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ϵ-EFFICIENT SOLUTIONS IN SEMI-INFINITE MULTIOBJECTIVE OPTIMIZATION

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Abstract. In this paper we apply some tools of nonsmooth analysis and scalarization method due to Chankong–Haimes to find ϵ-efficient solutions of semi-infinite multiobjective optimization problems (MP). We establish ϵ-optimality conditions of Karush–Kuhn–Tucker (KKT) type under Farkas–Minkowski (FM) constraint qualification by using ϵ-subdifferential concept. In addition we propose mixed type dual problem (including dual problems of Wolfe and Mond–Weir types as special cases) for ϵ-efficient solutions and investigate relationship between mentioned (MP) and its dual problem as well as establish several ϵ-duality theorems.

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1. Introduction

Multiobjective optimization is a process of simultaneous optimization of more than one objective function in the given domain. This discipline along with decision-making theory has been applied in such fields of science as engineering, economics, logistics and etc where optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives. It is reasonable to use scalarization methods for finding solutions of multiobjective problems, for example, see Chankong and Haimes [1] and references therein. Our aim of this paper is to establish results on ϵ-optimality conditions and ϵ-duality theorems for a multiobjective nonconvex optimization problem which has an infinite number of constraints by solving the corresponding scalar problem.

First, ϵ-optimality conditions for multiobjective problems have been studied by Kutateladze [10] and independently by Loridan [14]. Later, ϵ-solutions in vector optimization problems got a keen interest by a lot of authors, for example, see [5, 11, 13]. In this paper we explore ϵ-efficient solutions by establishing ϵ-optimality conditions. This concept was extended from the one for scalar optimization problem given by Strodiot *et al.* [17]. Later, Liu [13] considered multiobjective programming problems by using well-known weighted-sum scalarization method. However, the mentioned method is used for exploring properly efficient solutions but not efficient ones. Motivated by this fact, we suggest another scalarization method to establish ϵ-optimality conditions for multiobjective optimization problem by providing relationship between its ϵ-efficient solution and corresponding ϵ_j-optimal solution and using this equivalence. Since, one of the main tools for establishing ϵ-optimality conditions is ϵ-subdifferential concept, we would like to refer the reader to Dhara and Dutta [3] and Hiriart-Urruty [8] for better understanding.

Keywords and phrases: ϵ-Efficiency, semi-infinite optimization, ϵ-optimality conditions, ϵ-duality.

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In more recent time optimization problems including infinite number of constraints have been studied by several authors (see [2, 4, 6]). Moreover, ϵ-duality theorems of a class of nonconvex problems with an infinite number of constraints were established in [16].

We now describe the contents of the paper. Logically, our results can be divided into two parts. In the first one, we would like to propose ϵ-optimality conditions for semi-infinite multiobjective programming problems using scalar problem due to Chankong and Haimes [1]. It should be mentioned that in spite the fact that Chankong–Haimes scalarization method was described in 1983 in [1], there are not so many papers focused on it. Another ϵ-sufficient optimality condition under generalized convexity assumption was also considered. Moreover, for our semi-infinite programming problem we use Farkas–Minkowski constraint qualification described in [7] and, later, extended in [4]. The second part of this paper is dedicated to ϵ-duality theorems for mixed type of Wolfe and Mond–Weir types dual problem.

The paper is organized as follows. In Section 2, problem statement and main notions are described. Section 3 deduces ϵ-optimality conditions to semi-infinite multiobjective optimization problem, which are meant to be our main result. Section 4 is devoted to describing duality relations. Namely, both weak and strong ϵ-duality theorems for mixed type dual problem, including dual problems of Wolfe type and Mond–Weir type as special cases (see [12, 15]) are considered. Finally, Section 5 provides conclusions in brief.

2. Preliminaries

Let us consider the following semi-infinite multiobjective optimization problem:

$$\begin{array}{lll}(\mathrm{MP}) & \text{Minimize} & f(x) := \big(f_1(x), f_2(x), \ldots, f_m(x)\big) \\ & \text{subject to} & g_t(x) \leqq 0, t \in T, \\ & & x \in C,\end{array}$$

where $f_i(x) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in M := \{1, 2, \ldots, m\}$ and $g_t(x) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $t \in T$ (possible infinite) are proper lower semiconscious functions (l.s.c.), and C is a closed convex subset of $\mathbb{R}^n$. The feasible set of (MP) is denoted by $F_M := \{x \in C \mid g_t(x) \leqq 0, t \in T\}$.

Due to Chankong–Haimes method for $j \in M$ and $\bar{x} \in C$ we associated to (MP) the following scalar problem,

$$\begin{array}{lll}(\mathrm{P_j(\bar{x})}) & \text{Minimize} & f_j(x) \\ & \text{subject to} & f_i(x) \leqq f_i(\bar{x}), i \in M^j := M \setminus \{j\}, \\ & & g_t(x) \leqq 0, t \in T \\ & & x \in C.\end{array}$$

For the problem

$$\min\{f_j(x) \mid x \in C, G_t(x) \leqq 0, t \in \overline{T}\}$$

we define G_t as follows (with the assumption that $T \cap M = \emptyset$):

$$G_t(\cdot) = \begin{cases} f_t(\cdot) - f_t(\bar{x}), & t \in M^j, \\ g_t(\cdot), & t \in T, \end{cases} \quad \text{and } \overline{T} = T \cup M^j. \tag{2.1}$$

We now give some basic concepts and notions. The following linear space is used for semi-infinite programming [6].

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \mid \lambda_t = 0, \quad \text{for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $T(\lambda) = \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T.

The nonnegative cone of $\mathbb{R}^{(T)}$ is denoted by:

$$\mathbb{R}_+^{(T)} = \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geqq 0, t \in T\}.$$

With $\lambda \in \mathbb{R}^{(T)}$ and $g_t,\ t \in T$, we understand that

$$\sum_{t\in T} \lambda_t g_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t g_t, & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0, & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

To establish ϵ-optimality conditions of KKT-type we need the following notions.

Definition 2.1. Let $\phi : \mathbb{R}^n \to \mathbb{R} \bigcup \{+\infty\}$ be a proper convex function. The ϵ-subdifferential of ϕ at $\bar{x} \in \text{dom } \phi$ is the set $\partial_\epsilon \phi(\bar{x})$ defined by

$$\partial_\epsilon \phi(\bar{x}) = \{\xi \in \mathbb{R}^n \mid \phi(x) \geqq \phi(\bar{x}) - \epsilon + \langle \xi, x - \bar{x}\rangle, \quad \forall x \in \text{dom } \phi\}.$$

In particular, if $\epsilon = 0$, then $\partial_0 \phi = \partial \phi$.

