

**DIFFERENTIAL APPROXIMATION
OF NP-HARD PROBLEMS
WITH EQUAL SIZE FEASIBLE SOLUTIONS**

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Abstract. In this paper, we focus on some specific optimization problems from graph theory, those for which all feasible solutions have an equal size that depends on the instance size. Once having provided a formal definition of this class of problems, we try to extract some of its basic properties; most of these are deduced from the equivalence, under differential approximation, between two versions of a problem π which only differ on a linear transformation of their objective functions. This is notably the case of maximization and minimization versions of π , as well as general minimization and minimization with triangular inequality versions of π . Then, we prove that some well known problems do belong to this class, such as special cases of both spanning tree and vehicles routing problems. In particular, we study the strict rural postman problem (called *SRPP*) and show that both the maximization and the minimization versions can be approximately solved, in polynomial time, within a differential ratio bounded above by $1/2$. From these results, we derive new bounds for standard ratio when restricting edge weights to the interval $[a, ta]$ (the *SRPP* $[t]$ problem): we respectively provide a $2/(t+1)$ - and a $(t+1)/2t$ -standard approximation for the minimization and the maximization versions.

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1. INTRODUCTION

Routing design problems are of major importance in combinatorial optimization, and related works have been numerous during the last twenty years (see Christofides [7], Fisher *et al.* [18], Kosaraju *et al.* [35], Engebretsen and Karpiniski [17]). The purpose of this paper is to study a class of graph-theoretic problems, called *FGNPO*², including *Depth-constrained Spanning Tree* and *Traveling Salesman* problems. The feasible solutions of each problem of *FGNPO* are subsets of edges having an equal size (depending on the instance), and the cost of a solution is the sum of the weights on its edges. For example, the minimum spanning tree problem is typically an *FGNPO* problem: each tree has exactly $n-1$ edges and the cost of a solution is the sum of the weights on the tree's edges. On the other hand, the minimum Steiner tree problem is not in *FGNPO* since the number of edges in a Steiner tree is not uniquely determined by the input size. Other close problem families (sharing the fixed solution size property) have been widely studied over the past ten years; for instance, the problems of finding *minimum-cost subsets of a combinatorial structure and a given size* are the most common (see Aggarwal *et al.* [1], Ravi *et al.* [46]). In particular, concerning the problem *Min k - MST*, the performance ratio has been recently improved to the best possible for general graphs and Euclidean graphs (see Garg [21] and Mitchell [38]). Also problems of finding *subsets maximizing minimum structures* such as *Remote k - MST* and *Remote k - TSP* have been studied by Halldorsson *et al.* [26].

We observe an important fact on these problems: the fixed-size property of solutions induces an homogeneous complexity between maximization and minimization versions since both versions of each problem are equivalent from an optimization point of view (*i.e.*, they are Karp-reducible [32]).

We focus on the design of approximation algorithms with guaranteed performance ratios. Usually, one compares the worst-case ratio (here called standard ratio) of the cost of the solution generated by the algorithm to the optimal cost. However, we mainly refer in this article to another ratio called *differential ratio* which measures the worst ratio of, on the one hand, the difference between the cost of the solution generated by the algorithm and the worst cost, and on the other hand, the difference between the optimal cost and the worst cost. This measure, studied by Aiello *et al.* [2], Ausiello *et al.* [6], Fisher *et al.* [18], Vavasis [48] (in the context of non-linear programming), Zemel [49] and more recently by Demange and Paschos [15] and Hassin and Khuller [27], leads to new algorithms taking into account the extreme solutions of the instance, and provides the opportunity to better understand these problems. In this paper, we show that there are tight links between both measures for the studied problems. Most of the positive

²Fixed-size-solution Graph NPO.

approximation results are obtained by getting, for each problem, several feasible solutions x_1, \dots, x_q and by choosing the best one among them; each of these individual solutions has a differential approximation ratio tending towards zero when the instance size increases. In other words, for each solution x_i , there exists a subset of instances S_i (called *Family of critical instances* in Monnot [39]) illustrating this worst-case behavior. Our approximate algorithms have the property that, for each instance in S_i , there is another solution x_j which has a good approximation ratio. These notions of families of critical and strongly critical instances are well known in the polynomial approximation community, but they have never been finalized until now. However in Monnot [39], they have been largely studied and commented.

We now give some standard definitions in the field of optimization and approximation theory. For a more detailed statement of this theory, we refer the reader to Ausiello *et al.* [4, 5], Hochbaum [29].

Definition 1.1. An *NP-optimization problem* (*NPO-problem*) π is a five-tuple $(D, sol, m, Triv, goal)$ such that:

- (i) D is the set of instances and is recognizable in polynomial-time;
- (ii) given an instance $I \in D$, $sol[I]$ is the set of feasible solutions of I ; moreover, there exists a polynomial P such that, for any $x \in sol[I]$, $|x| \leq P(|I|)$; furthermore, it is decidable in polynomial time whether $x \in sol[I]$ for any I and for any x such that $|x| \leq P(|I|)$. Finally, there is a feasible solution $Triv(I)$ ³ computable in polynomial-time for any I ;
- (iii) given an instance I and a solution x of I , $m[I, x]$ denotes the positive integer value of x . The function m is computable in polynomial time and is also called the objective function;
- (iv) $goal \in \{\text{Max}, \text{Min}\}$.

We call \overline{goal} ⁴ the complementary notion of goal and $\overline{\pi}$ the *NPO-problem* $(D, sol, m, Triv, \overline{goal})$. The goal of an *NPO-problem* with respect to an instance I is to find an *optimum solution* x^* such that $OPT(I) = m[I, x^*] = goal\{m[I, x] : x \in sol[I]\}$. Another important solution of π is a *worst solution* x_* defined by: $WOR(I) = m[I, x_*] = \overline{goal}\{m[I, x] : x \in sol[I]\}$. A worst solution for π is an optimal solution for $\overline{\pi}$ and *vice versa*. In Ausiello *et al.* [6], the term *trivial solution* referred to as *worst solution* and all the exposed examples have the property that a worst solution can be trivially computed in polynomial-time. For example, this is the case of the maximum cut problem where, given a graph, the worst solution is the empty edge-set, and the bin-packing problem where we can trivially put the items using a distinct bin per item. It is not the case however in this paper, where the computation of a worst solution is far from being trivial.

