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FUZZY INTEGER-VALUED DATA ENVELOPMENT ANALYSIS☆

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Abstract. In conventional data envelopment analysis (DEA) models, the efficiency of decision making units (DMUs) is evaluated while data are precise and continuous. Nevertheless, there are occasions in the real world that the performance of DMUs must be calculated in the presence of vague and integer-valued measures. Therefore, the current paper proposes fuzzy integer-valued data envelopment analysis (FIDEA) models to determine the efficiency of DMUs when fuzzy and integer-valued inputs and/or outputs might exist. To illustrate, fuzzy number ranking and graded mean integration representation methods are used to solve some integer-valued data envelopment analysis models in the presence of fuzzy inputs and outputs. Two examples are utilized to illustrate and clarify the proposed approaches. In the provided examples, two cases are discussed. In the first case, all data are as fuzzy and integer-valued measures while in the second case a subset of data is fuzzy and integer-valued. The results of the proposed models indicate that the efficiency scores are calculated correctly and the projections of fuzzy and integer factors are determined as integer values, while this issue has not been discussed in fuzzy DEA, and projections may be estimated as real-valued data.

Mathematics Subject Classification. 90C10, 90C70

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1. Introduction

Data envelopment analysis (DEA), initially proposed by Charnes *et al.* [4], is a methodology for measuring the relative efficiency of decision making units (DMUs) that consume multiple inputs and produce multiple outputs. In traditional DEA models, inputs and outputs are usually considered as precise and continuous factors. Nevertheless, in real world applications, situations exist in which the performance of DMUs must be evaluated in the presence of imprecise and integer-valued data. For instance, the number of employees and the number of the correct operations in banks can be considered as fuzzy and integer numbers. Uncertainty in the number of bank employees can occur, considering ongoing reforms in banks. Actually, these situations happen in imprecise-knowledge-based systems. As another example, in evaluating the efficiency of suppliers in sustainable supply chain management, factors such as the number of shipments to arrive on time and the number of bills received from the supplier without errors can be taken as fuzzy and integer numbers. Also, factors like the number of doctors, the number of nurses, and the number of beds can be considered as fuzzy and integer-valued

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measures in evaluating the performance of health care systems. Indeed, as Dotoli *et al.* [5] argued several reforms of local health care systems lead to a significant uncertainty about data. To illustrate, Dotoli *et al.* [5] proposed a cross-efficiency fuzzy DEA method for evaluating the performance of DMUs without considering the number of doctors, the number of nurses, and the number of beds as integer-valued data. Thus and as mementioned before, there are many situations in the real world in which the performance of organizations and sectors must be evaluated in the presence of uncertain and integer-valued data.

In the DEA literature, there are studies that deal with integer-valued inputs and outputs. Ehrgott and Tind [6] proposed a column generation technique for solving the free replicability (FR) model. To explain, the FR model that has been suggested by Tulknes [35] is a mixed integer programming problem (a DEA model) with integer variables and one continuous variable. Lozano and Villa [27] introduced new DEA concepts and models for investigating integer-valued inputs and outputs. Afterwards, Kuosmanen and Kazemi Matin [22] suggested a new axiomatic foundation for integer-valued DEA models and indicated consistency between the production possibility set (PPS) of Lozano and Villa [27] and the new set of axioms. Moreover, they provided a mixed integer linear programming problem to determine the efficiency under constant returns to scale assumption. Then, Kazemi Matin and Kuosmanen [19] extended the axiomatic foundation for the integer-valued DEA model under different assumptions on returns to scale; *i.e.*, variable returns to scale, non-decreasing returns to scale, and non-increasing returns to scale. Kazemi Matin and Emrouznejad [18] also introduced an integer-valued data envelopment analysis model when outputs are bounded. Khezrimotlagh *et al.* [20] claimed that Kuosmanen and Kazemi Matin's models [19, 22] may not be stronger than Lozano and Villa's model [27]. Nevertheless, Jie *et al.* [15] revised and improved Kuosmanen and Kazemi Matin's model [22] and stated that the model can perform the calculations correctly. In this study, Jie *et al.*'s models [15] are extended for incorporating fuzzy factors.

In the DEA context, models with imprecise and fuzzy measures can be found. Hatami-Marbini *et al.* [11] provided a review and taxonomy of such models. They categorized the fuzzy DEA papers published into four groups, namely as the tolerance approach, the α-level based approach, the fuzzy ranking approach, and the possibility approach. Afterwards, Emrouznejad *et al.* [7] extended the mentioned categories and added two new groups including the fuzzy arithmetic and the fuzzy random/type-2 fuzzy set. The tolerance approach is one of the initial fuzzy DEA models [16, 32, 33]. Lack of consideration of fuzzy coefficients is the main disadvantage of this method. Also, several studies have used the α-level based approach for investigating fuzzy factors in DEA models [1, 2, 17, 28, 29]. Furthermore, the fuzzy ranking approach, originally proposed by Guo and Tanaka [9], is another approach for handling fuzzy measures that is a popular fuzzy DEA technique among researchers [12, 21, 25, 34, 37]. Guo and Tanaka's approach [9] computes two linear programming problems. Alternatively, the possibility approach is also found in some previous studies [10, 26]. In addition to the aforementioned methods, substitute techniques can be found, see [7, 30, 31, 39, 42]. Nevertheless, the majority of models built based on these methodologies require substantial computational efforts. For example, as mentioned in [8], α-level-based and fuzzy ranking approaches require users to solve a sequence of linear programming models due to different optimal solutions for each α-level or the possibility approach introduced by Lertworasirikul *et al.* [26] that shows all fuzzy constraints with different possibility levels. Therefore, we apply two approaches in the current study, a fuzzy number ranking method and the graded mean integration representation method, due to their reasonable computational efforts and simplicity.

Although some research has been carried out on fuzzy DEA models, no study has been found concerning fuzzy integer-valued inputs and outputs in DEA models. Hence, in the current research here, the fuzzy number ranking method and the graded mean integration representation method are utilized to evaluate the efficiency of DMUs in the presence of fuzzy and integer-valued factors.

Fuzzy integer linear programming problems have been studied by some authors. Herrera and Verdegay [13] considered integer programming problems in which some lack of precision exist; they focused on Boolean fuzzy linear programming problems. Then, in another study [14] they pointed to some models for investigating fuzzy integer linear programming. However, as far we know, there is no research taking into account fuzzy and integer inputs and/or outputs in the DEA context.

In these situations, projections of DMUs (virtual DMUs) must be obtained as integer values. For this reason, the current paper attempts to provide models and methods for estimating the relative efficiency of DMUs where fuzzy data with integer values exist. Triangular fuzzy numbers are used to estimate uncertain inputs and outputs. Indeed, fuzzy data are taken in some integer-valued DEA models and different approaches; a fuzzy number ranking method and the graded mean integration representation method are used for solving fuzzy integer-valued DEA models. Due to the kind of methods used to investigate fuzzy data, our proposed approaches have reasonable computational efforts, and it is only needed to solve a mixed integer linear programming problem for evaluating the efficiency of each DMU. Furthermore, previous studies of fuzzy DEA have not handled integer-valued measures and their projections whereas integer-valued projected targets are obtained for fuzzy integer-valued factors in this study. To demonstrate the potential of the approach and the suitability for application, two examples are investigated. Two cases are deemed: first, all data are considered as fuzzy integer-valued factors; second, the subset of fuzzy integer-valued measures is taken.

