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OPTIMAL PRODUCTION MODEL WITH QUALITY SENSITIVE MARKET DEMAND, PARTIAL BACKLOGGING AND PERMISSIBLE DELAY IN PAYMENT

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Abstract. In this study, we consider an imperfect production inventory system with quality of the products dependent market's demand structures and allowable delay in payments. Two alternative approaches of trade credit policies have been discussed when the manufacturer could not pay the due amount to the supplier within the credit period offered. Here, a new cycle is begun with new production when the manufacturer's inventory touches to a certain level of shortages. The cycle also ends when backlogged inventory level is reached a certain level. The backlogging rate for the player is dependent on waiting time. The production cost of the manufacturer varies with ordering lot size and quality of product. The behavior of the model under integrated system is analyzed. The sensitivity of the key parameters is examined to test feasibility of the model. Finally, a numerical example is provided to investigate the proposed model.

Mathematics Subject Classification. 90B05.

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1. Introduction

In the era of globalization, every business organization needs an effective inventory management strategy to survive from competitive marketing situation. Production planning is one of the most important parts to the companies for their business strategy. Careful production planning is necessary to ensure good deliveries and productive efficiencies. Without proper planning, the firm will face "overages" or "shortages" problem, which has an negative effect on the business. Product quality is crucial for the manufacturing company as quality products help to maintain customer satisfaction and loyalty in the business. In the era of social media, customers can easily share their favorable opinions and criticism of product quality on the networking forums. So, quality of product can be an important differentiator in the market. Another key business policy is: what is the company's credit period policies for the customers? Credit period is very important component in the business because

Keywords. Imperfect production inventory model, quality, partially backlogging, permissible delay in payment.

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companies give you the chance to buy the items or necessities when you do not have the cash but you have to pay the actual amount before the mentioned time or you have to pay interest for late payment. Another side of offering credit period is to build up goodwill and reputation with the customer that can help someone's business security.

Now, our aim is to develop a production inventory model considering the quality control, permissible delay in payment policy, partially backlogging situation, *etc.* First, we shall review the existing literature related to the study. Aggarwal and Jaggi [1] developed an economic ordering quantity (EOQ) model of deteriorating items under a permissible delay in payments. Afterward, Abad [3] studied pricing and production planning problem for perishable products. They presented an integrated price sensitive model for waiting time dependent partial backordering and lost sale environment. Thereafter, Chung *et al.* [12] studied an economic order quantity model considering permissible delay in payments. They assumed delay in payments dependent on ordered quantity. Later, Wee *et al.* [31] formulated an optimal inventory model for items with imperfect quality and shortage backordering. Later, Skouri *et al.* [22] studied inventory model for deteriorating items with ramp type time dependent demand rate. They also considered the partially backlogging rate which was any non-increasing function of the waiting time up to the next replenishment. Afterward, Zhang *et al.* [32] developed a two-item EOQ model with identical order cycles. In this study, they assumed that the unmet demand of the major item can be partially backordered with lost sales whereas the demand of the minor item must be met without stock-outs. Further, they extended the model with the order cycle of the major item is an integer multiple of that of the minor item. In the same time, Toews *et al.* [29] extended existing partial backordering EPQ model considering linearly increasing backordering rate as the time until delivery decreases. They showed how this model can be modified to determine the optimal policy for the EOQ with partially back-ordering. Later, Sicilia *et al.* [23] studied an inventory model with partially backlogging, where the fraction of backlogged demand was dependent exponentially on the waiting time, and the backorder and lost sale costs were proportional to the length of the shortage period. They considered two types of stock-out related cost: a fixed cost and a cost proportional to the length of the shortage period. On the other hand, Ma *et al.* [15] derived quality and marketing effort-sensitive the supply channel model. They analyzed the behavior of the model under manufacturer Stackelberg, retailer Stackelberg, and vertical Nash structures. Thereafter, Taleizadeh [25] developed an EOQ model with partial multiple prepayments and partial backordering. In their model, the retailers offered the customer to pay all or a fraction of the purchasing cost in advance. Also, the retailer may allow them to divide the prepayment into several equal-sized parts. At the same time, an EOQ model for perishable items considering back-ordering and delayed payment over the finite horizon planning were proposed by Taleizadeh and Nematollahi [26]. During same time, Taleizadeh *et al.* [27] studied an imperfect production inventory model for multiple products where shortages were permitted and fully back-ordered for a multi products-single machine system. Recently, Bhunia *et al.* [6] formulated two-warehouse inventory model for single deteriorating items. They studied the model under inventory follows shortage and shortage follows inventory policies with partially backlogged shortages. Selling price dependent demand and alternative approach for trade credit financing also discussed there. Among other researches in this direction, the noteworthy works of Huang [13], Papachristos and Skouri [19], Alfares *et al.* [2], Papachristos and Konstantaras [20], Cardenas–Barron [9, 10], Chang *et al.* [11], Sana [21], Jeang [14], Wee *et al.* [30], Pal *et al.* [17, 18], Bhunia *et al.* [5], Tsao [28], Stojkovska [24], Bhunia *et al.* [6–8], Mishra *et al.* [16] should be mentioned.

In this model, an imperfect production inventory model with quality of produced products sensitive market demand rate has been formulated. Production cast for the manufacturer is assumed to be dependent on lot size and quality of product. In this study, the manufacturer inventory level may face partially backlogged situation in the beginning as well as end of the cycle and the backlogged rate is dependent on the length of the waiting time up to the availability of the new product. Here, we develop the model applying two alternative approaches of trade credit policies if the manufacturer could not pay the due amount to the supplier within offering credit period. We analyzed the model in integrated system and find the best strategies for the manufacturer. Finally, the sensitivity analysis have been discussed to study the effect of changes of different key parameters on the optimal values of ordering lot size, quality of product and maximum back-ordered level of the manufacturer.