³The common definition of class *NPO* does not require the existence of a trivial solution.

⁴If $goal = \text{Max}$, then $\overline{goal} = \text{Min}$ and $\overline{\overline{goal}} = \text{Max}$.

1.1. APPROXIMATE ALGORITHMS AND REDUCTIONS

In order to study algorithm performances, there are two known measures: *standard ratio*, see Garey and Johnson [20], Ausiello *et al.* [5], Crescenzi and Kann [9] and *differential ratio*, see Demange and Paschos [15], Ausiello *et al.* [6], Cornuejols *et al.* [8], Zemel [49].

Definition 1.2. Let π be an *NPO* problem and x be a feasible solution of an instance I . We define the performance ratios of x with respect to the instance I as

- **(standard ratio)** $\rho[\pi](I, x) = \text{Min} \left\{ \frac{m[I, x]}{OPT(I)}, \frac{OPT(I)}{m[I, x]} \right\}$
- **(differential ratio)** $\delta[\pi](I, x) = \frac{WOR(I) - m[I, x]}{WOR(I) - OPT(I)}$.

The performance ratio is a number less than or equal to 1, and is equal to 1 if and only if $m[I, x] = OPT(I)$. Remark that we have inverted the usual standard performance ratio in the case of minimization problems so that the ratio value is always between 0 and 1. Let π be an *NPO* problem. For any instance I of π , a polynomial time algorithm A returns a feasible solution x^A . The performance of A with respect to $R \in \{\delta, \rho\}$ on the instance I is the quantity $R_A[\pi](I) = R[\pi](I, x^A)$. We say that A is an *r-approximation algorithm* with respect to R if for any instance I , we have $R_A[\pi](I) \geq r$.

Definition 1.3. For any performance ratio $R \in \{\delta, \rho\}$,

- an *NPO* problem belongs to the class $APX(R)$ if there exists an *r-approximation algorithm* with respect to R for some constant $r \in]0; 1]$;
- an *NPO* problem belongs to the class $PTAS(R)$ if there exists an *r-approximation algorithm* A_r for any constant $r \in]0; 1[$. The family $\{A_r\}_{0 < r < 1}$ is said to be an *approximation scheme*;
- an *NPO* problem belongs to the class $FPTAS(R)$ where $R \in \{\delta, \rho\}$ if there exists a *r-approximation algorithm* with respect to R , for any constant $r \in]0; 1[$ and the time-complexity of this algorithm is $O[P(|I|; \frac{1}{1-r})]$ where P is a polynomial of two variables.

Clearly, the following inclusions hold for any performance ratio $R \in \{\delta, \rho\}$: $FPTAS(R) \subseteq PTAS(R) \subseteq APX(R)$. As it is usually done, we will denote by APX , $PTAS$ and $FPTAS$, respectively, the classes $APX(\rho)$, $PTAS(\rho)$ and $FPTAS(\rho)$. As shown in Demange *et al.* [12], many problems such as the Vertex Covering or the Dominating Set problems can have different behavior patterns depending on whether the differential or standard ratio is chosen. On the other hand, for several problems we may establish some connections between the differential and the standard ratios, like Bin Packing (see Demange *et al.* [13]) or Steiner Tree

problem with distance one and two (see Demange *et al.* [14]) and see Zemel [49] for motivations and complementarity links between the two measures. Besides for the problems that we consider here, we show that there are tight links between the two measures when the ratio d_{\max}/d_{\min} is upper bounded (where d_{\max} and d_{\min} denote respectively the maximum and the minimum cost of the edges).

Consider the following approximation preserving reductions between pairs (π, R) :

Definition 1.4. For $\pi_i \in NPO$ and $R_i \in \{\delta, \rho\}$, $i = 1, 2$,

- an *A-reduction* from the pair (π_1, R_1) to the pair (π_2, R_2) , denoted by $(\pi_1, R_1) \leq^A (\pi_2, R_2)$, is a triplet (α, f, c) such that:
 - (i) $\alpha: D_{\pi_1} \mapsto D_{\pi_2}$, transforms an instance of π_1 into an instance of π_2 in polynomial time;
 - (ii) $f: sol_{\pi_2}[\alpha(I)] \mapsto sol_{\pi_1}[I]$, transforms solutions for π_2 into solutions for π_1 in polynomial time;
 - (iii) $c: [0; 1] \mapsto [0; 1]$ (called *expansion* of the A-reduction) is a function verifying $c^{-1}(0) \subseteq \{0\}$, and $\forall \varepsilon \in [0; 1], \forall I \in D_{\pi_1}, \forall x \in sol_{\pi_2}[\alpha(I)]: R_2[\pi_2](\alpha(I), x) \geq \varepsilon \implies R_1[\pi_1](I, f(x)) \geq c(\varepsilon)$;
- an *A * P-reduction* from the pair (π_1, R_1) to the pair (π_2, R_2) , denoted by $(\pi_1, R_1) \leq^{A*P} (\pi_2, R_2)$, is an *A-reduction* from (π_1, R_1) to (π_2, R_2) such that the restriction of function c to an interval $[a; 1]$ is bijective and $c(1) = 1$ (note that c does not necessarily verify $c^{-1}(0) \subseteq \{0\}$).