Definition 2.2. Consider a function $\phi : \mathbb{R}^n \to \mathbb{R} \bigcup \{+\infty\}$. The conjugate of $\boldsymbol{\phi}$, $\phi^* : \mathbb{R}^n \to \mathbb{R} \bigcup \{+\infty\}$ is defined as

$$\phi^*(\xi) = \sup_{x\in\mathbb{R}^n} \{\langle \xi, x\rangle - \phi(x)\}.$$

The ϵ-subdifferential definition in term of *conjugate function* ϕ^* of ϕ is as follows:

$$\partial_\epsilon \phi(\bar{x}) = \{\xi \in \mathbb{R}^n \mid \phi^*(\xi) + \phi(\bar{x}) \leqq \langle \xi, \bar{x}\rangle + \epsilon\}.$$

Definition 2.3. The epigraph of a function $\phi : \mathbb{R}^n \to \mathbb{R} \bigcup \{+\infty\}$ is defined by

$$\text{epi } \phi = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid \phi(x) \leqq r\}.$$

It is worth to observe that if ϕ is a proper l.s.c. convex function and $\bar{x} \in \text{dom } \phi$, then [9]

$$\text{epi } \phi^* = \bigcup_{\epsilon \geqq 0} \{(\xi, \langle \xi, \bar{x}\rangle + \epsilon - \phi(\bar{x})) \mid \xi \in \partial_\epsilon \phi(\bar{x})\}.$$

Definition 2.4. Let C be a subset of $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be:

(i) convex at $x \in C$ if

$$f(y) - f(x) \geqq u\langle y - x\rangle, \quad u \in \partial f(x),\ y \in C.$$

And the function f is said to be convex on C if it is convex at every $x \in C$.

(ii) pseudoconvex at $x \in C$ if

$$f(y) < f(x) \Rightarrow u\langle y - x\rangle < 0, \quad u \in \partial f(x),\ y \in C,$$

equivalently,

$$u\langle y - x\rangle \geqq 0 \Rightarrow f(y) \geqq f(x), \quad u \in \partial f(x),\ y \in C.$$

And the function f is said to be pseudoconvex on C if it is pseudoconvex at every $x \in C$. Moreover, the function f is said to be strictly pseudoconvex at $x \in C$ if

$$u\langle y - x\rangle \geqq 0 \Rightarrow f(y) > f(x), \quad u \in \partial f(x),\ y \neq x,\ y \in C.$$

(iii) quasiconvex at $x \in C$ if

$$f(y) \leqq f(x) \Rightarrow u\langle y - x\rangle \leqq 0, \quad u \in \partial f(x),\ y \in C,$$

equivalently,

$$u\langle y - x\rangle > 0 \Rightarrow f(y) > f(x), \quad u \in \partial f(x),\ y \in C.$$

And the function f is said to be quasiconvex on C if it is quasiconvex at every $x \in C$.

Minimization means obtaining efficient solutions in the following sense. A point $\bar{x} \in F_M$ is said to be an efficient solution for (MP) if there is no $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}), \quad \text{for all } i \in M.$$

with at least one strict inequality.

Let ϵ be an element of $\mathbb{R}^m_+$. A point $\bar{x} \in F_M$ is said to be an ϵ-efficient solution for (MP) if there is no $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \epsilon_i, \quad \text{for all } i \in M.$$

with at least one strict inequality.

3. ϵ-Optimality conditions

Definition 3.1. The indicator function δ_K of a subset $K \subset \mathbb{R}^n$ is the function defined as follows:

$$\delta_K = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if } x \in \mathbb{R}^n \backslash K. \end{cases}$$

Definition 3.2. Let C be a nonempty closed convex subset of $\mathbb{R}^n$, $\epsilon > 0$, $\bar{x} \in C$. The ϵ-normal set of C at $\bar{x}$ is the set

$$N_\epsilon(C; \bar{x}) = \{\xi \in \mathbb{R}^n \mid \langle \xi, x - \bar{x}\rangle \leqq \epsilon, \quad \forall x \in C\}.$$

If $\epsilon = 0$, the ϵ-normal set reduces to the normal cone $N(C; \bar{x})$ to C at $\bar{x}$ that is

$$N(C; \bar{x}) = \{\xi \in \mathbb{R}^n \mid \langle \xi, x - \bar{x}\rangle \leqq 0, \quad \forall x \in C\}.$$

Let us define the following sets:

$$S_i = \{x \in \mathbb{R}^n \mid f_i - f_i(\bar{x}) \leqq 0\}, \quad \text{for } i \in M^j,$$
$$S_t = \{x \in \mathbb{R}^n \mid g_t(x) \leqq 0\}, \quad \text{for } t \in T.$$

It is easy to check that

$$\partial_\epsilon \delta_C(\bar{x}) = N_\epsilon(C;\bar{x}).$$

Using the indicator functions $\delta_i, \delta_t, \delta_C$ of the subsets $S_i,\ i \in M^j, S_t, t \in T(v)$ and C we can rewrite our problem $(P_j(\bar{x}))$ as unconstrained problem $P_j^0(\bar{x}))$ in the entire space $\mathbb{R}^n$ as follows:

$$(\mathrm{P}_\mathrm{j}^0(\bar{\mathrm{x}})) \quad \begin{array}{ll} \text{Minimize} & f_j(x) + \sum_{i\in M^j} \delta_{S_i} + \sum_{t\in T(v)} \delta_{S_t}(x) + \delta_C(x) \\ \text{subject to} & x \in \mathbb{R}^n. \end{array}$$

We can obtain the following lemma for establishing ϵ-complementary slackness condition by using (Prop. 2.2. in Strodiot *et al.* [17]) (possible since G_t is defined by (2.1)):

Lemma 3.3. *Let $\epsilon \geqq 0$ and suppose f and g are convex functions. Let $\bar{x} \in S = \left(\cap_{t\in T(v)} S_t\right) \cap \left(\cap_{i\in M^j} S_i\right)$ and the following constraint qualification of the Slater type holds true:*

$$(CQ), \quad \exists x_0 \in C: \ G(x_0) < 0, \qquad x_0 \in C,$$

where $G = \sup_{t\in \overline{T}} G_t$.