TABLE 1. A comparison of the present work with related previous works.

References	Imperfect production system	Partially backlogged situation in the beginning as well as end of the cycle	Allowable delay in payments.	Alternative approaches of trade credit	Quality dependent customers demand.
Aggarwal and Jaggi [1]	×	×	√	×	×
Huang [13]	×	×	√	×	×
Abad [3]	×	×	√	×	×
Chung *et al.* [12]	×	×	√	×	×
Papachristos *et al.* (2006)	√	×	×	×	×
Wee *et al.* [31]	√	×	×	×	×
Cardenas–Barron [9]	√	×	×	×	×
Cardenas–Barron [10]	√	×	×	×	×
Skouri *et al.* [22]	×	√	×	×	×
Sana [21]	√	×	×	×	×
Toews *et al.* [29]	√	×	×	×	×
Jeang [14]	×	√	×	×	√
Pal *et al.* [17]	√	×	×	×	×
Wee *et al.* [30]	√	√	×	×	×
Ma *et al.* [15]	×	×	×	×	√
Pal *et al.* [18]	√	×	×	×	×
Taleizadeh [25]	×	√	×	√	√
Taleizadeh *et al.* [27]	√	√	×	×	×
Bhunia *et al.* [5]	×	×	√	√	×
Taleizadeh *et al.* (2014)	×	×	√	×	×
Bhunia *et al.* [6]	×	×	√	√	×
This paper	√	√	√	√	√

2. Fundamental assumptions and notation

2.1. Assumptions

The following assumptions are adopted to develop the model for single item:

(i) Manufacturer produces the units of product at a constant rate where production rate of perfect items is greater than customer's demand rate.
(ii) The production system of manufacturer is not perfect which also produces defective product in a constant rate.
(iii) Production cost per unit item varies with lot size and quality of product.
(iv) Demand rate of the markets are dependent on quality of the product.
(v) The manufacturer inventory level may face partially backlogged situation in the beginning as well as end of the cycle. The backlogged rate is dependent on the length of the waiting time up to the availability of the new product.
(vi) All the defective units of item for the manufacturer are sold to the out side markets in a lot.
(vii) The supplier provides a fixed credit period M to the manufacturer to settle the accounts.

2.2. Notations

The following notations are used throughout the paper

Q	Raw materials lot size for production system.
P	Production rate for the manufacturer.
D_c	Demand rate for the customers.
p_m	Selling price (\$/ unit) for the manufacturer.
w	Selling price (\$/ unit) for the raw materials supplier.
θ	Quality parameter of the product which lies between $\theta_{\min}$ and $\theta_{\max}$ where $0 < \theta_{\min} < \theta_{\max} < 1$.
S_m	Maximum backlogged level of the manufacturer.
$0 < \delta_m < 1$	Backlogging parameter.
α	The fraction of defective items in the production run-time of manufacturer.
$C_p(Q, \theta)$	Production cost (\$) per unit item for the manufacturer.
C_{hm}	Holding cost for manufacturer per unit per unit time.
C_{bm}	Backlogged cost for manufacturer per unit per unit time.
C_{pm}	Penalty cost for manufacturer per unit per unit time.
M	Manufacturer's trade credit period offered by the supplier in years.
S_{v_m}	Salvage value of the per unit product for the manufacturer.
I_e	Interest that can be earned per \$ per unit time.
I_p	Interest that can be paid per \$ per unit time.
T	Total cycle run-time of the chain.

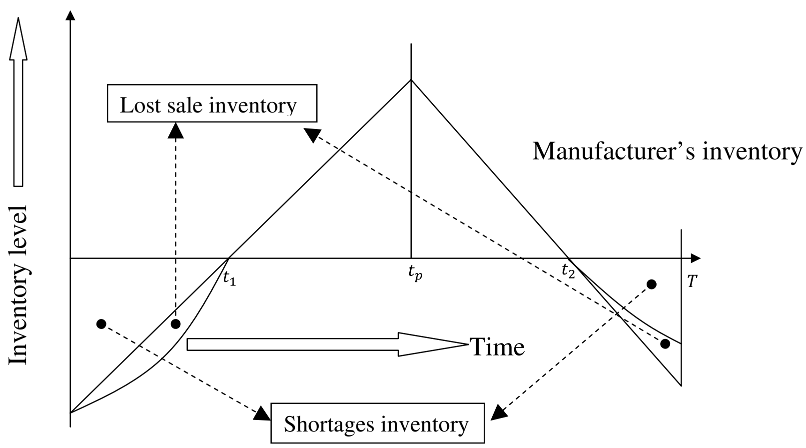

FIGURE 1.

3. Formulation of the model

In this paper, an imperfect production inventory model has been formulated, where the manufacturer's production rate is constant but the production system is not perfect. The production systems produces the good finished product lot at the rate $(1-\alpha)P$ which is greater than the customers' demand rate $D_c(\theta)$. The inventory level of the manufacturer may face partially backlogged shortages, where the backlogging rate is dependent on waiting time. In the proposed model, inventory level for manufacturer starts with shortage and then production. In this time, the manufacturer first delivers the new demand, and then supplies the backlogged demand with excess production considering the factor of new customers goodwill. In this time, the backlogging rate is considered as exponentially waiting time dependent $\mathrm{e}^{\frac{\delta_m I_{m_1}(t)}{(1-\alpha)P-D_c}}$. After time t_1, the production inventory level recovers from shortages, and stores the inventory at $(1-\alpha)P-D_c$ rate up to time t_p. The stored inventory of manufacturer satisfies the demand of the customers up to t_2 time, and then it again falls into shortages and continues up to end of manufacturer cycle time t_3 with maximum backlogged level S_m. In this shortages period, the backlogging rate is $\mathrm{e}^{-\delta_m(t_3-t)}$. The unit production cost of the manufacturer is considered as ordering lot and quality of the product dependent. All the defective items are sold to the outer markets in a lot after the production run.