Definition 1.5. If $(\pi_1, R_1) \leq^{A*P} (\pi_2, R_2)$ and $(\pi_2, R_2) \leq^{A*P} (\pi_1, R_1)$ with $c(\varepsilon) = \varepsilon$, we say that (π_1, R_1) is *equivalent* to (π_2, R_2) .

The *A-reduction* preserves constant ratio approximation while the *A * P-reduction* preserves approximation schemes. They are natural generalizations of reductions described by Orponen and Mannila [45], Ausiello *et al.* [5], Crescenzi and Panconesi [10].

The differential ratio measures how the value of an approximate solution is located in the interval between $WOR(I)$ and $OPT(I)$ whereas for a maximization problem, the standard ratio measures how the value of an approximate solution is placed in the interval between 0 and $OPT(I)$. Hence in this case, we have an *A * P-reduction* from the standard ratio to the differential ratio:

Lemma 1.6. *If $\pi = (D, sol, m, Triv, Max) \in NPO$, then $(\pi, \rho) \leq^{A*P} (\pi, \delta)$ with $c(\varepsilon) = \varepsilon$.*

Proof. Let I be an instance of π and x be a feasible solution. If $m[I, x] \geq \varepsilon OPT(I) + (1 - \varepsilon)WOR(I)$ then $m[I, x] \geq \varepsilon OPT(I)$ since $WOR(I) \geq 0$. \square

Remark that, in general, there is no evident transfer of a positive or negative result from one framework to the other for a minimization problem. For instance, we have proved in Demange *et al.* [11] that the best achievable standard ratio is

7/8 for a version of weighted minimum coloring in bipartite graphs whereas we have built a differential approximation scheme; in this coloring version, the cost of a stable set is given by the maximum of the vertex weights in this stable set.

The rest of this paper is organized as follows. In Section 2, we give a general definition of the *FGNPO* class and some elementary properties are presented. In Section 3, we study the approximation of *depth-constrained Spanning Tree problems*. In Section 4, we give approximation results of *strict rural postman problems* and some results are given for derived problems, namely the *traveling salesman*, *Hamiltonian path with specified endpoints* and *clustered traveling salesman*. Finally, in Section 5, we conclude by some prospects in order to obtain negative differential results on the *FGNPO* class.

2. THE *FGNPO* PROBLEMS

We study a class of graph *NPO*-problems having the property that for any couple (x, y) of feasible solutions, the size of x and y are the same (*i.e.* $|x| = |y|$); the feasible solutions are subsets of edges of equal cardinality and the cost of a solution is the sum of the weights on its edges. Many well-known *NP*-hard problems are in this class as the Traveling Salesman problem or tree-problems with additional constraints (see Magnanti and Wolsey [36]) but also all the problems of partitioning into a given sub-graph (an approximation algorithm is given in Hassin and Rubinstein [28] when the sub-graph is 3-edge path). The latter problems called *H-partition* in Kirkpatrick and Hell [33], can be formulated as follows: a *H-partition* of a complete graph K_n with edges valued by a function d is a set of sub-graphs (G_1, \dots, G_k) where $G_i = (V_i, E_i)$ such that each G_i is isomorphic to H and the set(s) V_1, \dots, V_k partition V ; the goal is to optimize $\sum d(E_i)$.

Formally, we can define these problems in the following way:

Definition 2.1. Let π be an *NPO* problem; π is called *FGNPO*⁵ if it satisfies the following properties:

- (i) an instance I is a complete graph $K_n = (V, E)$, edge-valued by a function $d : E \rightarrow \mathbb{N}$;
- (ii) a feasible solution is a subset $E' \subseteq E$ of edges such that $pred(E')$ is true, where $pred$ is a suitable polynomial-time decidable predicate. Moreover, there exists an integer function g such that $|E'| = g(|V|)$;
- (iii) the objective function verifies $m[I, E'] = \sum_{e \in E'} d(e)$.

If $goal = \text{Max}$, the problem is called $\text{Max } \pi$, else $\text{Min } \pi$. We use notation π with no prefix when we consider without distinction the case $goal = \text{Max}$ or $goal = \text{Min}$. Finally, $\pi[t]$ for some $t > 1$ indicates the sub-problem verifying $d_{\max} \leq td_{\min}$ where

⁵Fixed-size-solution Graph *NPO*.

$d_{\max} = \max_{e \in E} d(e)$ and $d_{\min} = \min_{e \in E} d(e)$. For sake of simplicity, we identify π with $\pi[\infty]$.

The *FGNPO*-problems verify the following properties:

For any $t > 1$, $\pi[t] \in \text{FGNPO}$ is *NP-hard* if and only if so is $\overline{\pi[t]}$ where we remember that $\bar{\pi} = (D, S, m, \text{Triv}, \text{Min})$ when $\pi = (D, S, m, \text{Triv}, \text{Max})$ and $\bar{\bar{\pi}} = \pi$; in other words, computing a worst solution of $\pi[t]$ is as hard as computing an optimal one. When $\pi[t]$ is *NP-hard* for some constant $t > 1$, $\text{Max } \pi[t]$ and $\text{Min } \pi[t]$ are trivially in *APX* since any solution is at least a $1/t$ -standard approximation; in this case, the standard ratio may not be very meaningful since even a worst solution yields a constant standard approximation. On the other hand, when t tends towards infinity (in other words, we are interested in the general problem), the standard approximation results are very different for the two versions (maximization and minimization); $\text{Max } \pi$ seems to be usually in *APX* (we do not know a problem $\pi \in \text{FGNPO}$ such that $\text{Max } \pi \notin \text{APX}$) whereas generally, the problems $\text{Min } \pi$ are not in *APX*. This last assertion is a natural generalization which comes from the well-known result established by Sahni and Gonzalez [47] for *TSP*: if $\text{Min } \pi \in \text{APX}$, then the following decision problem is polynomial. Given a simple graph $G = (V, E)$, does $E' \subseteq E$ exist such that $|E'| = g(|V|)$ and $\text{pred}(E')$? (we call it the decision problem associated to π).