The paper is unfolded as follows. Section 2 reviews the basic ideas of integer-valued DEA and fuzzy integer-valued numbers. In Section 3, some fuzzy integer-valued DEA models are introduced, and approaches are stated for solving them. Section 4 provides two examples to clarify and validate approaches. Conclusions are presented in Section 5.

2. Preliminaries

Firstly, the definition of integer-valued DEA is clarified in this section. Secondly, basic concepts and fuzzy integer-valued numbers are explained and described.

2.1. Integer-valued DEA

Assume there are n DMUs, $DMU_j \;\; (j = 1, \ldots, n)$, with m inputs x_{ij} $(i = 1, \ldots, m)$ and s outputs y_{rj} $(r = 1, \ldots, s)$. In conventional DEA models, all input and output measures are deemed as real-valued factors. Therefore, the efficiency scores of DMUs are estimated while the projections of DMUs (virtual DMUs) obtain real values. However, factors like the number of nurses, number of buses, and number of transactions are integer-valued. Thus, with considering x_{ij} $(i = 1, \ldots, m)$ and y_{rj} $(r = 1, \ldots s)$ as integer-valued factors of $DMU_j \;\; (j = 1, \ldots, n)$, some DEA models have been extended and modified for obtaining integer targets for integer-valued factors by [15, 18, 19, 22]. In the current study, the approach of Jie *et al.* [15] is generalized for situations in which fuzzy integer-valued factors are present. In the next subsection, a fuzzy integer-valued number is defined.

2.2. Fuzzy integer-valued numbers and basic concepts

Let R be the real number set and I be the integer number set.

Definition 2.1. A fuzzy set $u : R \to [0, 1]$ is called a fuzzy integer if its support is a closed integer interval (denoted as $\langle \underline{u}(0), \bar{u}(0)\rangle$) and satisfies the following [38]

(1) u is normal; *i.e.*, there exists $x' \in \langle \underline{u}(0), \bar{u}(0)\rangle$such that $u(x') = 1$,
(2) $u(x_i) \leq u(x_j)$for any $x_i, x_j \in \langle \underline{u}(0), x'\rangle$with $x_i \leq x_j$,
(3) $u(x_i) \geq u(x_j)$for any $x_i, x_j \in \langle x', \bar{u}(0)\rangle$with $x_i \leq x_j$.

Notice that a closed integer interval is denoted by $\langle s_1, s_2\rangle = \{x \in I | s_1 \leq x \leq s_2\}$ for any $s_1, s_2 \in I$ and $s_1 \leq s_2$. Also, the collection of all fuzzy integers is shown by FI.

Remark 2.2. Consider the closed integer interval $[u]^r = \langle \underline{u}(r), \bar{u}(r)\rangle$. If $u, v \in FI, k \in R$, then for any $r \in [0, 1]$ [38],

(1) $[u + v]^r = [u]^r + [v]^r$,
(2) $[ku]^r = k\, [u]^r$,
(3) $[uv]^r = [u]^r\, [v]^r$.

Remark 2.3. If $u, v \in FI$, $k \in R$; [38] then

$$\begin{aligned} &u+v \in FI,\\ &ku \in FI,\\ &uv \in FI \end{aligned}$$

that for any $r \in [0,1]$, it is defined

$$\begin{aligned} u+v &= \langle (\underline{u+v})(r), (\overline{u+v})(r)\rangle = \langle (\underline{u})(r) + (\underline{v})(r), (\bar{u})(r) + (\bar{v})(r)\rangle \\ ku &= \langle \lfloor k\underline{u}(r)\rceil, \lfloor k\bar{u}(r)\rceil\rangle fork \geq 0,\\ ku &= \langle \lfloor k\bar{u}(r)\rceil, \lfloor k\underline{u}(r)\rceil\rangle fork < 0,\\ uv &= \langle \min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}\rangle \end{aligned}$$

in which $\lfloor x\rceil$ indicates the integer number that is obtained by arithmetic rounding to x.

Definition 2.4. Assume s_0, s_1, t_1 and $t_0 \in I$ with $s_0 \leq s_1 \leq t_1 \leq t_0$, and $\underline{m}, \bar{m} \in I$ [38]. If the fuzzy set $u : R \to [0,1]$ is defined as

$$u(x) = \begin{cases} 1 & \text{if } x \in \langle s_1, t_1\rangle \\ \frac{x-s_0}{s_1-s_0} & \text{if } x \in \langle \underline{m}, s_1\rangle \\ \frac{t_0-x}{t_0-t_1} & \text{if } x \in \langle t_1, \bar{m}\rangle \\ 0 & \text{if } x \in \langle \underline{m}, \bar{m}\rangle \end{cases}$$

where $s_0 \leq \underline{m} \leq s_1$ and $t_1 \leq \bar{m} \leq t_0$; then u is a trapezoidal fuzzy integer. If $s_1 = t_1$, then we have a triangular fuzzy integer. Readers can refer to [38] for more information.

3. Fuzzy integer-valued DEA models

In this section, models are proposed for evaluating the efficiency of DMUs where fuzzy integer-valued factors are present. Suppose n DMUs, DMU_j $(j = 1, \ldots, n)$, exist that use m inputs x_{ij} $(i = 1, \ldots, m)$and produce s outputs y_{rj} $(r = 1, \ldots, s)$. Charnes *et al.* [4] proposed the following model, referred to as the CCR model, for evaluating the efficiency of DMUs with continuous and precise data.

$$\begin{aligned} &\text{Min } \theta \\ &\text{s.t. } y_{ro} \leq \sum_{j=1}^{n} y_{rj}\lambda_j,\ r = 1, \ldots, s,\\ &\qquad \theta x_{io} \geq \sum_{j=1}^{n} x_{ij}\lambda_j,\ i = 1, \ldots, m, \quad \lambda_j \geq 0,\ j = 1, \ldots, n. \end{aligned} \tag{3.1}$$

in which $\lambda_j (j = 1, \ldots, n)$are intensity variables. x_{io} and y_{ro} are inputs and outputs of unit under evaluation, DMU_o. θ is a measure of efficiency. In the presence of imprecise inputs and outputs as triangular fuzzy numbers, *i.e.* $\tilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3})$, $\tilde{y}_{rj} = (y_{rj1}, y_{rj2}, y_{rj3})$ in the model (3.1) that $x_{ij1} \geq 0$ and $y_{rj1} \geq 0$, a fuzzy method should be used to evaluate the efficiency of entities. Note that in this case inputs and outputs of DMU_o are indicated by $\tilde{x}_{io}$ and $\tilde{y}_{ro}$, respectively.

For handling fuzzy measures and for computing the DEA models with fuzzy inputs and outputs, we use the graded mean integration representation method and the fuzzy number ranking method that are described as follows:

Definition 3.1. Consider a triangular fuzzy number $\tilde{A} = (a, b, c)$, the graded mean integration representation $\tilde{A}$ can be defined as $(a + 4b + c)/6$.