The governing differential equation of inventory level of this case is

$$\frac{I_{m_1}(t)}{\mathrm{d}t} = \{(1-\alpha)P - D_c\} - \mathrm{e}^{\frac{\delta_m I_{m_1}(t)}{(1-\alpha)P-D_c}} S_m, \text{ with } I_m(0) = -S_m,\ 0 \le t < t_1 \tag{3.1}$$

$$\frac{I_{m_2}(t)}{\mathrm{d}t} = (1-\alpha)P - D_c, \text{ with } I_{m_2}(t_1) = I_{m_1}(t_1) = 0,\ t_1 \le t < t_p \tag{3.2}$$

$$\frac{I_{m_3}(t)}{\mathrm{d}t} = -D_c, \text{ with } I_{m_3}(t_p) = I_{m_2}(t_p),\ t_p \le t < t_2 \tag{3.3}$$

$$\frac{I_{m_4}(t)}{\mathrm{d}t} = -\mathrm{e}^{-\delta_m(T-t)} D_c, \text{ with } I_{m_4}(t_2) = 0,\ t_2 \le t < T \tag{3.4}$$

Using the boundary conditions, we have from equation (3.1) to equation (3.4)

$$I_{m_1}(t) = -\frac{((1-\alpha)P - D_c)}{\delta_m} \log\left[\frac{S_m\left(\mathrm{e}^{-\delta_m t} - 1\right)}{(1-\alpha)P - D_c} + \mathrm{e}^{-\left(t - \frac{S_m}{(1-\alpha)P-D_c}\right)\delta_m}\right],\ 0 \le t < t_1 \tag{3.5}$$

$$I_{m_2}(t) = ((1-\alpha)P - D_c)(t - t_1),\ t_1 \le t < t_p \tag{3.6}$$

$$I_{m_3}(t) = \{(1-\alpha)P - D_c\}\left(\frac{Q}{P} - t_1\right) - D_c(t - t_p),\ t_p \le t < t_2 \tag{3.7}$$

$$I_{m_4}(t) = -\frac{D_c \mathrm{e}^{-T\delta_m}\left(\mathrm{e}^{\delta_m t} - \mathrm{e}^{\delta_m t_2}\right)}{\delta_m},\ t_2 \le t < T \tag{3.8}$$

$$I_{m_1}(t_1) = 0 \Rightarrow t_1 = \frac{S_m}{(1-\alpha)P - D_c} + \frac{1}{\delta_m}\log\left[\frac{(1-\alpha)P - D_c + S_m \mathrm{e}^{-\frac{S_m\delta_m}{(1-\alpha)P-D_c}}}{(1-\alpha)P - D_c + S_m}\right] \tag{3.9}$$

$$I_{m_3}(t_2) = 0 \Rightarrow t_2 = \frac{1}{D_c}\left(\{(1-\alpha)P - D_c\}\left(\frac{Q}{P} - t_1\right) + \frac{QD_c}{P}\right) \tag{3.10}$$

$$I_{m_3}(T) = -S_m \Rightarrow T = t_2 - \frac{1}{\delta_m}\log\left[1 - \frac{S_m\delta_m}{D_c}\right] \tag{3.11}$$

Inventory holding cost for good items, using equation (3.6) and (3.7), is

$$\begin{aligned} IC_m &= C_{hm}\left[\int_{t_1}^{t_p} I_{m_2}(t)\ \mathrm{d}t + \int_{t_p}^{t_2} I_{m_3}(t)\ \mathrm{d}t\right] \\ &= C_{hm}\left[-\frac{1-\alpha}{2P}Q^2 + (1-\alpha)Qt_2 - \frac{1}{2}D_c(t_2 - t_1)^2 - \frac{1}{2}(1-\alpha)(2t_2 - t_1)Pt_1\right] \end{aligned} \tag{3.12}$$

Simplifying (3.12) using the values of $t_p = \frac{Q}{P}$ and t_2 from expression (3.10), we have

$$IC_m = \frac{(1-\alpha)C_{hm}}{2D_c P}\left[(Q - Pt_1)^2\{(1-\alpha)P - D_c\}\right] \tag{3.13}$$

Back-ordering cost and penalty costs, , using equation (3.5) and (3.8), are respectively

$$BC_m = C_{bm}\left[\int_0^{t_1}(-I_{m_1}(t))\ \mathrm{d}t + \int_{t_2}^{T}(-I_{m_4}(t))\ \mathrm{d}t\right] \tag{3.14}$$

$$\begin{aligned} PC_m &= C_{pm}\left[\int_0^{t_1}(1-\mathrm{e}^{\frac{\delta_m I_{m_1}(t)}{(1-\alpha)P-D_c}})S_m\ \mathrm{d}t + \int_{t_2}^{T}(1-\mathrm{e}^{-\delta_m(T-t)})Dr\ \mathrm{d}t\right]\\ &= C_{pm}\left[S_m t_1 - \frac{1}{\delta_m}((1-\alpha)P-D_c)\log\left[1+\frac{S_m\left(\mathrm{e}^{t_1\delta_m}-1\right)\mathrm{e}^{-\frac{S_m\delta_m}{(1-\alpha)P-D_c}}}{((1-\alpha)P-D_c)}\right]\right.\\ &\left.+(T-t_2)D_c - \frac{D_c\left(1-\mathrm{e}^{-(T-t_2)\delta_m}\right)}{\delta_m}\right]. \end{aligned} \tag{3.15}$$