Proposition 2.2. *Let π be an FGNPO problem. If the decision problem associated to π is NP-Complete then $\text{Min } \pi$ is not in APX unless $P = NP$.*

Proof. The proof of this claim is similar to the one in Sahni and Gonzalez [47]. Suppose that the algorithm A yields a ε -standard approximation for $\text{Min } \pi$. We could decide if a graph G verifies the property P by the following way: from $G = (V, E)$, we build the instance $I = (K_n, d)$ of π by taking $d(e) = 1$ if $e \in E$ else $d(e) = \lceil \frac{g(|V|)}{\varepsilon} \rceil$ and we evaluate the quantity $A(I)$; we affirm that G verifies P if and only if $A(I) \leq \frac{g(|V|)}{\varepsilon}$. Indeed, if the graph G verifies P then by construction $\text{OPT}(I) = g(|V|)$ and $A(I) \leq \frac{\text{OPT}(I)}{\varepsilon} = \frac{g(|V|)}{\varepsilon}$. Conversely, if the graph G does not verify P , then $A(I) \geq \text{OPT}(I) > \frac{g(|V|)}{\varepsilon}$ since we can easily suppose that $g(n) \geq 2$. □

We deduce that $\text{Min } \text{TSP}$, $\text{Min } \text{HPP}_{s,p}$ ⁶, $\text{Min } \text{H-Partitioning}$ are not in *APX* unless $P = NP$, because the associated decision problems are *NP-Complete*. The reverse is not true since the decision problem associated to $\text{Min } k\text{-depthSTP}_r$ ⁷ is polynomial and $\text{Min } k\text{-depthSTP}_r \notin \text{APX}$.

This asymmetry in the approximability of both versions can be considered as somewhat strange given the structural symmetry existing between them. Since

⁶Hamiltonian Path with specified endpoints s and p .

⁷ $k\text{-depthSTP}_r$ consists in finding a Tree T where the length of any path in T from r is at most k .

differential approximation is stable under affine transformation of the objective function, $\text{Max } \pi$ and $\text{Min } \pi$ are differential-equivalent⁸. Besides, another difference with the standard ratio is that π is not more difficult than the same problem with triangular inequality (called *metric* π). Finally, for the restriction where the weight function is bivalued (called $\pi(a, b)$), we can always boil down to the case $a = 1$ and $b = 2$.

Proposition 2.3. *For a problem $\pi \in \text{FGNPO}$, the following assertions hold:*

- (i) *for any $t \in]1; \infty]$, $\text{Min } \pi[t]$ is differential-equivalent to $\text{Max } \pi[t]$;*
- (ii) *π is differential-equivalent to metric π ;*
- (iii) *$\pi(a, b)$ ⁹ is differential-equivalent to $\pi(1, 2)$.*

Proof. • For (i): Let $I = (G, d)$ be an instance of $\pi[t]$, we transform I into instance $\alpha(I) = (G, d')$ of $\bar{\pi}$ defined by: $\forall e \in E, d'(e) = d_{\max} + d_{\min} - d(e)$. We observe that $\alpha(I)$ is also an instance of $\pi[t]$. We obtain for any feasible solution E' : $m[\alpha(I), E'] = g(|V|)(d_{\max} + d_{\min}) - m[I, E']$. Since $g(|V|)(d_{\max} + d_{\min})$ is constant, the differential ratio is the same for $\pi[t]$ and $\bar{\pi}[t]$ (which also holds for the proofs of (ii) and (iii)).

- For (ii), the proof is similar, except that function d' is now defined by $d'(e) = d_{\max} + d(e)$. We remark that d' satisfies the triangle inequality and we have for any solution E' : $m[\alpha(I), E'] = g(|V|)d_{\max} - m[I, E']$.
- Finally, for (iii), we consider $\alpha(I)$ of $\pi(1, 2)$ defined by: $\alpha(I) = (G, d')$ with $d'(e) = 1$ iff $d(e) = a$. We obtain for any solution E' : $m[\alpha(I), E'] = \frac{m[I, E'] - (2a - b)g(|V|)}{b - a}$.

□

All the conditions defining the *FGNPO* class are important for establishing the previous properties, not only the fixed-size conditions. Some problems with fixed-size solutions are not in *FGNPO* and do not verify the differential-equivalence properties of Proposition 2.3. For example, $\text{Min } k - \text{MST}$ (see Ravi *et al.* [46]) consisting in computing a minimum spanning tree among those having $k - 1$ edges, is a polynomial problem when the edge weights are bi-valued (called $\text{Min } k - \text{MST}(a, b)$) whereas its maximization version $\text{Max } k - \text{MST}(a, b)$, also called *Remote* $k - \text{MST}(a, b)$ in Halldorsson *et al.* [26] has been proved *NP-hard*.

There are some tight links for these problems between differential and standard ratios. For example, most of these problems are strongly *NP-hard* and cannot have a fully polynomial time differential or standard approximation scheme unless $P = NP$. Moreover, when we deal with the case $d_{\max} \leq td_{\min}$ (a case that seems to be very common in practical applications) we can establish a bridge between

⁸See Definition 1.5.

⁹Restriction where all edge weights are a or b .

differential and standard ratios:

Theorem 2.4. *If $\pi \in FGNPO$, then for any $t \in]1; \infty[$, $(goal \ \pi[t], \rho) \leq^{A^*P} (goal \ \pi[t], \delta)$ with the expansion verifying:*

$$c(\varepsilon) = \frac{(t-1)\varepsilon + 1}{t} \text{ if } goal = \text{Max} \qquad c(\varepsilon) = \frac{1}{t - (t-1)\varepsilon} \text{ if } goal = \text{Min}.$$

Proof. We only demonstrate the Max $\pi[t]$ case. Let I be an instance and x be a feasible solution. If $m[I, x] \geq \varepsilon OPT(I) + (1 - \varepsilon)WOR(I)$ then $m[I, x] \geq c(\varepsilon)OPT(I)$ since $WOR(I) \geq \frac{1}{t}OPT(I)$. \square

Observe that if t tends towards infinity, we find again the Lemma 1.6 for $goal = \text{Max}$ and problems of $FGNPO$, whereas for $goal = \text{Min}$, the expansion is $c(\varepsilon) = 0$ (in other words, we can not transfer a positive differential result to a standard result by this way). When $\pi[t]$ is APX -complete for some $t \in]1; \infty[$, we can deduce from the previous theorem that the hardness thresholds for standard and differential framework are identical.