Definition 3.2. Given a triangular fuzzy number $\tilde{A} = (a, b, c)$, according to a ranking function that is the first index of Yager [40, 41], the fuzzy number can be estimated by the following crisp number:

$$b + (d_3 - d_1)/3$$

in which $d_3 = c - b$, $d_1 = b - a$.

Indeed, the aforementioned methods are used due to rational computational efforts and simplicity. Thus, according to the Definition 3.1, the CCR model with fuzzy measures can be substituted with model (3.2) as follows:

$$\begin{aligned}
&\text{Min } \theta \\
&\text{s.t. } \left(\frac{4y_{ro2} + y_{ro1} + y_{ro3}}{6}\right) \leq \sum_{j=1}^{n} \left(\frac{4y_{rj2} + y_{rj1} + y_{rj3}}{6}\right) \lambda_j,\ r = 1, \ldots, s, \\
&\quad \theta \left(\frac{4x_{io2} + x_{io1} + x_{io3}}{6}\right) \geq \sum_{j=1}^{n} \left(\frac{4x_{ij2} + x_{ij1} + x_{ij3}}{6}\right) \lambda_j,\ i = 1, \ldots, m, \quad \lambda_j \geq 0,\ j = 1, \ldots, n.
\end{aligned} \tag{3.2}$$

Also, due to Definition 3.2, $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ in the fuzzy CCR model can be changed to $x_{ij2} + (d_{x_{ij3}} - d_{x_{ij1}})/3$ and $y_{rj2} + (d_{y_{rj3}} - d_{y_{rj1}})/3$ that $d_{x_{ij3}} = x_{ij3} - x_{ij2}$, $d_{x_{ij1}} = x_{ij2} - x_{ij1}$, $d_{y_{rj3}} = y_{rj3} - y_{rj2}$ and $d_{y_{rj1}} = y_{rj2} - y_{rj1}$. Therefore, the fuzzy CCR model can be rewritten as follows:

$$\begin{aligned}
&\text{Min } \theta \\
&\text{s.t. } \left(y_{ro2} + \frac{d_{y_{ro3}} - d_{y_{ro1}}}{3}\right) \leq \sum_{j=1}^{n} \left(y_{rj2} + \frac{d_{y_{rj3}} - d_{y_{rj1}}}{3}\right) \lambda_j,\ r = 1, \ldots, s, \\
&\quad \theta \left(x_{io2} + \frac{d_{x_{io3}} - d_{x_{io1}}}{3}\right) \geq \sum_{j=1}^{n} \left(x_{ij2} + \frac{d_{x_{ij3}} - d_{x_{ij1}}}{3}\right) \lambda_j,\ i = 1, \ldots, m, \quad \lambda_j \geq 0,\ j = 1, \ldots, n.
\end{aligned} \tag{3.3}$$

Notice that in the presence of integer variables in the fuzzy linear programming, Definitions 3.1 and 3.2 will be likewise correct. Readers can refer to [14, 23, 36] for more information.

Nevertheless, models (3.2) and (3.3) are not suitable for calculating the efficiency of DMUs where fuzzy and integer measures are present. Actually, the projection of a DMU with integer inputs/outputs may be obtained as non-integer values. The purpose of providing models (3.2) and (3.3) is to compare their results with models with fuzzy and integer factors that will be proposed in this study. Therefore, we focus on approaches and models to assess the efficiency of DMUs with integer fuzzy data.

For this purpose, we deem all inputs and outputs are as fuzzy (triangular fuzzy numbers) and integer-valued measures in this stage. To illustrate, we consider n DMUs, DMU_j $(j = 1, \ldots, n)$, with m triangular fuzzy inputs $\tilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3})(i = 1, \ldots, m)$ and s triangular fuzzy outputs $\tilde{y}_{rj} = (y_{rj1}, y_{rj2}, y_{rj3})$ $(r = 1, \ldots, s)$. Furthermore, $\tilde{x}_{io}$ and $\tilde{y}_{ro}$ are used to denote inputs and outputs of the unit under evaluation, DMU_o. Thus, for analyzing the performance of firms in the presence of fuzzy integer-valued data, the following problem is suggested:

$$
\begin{aligned}
&\text{Min } \theta - \varepsilon \left(\sum_{r=1}^{s} s_r^{I+} + \sum_{i=1}^{m} s_i^{I-} - \sum_{i=1}^{m} s_i^{-} - \sum_{r=1}^{s} s_r^{+} \right),\\
&\text{s.t. } y_r + s_r^{+} = \sum_{j=1}^{n} \tilde{y}_{rj} \lambda_j,\ r = 1, \ldots, s,\\
&\tilde{y}_{ro} + s_r^{I+} = y_r,\ r = 1, \ldots, s,\\
&x_i - s_i^{-} = \sum_{j=1}^{n} \tilde{x}_{ij} \lambda_j,\ i = 1, \ldots, m,\\
&\theta \tilde{x}_{io} - s_i^{I-} = x_i,\ i = 1, \ldots, m,\\
&x_i, y_r \in \mathbb{Z}_+,\ \lambda_j \geq 0,\\
&s_r^{+} \geq 0, s_i^{-} \geq 0, s_i^{I-} \geq 0, s_r^{I+} \geq 0,\\
&i = 1, \ldots, m, r = 1, \ldots, s, j = 1, \ldots, n.
\end{aligned} \tag{3.4}
$$

in which, s_r^{I+}, s_i^{I-}, s_i^{-} and s_r^{+} are slack variables. ε is a non-Archimedean infinitesimal. x_i and y_r are positive integer-valued variables that show integer-valued reference points for inputs and outputs, respectively. θ is an efficiency measure. Also, λ_j indicates intensity weights. Model (3.4) is an extension of Jie *et al.*'s method [15], evaluating the efficiency of DMUs when integer-valued imprecise data exist.

For transforming the above fuzzy mixed integer linear programming to a mixed integer linear programming problem, we utilize Definitions 3.1 and 3.2 due to reasonable computational efforts and simplicity. However, the approach can be adapted to use different fuzzy numbers ranking methods.