Then $x^* \in N_\epsilon(S;\bar{x})$, iff there exist $v \geqq 0$ and $\bar{\epsilon} \geqq 0$ such that

$$x^* \in \partial_{\bar{\epsilon}}(vG)(\bar{x}) \quad \text{and} \quad \bar{\epsilon} - \epsilon \leqq (vG)(\bar{x}) \leqq 0.$$

Now we would like to derive one useful lemma for obtaining our main result viz ϵ-optimality condition for (MP). It should be noticed that the proof method is similar to Strodiot *et al.* [17] or Liu [13] but our goal is to establish ϵ-optimality condition for $(P_j(\bar{x}))$ problem with infinite number of constraints, which is the main difference. But first, Slater type (CQ) should be replaced by another one suitable for semi-infinite programming (see [7]).

Definition 3.4. The convex semi-infinite programming problem is said to satisfy the Farkas–Minkowski (FM) constraint qualification if

$$\{v_t g_t(x), t \in T(v), \quad x \in C\}$$

is a (FM) system, *i.e.* its characteristic cone $K := \operatorname{cone}\{\bigcup_{t\in T(v)} \operatorname{epi}(v_t g_t)^* + \operatorname{epi}\delta_C^*\}$ is closed.

Remark 3.5. According to (Prop. 11.16 in [3]) if (CQ) holds then (FM) is also satisfied.

Lemma 3.6. *Let $\epsilon_j \geqq 0$, $\bar{x}$ be a feasible point of $(P_j(\bar{x}))$ and $f_i,\ i \in M$ and $g_t,\ t \in T$ be convex functions. Suppose that (FM) holds then $\bar{x}$ is an ϵ_j-optimal solution to $(P_j(\bar{x}))$ iff there exist scalars $\overline{\epsilon_{0j}} \geqq 0$, $\overline{\epsilon_{0i}} \geqq 0$ for $i \in M^j$, $\overline{\epsilon_{1t}} \geqq 0$ for $t \in T$, $\overline{\epsilon_q} \geqq 0$, $\overline{\lambda_i} > 0$ for $i \in M$ and $\overline{v_t} \in \mathbb{R}_+^{(T)}$, such that*

$$0 \in \partial_{\overline{\epsilon_{0j}}} f_j(\bar{x}) + \sum_{i\in M^j} \partial_{\epsilon_{0i}}(\overline{\lambda_i} f_i)(\bar{x}) + \sum_{t\in T(v)} \partial_{\overline{\epsilon_{1t}}}(\overline{v_t} g_t)(\bar{x}) + N_{\overline{\epsilon_q}}(C;\bar{x}), \tag{3.1}$$

$$\overline{\epsilon_{0j}} + \sum_{i\in M^j} \overline{\lambda_i}\overline{\epsilon_{0i}} + \sum_{t\in T(v)} \overline{v_t \epsilon_{1t}} + \overline{\epsilon_q} - \epsilon_j \leqq \sum_{t\in \overline{T}(v)} \overline{v_t} G_t(\bar{x}) \leqq 0. \tag{3.2}$$

(We call the condition (3.2) the ϵ-complementary slackness condition.)

Proof. It is obviously that $(P_j(\bar{x}))$ and $(P_j^0(\bar{x}))$ have the same ϵ_j-solutions. $\bar{x}$ is an ϵ_j-optimal solution if and only if

$$0 \in \partial_{\epsilon_j}\left(f_j + \sum_{i\in M^j}\delta_{S_i} + \sum_{t\in T(v)}\delta_{S_t}(x) + \delta_C\right)(\bar{x}).$$

Since there is at least one point $x_0 \in int\ S_i \bigcap int\ S_t \bigcap int\ C$ and (FM) holds then, according to Hirriart-Urruty [8],

$$\begin{aligned}&\partial_{\epsilon_j}\left(f_j + \sum_{i\in M^j}\delta_{S_i} + \sum_{t\in T(v)}\delta_{S_t}(x) + \delta_C\right)(\bar{x})\\ &\quad= \bigcup_{\substack{\overline{\epsilon_{0j}}\geqq 0,\overline{\epsilon_{0i}}\geqq 0,\overline{\epsilon_{1t}}\geqq 0,\overline{\epsilon_q}\geqq 0\\ \overline{\epsilon_{0j}}+\sum_{i\in M^j}\overline{\epsilon_{0i}}+\sum_{t\in T(v)}\overline{\epsilon_{1t}}+\overline{\epsilon_q}=\epsilon_j}}\left\{\partial_{\overline{\epsilon_{0j}}}f_i(\overline{x}) + \sum_{i\in M^j}\partial_{\overline{\epsilon_{0i}}}\delta_{S_i}(\bar{x}) + \sum_{t\in T(v)}\partial_{\overline{\epsilon_{1t}}}\delta_{S_t}(\bar{x}) + \partial_{\overline{\epsilon_q}}\delta_C(\bar{x})\right\},\end{aligned}$$

(3.2) follows from Lemma 2.3. □

Now, for obtaining ϵ-optimality conditions for (MP) we would like to mention the following lemma, which is an approximate version of Chankong–Haimes characterization [1].

Lemma 3.7. *Let $\bar{x} \in C$ and $\epsilon \in \mathbb{R}^m_+$. A feasible point $\bar{x}$ is an ϵ-efficient solution of (MP) if and only of $\bar{x}$ is an ϵ_j-optimal solution of $(P_j(\bar{x}))$ for each $j \in M$.*

Proof. Let $\bar{x}$ be an ϵ_j-optimal solution of $(P_j(\bar{x}))$ for each $j \in M$. Hence,

$$f_j(\bar{x}) \leqq f_j(x) + \epsilon_j, \quad \text{for all } j \in M.$$

If $\bar{x}$ is not an ϵ-efficient solution of (MP) then there exists $x \in F_M$ such that:

$$f_i(x) \leqq f_i(\bar{x}) - \epsilon_i, \quad \text{for all } i \in M$$

with at least one strict inequality. Suppose that the strict inequality takes place at k. We get $f_k(x) < f_k(\bar{x}) - \epsilon_k$, *i.e.* $f_k(x) + \epsilon_k < f_k(\bar{x})$. Hence, there exists $k \in M$ such that $\bar{x}$ is not an ϵ_k-optimal solution of $(P_k(\bar{x}))$ that is a contradiction.

Conversely, let $\bar{x}$ be an ϵ-efficient solution of (MP). Hence, there exists no such $x \in F_M$ that

$$f_i(x) \leqq f_i(\bar{x}) - \epsilon_i, \quad \text{for all } i \in M$$

with at least one strict inequality. If there exists $j \in M$ such that $\bar{x}$ is not an ϵ_j-optimal solution of $(P_j(\bar{x}))$ then there exists $x \in F_j(\bar{x})$ such that

$$f_j(x) + \epsilon_j < f_j(\bar{x}),$$

which is a contradiction. □

Now, we give an example to illustrate the aforesaid lemma.

