under a linear transformation [18, Th. 9.1] as well as [2,15], and for the image of a closed set under linear or nonlinear mappings in the presence of coercivity conditions [20, Th. 3.10, exercise 3.16].

Recall that, for a nonempty convex set $\bar{C}$, the linearity space of $\bar{C}$ is, by definition [18, p. 65], $(-\bar{C}^\infty) \cap \bar{C}^\infty$; for a proper lower semi-continuous convex function h, the constancy space is

$$M_h = \{x \in \mathbb{R}^n \mid h^\infty(x) \leq 0, \quad h^\infty(-x) \leq 0\},$$

which is the largest subspace contained in the recession cone of h. The recession cone of h, by definition, is the set $\{x \in \mathbb{R}^n \mid h^\infty(x) \leq 0\}$. The following theorem provides a general sufficient condition for the closeness of the image of a convex set under R^m_+- convex mappings. This result is interesting in its own right.

Theorem 4.2. *Let $\bar{C}$ is a nonempty convex set, and $F = (f_1, f_2, \ldots, f_m, l_1, l_2, \ldots, l_p)$, where each f_i is a finite convex function, and each $l_k(x) = a_k^T x + \alpha_k$ is an affine function. If*

$$cl\ \left(\bar{C}\right)^\infty \cap (\cap_{i=1}^m\ \{x \in \mathbb{R}^n\ |f_i^\infty(x) \leq 0\}) \cap (\cap_{k=1}^p a_k^\perp)\ \textit{is a subspace,} \tag{17}$$

then $cl\ \left(F(\bar{C})\right) = F(cl\ (\bar{C}))$.

Proof. The inclusion $cl\ \left(F(\bar{C})\right) \supset F(cl\ (\bar{C}))$ follows easily from the continuity of F.

We now show the reverse inclusion. Denote the subspace in (17) by $L^\perp$, where $L^\perp$ is the orthogonal complement of the subspace $L \subset \mathbb{R}^n$. Let $y \in cl\ \left(F(\bar{C})\right)$ and let $\{x(n)\} \subset cl\ \left(\bar{C}\right)$ be a sequence with $F(x(n)) \to y$ as $n \to \infty$. By (17), $L^\perp \subset (-cl\ \left(\bar{C}\right)^\infty) \cap cl\ \left(\bar{C}\right)^\infty$. Write $x(n) = x_L(n) + x_{L^\perp}(n)$ with $x_L(n) \in L$

and $x_{L^\perp}(n) \in L^\perp$. Since $-x_{L^\perp}(n) \in L^\perp \subset (-cl\ (\bar{C})^\infty) \cap cl\ (\bar{C})^\infty$, $x_L(n) = x(n) - x_{L^\perp}(n) \in cl\ (\bar{C})$ by [18, Th. 8.1]. By [18, Cor. 8.6.1], $f_i(x(n)) = f_i(x_L(n))$ for each $i \in [1, m]$ since $x_{L^\perp}(n) \in M_{f_i}$. The same argument also implies that $l_k(x(n)) = l_k(x_L(n))$ for each $k \in [1, p]$. We claim that

$$\{x_L(n)\} \text{ is a bounded sequence.}$$

To see this, suppose that $\{x_L(n)\}$ is unbounded. Without loss of generality, suppose that

$$x_L(n)/||x_L(n)|| \to d \in L.$$

It is clear that $d \in cl\ (\bar{C})^\infty$. Since $\{F(x_L(n))\} = \{F(x(n))\}$ is a convergent sequence, it follows that d is a direction of recession for each f_i with $i \in [1, m]$. So $f_i^\infty(d) \le 0$ for each $i \in [1, m]$. We can refine this observation for each l_k by the hypothesis. For each $k \in [1, p]$, a simple computation shows that $d \in a_k^\perp$ since $l_k(x_L(n))$ is convergent and l_k is affine. This shows that $d \in L^\perp$. Therefore $0 \ne d \in L \cap L^\perp = \{0\}$. The contradiction shows that $\{x_L(n)\}$ must be a bounded sequence. Without loss of generality, suppose that $x_L(n) \to \bar{x} \in cl\ (\bar{C})$. By the continuity of F at $\bar{x}$, $y \in F(cl\ (\bar{C}))$. This completes the proof. □