First, the graded mean integration representation method is applied for converting the above model to a mixed integer linear programming problem. Therefore, model (3.4) can be substituted with the following mixed integer linear programming.

$$
\begin{aligned}
&\text{Min } \theta - \varepsilon \left(\sum_{r=1}^{s} s_r^{I+} + \sum_{i=1}^{m} s_i^{I-} - \sum_{i=1}^{m} s_i^{-} - \sum_{r=1}^{s} s_r^{+} \right),\\
&\text{s.t. } y_r + s_r^{+} = \sum_{j=1}^{n} \left(\frac{4y_{rj2} + y_{rj1} + y_{rj3}}{6} \right) \lambda_j,\ r = 1, \ldots, s,\\
&\left(\frac{4y_{ro2} + y_{ro1} + y_{ro3}}{6} \right) + s_r^{I+} = y_r,\ r = 1, \ldots, s,\\
&x_i - s_i^{-} = \sum_{j=1}^{n} \left(\frac{4x_{ij2} + x_{ij1} + x_{ij3}}{6} \right) \lambda_j,\ i = 1, \ldots, m,\\
&\theta \left(\frac{4x_{io2} + x_{io1} + x_{io3}}{6} \right) - s_i^{I-} = x_i,\ i = 1, \ldots, m,\\
&x_i, y_r \in \mathbb{Z}_+,\ \lambda_j \geq 0,\\
&s_r^{+} \geq 0, s_i^{-} \geq 0, s_i^{I-} \geq 0, s_r^{I+} \geq 0,\\
&i = 1, \ldots, m, r = 1, \ldots, s, j = 1, \ldots, n.
\end{aligned} \tag{3.5}
$$

Indeed, fuzzy sets $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ are substituted with $(4x_{ij2} + x_{ij1} + x_{ij3})/6$ and $(4y_{rj2} + y_{rj1} + y_{rj3})/6$, respectively. To explain, the weighted average of the most possible value, the pessimistic and optimistic values are applied to represent the fuzzy consumed inputs and the fuzzy produced outputs according to [24]; that is,

$w_1 x_{ij2} + w_2 x_{ij1} + w_2 x_{ij3}$ and $w_1 y_{rj2} + w_2 y_{rj1} + w_3 y_{rj3}$ where $w_1 + w_2 + w_3 = 1$. We use $w_1 = 1/6$, $w_2 = 4/6$, $w_3 = 1/6$ in which the weights can be changed subjectively. As mentioned in [24], the reason of using the above weighted average values is x_{ij1} and y_{rj1} are too pessimistic and x_{ij3} and y_{rj3} are too optimistic. Of course, these boundary values provided us boundary solutions. Besides, the most possible values are often the most important ones. Thus, more weights should be assigned.

As another approach, according to the first index of Yager [40, 41] and Definition 3.2, model (3.4) can be transformed into the following mixed integer linear programming problem:

$$
\begin{aligned}
&\text{Min } \theta - \varepsilon \left(\sum_{r=1}^{s} s_r^{I+} + \sum_{i=1}^{m} s_i^{I-} - \sum_{i=1}^{m} s_i^{-} - \sum_{r=1}^{s} s_r^{+} \right), \\
&\text{s.t. } y_r + s_r^{+} = \sum_{j=1}^{n} \left(y_{rj2} + \frac{d_{y_{rj3}} - d_{y_{rj1}}}{3} \right) \lambda_j, \ r = 1, \ldots, s, \\
&\left(y_{ro2} + \frac{d_{y_{ro3}} - d_{y_{ro1}}}{3} \right) + s_r^{I+} = y_r, \ r = 1, \ldots, s, \\
&x_i - s_i^{-} = \sum_{j=1}^{n} \left(x_{ij2} + \frac{d_{x_{ij3}} - d_{x_{ij1}}}{3} \right) \lambda_j, \ i = 1, \ldots, m, \\
&\theta \left(x_{io2} + \frac{d_{x_{io3}} - d_{x_{io1}}}{3} \right) - s_i^{I-} = x_i, \ i = 1, \ldots, m, \\
&x_i, y_r \in \mathbb{Z}_+, \ \lambda_j \geq 0, \\
&s_r^{+} \geq 0, s_i^{-} \geq 0, s_i^{I-} \geq 0, s_r^{I+} \geq 0, \\
&i = 1, \ldots, m, r = 1, \ldots, s, j = 1, \ldots, n.
\end{aligned} \tag{3.6}
$$

that $d_{x_{ij3}} = x_{ij3} - x_{ij2}$, $d_{x_{ij1}} = x_{ij2} - x_{ij1}$, $d_{y_{rj3}} = y_{rj3} - y_{rj2}$ and $d_{y_{rj1}} = y_{rj2} - y_{rj1}$. Actually, $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ are replaced with $x_{ij2} + (d_{x_{ij3}} - d_{x_{ij1}})/3$ and $y_{rj2} + (d_{y_{rj3}} - d_{y_{rj1}})/3$, respectively.

Note that by considering (a, b, c) as a triangular integer fuzzy number, provided that $z_1 = (a + 4b + c)/6$ and $z_2 = b + ((c - b) - (b - a))/3$ are obtained as non-integer values, we will round them to the closest integer values. Actually, the influence of rounding to z_1 and z_2 is almost negligible and we consider $\lfloor z_1 \rceil$ and $\lfloor z_2 \rceil$.

Proposition 3.3. *The solution set is the same for both models* (3.4) *and* (3.5)*.*

Proof. Assume S_1 and S_2 be the set of all feasible solutions of models (3.4) and (3.5), respectively. $P \in S_1$ if and only if

$$
\begin{aligned}
&y_r \leq \sum_{j=1}^{n} \tilde{y}_{rj} \lambda_j, \ r = 1, \ldots, s, \\
&\tilde{y}_{ro} \leq y_r, \ r = 1, \ldots, s, \\
&x_i \geq \sum_{j=1}^{n} \tilde{x}_{ij} \lambda_j, \ i = 1, \ldots, m, \\
&\theta \tilde{x}_{io} \geq x_i, \ i = 1, \ldots, m, \\
&x_i, y_r \in \mathbb{Z}_+, \ \lambda_j \geq 0, \\
&i = 1, \ldots, m, r = 1, \ldots, s, j = 1, \ldots, n
\end{aligned}
$$

if and only if

$$\begin{aligned}
&(y_r, y_r, y_r) \leq \sum_{j=1}^{n} (y_{rj1}, y_{rj2}, y_{rj3})\lambda_j,\ r = 1, \ldots, s,\\
&(y_{ro1}, y_{ro2}, y_{ro3}) \leq (y_r, y_r, y_r),\ r = 1, \ldots, s,\\
&(x_i, x_i, x_i) \geq \sum_{j=1}^{n} (x_{ij1}, x_{ij2}, x_{ij3})\lambda_j,\ i = 1, \ldots, m,\\
&\theta(x_{io1}, x_{io2}, x_{io3}) \geq (x_i, x_i, x_i),\ i = 1, \ldots, m,\\
&x_i, y_r \in \mathbb{Z}_+,\ \lambda_j \geq 0,\\
&i = 1, \ldots, m, r = 1, \ldots, s, j = 1, \ldots, n
\end{aligned}$$

in which $y_r = (y_r, y_r, y_r)$ and $x_i = (x_i, x_i, x_i)$, if and only if

$$\begin{aligned}
&(4y_r + y_r + y_r)/6 \leq \sum_{j=1}^{n} ((4y_{rj2} + y_{rj1} + y_{rj3})/6)\lambda_j,\ r = 1, \ldots, s,\\
&((4y_{ro2} + y_{ro1} + y_{ro3})/6) \leq (4y_r + y_r + y_r)/6,\ r = 1, \ldots, s,\\
&(4x_i + x_i + x_i)/6 \geq \sum_{j=1}^{n} ((4x_{ij2} + x_{ij1} + x_{ij3})/6)\lambda_j,\ i = 1, \ldots, m,\\
&\theta((4x_{io2} + x_{io1} + x_{io3})/6) \geq (4x_i + x_i + x_i)/6,\ i = 1, \ldots, m,\\
&x_i, y_r \in \mathbb{Z}_+,\ \lambda_j \geq 0,\\
&i = 1, \ldots, m, r = 1, \ldots, s, j = 1, \ldots, n
\end{aligned}$$

if and only if $P \in S_2$. Thus, $S_1 = S_2$. □

Correspondingly, it can be shown that the solution set is equivalent for models (3.4) and (3.6).