Corollary 2.5. *If $\pi[t] \in FGNPO$ and is APX -complete for some $t \in]1; \infty[$, then $\pi \notin PTAS(\delta)$ unless $P = NP$.*

3. THE DEPTH-CONSTRAINED MINIMUM SPANNING TREE

In this section we consider the f -depth constrained spanning tree problem also called the *Hop tree problem* in Gouveia [23], Gouveia and Janssen [24] or the *Shallow-Light spanning tree problem* in Naor and Schieber [43], Kortsarz and Peleg [34], formally defined as follows. Let f be an integer function, we suppose w.l.o.g. that $f(n)$ is always an integer between 2 and n where $n = |V|$.

Definition 3.1. Consider a complete graph K_n and $r \in V$ with non-negative cost $d(e)$ for each edge $e \in E$. The problem of finding an optimal-cost spanning tree T with root r such that the depth¹⁰ of T is at most $f(|V|)$ is called f -depthSTP $_r$. The restriction of f -depthSTP $_r$ when f is constant (i.e., $f(n) = k \ \forall n$), is called k -depthSTP $_r$.

The Min k -depthSTP $_r$ problem has been extensively studied by Gouveia [22–24] and arises in many applications such as telecommunication network design and facility location. It is NP -hard for any $f(n) = k \geq 2$, even if the distances are one and two (see Alfandari and Paschos [3], Manyem and Stallmann [37]) and it is generally not in APX . For $k = 2$, this problem is $O(1/\ln n)$ -approximable under the standard ratio and is equivalent to a version of the single-source uncapacitated facility location problem whereas its restriction with distances one and two is

¹⁰The depth of a tree T with root r is the maximum number of edges of a path in T with endpoint r .

4/5-standard approximable and APX-complete (see Alfandari and Paschos [3]). Moreover, it has been proved by Kortsarz and Peleg [34] that $\text{Min } k\text{-depthSTP}_r$ is $O(1/\ln n)$ -standard approximable for every constant $k \geq 2$ and $\text{Min } f\text{-depthSTP}_r$ is $O(1/n^\epsilon)$ -standard approximable for any fixed $0 < \epsilon < 1$.

To our knowledge, no standard approximation result has been established for $\text{Max } f\text{-depthSTP}_r$, even when the depth does not depend on the graph order. Nevertheless, when the graph is directed, the bound $(\liminf f - 1)/\liminf f$ has been proved in Monnot [40]. So, the following result is presented for the sake of illustration.

We show that $\text{Max } f\text{-depthSTP}_r$ is $(\liminf f - 1)/\liminf f$ -approximable in the differential framework. Then, we deduce the same results for standard ratio.

3.1. THE ALGORITHM

For a spanning tree T with root r of $G = (V, E)$, let p be the “father” function (i.e., $p(v)$ is the vertex adjacent to v in the unique path connecting v to r in T) of the tree and let (V_0, V_1, \dots, V_q) be the partition of V where $V_i = \{v : r = p^i(v)\}$ is the i -th level vertices subset and q is the depth of T . So, we have in particular $V_0 = \{r\}$ and all vertices of V_q are leaves of T . In order to keep it simple, we suppose that q is a multiple of $f(n)$ (we could always add some $V_i = \emptyset$). Finally, let T_0 be the tree defined by $T_0 = \cup_{v \in V \setminus \{r\}} \{(r, v)\}$, which has a depth equal to 1. The main idea of the following algorithm consists in discarding some edges of T and connecting each resulting tree to the root r in order to obtain a tree with depth $f(n)$.

[shorteningtree_f]

input : A complete graph $K_n = (V, E)$ with $r \in V$, edge-valued by d

output : A tree sol_1 of depth at most $f(n)$ and root r ;

 Compute a maximum weight spanning tree T^* ;

 Build (V_0, V_1, \dots, V_q) the partition associated with T^* ;

 For $i = 1$ to $f(n)$ do

$$V^i = \cup_{j \leq q/f(n)-1} V_{j f(n)+i};$$

$$\text{build } T_i = T^* \cup \{(v, r) : v \in V^i\} \setminus \{(v, p(v)) : v \in V^i\};$$

 End for i

$$sol_1 = \text{argmax}\{d(T_i), i = 1, \dots, f(n)\}.$$

It is easy to see that for any $i \leq f(n)$, the tree T_i is a feasible solution and that the time-complexity of this algorithm is polynomial with respect to the encoding of the input since $f(n) \leq n$.

Theorem 3.2. *The algorithm [shorteningtree_f] is a $\frac{\liminf f - 1}{\liminf f}$ -differential approximation for $\text{Max } f\text{-depthSTP}_r$ and this ratio is tight.*

Proof. Let $I = (K_n, d)$ be an instance of Max f -depthSTP $_r$. We just have to consider the following equality:

$$\sum_{i \leq f(n)} d(T_i) = (f(n) - 1)d(T^*) + d(T_0) \tag{3.1}$$

and observe that solution sol_1 satisfies

$$f(n)d(sol_1) \geq \sum_{i \leq f(n)} d(T_i). \tag{3.2}$$

On the other hand, regarding an optimal solution, we have:

$$OPT(I) \leq d(T^*) \tag{3.3}$$

and for a worst solution

$$WOR(I) \leq d(T_0). \tag{3.4}$$

Thus, using (3.1–3.3) and (3.4), we obtain the expected result since we are interested here only in asymptotic bounds.