Example 3.8.

$$\begin{array}{lll}\text{(MP)} & \text{Minimize} & \big(f_1(x), f_2(x)\big)\\ & \text{subject to} & g_t(x) \leqq 0, t \in T := [1,2],\\ & & x \in \mathbb{R},\end{array}$$

where $f_1 = x$, $f_2 = \frac{1}{2}x^2$ and $g_t = tx^2 - 2tx$. Let us put $\epsilon_1 = 1$ and $\epsilon_2 = \frac{1}{2}$.

Hence, the feasible set of (MP) is equal to $[0,2]$. Since $f_1 = x$ and $f_2 = \frac{1}{2}x^2$, we have $f_2(f_1) = \frac{1}{2}(f_1)^2$. We can check that ϵ-solution set is as follows: $(-\infty, 1] \cap [0,2] = [0,1]$. And set of ϵ-objective values

$$\left\{(f_1, f_2) \in \mathbb{R}^2 \mid f_1 \in [0,1],\ f_2 = \frac{1}{2}(f_1 - 1)^2 + \frac{1}{2}\right\}.$$

We can easily check that the point $\bar{x} = 1$ is the ϵ-efficient solution of (MP) by using scalarization method due to Chankhong–Haimes. However, if one choose point $\bar{x} = 1.01$ it is not ϵ-efficient solution. For example, from the definition of ϵ_j-solution for $(P_1(1))$ we can check that

$$1.01 \leqq x + 1, \qquad \text{for any } x \text{ in feasible set}$$

fails if $x = 0$.

By using Lemmas 3.6 and 3.7 we can derive the following theorem, which is meant to be our main result.

Theorem 3.9 (ϵ-Optimality condition)**.** *Let $\bar{x} \in C$ and $\epsilon \in \mathbb{R}^m_+$. Suppose that f_i, $i \in M$ and g_t, $t \in T$ are convex functions and (FM) holds then $\bar{x}$ is an ϵ-efficient solution for (MP) iff there exist scalars $\widetilde{\epsilon_{0i}} \geqq 0$ and $\widetilde{\lambda_i} > 0$ for $i \in M$, $\sum_{i\in M} \widetilde{\lambda_i} = 1$, $\widetilde{\epsilon_{1t}} \geqq 0$, for $t \in T$, $\widetilde{\epsilon_q} \geqq 0$ and $\widetilde{v_t} \in \mathbb{R}^{(T)}_+$, such that*

$$0 \in \sum_{i\in M} \widetilde{\lambda_i}\partial_{\widetilde{\epsilon_{0i}}} f_i(\bar{x}) + \sum_{t\in T(v)} \widetilde{v_t}\partial_{\widetilde{\epsilon_{1t}}} g_t(\bar{x}) + N_{\widetilde{\epsilon_q}}(C;\bar{x}), \tag{3.3}$$

$$\sum_{i\in M} \widetilde{\lambda_i}\widetilde{\epsilon_{0i}} + \sum_{t\in T(v)} \widetilde{v_t}\widetilde{\epsilon_{1t}} + \widetilde{\epsilon_q} - \widetilde{\lambda}^T\epsilon \leqq \sum_{t\in T(v)} \widetilde{v_t} g_t(\bar{x}) \leqq 0. \tag{3.4}$$

Proof. By Lemma 3.7, $\bar{x}$ is an ϵ-efficient solution to (MP) iff $\bar{x}$ is an ϵ_j-optimal solution for $(P_j(\bar{x}))$ for all $j \in M$. According to Lemma 3.6 there exist $\overline{\epsilon_{0j}} \geqq 0$, $\overline{\epsilon_{0i}} \geqq 0$, $i \in M^j$, $\overline{\epsilon_{1t}} \geqq 0$, $t \in T$, $\overline{\epsilon_q} \geqq 0$, $\overline{v_t} \in \mathbb{R}^{(T)}_+$ and $\overline{\lambda_i}$, $i \in M^j$ such that (3.1) and (3.2) holds.

First, let us focus on (3.1). Due to scalar product rule (Thm. 2.117 in [3]) $\partial_\epsilon \lambda\phi(\xi) = \lambda\partial_{\epsilon/\lambda}\phi(\xi)$:

$$0 \in \partial_{\overline{\epsilon_{0j}}} f_j(\bar{x}) + \sum_{i\in M^j} \overline{\lambda_i}\partial_{\overline{\epsilon_{0i}}/\overline{\lambda_i}} f_i(\bar{x}) + \sum_{t\in T(v)} \overline{v_t}\partial_{\overline{\epsilon_{1t}}/\overline{v_t}} g_t(\bar{x}) + N_{\overline{\epsilon_q}}(C,\bar{x}).$$

It implies that

$$0 \in \frac{1}{1 + \sum_{i\in M^j} \overline{\lambda_i}}\left(\partial_{\overline{\epsilon_{0j}}} f_j(\bar{x}) + \sum_{i\in M^j} \overline{\lambda_i}\partial_{\overline{\epsilon_{0i}}/\overline{\lambda_i}} f_i(\bar{x}) + \sum_{t\in T(v)} \overline{v_t}\partial_{\overline{\epsilon_{1t}}/\overline{v_t}} g_t(\bar{x}) + N_{\overline{\epsilon_q}}(C,\bar{x})\right). \tag{3.5}$$

Note that $N_{\widetilde{\epsilon_q}}(C;\bar{x}) = \dfrac{1}{1+\sum_{i\in M^j}\overline{\lambda_i}} N_{\overline{\epsilon_q}}(C;\bar{x}) \subset N_{\overline{\epsilon_q}}(C;\bar{x})$ and set

$$\begin{aligned}
&\widetilde{\lambda_j} = \frac{1}{1+\sum_{i\in M^j}\overline{\lambda_i}};\\
&\widetilde{\lambda_i} = \frac{\overline{\lambda_i}}{1+\sum_{i\in M^j}\overline{\lambda_i}}, \quad i\in M^j,\\
&\widetilde{v_t} = \frac{\overline{v_t}}{1+\sum_{i\in M^j}\overline{\lambda_i}}, \quad t\in T,\\
&\overline{\epsilon_{0j}} = \widetilde{\epsilon_{0j}},\\
&\overline{\epsilon_{0i}}/\overline{\lambda_i} = \widetilde{\epsilon_{0i}}, \quad i\in M^j,\\
&\overline{\epsilon_{1t}}/\overline{v_t} = \widetilde{\epsilon_{1t}}, \quad t\in T,
\end{aligned} \tag{3.6}$$

from (3.5) we deduce

$$0 \in \sum_{i\in M}\widetilde{\lambda_i}\partial_{\widetilde{\epsilon_{0i}}} f_i(\bar{x}) + \sum_{t\in T(v)}\widetilde{v_t}\partial_{\widetilde{\epsilon_{1t}}} g_t(\bar{x}) + N_{\widetilde{\epsilon_q}}(C;\bar{x}).$$

It is easy to check that $\sum_{i\in M}\widetilde{\lambda_i} = 1$.