Definition 3.4. DMU_o in models (3.5) and (3.6) is said to be the efficient if no other integer-valued point dominates it.

Proposition 3.5. *DMU_o in models proposed, models (3.5) and (3.6), is the efficient if and only if $\theta^* = 1$, $s_r^{*I+} = 0$, $s_r^{*+} = 0, \forall r, s_i^{*I-} = 0$ and $s_i^{*-} = 0$, $\forall i$.*

Proof. At first we assume DMU_o is efficient but $\theta_o^* < 1$ or $\exists r, s_r^{*I+} > 0$, or $\exists r, s_r^{*+} > 0$, or $\exists i, s_i^{*I-} > 0$ or $\exists i, s_i^{*-} > 0$. Thus, the integer-valued target obtained will dominate DMU_o. So, DMU_o will not be efficient according to Definition 3.4 that is a contradiction.

Next, we deem $\theta_o^* = 1$, $s_r^{I+} = 0$, $s_r^{+} = 0, \forall r, s_i^{I-} = 0$ and $s_i^{-} = 0$, $\forall i$, but DMU_o is inefficient. Therefore, there is an integer-valued target point like $(x', y') \neq (x_o, y_o)$ that dominates DMU_o, *i.e.* $x' \leq x_o, y' \geq y_o$. Also, (x', y') belongs to the possibility production set. Thus, a vector like $\lambda_j, (j = 1, \ldots, n)$exists such that $x'_i - s_i^- = \sum_j \lambda_j x''_{ij} (i = 1, \ldots, m)$ and $y'_r + s_r^+ = \sum_j \lambda_j y''_{rj} (r = 1, \ldots, s)$and

$$\begin{aligned}
&y'_r = y''_{ro} + s_r^{I+},\ r = 1, \ldots, s,\\
&x'_i = \theta' x''_{io} - s_i^{I-},\ i = 1, \ldots, m,
\end{aligned}$$

Notice that in model (3.5) $y''_{rj} = (4y_{rj2} + y_{rj1} + y_{rj3})/6$ and in model (3.6) $y''_{rj} = (y_{rj2} + (d_{y_{rj3}} - d_{y_{rj1}})/3)$. In similar ways x''_{ij} can be defined in models (3.5) and (3.6). As mentioned (x', y') are integer-valued and

$(x', y') \neq (x_o, y_o)$. Therefore, $\theta' < 1$ and/or $\exists r, s_r^{'I+} > 0$, $\exists r, s_r^{'+} > 0$, $\exists i, s_i^{'I-} > 0$, $\exists i, s_i^{'-} > 0$. It is clear that

$$\theta' - \varepsilon \left(\sum_{r=1}^{s} s_r^{'I+} + \sum_{i=1}^{m} s_i^{'I-} - \sum_{i=1}^{m} s_i^{'-} - \sum_{r=1}^{s} s_r^{'+} \right) < \theta_o^* - \varepsilon \left(\sum_{r=1}^{s} s_r^{*I+} + \sum_{i=1}^{m} s_i^{*I-} - \sum_{i=1}^{m} s_i^{*-} - \sum_{r=1}^{s} s_r^{*+} \right),$$

It means that a feasible solution exists that has a better objective function in comparison with the optimal solution that is a contradiction. As a result, reduction ad absurdum is invalid and this completes the proof. □

Proposition 3.6. *Models (3.5) and (3.6) are always feasible and their objective function values are bounded.*

Proof. Consider an arbitrary solution for model (3.5) as follows: $\theta_o = 1$, $s_r^{I+} = 0$, $s_r^{+} = 0, \forall r, s_i^{I-} = 0$ and $s_i^{-} = 0$, $\forall i$, $\lambda_o = 1$, $\lambda_j = 0, j \neq o$, $x_i = \lfloor (4x_{io2} + x_{io1} + x_{io3})/6 \rceil$ and $y_r = \lfloor (4y_{ro2} + y_{ro1} + y_{ro3})/6 \rceil$. It is obvious that model (3.5) is always feasible. Also, the objective function of model (3.5) is the minimization form. Thus, the optimal value of model (3.5) that is θ_o^* is not greater than the feasible solution $\theta_o = 1$. In other words, $\theta_o^* \leq \theta_o = 1$. Moreover, $0 < \theta_o^*$. This is because the input and output vectors have at least a nonzero component. Assume $\theta_o = 0$, from the constraints of model (3.5) $\lambda_j = 0$ and $y_r \leq 0$ is obtained. But we have $y_r \in \mathbb{Z}_+$. Thus, $y_r = 0$, while it has been assumed input and output vectors are nonzero, at least in one component. As a result, reduction ad absurdum is invalid and $0 < \theta_o^*$. So $0 < \theta_o^* \leq 1$ that means model (3.5) is bounded. Similarly, it can be proved that model (3.6) is feasible and bounded. □

Models (3.5) and (3.6) are used only when fuzzy integer-valued data exist. Nevertheless, there are situations in the real world that both integer-valued factors and real-valued factors are present in a fuzzy environment. For addressing these cases, the following model is proposed:

$$\begin{aligned}
&\mathrm{Min}\theta - \varepsilon \left(\sum_{r \in O^I} s_r^{I+} + \sum_{i \in I^I} s_i^{I-} + \sum_{i \in I^{NI}} s_i^{-} + \sum_{r \in O^{NI}} s_r^{+} - \sum_{i \in I^I} s_i^{-} - \sum_{r \in O^I} s_r^{+} \right) \\
&\text{s.t.}\, \tilde{y}_{ro} + s_r^{+} = \sum_{j=1}^{n} \tilde{y}_{rj} \lambda_j, \;\; r \in O^{NI}, \\
&y_r + s_r^{+} = \sum_{j=1}^{n} \tilde{y}_{rj} \lambda_j, r \in O^I, \\
&\tilde{y}_{ro} + s_r^{I+} = y_r, \;\; r \in O^I, \\
&\theta \tilde{x}_{io} - s_i^{-} = \sum_{j=1}^{n} \tilde{x}_{ij} \lambda_j, \;\; i \in I^{NI}, \\
&x_i - s_i^{-} = \sum_{j=1}^{n} \tilde{x}_{ij} \lambda_j, \;\; i \in I^I, \\
&\theta \tilde{x}_{io} - s_i^{I-} = x_i, \;\; i \in I^I, \\
&x_i, y_r \in \mathbb{Z}_+, i \in I^I, r \in O^I, \lambda_j \geq 0, \\
&s_r^{+} \geq 0, s_i^{-} \geq 0, s_i^{I-} \geq 0, s_r^{I+} \geq 0, \\
&i = 1, \ldots, m, r = 1, \ldots, s, j = 1, \ldots, n.
\end{aligned} \tag{3.7}$$

Terms applied in model (3.7) are similar to model (3.4). Inputs and outputs are just divided into integer-valued and real-valued measures. In other words, $I = I^I \cup I^{NI}$ and $O = O^I \cup O^{NI}$.