Let us show that this ratio is tight: consider the following instance $I_n = (K_n, r, d)$ such that the sub-graph induced by $V \setminus \{r\}$ is a complete graph of $n \liminf f$ vertices and the edges-weights are two. Moreover, $d(r, v) = 1, \forall v \in V \setminus \{r\}$. So, T^* is a Hamiltonian path from r to an arbitrary vertex v and: $d(sol_1) = d(T_1) = 2n \liminf f - n$, $WOR(I_n) = n \liminf f$, $OPT(I_n) = 2n \liminf f - 1$. Hence, we obtain $\delta_{shorteningtree_f}(I_n) \longrightarrow \frac{\liminf f - 1}{\liminf f}$. \square

For the standard ratio, we deduce new improved results from Theorem 3.2 and Theorem 2.4.

Corollary 3.3. *We have the following results:*

- Max f -depthSTP $_r[t]$ is $\frac{t(\liminf f - 1) + 1}{t \liminf f}$ -standard approximable;
- Min f -depthSTP $_r[t]$ is $\frac{\liminf f}{\liminf f + (t - 1)}$ -standard approximable.

Notice that, if $\liminf f$ is infinite, then f -depthSTP $_r \in PTAS(\delta)$, Max f -depthSTP $_r \in PTAS$ and Min f -depthSTP $_r[t] \in PTAS$ for any $t \in]1; \infty[$. In the paper Monnot [40], we improved the differential ratio to 3/4 for the sub-problem 2-depthSTP $_r(1, 2)$ (the edge-weights are one or two and $f(n) = 2, \forall n$) and we have derived a 4/5-standard approximation for Min 2-depthSTP $_r(1, 2)$ and a 7/8-standard approximation for Max 2-depthSTP $_r(1, 2)$.

4. THE STRICT RURAL POSTMAN PROBLEM

In this section we study routing problems with additional constraints on edges. Usually the traveling salesman problem is defined as finding a tour

(i.e., Hamiltonian cycle) of minimal length, that a salesman can take to visit n cities, each exactly once and returning to his starting city whereas the rural postman problem requires that some specified edges are traversed at least once in a solution. Hence, in the latter case the feasible solutions are not necessarily Hamiltonian cycles; it depends on the structure of the specified edges. In our version any solution must be a tour and then, we can assume that the predefined edges belong to vertex disjoint paths; indeed without loss of generality, we can suppose that the paths are shrunk to a matching. This problem called *Strict Rural Postman problem* and shortly noted *SRPP*, is a sub-case of the one described by Orloff [44] and models the problem of mail-carrier where each edge corresponds to a street along which the mail must be delivered. More generally, *the rural postman problem* underlies applications in contexts where roads have to be traversed for maintenance, garbage collection, school bus transportation, electrical lines and gas mains inspection, etc. like indicated in Eiselt *et al.* [16]. From a standard measure point of view, the metric *Min SRPP* is $2/3$ -standard approximable and the approximation algorithm is obtainable by using an algorithm similar to the Christofides's one [19]. To our knowledge, no standard approximation result exists for *Max SRPP*.

When using the differential measure, the expected results will be less optimistic, this problem containing several other particular problems which are difficult to approximate. Among those, we mention the *Hamiltonian path with specified endpoints problem* (denoted by *HPP_{s,p}*), the *traveling salesman problem* (in short *TSP*) and the *clustered traveling salesman problem*¹¹ (called *CTSP_{s_i,p_i}*); hence from this point of view, the problem seems to be a central routing problem. We show that this problem can be approximated with a differential ratio bounded above by $1/2$ and like previously, we improve the standard bounds for the case where all the edge weights are within an interval $[a, ta]$. Hence, we will deduce that *Min CTSP_{s_i,p_i}[t]*, *Min HPP_{s,p}[t]* are $2/(t+1)$ -standard approximable, and *Max CTSP_{s_i,p_i}[t]*, *Max HPP_{s,p}[t]* are $(t+1)/2t$ -standard approximable.

The problem is formally defined as follows:

Definition 4.1. Consider a complete graph $K_n = (V, E)$ with non-negative cost $d(e)$ for each edge $e \in E$ and a matching $E' \subset E$. The problem is to compute an optimal-cost Hamiltonian cycle of K_n containing E' . This problem is called *Strict Rural postman* and noted *SRPP*.

We exploit the structural properties of solutions to approximate this problem. The principle of our algorithm is to generate two feasible solutions following the method consisting in finding a maximum weight 2-matching among those containing E' , then discarding some edges (not belonging to E') and arbitrarily connecting

¹¹This problem consists in computing a tour of K_n in which the vertices of each cluster V_i are visited consecutively from s_i to p_i where V is split into V_1, \dots, V_k .

the resulting paths to form a Hamiltonian cycle. In order to build an optimal 2-matching M containing E' , we substitute $|V|d_{\max} + 1$ for the cost of each edge of E' and we compute a maximum 2-matching M in this new graph. We suppose that M contains cycles Γ_i , $i = 1, \dots, p$ and each cycle Γ_i has at least an edge $e_i \in E'$ (otherwise we take arbitrarily an edge and consider like belonging to E'). Formally, for each cycle Γ_i , we always consider four consecutive vertices $x_1^i, x_2^i, x_3^i, x_4^i$, such that $(x_2^i, x_3^i) \in E'$. Indeed, if $|\Gamma_i| = 3$ then $x_4^i = x_1^i$ and we must have in any case $(x_1^i, x_2^i) \notin E'$ and $(x_3^i, x_4^i) \notin E'$.