Per unit production cost for the manufacturer is

$$C_p(Q,\theta) = \epsilon_m + \frac{L}{Q^{\nu}} + B(1+\theta)^{\lambda} \tag{3.16}$$

3.1. Different cases for trade credit periods

In this study, we assume here that supplier initially offers to the manufacturer a permissible delay period M at the beginning. The permissible delay period M is considered as greater than beginning shortages time t_1 because the manufacturer needs at least some time to pay all the payment together. The manufacturer has to pay total amount $\$w_sQ$ at time M. If he can not pay the full amount of payment at time M, the supply may agree to the partial payment at time M and gives time to pay the rest of payment any time after $t = M$ with proper interest or he may offer to the manufacturer for full amount of payment any time after $t = M$ with proper interest. According to the values of M, the following cases may occur:

3.1.1. Case I: $t_1 \le M < t_2$

In this scenario, the manufacturer collects revenue from the customers from the time period 0 to M and earns the interest of revenue at rate I_e during time period 0 to M. So, total revenue of the manufacturer due to sale and interest earned at time M is given by

$$\begin{aligned} R\pi_m =& \text{revenue and earned interest for running demand} + \text{revenue and earned interest}\\ &\text{for back-ordered demand.}\\ =& p_m D_c M\left(1+\frac{1}{2}MI_e\right) + t_1 p_m((1-\alpha)P-D_c)\left(1+\frac{1}{2}t_1 I_e\right)\{1+I_e(M-t_1)\}. \end{aligned} \tag{3.17}$$

Now, according to the value of $R\pi_m$, two sub cases may arise:

Case Ia: $R\pi_m \ge w_sQ$

According to the collection of total amount, the manufacturer will pay all the due amount $\$w_sQ$ to the supplier at time M, and he will earn interest of the excess amount $R\pi_m - w_sQ$ through out the time interval $[M\ T]$. After time $t = M$, the manufacturer will collect the revenue on the sales of the product and continuously earn interest on the revenue. Hence, total revenue of the manufacturer with earned interest during time interval $[M\ T]$ is given by

$$R\pi'_m = p_m D_c(t_2 - M)\left(1+\frac{1}{2}I_e(t_2-M)\right)\{1+I_e(T-t_2)\}$$

Therefore, average profit of the manufacturer is

$$E\pi_{m_1} = \frac{1}{Q}\Big[R\pi'_m + \text{interest earned during } [M\ T] \text{ for excess amount } \$(R\pi_m - w_sQ) + \text{Salvage value}$$
$$- \text{production cost} - \text{inventory holding cost} - \text{back-ordering cost} - \text{penalty costs} - \text{setup costs}\Big]$$
$$= \frac{1}{Q}\Big[R\pi'_m + (R\pi_m - w_sQ)(1 + I_e(T-M)) + S_{v_m}\alpha Q - QC_p(Q,\theta) - IC_m - BC_m - PC_m - A_m\Big]. \tag{3.18}$$

Case Ib: $R\pi_m < w_sQ$ with partial payment
According to the collection of total amount, the manufacturer can not give total amount of due payment $\$w_sQ$ to the supplier at time M. Then, the supplier offers to the manufacturer to pay partial payment at time M, and the rest of payment will be payed any time after $t = M$ together with interest. Let us consider, the manufacturer will pay the rest of amount $\$(w_sQ - R\pi_m)$ with interest at time $M + t_{M_1}$. Hence, the total payable amount to the supplier at time $M + t_{M_1}$ equal to the total available revenue to the manufacturer at time t_{M_1}, *i.e.*,

$$(w_sQ - R\pi_m)\{1 + I_p(t_{M_1} - M)\} = p_mD_c(t_{M_1} - M)\left(1 + \frac{1}{2}I_e(t_{M_1} - M)\right)$$

Solving the above equation, we have

$$t_{M_1} = \frac{1}{D_cI_ep_m}\{D_c(I_eM - 1)p_m + I_p(Qw_s - R\pi_m)$$
$$+ \sqrt{I_p^2(Qw_s - R\pi_m)^2 - 2D_cI_pp_m(Qw_s - R\pi_m) + D_cp_m(D_cp_m - 2I_eR\pi_m + 2I_eQw_s)}\}$$

After the time $t = M + t_{M_1}$, the manufacturer will collect the revenue on the sales of the product and continuously earn interest on the revenue. Therefore, average profit of the manufacturer is

$$E\pi_{m_2} = \frac{1}{Q}\Big[\text{sale revenue during } [t_{M_1}\ T] + \text{interest earned of sale revenue during } [t_{M_1}\ T] + \text{Salvage value}$$
$$- \text{production cost} - \text{inventory holding cost} - \text{back-ordering cost} - \text{penalty costs} - \text{setup costs}\Big]$$
$$= \frac{1}{Q}\Big[p_mD_c(t_2 - t_{M_1})\left\{1 + \frac{1}{2}I_e(t_2 - t_{M_1})\right\}\{1 + I_e(T - t_2)\} + S_{v_m}\alpha Q - QC_p(Q,\theta) - IC_m$$
$$- BC_m - PC_m - A_m\Big]. \tag{3.19}$$