The above model is a fuzzy mixed integer linear programming problem. Similar to prior cases (models (3.5) and (3.6)), two methods, the graded mean integration representation method and the fuzzy number ranking method, are used for solving the fuzzy mixed integer linear programming problem. Thus, by using the graded mean integration representation method model (3.7) can be rewritten as follows:

$$
\begin{aligned}
&\text{Min } \theta-\varepsilon\left(\sum_{r\in O^I} s_r^{I+}+\sum_{i\in I^I} s_i^{I-}+\sum_{i\in I^{NI}} s_i^{-}+\sum_{r\in O^{NI}} s_r^{+}-\sum_{i\in I^I} s_i^{-}-\sum_{r\in O^I} s_r^{+}\right)\\
&\text{s.t. } \left(\frac{4y_{ro2}+y_{ro1}+y_{ro3}}{6}\right)+s_r^{+}=\sum_{j=1}^{n}\left(\frac{4y_{rj2}+y_{rj1}+y_{rj3}}{6}\right)\lambda_j,\ r\in O^{NI},\\
&y_r+s_r^{+}=\sum_{j=1}^{n}\left(\frac{4y_{rj2}+y_{rj1}+y_{rj3}}{6}\right)\lambda_j,\ r\in O^{I},\\
&\left(\frac{4y_{ro2}+y_{ro1}+y_{ro3}}{6}\right)+s_r^{I+}=y_r,\ r\in O^I,\\
&\theta\left(\frac{4x_{io2}+x_{io1}+x_{io3}}{6}\right)-s_i^{-}=\sum_{j=1}^{n}\left(\frac{4x_{ij2}+x_{ij1}+x_{ij3}}{6}\right)\lambda_j,\ i\in I^{NI},\\
&x_i-s_i^{-}=\sum_{j=1}^{n}\left(\frac{4x_{ij2}+x_{ij1}+x_{ij3}}{6}\right)\lambda_j,\quad i\in I^{I},\\
&\theta\left(\frac{4x_{io2}+x_{io1}+x_{io3}}{6}\right)-s_i^{I-}=x_i,\ \ i\in I^I,\\
&x_i,y_r\in\mathbb{Z}_+,\ i\in I^I,r\in O^I,\lambda_j\geq 0,\\
&s_r^{+}\geq 0,s_i^{-}\geq 0,s_i^{I-}\geq 0,s_r^{I+}\geq 0,\\
&i=1,\ldots,m,r=1,\ldots,s,j=1,\ldots,n.
\end{aligned}
\tag{3.8}
$$

Also, by utilizing the ranking function, the first index of Yager [40, 41], model (3.7) is substituted with the following model:

$$
\begin{aligned}
&\text{Min } \theta-\varepsilon\left(\sum_{r\in O^I} s_r^{I+}+\sum_{i\in I^I} s_i^{I-}+\sum_{i\in I^{NI}} s_i^{-}+\sum_{r\in O^{NI}} s_r^{+}-\sum_{i\in I^I} s_i^{-}-\sum_{r\in O^I} s_r^{+}\right)\\
&\text{s.t. } \left(y_{ro2}+\frac{d_{y_{ro3}}-d_{y_{ro1}}}{3}\right)+s_r^{+}=\sum_{j=1}^{n}\left(y_{rj2}+\frac{d_{y_{rj3}}-d_{y_{rj1}}}{3}\right)\lambda_j,\ r\in O^{NI},\\
&y_r+s_r^{+}=\sum_{j=1}^{n}\left(y_{rj2}+\frac{d_{y_{rj3}}-d_{y_{rj1}}}{3}\right)\lambda_j,\ r\in O^{I},\\
&\left(y_{ro2}+\frac{d_{y_{ro3}}-d_{y_{ro1}}}{3}\right)+s_r^{I+}=y_r,\ r\in O^I,\\
&\theta\left(x_{io2}+\frac{d_{x_{io3}}-d_{x_{io1}}}{3}\right)-s_i^{-}=\sum_{j=1}^{n}\left(x_{ij2}+\frac{d_{x_{ij3}}-d_{x_{ij1}}}{3}\right)\lambda_j,\ i\in I^{NI},\\
&x_i-s_i^{-}=\sum_{j=1}^{n}\left(x_{ij2}+\frac{d_{x_{ij3}}-d_{x_{ij1}}}{3}\right)\lambda_j,\quad i\in I^{I},\\
&\theta\left(x_{io2}+\frac{d_{x_{io3}}-d_{x_{io1}}}{3}\right)-s_i^{I-}=x_i,\ \ i\in I^I,\\
&x_i,y_r\in\mathbb{Z}_+,\ i\in I^I,r\in O^I,\lambda_j\geq 0,
\end{aligned}
$$

$$s_r^+ \geq 0, s_i^- \geq 0, s_i^{I-} \geq 0, s_r^{I+} \geq 0,$$
$$i = 1, \ldots, m, r = 1, \ldots, s, j = 1, \ldots, n. \tag{3.9}$$

Clearly, in models (3.7)–(3.9), subsets of integer-valued inputs and integer-valued outputs are indicated by I^I and O^I, respectively. Furthermore, I^{NI} and O^{NI} denote subsets of real-valued inputs and real-valued outputs. Note that $I^I \cap I^{NI} = \emptyset$ and $O^I \cap O^{NI} = \emptyset$ in all aforementioned models.
Similar to the proof of proposition 3.3, it can be conveniently indicated that the solution set of model (3.7) is equal to that of models (3.8) and (3.9).

Definition 3.7. DMU_o in models (3.8) and (3.9) is said to be the efficient if no other integer-valued point for integer factors and no other real-valued point for real factors dominates it.

Proposition 3.8. *DMU_o in models proposed, models (3.8) and (3.9), is the efficient if and only if $\theta^* = 1, s_r^{*I+} = 0, \forall r \in O^I, s_r^{*+} = 0, \forall r \in O^I \cup O^{NI}, s_i^{*I-} = 0, \forall i \in I^I$ and $s_i^{*-} = 0$, $\forall i \in I^I \cup I^{NI}$.*

Proof. Similar to the proof of proposition 3.5. □

The following proposition shows models (3.8) and (3.9) are always feasible and bounded.

Proposition 3.9. *Models (3.8) and (3.9) are always feasible and bounded.*

Proof. Consider the following arbitrary solution for model (3.8): $\theta_o = 1, s_r^{I+} = 0, s_r^+ = 0, \forall r, s_i^{I-} = 0$ and $s_i^- = 0$, $\forall i, \lambda_o = 1$, $\lambda_j = 0, j \neq o$, $x_i = \lfloor (4x_{io2} + x_{io1} + x_{io3})/6 \rceil$ for $\forall i \in I^I$ and $y_r = \lfloor (4y_{ro2} + y_{ro1} + y_{ro3})/6 \rceil$ for $\forall r \in O^I$. It is clear to see that model (3.8) is always feasible.