[Patching.SRP]

input: A complete graph K_n edge valued by d and a matching E' ;

output : A Hamiltonian cycle SOL of G containing E' ;

Change the cost of $e \in E'$ by $|V|d_{\max} + 1$. Call d' this new weight function;

Compute a maximum weight 2-matching $M = \{\Gamma_i, i = 1, \dots, p\}$ of (G, d') ;

if $p = 1$ then $SOL = M$ and exit;

build $S_1 = \cup_{j=1}^p \{(x_1^j, x_2^j)\}$;

build $S_2 = \cup_{j=1}^p \{(x_3^j, x_4^j)\}$;

if p is even then

$$sol_1 = [M \setminus S_1] \cup \{(x_2^{p-1}, x_2^p), (x_1^p, x_1^1)\} \cup_{j=1}^{(p-2)/2} \{(x_2^{2j-1}, x_2^{2j}), (x_1^{2j}, x_1^{2j+1})\};$$

$$sol_2 = [M \setminus S_2] \cup \{(x_4^1, x_3^p), (x_4^2, x_3^{p-1})\} \cup_{j=1}^{p-2} \{(x_3^j, x_4^{j+2})\};$$

End if;

if p is odd then

$$sol_1 = [M \setminus S_1] \cup \{(x_2^p, x_1^1)\} \cup_{j=1}^{(p-1)/2} \{(x_2^{2j-1}, x_2^{2j}), (x_1^{2j}, x_1^{2j+1})\};$$

$$sol_2 = [M \setminus S_2] \cup \{(x_4^1, x_3^{p-1}), (x_4^2, x_3^p)\} \cup_{j=1}^{p-2} \{(x_3^j, x_4^{j+2})\};$$

End if;

$$SOL = argmax\{d(sol_1), d(sol_2)\}.$$

This algorithm is polynomial and we can easily observe that the solutions sol_1 and sol_2 are Hamiltonian cycles containing E' .

Theorem 4.2. *The algorithm [Patching.SRP] is a $\frac{1}{2}$ -differential approximation for Max SRPP and this ratio is tight.*

Proof. Let $I = (K_n, E', d)$ be an instance and let SOL^* be an optimal Hamiltonian cycle containing E' . We denote $loss_i$, $i = 1, 2$, the quantity $d(sol_i) - d(M)$. Obviously, $loss_i \leq 0$ and we have

$$d(SOL) \geq d(M) + \frac{1}{2}(loss_1 + loss_2). \tag{4.1}$$

Moreover, the following structural property holds:

$$SOL_* = \cup_{j=1,2}(sol_j \setminus M) \cup M \setminus (S_1 \cup S_2) \text{ is a Hamiltonian cycle containing } E'. \tag{4.2}$$

Hence, SOL_* is a feasible solution of our problem and approximates well the value of a worst solution. Indeed, when sol_i is better than sol_{3-i} then from sol_{3-i} we can build a tour (the solution SOL_*) which is relatively worst. The cost of this solution is $d(M) + loss_1 + loss_2$ since we have $d(sol_j) = loss_j + d(M)$ and $d(M \setminus (S_1 \cup S_2)) = d(M_{r^*}) - d(S_1) - d(S_2)$. Thus, we deduce

$$WOR(I) \leq d(M) + loss_1 + loss_2. \tag{4.3}$$

Since SOL_* is a particular 2-matching containing E' , we have:

$$OPT(I) \leq d(M). \tag{4.4}$$

Finally, combining (4.1, 4.3) and (4.4) we obtain:

$$d(SOL) \geq \frac{1}{2}WOR(I) + \frac{1}{2}OPT(I).$$

To show that the bound is tight, consider the following instances $I_n = (K_n, E', d)$ with $V = \{x_i^j, 1 \leq j \leq 2n, 1 \leq i \leq 4\}$, $E' = \{(x_2^j, x_3^j), (x_4^j, x_1^j), 1 \leq j \leq 2n\}$, $d(x_1^j, x_2^j) = d(x_2^j, x_3^j) = d(x_3^j, x_4^j) = d(x_4^j, x_1^j) = 2$, $d(x_1^{2j-1}, x_1^{2j}) = d(x_2^{2j}, x_2^{2j+1}) = 2$, $j = 1, \dots, n - 1$, and $d(x_1^{2n-1}, x_1^{2n}) = d(x_2^{2n}, x_2^1) = 2$. Let the cost of all other edges be one. Thus, we obtain:

$$d(SOL) = 3n, \quad WOR(I_n) = 2n, \quad OPT(I_n) = 4n$$

and thus $\delta_{Patching.SRP}(I_n) = \frac{1}{2}$.

□

4.1. SOME LINKS WITH OTHER ROUTING PROBLEMS

Now, we study some particular cases of this problem which are known to be *NP*-hard problems in the literature. For instance, *the traveling salesman problem* can be viewed as a special case of *SRPP* in which there is no specified edge (take $E' = \emptyset$). Nevertheless, this problem is as well approximable as *TSP* in its minimization metric version since there is a 2/3-standard approximation and it is the best known ratio until today.

Since we have proved in Monnot *et al.* [41] that no polynomial time algorithm can guarantee a differential approximation ratio greater than 741/742 for *TSP*[2],

we can deduce the following negative results:

Proposition 4.3. *If $P \neq NP$ then $SRPP[t]$ is not approximable with differential ratio greater than $\frac{741}{742}$, for any $t \geq 2$.*

This problem also contains *Hamiltonian path problem with two specified endpoints* and *clustered traveling salesman problem with specified endpoints*; the first one consists in finding, in a complete graph K_n valued on edges, a Hamiltonian path from s to p of optimum weight whereas the second one consists in computing a tour of K_n in which the vertices of each cluster V_i are visited consecutively from s_i to p_i , where V is split into V_1, \dots, V_k . These properties seem not to hold for the standard framework; when considering the minimization metric version, several authors estimate that $HPP_{s,p}$ and $CTSP_{s_i,p_i}$ are more difficult to approximate than TSP or $SRPP$. Already in the middle of the eighties, Johnson and Papadimitriou [31] raised the question of the relative hardness of the Hamiltonian path specified endpoint version compared to the traveling salesman. These conjectures are strengthened by the positive results given on these problems since the best-known standard ratios are respectively $3/5$ and $11/21$ for *Min metric – $HPP_{s,p}$* (see Hoogeveen [30]) and *Min metric – $CTSP_{s_i,p_i}$* (see Guttmann–Beck *et al.* [25]) whereas it is $2/3$ for *Min metric – $SRPP$* .