Case Ic: $R\pi_m < w_sQ$ without partial payment In this subcase, the manufacturer can not give total amount of due payment $\$w_sQ$ to the supplier at time M. Then, the supplier offers to the manufacturer to pay full payment at any time after $t = M$ together with interest. Let us consider, the manufacturer will pay the rest of amount $\$w_sQ$ with interest at time $M + t_{M_2}$. Hence, the total payable amount to the supplier at time t_{M_2} equal to the total available revenue to the manufacturer at time $M + t_{M_2}$, *i.e.*,

$$w_sQ\{1 + I_p(t_{M_2} - M)\} = R\pi_mI_e(t_{M_2} - M) + p_mD_c(t_{M_2} - M)\left\{1 + \frac{1}{2}I_e(t_{M_2} - M)\right\}$$

Solving the above equation, we have

$$t_{M_2} = \frac{1}{D_c I_e p_m}\Big\{D_c(I_e M - 1)p_m - I_e R\pi_m + I_p Q w_s + \sqrt{D_c^2 p_m^2 + (I_e R\pi_m - I_p Q w_s)^2 + 2D_c p_m(I_e(R\pi_m + Qw_s) - I_p Q w_s)}\Big\}$$

After the time $t = M + t_{M_2}$, the manufacturer will collect the revenue on the sales of the product and continuously earn interest on the revenue. Therefore, average profit of the manufacturer is

$$\begin{aligned} E\pi_{m_3} =& \frac{1}{Q}\Big[\text{sale revenue during } [t_{M_2}\ T] + \text{interest earned of sale revenue during } [t_{M_2}\ T] + \text{Salvage value} \\ & - \text{production cost} - \text{inventory holding cost} - \text{back-ordering cost} - \text{penalty costs} - \text{setup costs}\Big] \\ =& \frac{1}{Q}\Big[p_m D_c(t_2 - t_{M_2})\left\{1 + \frac{1}{2}I_e(t_2 - t_{M_2})\right\}\{1 + I_e(T - t_2)\} + S_{v_m}\alpha Q - QC_p(Q,\theta) - IC_m \\ & - BC_m - PC_m - A_m\Big]. \end{aligned} \tag{3.20}$$

3.1.2. Case II: $t_2 \le M \le T$

In this case, the manufacturer collects revenue from the customers from the time period 0 to t_2 and earns the interest of revenue at rate I_e during time period 0 to M. So, total revenue of the manufacturer due to sale and interest earned at time M is given by

$$\begin{aligned} R\pi_{m_2} =& \text{revenue and earned interest for running demand} + \text{revenue and earned interest} \\ & \text{for back-ordered demand.} \\ =& p_m D_c t_2\left(1 + \frac{1}{2}t_2 I_e\right)(1 + I_e(M - t_2)) + t_1 p_m((1-\alpha)P - D_c)\left(1 + \frac{1}{2}t_1 I_e\right)\{1 + I_e(M - t_1)\}. \end{aligned} \tag{3.21}$$

Total revenue and earned interest for the manufacturer must be greater than total payment of the supplier as the manufacturer collects all his revenue from the market for the running cycle. Hence, average profit of the manufacturer, using equations (3.13) to (3.16) and (3.21), is

$$\begin{aligned} E\pi_{m_4} =& \frac{1}{Q}\Big[(R\pi_{m_2} - w_s Q) + \text{interest earned during } [M\ T] \text{ for the excess amount } \$(R\pi_{m_2} - w_s Q) \\ & + \text{Salvage value} - \text{production cost} - \text{inventory holding cost} - \text{back-ordering cost} \\ & - \text{penalty costs} - \text{setup costs}\Big] \\ =& \frac{1}{Q}\Big[(R\pi_{m_2} - w_s Q)(1 + I_e(T - M)) + S_{v_m}\alpha Q - QC_p(Q,\theta) - IC_m - BC_m - PC_m - A_m\Big]. \end{aligned} \tag{3.22}$$

3.2. Solution and optimality test

The aim of the model is to optimize the average profits for above mentioned different cases with respect to Q, θ, and S_m. *i.e.*, Maximize $E\pi_{m_i}(Q, \theta, S_m)$, $i = 1, 2, 3, 4$ subject to $(1 - \alpha)P > D_c(\theta)$ and $0 < \theta < 1$.

Now, equating the first order partial derivatives of $E\pi_{m_i}(Q, \theta, S_m)$ with respect to Q, θ and S_m with zero and solving these equations, we get a solution $Q = Q_c^*$, $\theta = \theta_c^*$ and $S_m = S_{m_c}^*$. These solutions will be optimal if all the eigenvalues of the Hessian matrices of the total average profits of the model are negative. Due to the complexity of the integrated profit, analytical discussions may not be possible, we have verified these conditions numerically.

4. Numerical example

Here, we illustrate our model numerically to gain the insight behavior of the model. Demand rate of customer is considered as $D_c(\theta) = U(1 + \theta)^{k_m}$ where U is based demand and demand rate is increasing with higher values of quality parameter with elasticity parameter $k_m > 0$.

Table 2. Sensitivity analysis of Example 1.