Since the feasible set of (MP) is $F_M := \{x\in C \mid g_t(x) \leqq 0, t\in T\}$, without loss of generality, we can reduce $\sum_{t\in \overline{T}(v)}\overline{v_t}G_t$ to $\sum_{t\in T(v)}\overline{v_t}g_t$. Hence from (3.2) and (3.6) we have:

$$\widetilde{\epsilon_{0j}} + \sum_{i\in M^j}\overline{\lambda_i}\widetilde{\epsilon_{0i}} + \sum_{t\in T(v)}\overline{v_t}\widetilde{\epsilon_{1t}} + \overline{\epsilon_q} - \overline{\lambda}^T\epsilon \leqq \sum_{t\in T(v)}\overline{v_t}g_t(\bar{x}) \leqq 0.$$

Using the same method, we get

$$\sum_{i\in M}\widetilde{\lambda_i}\widetilde{\epsilon_{0i}} + \sum_{t\in T(v)}\widetilde{v_t}\widetilde{\epsilon_{1t}} + \widetilde{\epsilon_q} - \widetilde{\lambda}^T\epsilon \leqq \sum_{t\in T(v)}\widetilde{v_t}g_t(\bar{x}) \leqq 0,$$

where $\widetilde{\epsilon_q} = \dfrac{\overline{\epsilon_q}}{1+\sum_{i\in M^j}\overline{\lambda_i}}$. □

Remark 3.10. Condition (3.3) in Theorem 3.9 seems similar to (Thm. 4 in [13]) but we derived it using relationship between ϵ-efficient solutions of (MP) and ϵ_j-efficient solution of $P_j(\bar{x})$. Moreover, we deal with semi-infinite (MP). The other difference is in the fact that we consider not $(\lambda^T f)$ and $(v^T g)$ functions but move multipliers out of ϵ-subdifferential.

We establish another ϵ-sufficient optimality condition with generalized convexity assumption.

Theorem 3.11 (ϵ-Sufficient optimality condition)**.** *Let $\bar{x}\in C$ and $\epsilon\in R^m_+$. Assume that $\widetilde{\lambda}^T f$ is pseudoconvex and $\widetilde{v}^T g$ is quasiconvex functions. If there exist scalars $\widetilde{\epsilon_{0i}} \geqq 0$, $\widetilde{\lambda_i} \geqq 0$ for $i\in M$, $\sum_{i\in M}\widetilde{\lambda_i} = 1$, $\widetilde{\epsilon_{1t}} \geqq 0$ for $t\in T$, $\widetilde{\epsilon_q} \geqq 0$ and $\widetilde{v_t}\in\mathbb{R}^{(T)}_+$, such that (3.3) and (3.4) then $\bar{x}$ is an ϵ-efficient solution for (MP).*

Proof. Suppose that $\bar{x}$ is not ϵ-efficient solution to (MP). Then there is such $x\in C$ that

$$f_i(x) \leqq f_i(\bar{x}) - \epsilon_i, \quad \text{for all } i\in M$$

with at least one strict inequality. Multiplying by $\widetilde{\lambda}$ we have

$$\widetilde{\lambda}^T f(x) \leqq \widetilde{\lambda}^T f(\bar{x}) - \widetilde{\lambda}^T \epsilon.$$

If (3.3) holds then there exist $u_i \in \partial_{\widetilde{\epsilon_{0i}}} f_i,\ i \in M,\ \mu_t \in \partial_{\widetilde{\epsilon_{1t}}} g_t,\ t \in T(v)$ and $w \in N_{\widetilde{\epsilon_q}}(C;\bar{x})$ such that

$$\sum_{i\in M} \widetilde{\lambda}_i u_i(x-\bar{x}) + \sum_{t\in T(v)} \widetilde{v_t}\mu_t(x-\bar{x}) + w(x-\bar{x}) = 0.$$

By the definition of $N_{\widetilde{\epsilon_q}}(C;\bar{x})$ we have $w(x-\bar{x}) \leqq \widetilde{\epsilon_q}$ for all $x \in C$. Hence,

$$\sum_{i\in M} \widetilde{\lambda}_i u_i(x-\bar{x}) + \sum_{t\in T(v)} \widetilde{v_t}\mu_t(x-\bar{x}) \geqq -\widetilde{\epsilon_q}.$$

Since $\sum_{t\in T} \widetilde{v_t} g_t$ is quasiconvex and $g_t(\bar{x}) \leqq 0,\ t \in T$ and the definition of ϵ-subdifferential, we have

$$\sum_{t\in T(v)} \widetilde{v_t}(x-\bar{x}) \leqq \sum_{t\in T(v)} \widetilde{v_t}\widetilde{\epsilon_{1t}}.$$

Hence,

$$\sum_{i\in M} \widetilde{\lambda}_i u_i(x-\bar{x}) \geqq -\widetilde{\epsilon_q} - \sum_{t\in T(v)} \widetilde{v_t}\widetilde{\epsilon_{1t}} - \sum_{i\in M} \widetilde{\lambda}_i \widetilde{\epsilon_{0i}}.$$

By pseudoconvexity of $\widetilde{\lambda}^T f$ and the definition of ϵ-subdifferential it follows

$$\widetilde{\lambda}^T f(x) - \widetilde{\lambda}^T f(\bar{x}) \geqq -\widetilde{\epsilon_q} - \sum_{t\in T(v)} \widetilde{v_t}\widetilde{\epsilon_{1t}} - \sum_{i\in M} \widetilde{\lambda}_i \widetilde{\epsilon_{0i}}.$$