Remark. (1) We remark that the above theorem is an extension of [18, Th. 9.1] on the closeness of the image of a convex set under linear transformation to that of a convex set under R_+^m convex mappings. For a linear transformation $Ax = (a_1^T x, \ldots, a_m^T x)$, by letting $l_i(x) = a_i^T x$, we observe that $\cap_k^m a_k^\perp = ker(A)$. When $F(x) = Ax$, we recover Theorem 9.1 under the assumption (17). As well-known, $F(cl(\bar{C}))$ may not be closed even for $F(x) = Ax$ where A is a linear transformation from R^n to R^m if (17) is violated. See [18, p. 73] for such an example.
(2) We note that the image of a convex set under R_+^m convex mappings may not be convex.

Example 4.3. Let $F(x_1, x_2) = (x_1^2, x_2^2)$, and $C = co\{(0,0), (0,1), (1,0)\}$. Then $F(C)$ is not convex.

Recall that, for a given nonempty convex set $\bar{C} \subset I\!R^n$, the relative interior of $\bar{C}$, denoted by $\text{ri}(\bar{C})$, is defined as the interior which results when $\bar{C}$ is regarded as a subset of its affine hull. When F in Theorem 4.2 is a scalar convex function, condition (17) is actually a characterization of a well-known relative interior (or weak coercivity) condition, which has found many applications in convex optimization. We record this observation as the following proposition, which could also be derived from [18, Th. 27.1(b)].

Proposition 4.4. *Let $C \subset I\!R^n$ be a nonempty closed convex set, and f be a finite convex function. Then $0 \in ri[\ dom\ (f + I_C)^*]$, where $(f + I_C)^*$ is the convex conjugate function of $f + I_C$, if and only if*

$$C^\infty \cap \{x \in I\!R^n\ |f^\infty(x) \le 0\} \text{ is a subspace.} \tag{18}$$

Proof. We observe that $0 \in ri[\ dom\ (f + I_C)^*]$ if and only if the convex cone generated by $dom\ (f + I_C)^*$ is a subspace [4, Prop. 2.1, Lem. 2.2]. This is equivalent to saying that the polar of such a cone is a subspace. By [18, Th. 14.2], this subspace is

$$\{x \in I\!R^n \mid f^\infty(x) + I_{C^\infty}(x) \le 0\}\cdot \tag{19}$$

But $f^\infty(x) > -\infty$ for any $x \in I\!R^n$. So the set in (19) is the same one in (18). □

The above observations yield the following verifiable conditions for well-posedness of CVOPs.

Corollary 4.5. *Assume that $(\mathcal{P})$ is a CVOP, and that*

$$0 \in ri[\ dom\ (f_i + I_C)^*] \qquad \forall i \in [1, m]. \tag{20}$$

Then $F(E_w)$ is closed and $(\mathcal{P})$ is well-posed in the sense of (3).
In addition, if it is further assumed that

$$C^\infty \cap \{x \in I\!R^n \mid f_i^\infty(x) \le 0\} = L^\perp \qquad \forall i \in [1, m], \tag{21}$$

where $L^\perp$ is the orthogonal complement of the subspace $L \subset I\!R^n$, then $F(E_w)$ is bounded.

Proof. We note first that (20) implies (1) holds [18, Th. 27.1(b)]. By the hypothesis and Proposition 4.4, for each $i \in [1, m]$, $C^\infty \cap \{x \in I\!R^n \mid f_i^\infty(x) \le 0\}$ is a subspace. It follows that $C^\infty \cap (\cap_{i=1}^m(\{x \in I\!R^n \mid f_i^\infty(x) \le 0\}))$ is a subspace. By Theorem 4.2 and Lemma 4.1, both $F(C)$ and $F(E_w)$ are closed. This proves the first part. For each $\lambda \in \Lambda$, let $f_\lambda(x) = \lambda^T F(x)$. We claim that

$$0 \in ri\ [dom(f_\lambda + I_C)^*]. \tag{22}$$

To see this, let $I = \{i \in [1, m] \mid \lambda_i > 0\}$. Then $\lambda^T F(x) = \sum_{i \in I} \lambda_i f_i(x)$. Let $d \in I\!R^n$ be a member of the set

$$C^\infty \cap \left\{ x \in I\!R^n \;\middle|\; \sum_{i \in I} \lambda_i f_i^\infty(x) \le 0 \right\}\cdot \tag{23}$$