Parameter values		Case Ia: $R\pi_m \geq w_s Q$				Case Ib: $R\pi_m < w_s Q$ with partial payment				Case Ic: $R\pi_m < w_s Q$ without partial payment			
		Q	S_m	θ	$E\pi_{m_2}$	Q	S_m	θ	$E\pi_{m_3}$	Q	S_m	θ	$E\pi_{m_4}$
C_{hm}	1.0	2.86	−50.09	10.76	11.40	33.02	0.78	−40.13	9.32	33.51	0.46	−40.23	9.14
	1.5	6.93	−11.04	4.72	20.74	11.91	−0.48	−17.50	3.99	12.12	−0.54	−17.48	3.91
	2.5	−3.31	9.09	−3.59	−12.73	−7.80	0.94	14.51	−3.20	−7.98	0.93	14.42	−3.14
	3.0	−4.76	17.96	−6.50	−20.55	−13.35	1.99	27.06	−5.88	−13.68	1.96	26.85	−5.77
C_{bm}	1.50	−0.23	28.11	−4.59	0.60	−0.49	24.69	−4.65	0.52	−0.64	24.44	−5.04	0.54
	2.25	−0.20	12.01	−2.08	0.26	−0.23	10.80	−2.07	0.23	−0.29	10.70	−2.24	0.24
	3.75	0.25	−9.38	1.74	−0.22	0.20	−8.68	1.70	−0.19	0.25	−8.62	1.84	−0.20
	4.50	0.52	−16.95	3.23	−0.39	0.37	−15.84	3.13	−0.36	0.47	−15.73	3.39	−0.37
C_{pm}	1.750	3.90	33.60	0.88	0.53	1.01	23.76	−1.26	0.38	0.87	23.26	−1.49	0.38
	2.620	1.26	13.15	−0.01	0.22	0.37	9.97	−0.67	0.17	0.31	9.79	−0.76	0.17
	4.375	−0.68	−9.41	0.32	−0.17	−0.22	−7.64	0.66	−0.13	−1.18	−7.54	0.75	−0.13
	5.250	−1.08	−16.59	0.76	−0.31	−0.36	−13.74	1.29	−0.24	−0.29	−13.57	1.45	−0.25
δ_m	0.2250	−2.72	7.97	−3.18	0.07	−1.12	10.00	−2.28	0.11	−1.14	10.05	−2.41	0.11
	0.3375	−1.42	3.20	−1.65	0.03	−0.56	4.33	−1.14	0.05	−0.57	4.37	−1.20	0.05
	0.5625	1.52	−2.38	1.75	−0.016	0.55	−3.59	1.14	−0.04	0.56	−3.64	1.20	−0.05
	0.5760	1.70	−2.63	1.97	−0.017	1.07	−6.76	2.28	−0.08	1.09	−6.86	2.41	−0.09
k_m	0.40	−24.47	−2.24	−43.60	−1.91	−15.59	2.25	−38.38	−1.61	−15.15	2.46	−38.60	−1.53
	0.45	−15.87	−1.39	−24.99	−1.10	−9.11	1.50	−20.06	−0.90	−8.81	1.61	−20.10	−0.85
	0.55	35.58	−0.42	42.61	1.76	14.57	−2.97	24.64	1.13	13.82	−3.03	24.28	1.08
	0.60	40.12	−9.93	46.30	3.97	55.08	−11.26	72.35	2.69	47.03	−10.52	66.10	2.53
P	850	26.68	−44.85	38.55	8.65	126.99	−28.40	77.30	8.18	29.04	−30.46	82.87	7.84
	950	28.20	−3.82	29.40	2.45	13.51	−6.73	15.73	1.72	13.04	−6.89	15.43	1.64
	1050	−15.24	1.66	−16.95	−1.67	−8.05	5.18	−10.07	−1.38	−7.90	5.35	−9.98	−1.32
	1150	−28.48	5.96	−33.37	−4.07	−17.71	13.07	−22.98	−3.51	−17.44	13.56	−22.85	−3.36

Example 1. The values of different parameters in appropriate units are as follows:

$w_s = \$10$, $P = 1000$, $U = 500$, $k_m = 0.5, p_m = \$30$, $C_{hm} = \$2$, $C_{pm} = \$3.5$, $C_{bm} = \$3$, $\delta_m = 0.45, \alpha = 0.075$, $A_m = \$400$, $\epsilon_m = 1$, $L = 1500$, $B = 0.6$, $\lambda = 2$, $\nu = 0.75$, $S_{v_m} = \$15$, $I_e = 0.08$ per year, and $I_p = 0.12$ per year.

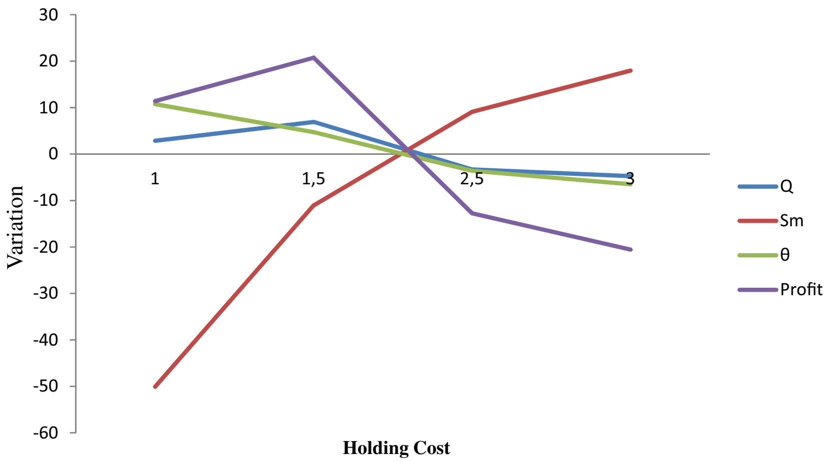

FIGURE 2.