Further, the form of objective function in model (3.8) is the minimization. Consequently, the optimal value of model (3.8), that is θ_o^*, is not greater that the feasible solution $\theta_o = 1$. It means that $\theta_o^* \leq \theta_o = 1$. Also, $0 < \theta_o^*$ due to the fact that the input and output vectors have at least a nonzero component. Thus, model (3.8) is bounded, *i.e.* $0 < \theta_o^* \leq 1$. This complete the proof. In a similar way, it can be shown that model (3.9) is feasible and bounded. □

In this research, we have applied two methods for defuzzification. As Brunelli and Mezei [3] mentioned, ranking methods are subjective, and as such a low agreement means that methods bring distinct evidence on the evaluation of a fuzzy number and in the presence of uncertainty a person may want to listen to a second option.

Furthermore, inputs and outputs in this study have been shown by triangular fuzzy numbers. Obviously, the proposed models can be extended for trapezoidal fuzzy integer-valued numbers.

4. Examples

Example 4.1. Suppose there are 9 DMUs with two fuzzy integer-valued inputs and one fuzzy integer-valued output that are denoted by triangular fuzzy numbers. Data can be found in Table 1. Columns 2 and 3 show inputs while column 4 indicates an output. The aim is to evaluate the efficiency of DMUs in the presence of aforementioned measures. For the purpose of analysis, models (3.2) and (3.3) are calculated. Table 2 provides the results obtained from models (3.2) and (3.3). The efficiency scores of models (3.2) and (3.3) are displayed in columns 2 and 6 from Table 2. Also, targets determined are present in columns 3–5 and 7–9. As can be seen two DMUs (DMU 3 and DMU 9) are efficient in both models. It is apparent from this table that the targets of integer-valued inputs and outputs are estimated as non-integer targets. Notice that we have first applied two methods for defuzzifying the fuzzy numbers and, then we have evaluated the efficiency of DMUs. Thus, the projected targets are estimated as crisp and continuous values.

Herein, models (3.5) and (3.6) (*i.e.* the proposed models) are computed. Results are shown in Table 3. The efficiency of model (3.5) is provided in column 2. Also, the results of model (3.6) can be seen in column 6.

Columns 3–5 show targets obtained by model (3.5) while targets obtained by model (3.6) are indicated in columns 7–9. It can be found all targets of integer-valued measures are resulted as integer targets in both

TABLE 1. Data of an example.

DMU	Input 1(x_1)	Input 2(x_2)	Output 1(y_1)
1	(11,15,19)	(455,480,510)	(77,95,103)
2	(12,12,15)	(475,510,525)	(71,75,93)
3	(5,10,13)	(400,420,435)	(85,90,100)
4	(15,18,21)	(520,600,645)	(67,80,97)
5	(5,7,8)	(495,520,565)	(45,50,56)
6	(6,10,15)	(450,500,560)	(63,70,81)
7	(9,12,17)	(515,550,605)	(69,75,87)
8	(10,14,18)	(540,570,585)	(52,55,69)
9	(7,8,9)	(420,450,470)	(85,90,113)

TABLE 2. Results of models (3.2) and (3.3).

DMU	Eff. of model (3.2)	Model (3.2)			Eff. of model (3.3)	Model (3.3)		
		x_1^*	x_2^*	y_1^*		x_1^*	x_2^*	y_1^*
1	0.90	9.933	430.703	93.333	0.87	9.333	418.333	91.667
2	0.70	8.230	356.869	77.333	0.72	8.112	363.570	79.667
3	1.00	9.667	419.167	90.833	1.00	9.333	418.333	91.667
4	0.63	8.585	372.251	80.667	0.63	8.281	371.176	81.333
5	0.63	4.315	241.843	50.167	0.63	4.194	234.190	50.333
6	0.67	6.764	333.750	70.667	0.65	6.735	328.083	71.333
7	0.64	7.867	352.950	76.000	0.63	7.840	351.400	77.000
8	0.46	6.048	262.268	56.833	0.47	5.973	267.733	58.667
9	1.00	8.000	448.333	93.000	1.00	8.000	446.667	96.000

TABLE 3. Results of the proposed models.

DMU	Eff. of model (3.5)	Model (3.5)			Eff. of model(3.6)	Model (3.6)		
		x_1^*	x_2^*	y_1^*		x_1^*	x_2^*	y_1^*
1	0.89	11.000	429.000	93.000	0.87	9.000	418.000	92.000
2	0.70	9.000	355.000	77.000	0.72	8.000	364.000	80.000
3	1.00	10.000	419.000	91.000	1.00	9.000	418.000	92.000
4	0.63	9.000	373.000	81.000	0.63	8.000	369.000	81.000
5	0.71	5.000	235.000	50.000	0.71	5.000	228.000	50.000
6	0.70	7.000	335.000	71.000	0.66	6.000	330.000	71.000
7	0.65	7.000	362.000	76.000	0.63	8.000	350.000	77.000
8	0.47	6.000	265.000	57.000	0.48	6.000	269.000	59.000
9	1.00	8.000	448.000	93.000	1.00	8.000	447.000	96.000

models. Also, DMUs 3 and 9 are efficient in each of the models. Moreover, DMU 8 is the most inefficient DMU in models (3.5) and (3.6). The results established by models (3.5) and (3.6) are approximately similar. Nevertheless, as can be seen in Table 3, each method has effects on the results. Actually, each method represents a different viewpoint on fuzzy numbers; thus, the results could be different. There are several possible explanations for these results.

In this stage we compare the obtained results from models (3.2) and (3.5). While two DMUs, DMU 3 and DMU 9, are efficient in both models, the scores of inefficient DMUs are not exactly the same. Actually, the efficiency scores of two models are different in five DMUs (56% of DMUs). Furthermore, if in estimating targets

TABLE 4. Data of inputs and fuzzy outputs.

No. supplier (DMU)	x_1	x_2	$\tilde{y}_1$	$\tilde{y}_2$
1	280	182	(100,121,160)	(160,182,195)
2	370	280	(180,210,232)	(145,156,160)
3	230	124	(102,120,136)	(150,175,190)
4	430	210	(150,170,190)	(50,60,70)
5	325	122	(102,130,160)	(280,286,293)
6	315	240	(190,213,234)	(72,85,94)
7	253	170	(130,151,167)	(260,275,286)
8	305	185	(201,225,246)	(76,87,93)
9	245	129	(130,146,160)	(230,242,251)
10	270	147	(125,147,170)	(176,181,189)
11	460	146	(180,205,231)	(110,117,124)
12	343	141	(132,152,160)	(132,140,149)
13	321	126	(104,112,135)	(282,287,298)
14	264	143	(132,145,160)	(171,182,189)
15	338	206	(106,130,147)	(280,287,296)

TABLE 5. Efficiency scores and targets of models (3.2) and (3.3).