From (3.4)

$$-\widetilde{\epsilon_q} - \sum_{t\in T(v)} \widetilde{v_t}\widetilde{\epsilon_{1t}} - \sum_{i\in M} \widetilde{\lambda}_i \widetilde{\epsilon_{0i}} \geqq -\widetilde{\lambda}^T \epsilon.$$

So we can rewrite

$$\widetilde{\lambda}^T f(x) \geqq \widetilde{\lambda}^T f(\bar{x}) - \widetilde{\lambda}^T \epsilon,$$

which contradicts our supposition that $\bar{x}$ is not an ϵ-efficient solution to (MP). □

4. ϵ-Duality

In this section, we introduce a mixed dual programming problem and establish weak and strong ϵ-duality theorems. Now we propose the mixed type dual problem due to (MP), which combines Wolfe type and Mond–Weir type as follows:

$$\begin{array}{lll}
\text{(MD)} & \text{Maximize} & f(y) + \sum_{t\in T} v_t g_t(y) e \\
& \text{subject to} & 0 \in \sum_{i\in M} \lambda_i \partial_{\widetilde{\epsilon_{0i}}} f_i(y) + \sum_{t\in T} (v_t + \nu_t) \partial_{\widetilde{\epsilon_{1t}}} g_t(y) + N_{\widetilde{\epsilon_q}}(C;y), \\
& & \sum_{i\in M} \lambda_i \widetilde{\epsilon_{0i}} + \sum_{t\in T} v_t \widetilde{\epsilon_{1t}} + \widetilde{\epsilon_q} - \lambda^T \epsilon \leqq 0 \\
& & \nu_t g_t(y) \geqq 0, \\
& & \lambda > 0, \lambda^T e = 1, e = (1, \ldots, 1) \in \mathbb{R}^m, \\
& & (y, \lambda, v, \nu) \in C \times \mathbb{R}^m \times \mathbb{R}_+^{(T)} \times \mathbb{R}_+^{(T)}.
\end{array}$$

Now we derive ϵ-weak duality theorem under convexity assumption.

Theorem 4.1 (ϵ-Weak duality)**.** *Let x and (y,λ,v,ν) be a feasible solution to (MP) and (MD) respectively. Assume that $f_i,\ i\in M$ and $g_t,\ t\in T$ are convex on C. Then the following cannot hold:*

$$f_i(x) \leqq f_i(y) + \sum_{t\in T} v_t g_t(y) - \epsilon_i, \quad \text{for all } i \in M$$

with at least one strict inequality.

Proof. Suppose contrary to result that it holds. Multiplying by $\lambda > 0$ with $\lambda^T e = 1$, we have

$$\lambda^T f(x) < \lambda^T f(y) + \sum_{t\in T(v)} v_t g_t(y) - \lambda^T \epsilon.$$

Hence $x \in C$ and $\sum_{t\in T(v)}(v_t+\nu_t)g_t(x) \leqq 0$ and $\nu_t g_t(y) \geqq 0$, we obtain $x \neq y$ and,

$$\lambda^T f(x) + \sum_{t\in T(v)} (v_t+\nu_t)g_t(x) < \lambda^T f(y) + \sum_{t\in T(v)} (v_t+\nu_t)g_t(y) - \lambda^T\epsilon.$$

Since (y,λ,v,ν) is a feasible solution to (MD), there exists $u_i \in \partial_{\widetilde{\epsilon_{0i}}} f_i(y)$, $i\in M$, $\mu_t \in \partial_{\widetilde{\epsilon_{1t}}} g_t,\ t\in T(v)$ and $w \in N_{\widetilde{\epsilon_q}}(C;y)$ such that:

$$\sum_{i\in M} \lambda_i u_i(x-y) + \sum_{t\in T(v)} (v_t+\nu_t)\mu_t(x-y) + w(x-y) = 0.$$

So, using the convexity of f_i, $i\in M$ and $g_t,\ t\in T$, we can obtain:

$$\begin{aligned}
&\lambda^T f_i(x) + \sum_{t\in T(v)} (v_t+\nu_t)g_t(x) - \left(\lambda^T f_i(y) + \sum_{t\in T(v)} (v_t+\nu_t)g_t(y) - \lambda^T\epsilon\right)\\
&= \sum_{i\in M} \lambda_i\Big(f_i(x)-f_i(y)\Big) + \sum_{t\in T(v)} (v_t+\nu_t)\Big(g_t(x)-g_t(y)\Big) + \lambda^T\epsilon\\
&\geqq \sum_{i\in M} \lambda_i u_i(x-y) + \sum_{t\in T(v)} (v_t+\nu_t)\mu_t(x-y) + \lambda^T\epsilon - \sum_{i\in M}\lambda_i\widetilde{\epsilon_{0i}} - \sum_{t\in T(v)} v_t\widetilde{\epsilon_{1t}}\\
&= -w(x-y) + \lambda^T\epsilon - \sum_{i\in M}\lambda_i\widetilde{\epsilon_{0i}} - \sum_{t\in T(v)} v_t\widetilde{\epsilon_{1t}} \geqq \lambda^T\epsilon - \sum_{i\in M}\lambda_i\widetilde{\epsilon_{0i}} - \sum_{t\in T(v)} v_t\widetilde{\epsilon_{1t}} - \widetilde{\epsilon_q} \geqq 0,
\end{aligned}$$

that is contradiction. □

Using Theorems 3.9 and 4.1, we establish ϵ-strong duality theorem.

Theorem 4.2 (ϵ-Strong duality)**.** *Let $\epsilon \in \mathbb{R}^m_+$. Assume that f_i, $i\in M$ and $g_t,\ t\in T$ are convex functions, (FM) and ϵ-weak duality hold. If $\bar{x}\in C$ is an ϵ-efficient solution of (MP) then $(\bar{x},\lambda,v,\nu)$ is 2ϵ-efficient solution for (MD).*

Proof. Since $\bar{x}$ is an ϵ-efficient solution for (MP), then by Theorem 3.9, there exist $\lambda_i,\ i \in M$ and $v \in \mathbb{R}_+^{(T)}$ such that

$$0 \in \sum_{i \in M} \lambda_i \partial_{\widetilde{\epsilon_{0i}}} f_i(\bar{x}) + \sum_{t \in T(v)} v_t \partial_{\widetilde{\epsilon_{1t}}} g_t(\bar{x}) + N_{\widetilde{\epsilon_q}}(C; \bar{x})$$
$$\subset \sum_{i \in M} \lambda_i \partial_{\widetilde{\epsilon_{0i}}} f_i(\bar{x}) + \sum_{t \in T(v)} (v_t + \nu_t) \partial_{\widetilde{\epsilon_{1t}}} g_t(\bar{x}) + N_{\widetilde{\epsilon_q}}(C; \bar{x})$$

holds, then $(\bar{x}, \lambda, v, \nu)$ is feasible for (MD).