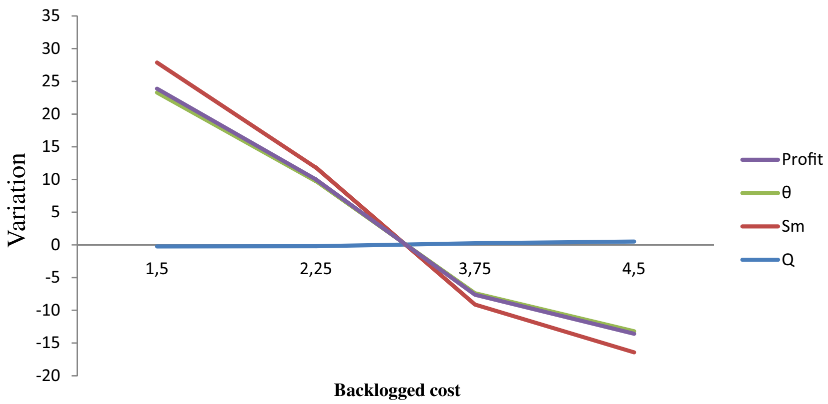

FIGURE 3.

Optimal results for $M = 4$ months of the imperfect production inventory model for Case Ia: $R\pi_m \geq w_s Q$ are:

$E\pi_{m_1} = \$13.52$, $Q = 6921.81$ unit, $S_m = 407.45$ unit, $\theta = 0.68$, $IC_m = \$2.19$ per unit, $BC_m = \$0.13$ per unit, $PC_m = \$0.10$ per unit, $t_1 = 0.71$ months, $t_p = 6.92$ months, $t_2 = 9.56$ months, and $T = 10.30$ months.

Eigenvalues of the Hessian matrix of profit function are: $-3.27, -3.47 \times 10^{-6}, -2.52 \times 10^{-8}$. Hence, the above results for Case Ia are optimal.

Optimal results for $M = 2$ months of the imperfect production inventory model for Case Ib: $R\pi_m < w_s Q$ with partial payment are:

$E\pi_{m_2} = \$13.07$, $Q = 5286.03$ unit, $S_m = 328.75$ unit, $\theta = 0.56$, $IC_m = \$1.85$ per unit, $BC_m = \$0.11$ per unit, $PC_m = \$0.08$ per unit, $t_1 = 0.59$ months, $t_p = 5.28$ months, $t_2 = 7.54$ months, $T = 8.14$ months, and $t_{M_1} = 2.31$.

Eigenvalues of the Hessian matrix of profit function are: $-3.23, -4.98 \times 10^{-6}, -7.75 \times 10^{-8}$. Hence, the above results of Case Ib are optimal.

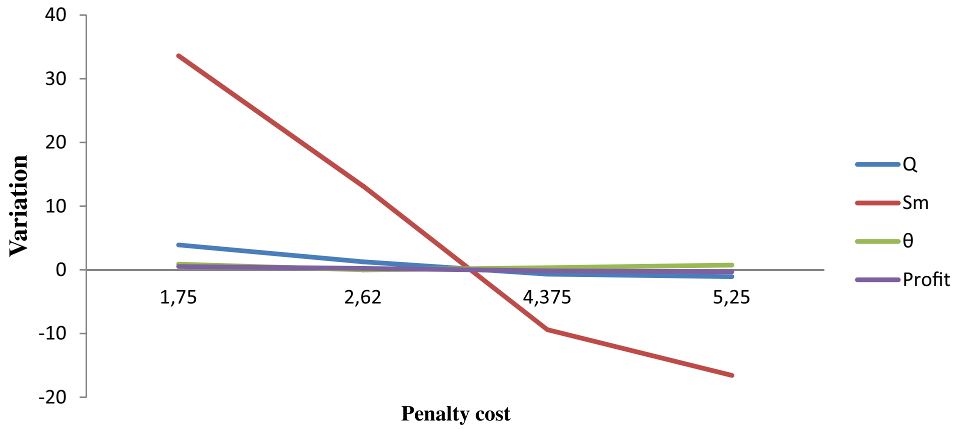

FIGURE 4.

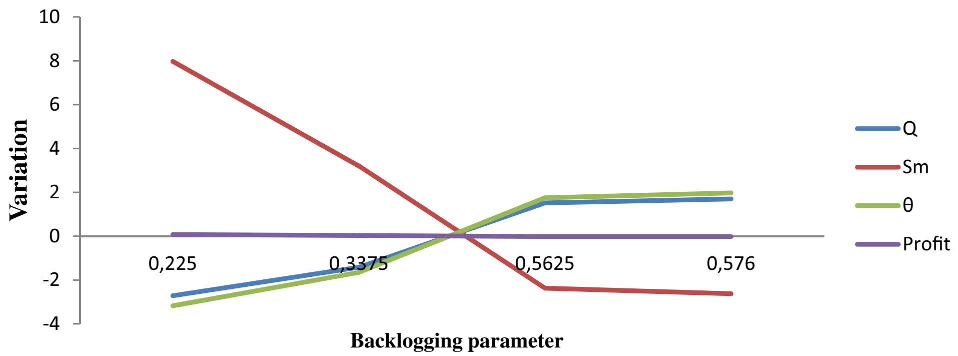

FIGURE 5.

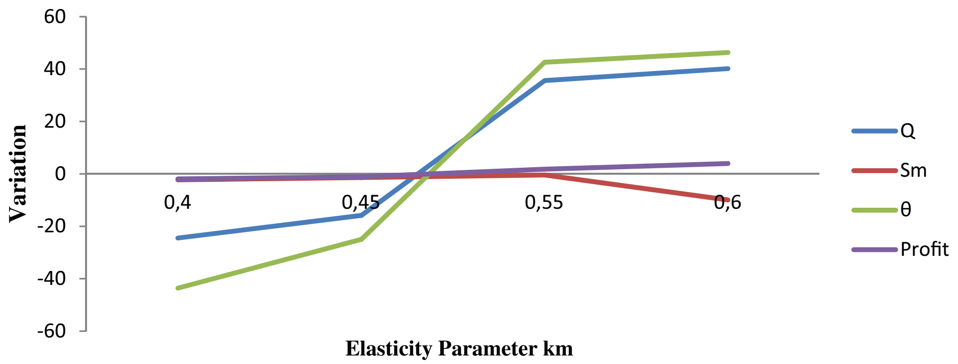

FIGURE 6.