No. supplier	Eff. of model (3.2)	Model (3.2)			Eff. of model (3.3)	Model (3.3)		
		x_2^*	y_1^*	y_2^*		x_2^*	y_1^*	y_2^*
1	0.71	129.180	124.000	180.500	0.73	131.954	127.000	179.000
2	0.81	187.907	208.667	154.833	0.81	187.288	207.300	153.667
3	0.85	105.055	119.667	173.333	0.85	104.830	119.300	171.667
4	0.64	134.447	170.000	74.606	0.64	134.626	170.000	74.260
5	1.00	122.000	130.333	286.167	1.00	122.000	130.667	286.333
6	0.92	175.827	212.667	84.333	0.92	175.963	212.333	83.667
7	1.00	170.000	150.167	274.333	1.00	170.000	149.333	273.667
8	1.00	185.000	224.500	86.167	1.00	185.000	224.000	85.333
9	1.00	129.000	145.667	241.500	1.00	129.000	145.333	241.000
10	0.86	127.062	147.167	181.500	0.87	127.517	147.333	182.000
11	1.00	146.000	205.167	117.000	1.00	146.000	205.333	117.000
12	0.84	118.405	150.000	140.167	0.83	117.049	148.000	140.333
13	1.00	126.000	114.500	288.000	1.00	126.000	117.000	289.000
14	0.88	125.484	145.333	181.333	0.88	125.984	145.667	180.667
15	0.82	167.984	149.000	287.333	0.82	168.396	149.140	287.667

by model (3.2) we round the obtained targets of integer-valued factors to the nearest numbers, different results are achieved in comparison with model (3.5). To illustrate, targets are unlike in 7 DMUs (78% of DMUs).

Next, we compare the results got from models (3.3) and (3.6). The obtained targets are rounded to the nearest values for handling targets of integer-valued measures in model (3.3). In both models, there are two efficient DMUs while the efficiency scores are dissimilar in three DMUs (33% of DMUs). Also, integer-valued targets are different in 5 DMUs (56% of DMUs). Nevertheless, as Lozano and Villa [27] mentioned heuristically rounding the continuous DEA projection may not be reasonable. Therefore, it seems the models introduced (models (3.5) and (3.6)) are suitable and rational for situations in which all data are fuzzy and integer-valued.

In the next investigation, we study occasions in which integer-valued and real-valued fuzzy data exist.

TABLE 6. Efficiency scores and targets of the proposed models.

No. supplier	Eff. of model (3.8)	Model (3.8)			Eff. of model (3.9)	Model (3.9)		
		x_2^*	y_1^*	y_2^*		x_2^*	y_1^*	y_2^*
1	0.71	129	124	181	0.73	132	127	179
2	0.81	189	209	155	0.81	188	207	154
3	0.85	105	120	173	0.85	105	119	173
4	0.64	135	170	79	0.64	135	170	78
5	1.00	122	130	286	1.00	122	131	286
6	0.92	177	213	84	0.92	176	212	84
7	1.00	170	150	274	1.00	170	149	274
8	1.00	185	224	86	1.00	185	224	85
9	1.00	129	146	242	1.00	129	145	241
10	0.86	127	147	184	0.87	128	147	190
11	1.00	146	205	117	1.00	146	205	117
12	0.84	119	150	150	0.84	117	148	140
13	1.00	126	115	288	1.00	126	117	289
14	0.88	125	145	181	0.89	127	146	188
15	0.82	168	163	287	0.82	168	148	288

Example 4.2. It is clear; selecting a supplier is an important issue in supply chain management. Here the efficiency of 15 suppliers is evaluated by using methods suggested. In this illustrative example, the total cost of shipments (x_1) and the number of shipments per month (x_2) are considered as inputs, and the number of shipments to arrive on time ($\tilde{y}_1$) and the number of bills received from the supplier without errors ($\tilde{y}_2$) are taken as outputs. Outputs are deemed as triangular fuzzy numbers. Input x_1 is a real-valued measure while input x_2 consists of an integer-valued measure. Outputs are given as fuzzy integer-valued factors.

Data can be found in Table 4. Columns 2 and 3 of Table 4 show inputs (x_1 and x_2). Outputs ($\tilde{y}_1$ and $\tilde{y}_2$) can be found in columns 4 and 5. Table 5 presents the results obtained from models (3.2) and (3.3). Columns 2 and 6 reveal six DMUs are efficient in both models. Furthermore, the projection points of integer-valued factors (estimated by models (3.2) and (3.3)) can be found in Table 5. The findings show the targets of integer measures are not obtained as integer numbers.

In the next stage, for measuring the efficiency of suppliers, models (3.8) and (3.9) are calculated. The results are given in Table 6. As can be seen, the second column of Table 6 shows the efficiency of model (3.8). Targets of second input (x_2^*) and outputs (y_1^* and y_2^*) are indicated in columns 3–5 of Table 6. Moreover, the results of evaluating the efficiency of model (3.9) can be found in the column 6 while the obtained targets are given in columns 7–9.

Results show both models (3.8) and (3.9) have 6 efficient DMUs, DMU5, DMU7, DMU8, DMU9, DMU11 and DMU13. Also, the fourth supplier has the least efficiency (*i.e.* 0.64) in comparison with other suppliers in both models (3.8) and (3.9). Furthermore, the results are approximately similar in both models. Nevertheless, there are some differences in some efficiency scores and targets. Indeed, different points of view on fuzzy numbers cause different results are obtained. Therefore, managers must focus on methods and select the method according to their preference.

Now, we round the calculated targets of integer measures in models (3.2) and (3.3). Analysis of the computed results shows the following:

(1) The efficiency scores calculated by models (3.2) and (3.8) are the same in this study.
(2) Targets of integer-valued factors are unlike in 7 DMUs (47% of suppliers).
(3) The efficiency scores are not exactly the same in models (3.3) and (3.9). Actually, scores of two DMUs are dissimilar.
(4) The integer-valued targets of models (3.3) and (3.9) are different in six DMUs (40% of suppliers)

One of the more significant findings to emerge from this study is to obtain integer targets for integer-valued measures. In summary, two examples have shown that the models suggested (*i.e.* models (3.5), (3.6), (3.8) and (3.9)) obtain integer-valued targets for integer-valued measures and calculate the efficiency scores accurately.

5. Conclusions

In real applications, there are systems that their performance must be evaluated while fuzzy and integer-valued measures are present. However, inputs and outputs of DMUs are usually considered as accurate and real-valued factors in conventional DEA models. For this reason, the current paper has introduced and extended DEA models for evaluating the efficiency of entities and determining targets where all input and output data are fuzzy and integer numbers.

Furthermore, models have been provided for assessing the efficiency of DMUs and for obtaining integer-valued input and output targets when a subset of data is fuzzy integer-valued, while others are real-valued in a fuzzy environment. Two methods, the fuzzy number ranking method and the graded mean integration representation method have been used for defuzzification of fuzzy inputs and outputs. Two examples have been presented to explain and to demonstrate approaches. It has been seen that different methods have effects on the results. To illustrate, in conventional fuzzy DEA approaches, targets of fuzzy integer-valued measures may be determined as non-integer factors. Therefore, the efficiency and targets may be obtained incorrectly. Nevertheless, the results have shown that the proposed models in this study overcome these drawbacks.

Further work will need to be done to determine imprecise and integer-valued targets for fuzzy integer-valued measures. Models proposed in this paper are under constant returns to scale assumption. It seems they can be generalized under variable returns to scale. Furthermore, the applications of models introduced in various fields such as artificial intelligence, computer science, and control theory are some interesting subjects for future researches.

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