Proposition 4.4. *The following results hold:*

- (i) $(HPP_{s,p}, \delta) \leq^{A^*P} (SRPP, \delta)$ with $c(\varepsilon) = \varepsilon$;
- (ii) $(CTSP_{s_i,p_i}, \delta) \leq^{A^*P} (SRPP, \delta)$ with $c(\varepsilon) = \varepsilon$.

Proof. Suppose that $SRPP$ contains a solution sol' which is an ε -differential approximation. □

For (i) When $E' = \{(s, p)\}$, sol' is a tour traversing (s, p) . Thus, the removal of this edge forms a Hamiltonian path from s to p (noted by sol) and we have $d(sol) = d(sol') - d(e)$. Moreover, the same property holds for the respective optimum and the respective worst values, so we obtain:

$$\begin{aligned} d(sol) &= d(sol') - d(e) \\ d(sol) &\leq \varepsilon(OPT_{SRPP}(I) - d(e)) + (1 - \varepsilon)(WOR_{SRPP}(I) - d(e)) \\ d(sol) &\leq \varepsilon OPT_{HPP}(I) + (1 - \varepsilon)WOR_{HPP}(I). \end{aligned}$$

For (ii): Let $I = (K_n, d)$ be an instance of *Min $CTSP_{s_i,p_i}$* such that (V_1, \dots, V_k) is a partition of V and $\{s_i, p_i\} \subseteq V_i$. Denote by G_i the sub-graph of K_n induced by V_i and G' the sub-graph of K_n induced by $\{s_i, p_i \mid 1 \leq i \leq k\}$. Consider $I_i = (G_i, d, s_i, p_i)$ and $I' = (G', d, E')$ with $E' = \{(s_i, p_i), 1 \leq i \leq k\}$ as being respectively an instance of $HPP_{s,p}$ and an instance of $SRPP$.

From (i), we know that $HPP_{s,p}$ is ε -differential approximable and if sol_i (resp. sol') is a feasible solution for HPP_{s_i,p_i} on I_i (resp. $SRPP$ on I'), then

$sol = (sol' \setminus E') \cup_{i=1, \dots, k} sol_i$ is a feasible solution of I for $CTSP_{s_i, p_i}$; this same partition of optimal solutions also hold and we have:

$$\begin{aligned} d(sol) &= d(sol') - d(E') + \sum_{i=1}^k d(sol_i) \\ d(sol) &\leq \varepsilon(OPT_{SRPP}(I') - d(E')) + (1 - \varepsilon)(WOR_{SRPP}(I') - d(E')) \\ &\quad + \sum_{i=1}^k [\varepsilon OPT_{HPP}(I_i) + (1 - \varepsilon)WOR_{HPP}(I_i)] \\ d(sol) &\leq \varepsilon OPT_{CTSP}(I) + (1 - \varepsilon)WOR_{CTSP}(I). \end{aligned}$$

That is the expected result.

We deduce from the previous results, thanks to Theorem 2.4 and Lemma 1.6:

Theorem 4.5. *For any $t \in]1; \infty]$, we have the following results:*

- Min $CTSP_{s_i, p_i}[t]$, Min $HPP_{s, p}[t]$ and Min $SRPP[t]$ are $\frac{2}{t+1}$ -standard approximable;
- Max $CTSP_{s_i, p_i}[t]$, Max $HPP_{s, p}[t]$ and Max $SRPP[t]$ are $\frac{t+1}{2t}$ -standard approximable.

5. CONCLUSION

In this article, we have established some basic properties on the approximability of *FGNPO*-problems with respect to the differential ratio. We also have exposed positive approximation results for some specific vehicles routing and spanning tree problems.

An open question of particular interest regarding to this class is the following: does any *FGNPO*-problem accept a differential approximation scheme? We conjecture that the answer is no, due to the structural properties on the size of any feasible solution that define *FGNPO* problems.

Moreover, for a class of problems close to *FGNPO*, namely when replacing the value $m[I, E'] = \sum_{e \in E'} d(e)$ of any solution E' by $m[I, E'] = \max_{e \in E'} d(e)$ and possibly turn the goal to minimize, the negative results become stronger. Actually, we denote by $NO APX(\Delta)$ the class of problems π from *NPO* which are not differential approximable better than Δ by any polynomial time algorithm, unless $P = NP$ (even when looking for a ratio depending on the size of the instance). In other words, $NO APX(\Delta)$ is constituted by the problems π which admit, for any polynomial time approximate algorithm A , an instance I_0 such that $\delta_A(I_0) = 0$. Thus, this class contains the worst approximation results we may expect (with respect to this measure), but still we have shown in Monnot *et al.* [42] that both the $\{0, 1\}$ -Linear Programming problem (with goal to minimize as well as to maximize) and a special weighted satisfiability problem are one of them. Now regarding to *FGNPO*, we have the following result, given in Monnot [39], similar to the Proposition 2.2 transposed to the differential framework.

Let π be an *FGNPO* problem and π' derived from π with the evaluating function given by $m[I, E'] = \max_{e \in E'} d(e)$ for any solution E' for π . If the decision problem associated to $\text{Min } \pi'$ is *NP-Complete*, then $\text{Min } \pi \in \text{NO APX}(\Delta)$.

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