Optimal results for $M = 2$ months of the imperfect production inventory model for Case Ic: $R\pi_m < w_s Q$ without partial payment are:

$E\pi_{m_3} = \$13.072$, $Q = 5130.87$ unit, $S_m = 329.19$ unit, $\theta = 0.55$, $IC_m = \$1.81$ per unit, $BC_m = \$0.11$ per unit, $PC_m = \$0.08$ per unit, $t_1 = 0.59$ months, $t_p = 5.13$ months, $t_2 = 7.34$ months, $T = 7.95$ months, and $t_{M_2} = 2.24$.

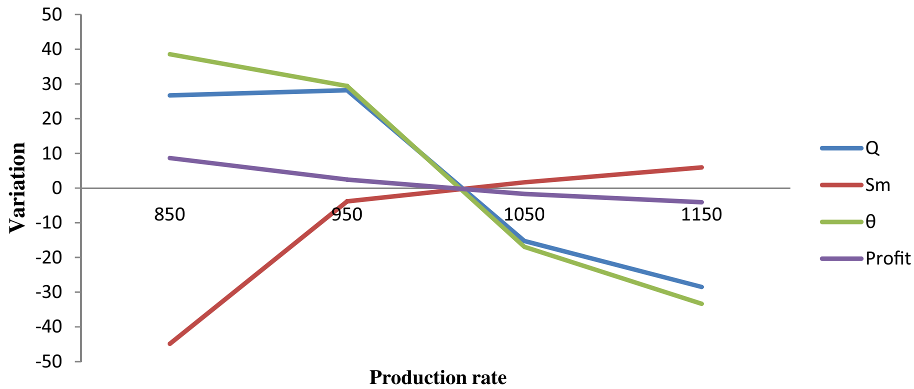

Figure 7.

Eigenvalues of the Hessian matrix of profit function are: $-3.21, -5.15\times10^{-6}, -8.27\times10^{-8}$. Hence, the above results of Case Ic are optimal.

Optimal results for $M = 7.75$ months of the imperfect production inventory model for Case 2: $t_2 \le M \le T$: $E\pi_{m_4} = \$13.44$, $Q = 5302.01$ unit, $S_m = 332.13$ unit, $\theta = 0.56$, $IC_m = \$1.86$ per unit, $BC_m = \$0.11$ per unit, $PC_m = \$0.08$ per unit, $t_1 = 0.59$ months, $t_p = 5.30$ months, $t_2 = 7.56$ months, $T = 8.17$ months, and $t_{M_2} = 2.24$.

Eigenvalues of the Hessian matrix of profit function are: $-3.24, -4.97\times10^{-6}, -7.66\times10^{-8}$. Hence, the above results of Case 2 are optimal.

5. Sensitivity analysis

In this section, we discuss the sensitivity of the key parameters for the Cases 1a, 1b and 1c and observe variation of the decision variables and expected profit for the different cases with varying key parameters.

- From the Table 2, we observe that C_{hm} is highly sensitive parameter. With the increasing value of the C_{hm}, the optimal ordering size and profit of the manufacturer decrease except Case 1a where the optimal ordering size and profit of the manufacturer first increase and then decrease. Again, manufacturer's maximum back-ordered level and quality of products increase with higher value of C_{hm} for Case 1a, and the quality of products increases for Case 1b and 1c. But, for the Cases 1b and 1c, manufacturer's maximum back-ordered level first decreases and then increases.

- The optimal lot size and average manufacturer's profit increase but, manufacturer's maximum back-ordered level and quality of products decreases with higher value of C_{bm} for all the cases.
- When the parameter C_{pm} is increased, the optimal ordering size, manufacturer's maximum back-ordered level, average manufacturer's profit decrease for all the cases but quality of the products increases except Case 1a where quality of the products first decreases and then increases.
- The optimal lot size and average manufacturer's profit increase but, manufacturer's maximum back-ordered level and quality of products decreases with higher value of δ_m for all the cases.
- When the parameter k_m is increased, the optimal lot size, quality of the products, and average manufacturer's profit increase for all the cases. The manufacturer's maximum back-ordered level decreases for Cases 1b and 1c but it first increases and then deceases for Case 1a with higher values of k_m.
- With the increasing value of the production rate P, the optimal lot size, quality of the products, and average manufacturer's profit decrease but manufacturer's maximum back-ordered level increase for all the cases.

We also illustrate the sensitivity of the parameters by graphically (See Fig. 2 to 7) for Case 1a.

6. Conclusion

In this model, we have studied an imperfect production inventory model with waiting time dependent partial backlogged shortages. This model has been formulated considering quality of the produced item dependent market demand. We have developed the model considering allowable delay in payments where two alternative approaches have been discussed when the manufacturer could not pay the due amount to the supplier within offering credit period. Here, the model has been analyzed with respect to the ordering lot size, quality of product, and maximum backlogged size for the manufacturer such that the average integrated profit is maximum.

The major contribution of this work is to study an imperfect production system for the manufacturer considering quality of products dependent customers' demand. For the first time, we have considered the partially backlogged situation in the beginning as well as end of the cycle for the manufacturer with allowable delay in payments. We study two alternative approaches if the manufacturer could not pay the due amount to the supplier within offering credit period.

For further investigation, one may extended this model in several ways. This can be done by considering stochastic demand, demand depending on time and/or selling price. Another different extensions can be made by introducing restrictions on storage, capital, etc.

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