Suppose that $(\bar{x}, \lambda, v, \nu)$ is not 2ϵ-efficient solution for (MD), then there exists $(x^*, \lambda^*, v^*, \nu^*)$ such that the following cannot hold:

$$f_i(x^*) + \sum_{t \in T(v)} (v_t^* + \nu_t^*) g_t(x^*) e - 2\epsilon_i \leqq f_i(\bar{x}) + \sum_{t \in T(v)} (v_t + \nu_t) g_t(\bar{x}) e,$$

with at least on strict inequality.

Taking strict inequality at jth place, we get

$$f_j(x^*) + \sum_{t \in T(v)} (v_t^* + \nu_t^*) g_t(x^*) - 2\epsilon_j > f_j(\bar{x}) + \sum_{t \in T(v)} (v_t + \nu_t) g_t(\bar{x})$$

Or $f_j(\bar{x}) + \sum_{t \in T(v)} (v_t + \nu_t) g_t(\bar{x}) - f_j(x^*) - \sum_{t \in T(v)} (v_t^* + \nu_t^*) g_t(x^*) < -2\epsilon_j$.

On the other hand, by ϵ-weak duality (Thm. 4.1)

$$f_j(\bar{x}) - \left(f_j(x^*) - \sum_{t \in T(v)} (v_t^* + \nu_t^*) g_t(x^*) \right) + \sum_{t \in T(v)} (v_t + \nu_t) g_t(\bar{x}) > -\epsilon_j + \sum_{t \in T(v)} (v_t + \nu_t) g_t(\bar{x}) > -\epsilon_j$$
$$+ \sum_{i \in M} \lambda_i \widetilde{\epsilon_{0i}} + \sum_{t \in T} v_t \widetilde{\epsilon_{1t}} + \widetilde{\epsilon_q} - \epsilon_i > -2\epsilon_j,$$

so we get contradiction. □

We can derive another ϵ-weak duality theorem under the generalized convexity assumptions.

Theorem 4.3 (ϵ-Weak duality)**.** *Let x and (y, λ, v, ν) be a feasible solution to (MP) and (MD) respectively. Assume that $\left(\lambda^T f + \sum_{t \in T} (v_t + \nu_t) g_t\right)$ is pseudoconvex on C, $f_i,\ i \in M$ and $g_t,\ t \in T$ are regular on C. Then the following cannot hold:*

$$f_i(x) \leqq f_i(y) + \sum_{t \in T} v_t g_t(y) - \epsilon_i, \quad \text{for all } i \in M$$

with at least one strict inequality.

Proof. Suppose contrary to result that it holds. Multiplying by $\lambda > 0$, we have

$$\lambda^T f(x) < \lambda^T f(y) + \sum_{t \in T(v)} v_t g_t(y) - \lambda^T \epsilon.$$

Hence $x \in C$ and $\sum_{t\in T(v)}(v_t+\nu_t)g_t(x) \leqq 0$ and $\nu_t g_t(y) \geqq 0$, we obtain $x \neq y$ and,

$$\lambda^T f(x) + \sum_{t\in T(v)} (v_t+\nu_t)g_t(x) < \lambda^T f(y) + \sum_{t\in T(v)} (v_t+\nu_t)g_t(y) - \lambda^T \epsilon. \tag{4.1}$$

Since (y,λ,v,ν) is a feasible solution to $(\mathrm{MD})_W$, there exists $u_i \in \partial_{\widetilde{\epsilon_{0i}}} f_i(y)$, $i \in M$, $\mu_t \in \partial_{\widetilde{\epsilon_{1t}}} g_t,\ t \in T(v)$ and $w \in N_{\widetilde{\epsilon_q}}(C;y)$ such that:

$$\sum_{i\in M} \lambda_i u_i(x-y) + \sum_{t\in T(v)} (v_t+\nu_t)\mu_t(x-y) + w(x-y) = 0.$$

By definition of $N_{\widetilde{\epsilon_q}}$, we get

$$\left(\sum_{i\in M} \lambda_i u_i + \sum_{t\in T(v)} (v_t+\nu_t)\mu_t\right)(x-y) \geqq -\widetilde{\epsilon_q}.$$

Since $\left(\lambda^T f + \sum_{t\in T}(v_t+\nu_t)g_t\right)$ is pseudoconvex and $f_i,\ i \in M$ and $g_t,\ t \in T$ are regular on C and using ϵ-subdifferential definition, we have:

$$\left(\lambda^T f_i + \sum_{t\in T(v)} (v_t+\nu_t)g_t\right)(x) \geqq \left(\lambda^T f_i + \sum_{t\in T(v)} (v_t+\nu_t)g_t\right)(y) - \widetilde{\epsilon_q} - \sum_{i\in M} \lambda_i \widetilde{\epsilon_{0i}} - \sum_{t\in T(v)} v_t \widetilde{\epsilon_{1t}}.$$

Hence $-\widetilde{\epsilon_q} - \sum_{i\in M} \lambda_i \widetilde{\epsilon_{0i}} - \sum_{t\in T(v)} v_t \widetilde{\epsilon_{1t}} \geqq -\lambda^T\epsilon$ we get

$$\lambda^T f_i(x) + \sum_{t\in T(v)} (v_t+\nu_t)g_t(x) \geqq \lambda^T f_i(y) + \sum_{t\in T(v)} (v_t+\nu_t)g_t(y) - \lambda^T\epsilon,$$

that is a contradiction. □

Special case 1. It is obvious that if $\nu = 0$, the problem (MD) is reduced to Wolfe type dual problem, which is denoted as follows:

$$
\begin{array}{lll}
(\mathrm{MD})_W & \text{Maximize} & f(y) + \sum_{t\in T} v_t g_t(y) e \\
 & \text{subject to} & 0 \in \sum_{i\in M} \lambda_i \partial_{\widetilde{\epsilon_{0i}}} f_i(y) + \sum_{t\in T} v_t \partial_{\widetilde{\epsilon_{1t}}} g_t(y) + N_{\widetilde{\epsilon_q}}(C;y), \\
 & & \sum_{i\in M} \lambda_i \widetilde{\epsilon_{0i}} + \sum_{t\in T} v_t \widetilde{\epsilon_{1t}} + \widetilde{\epsilon_q} - \lambda^T \epsilon \leqq 0 \\
 & & \lambda > 0, \lambda^T e = 1, e = (1,\ldots,1) \in \mathbb{R}^m, \\
 & & (y,\lambda,v) \in C \times \mathbb{R}^m \times \mathbb{R}_+^{(T)}.
\end{array}
$$

We can also obtain the following three theorems immediately.