If there is an $\bar{i} \in I$ such that $f_{\bar{i}}^\infty(d) > 0$, then there is an $\bar{j} \in I$ such that $f_{\bar{j}}^\infty(d) < 0$ by (23). This would imply that $d \in C^\infty \cap \{x \in I\!R^n \mid f_{\bar{j}}^\infty(x) \le 0\}$ and $f_{\bar{j}}^\infty(d) + I_{C^\infty}(d) < 0$, which contradicts the hypothesis that $0 \in ri[dom(f_{\bar{j}} + I_C)^*]$. So we must have $f_i^\infty(d) = 0$ for each $i \in I$. Consequently, for each $i \in I$, by Proposition 4.4, $-d \in C^\infty \cap \{x \in I\!R^n \mid f_i^\infty(x) \le 0\}$ since $0 \in ri[dom(f_i + I_C)^*]$. Therefore, $-d$ is a member of the set in (23). The claim then follows from Proposition 4.4. As well known, (22) implies that the problem: minimize $f_\lambda(x)$ subject to $x \in C$, is well-posed [1,3]. Since the parametric problem $\mathcal{P}(\lambda)$ is well-posed over Λ, $(\mathcal{P})$ is well-posed by Theorem 3.3.

When (21) holds, consider the following CVOP:

$$(\mathcal{Q}) \qquad \text{minimize } (f_1(x), \dots, f_m(x)) \text{ subject to } x \in C \cap L.$$

Denote the optimal solution set of $(\mathcal{Q})$ by $\bar{E}_w$. We claim that

$$C = L^\perp + (C \cap L).$$

For $x \in C$, write $x = x_L + x_{L^\perp}$ with $x_L \in L$ and $x_{L^\perp} \in L^\perp$. By [18, Th. 8.1], $x_L = x - x_{L^\perp} \in C$ since $-x_{L^\perp} \in L^\perp \subset (-C^\infty) \cap C^\infty$. This shows that $C \subset L^\perp + (C \cap L)$ and $C \cap L$ is nonempty. On the other hand, for any $x \in C \cap L \subset C$, by [18, Th. 8.1] again, $x + z \in C$ for all $z \in L^\perp$. This establishes the claim. Since $(C \cap L)^\infty = C^\infty \cap L$,

$$((C \cap L)^\infty \cap \{x \in I\!R^n \mid f_i^\infty(x) \le 0\} = L \cap L^\perp = \{0\} \qquad \forall i \in [1, m].$$

So each $f_i + I_{(C \cap L)}$ is level-coercive. By Theorem 2.1, $\bar{E}_w$ is nonempty and bounded. By the hypothesis, $f_i^\infty(z) \le 0$ and $f_i^\infty(-z) \le 0$ for each $i \in [1, m]$ and $z \in L^\perp$ since $L^\perp$ is contained in the recession cone of f_i. It follows from [18, Cor. 8.1] that

$$f_i(x + z) = f_i(x) \quad \forall i \in [1, m], \forall x \in I\!R^n, \forall z \in L^\perp.$$

So $E_w = \bar{E}_w + L^\perp$, and $F(E_w) = F(\bar{E}_w)$ is bounded. □

We note that condition (20) does not imply that $F(E_w)$ is bounded. In particular (20) does not imply (21). The following example illustrates this:

Example 4.6. Let $m = 2$, $n = 2$, $C = I\!R^2$, $f_1(x) = x_1^2$, and $f_2(x) = x_2^2$. Then $f_1^\infty(u) = \delta_{\{0\} \times I\!R}(u)$ and $f_2^\infty(u) = \delta_{I\!R \times \{0\}}(u)$. Clearly $E_w = (\{0\} \times I\!R) \cup (I\!R \times \{0\})$, and $F(E_w) = (\{0\} \times I\!R_+) \cup (I\!R_+ \times \{0\})$.

We also note that (21) can not be weaken to a general subspace assumption on the recession cone of E_w to ensure the boundedness of $F(E_w)$. Consider the following example:

Example 4.7. Let $C = R^2$, and $F(x) = (f_1(x), f_2(x))$ where $f_1(x_1, x_2) = x_1^2 + x_2^2$, and $f_2(x_1, x_2) = x_1^2$. Then $E_w = E_w^\infty = \{0\} \times R$, which is a subspace. But $F(E_w) = R_+ \times \{0\}$.

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