Theorem 4.4 (ϵ-Weak Duality)**.** *Let x and (y,λ,v) be a feasible solution to (MP) and $(MD)_W$ respectively. Assume that $f_i, i \in M$ and $g_t, t \in T$ are convex on C. Then the following cannot hold:*

$$f_i(x) \leqq f_i(y) + \sum_{t\in T} v_t g_t(y) - \epsilon_i, \quad \text{for all } i \in M$$

with at least one strict inequality.

Remark 4.5.

1. The Theorem 4.4 also holds true under generalized convexity assumptions, *i.e.* $\left(\lambda^T f + \sum_{t\in T} v_t g_t\right)$ is pseudoconvex on C, $f_i, i \in M$ and $g_t, t \in T$ are regular on C.
2. If $\lambda \geqq 0$, the pseudoconvex assumption mentioned above is replaced by strict pseudoconvexity.

Using Theorems 3.9 and 4.4, we establish ϵ-strong duality between (MP) and $(\text{MD})_W$.

Theorem 4.6 (ϵ-Strong duality)**.** *Let $\epsilon \in \mathbb{R}^m_+$. Assume that f_i, $i \in M$ and g_t, $t \in T$ are convex functions, (FM) and ϵ-weak duality hold. If $\bar{x} \in C$ is an ϵ-efficient solution of (MP) then $(\bar{x}, \lambda, v)$ is 2ϵ-efficient solution for $(MD)_W$.*

Special case 2. The other special case is $v = 0$. Then (MD) is equal to Mond–Weir type dual problem which is denoted as follows:

$$
\begin{array}{lll}
(\text{MD})_{\text{M}} & \text{Maximize} & f(y) \\
 & \text{subject to} & 0 \in \sum_{i\in M} \lambda_i \partial_{\widetilde{\epsilon_{0i}}} f_i(y) + \sum_{t\in T} v_t \partial_{\widetilde{\epsilon_{1t}}} g_t(y) + N_{\widetilde{\epsilon_q}}(C; y), \\
 & & \sum_{i\in M} \lambda_i \widetilde{\epsilon_{0i}} + \sum_{t\in T} v_t \widetilde{\epsilon_{1t}} + \widetilde{\epsilon_q} - \lambda^T \epsilon \leqq 0 \\
 & & v_t g_t(y) \geqq 0, \\
 & & \lambda > 0, \lambda^T e = 1, e = (1, \ldots, 1) \in \mathbb{R}^m, \\
 & & (y, \lambda, v) \in C \times \mathbb{R}^m \times \mathbb{R}^{(T)}_+.
\end{array}
$$

We can also obtain the following three theorems immediately.

Theorem 4.7 (ϵ-Weak duality)**.** *Let x and (y, λ, v) be a feasible solution to (MP) and $(\text{MD})_M$ respectively. Assume that f_i, $i \in M$ and g_t, $t \in T$ are convex on C. Then the following cannot hold*

$$
f_i(x) \leqq f_i(y) - \epsilon_i, \quad \text{for all } i \in M
$$

with at least one strict inequality.

Remark 4.8. Theorem 4.7 holds true under generalized convexity assumptions, *i.e.* $\lambda^T f$ is pseudoconvex and $\sum_{t\in T} v_t g_t$ is quasiconvex on C. It should be noticed that here we consider not sum of functions like in Theorem 4.3 but make different assumption for $\lambda^T f$ and $\sum_{t\in T} v_t g_t$.

Using Theorems 3.9 and 4.7, we can establish ϵ-strong duality. It should be mentioned that in contrast to ϵ-strong duality of mixed type and Wolfe type, $\bar{x}$ is an ϵ-efficient solution for $(MD)_M$, not 2ϵ-efficient. To show that, we provide the proof.

Theorem 4.9 (ϵ-Strong duality)**.** *Let $\epsilon \in \mathbb{R}^m_+$. Assume that f_i, $i \in M$ and g_t, $t \in T$ are convex functions, (FM) and ϵ-weak duality hold. If $\bar{x} \in C$ is an ϵ-efficient solution of (MP) then $(\bar{x}, \lambda, v)$ is an ϵ-efficient solution for $(MD)_M$.*

Proof. Since $\bar{x}$ is an ϵ-efficient solution for (MP), then by Theorem 3.9, there exist λ_i, $i \in M$ and $v \in \mathbb{R}^{(T)}_+$ such that

$$
0 \in \sum_{i\in M} \lambda_i \partial_{\widetilde{\epsilon_{0i}}} f_i(\bar{x}) + \sum_{t\in T(v)} v_t \partial_{\widetilde{\epsilon_{1t}}} g_t(\bar{x}) + N_{\widetilde{\epsilon_q}}(C; \bar{x})
$$

holds, then $(\bar{x}, \lambda, v)$ is feasible for $(\text{MD})_M$.

Suppose that $(\bar{x}, \lambda, v)$ is not ϵ-efficient solution for $(\text{MD})_M$, then there exists (x^*, λ^*, v^*) such that:

$$
f(x^*) - \epsilon \leqq f(\bar{x})
$$

cannot hold which contradicts ϵ-weak duality (Thm. 4.7). □

Remark 4.10. Compare to Liu [12] who established Wolfe type ϵ-duality, we derived mixed type ϵ-duality, that covers both Wolfe and Mond–Weir types. We established ϵ-weak duality and, by using our main result, *i.e.* Theorem 3.9, ϵ-strong duality for mentioned (MP) and (MD), including duality results for $(\text{MD})_W$ and $(\text{MD})_M$.

5. Conclusions

In this paper we established necessary and sufficient conditions for $(P_j(\bar{x}))$ and then, by using scalarization method due to Chankong and Haimes [1], we could derive ϵ-optimality conditions for (MP) as well as extended to semi-infinite case under FM constraint qualification due to Goberna *et al.* [7] and convexity assumptions. Moreover, we established another sufficient condition for (MP) with less strict assumption, *i.e.* generalized convexity. In addition we proposed both weak and strong ϵ-duality theorem under convexity assumption for mixed type dual problem. Wolfe type and Mond–Weir type dual problems were also considered as special cases. In addition, we gave another ϵ-weak duality theorems under generalized convexity assumption.

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