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OPTIMAL AND SIMPLE ALGORITHMS TO SOLVE INTEGRATED PROCUREMENT-PRODUCTION-INVENTORY PROBLEM WITHOUT/WITH SHORTAGE

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Abstract. This research work deals with an imperfect production system considering the purchasing of raw materials in order to study the economic production quantity (EPQ). This manufacturing system produces perfect and defective finished products; defectives are considered as scrap. A single product is manufactured from multiple raw materials which are purchased from outside suppliers. In the integrated procurement-production-inventory (IPPI) model, one of the principal decisions, in addition to determining the optimal lot size to produce, is to define the number of optimal orders of each raw material with respect to rate of consumption in the manufacturing of finished product. Two cases are considered: without shortage (first model) and with shortage (backordering, second model). In the first model, the purpose is to determine jointly the optimal lot size to manufacture and the optimal number orders of each raw material in order to minimize the total cost. The second model obtains the optimal number of orders of each raw material, the optimal lot size to manufacture and the optimal shortage level with aim to minimize the total cost. This research also shows that both of the proposed inventory models are a convex programming problem, so exact algorithms to solve these inventory problems are proposed.

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1. Introduction

Harris [23] presented the first inventory model to determine the optimal lot to purchase. Other common name of Harris [23]'s formula is economic order quantity (EOQ). As said by Cárdenas–Barrón *et al.* [11], Harris is considered as the Founding Father of Inventory Theory. According to Andriolo *et al.* [1] the EOQ formula was also developed individually by some researchers and it is also known by other names such as Wilson formula and Camp formula. Taft [54] extended Harris's formula including production rate, and this inventory model is conventionally known as economic production quantity (EPQ) or economic manufacturing quantity (EMQ).

Keywords. Economic production quantity, imperfect production system, shortage, lot sizing, optimization.

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Grubbström [20] and Cárdenas–Barrón [6] through an algebraic method derived the EOQ and EPQ inventory models without/with shortages respectively. In these inventory models it is supposed that only one type of product is considered. One of the first research for a manufacturing system taking into account a variety of products was introduced by Eilon [13] and Rogers [48].

The EPQ inventory models assume that all the items are manufactured with high quality and defective items are not produced. However, in fact, defective items appear in the most of manufacturing systems, in this sense, researchers have been developing EPQ inventory models for defective production systems. In these production systems, defective items are of two types: scrapped items and reworkable items. Salameh and Jaber [49] derived an EPQ inventory model for defective items. In this inventory model, manufactured items are inspected with 100% screening process and defective items are separated from perfect items. All defective items are sold. Hayek and Salameh [24] examined the effects of manufacturing defective items on an EPQ inventory model. They supposed that defective items after finishing the normal production cycle, are reworked by constant rate. Jamal *et al.* [26] proposed a production inventory model for an imperfect manufacturing system. They consider two ways for reworking defective items: (1) the rework of defective items is within same production cycle in which the products are manufactured, (2) accumulated defective items of Nproduction cycles are reworked in a separate production cycle. Cárdenas–Barrón [8] revisited Jamal *et al.* [26]'s research and he corrected the numerical examples. Then, Cárdenas–Barrón [9] offered a new simple solution to solve the inventory model in Jamal *et al.* [26]'s study. Thereafter, Haji *et al.* [22] considered an imperfect production system for several items that are manufactured by single machine. The defective items are reworked after the end of the regular production cycle. They consider setup times for regular process and rework process. Later, Pasandideh *et al.* [40] studied a manufacturing system in which defective items are reworked. At the same year, Mohammadi *et al.* [35] optimized an integrated manufacturing with inspection policy for a deteriorating manufacturing system with imperfect inspection. Afterwards, Nobil *et al.* [37] presented multi machines manufacturing system with defective items. At the same year, Manna *et al.* [33] investigated a production–inventory model for an imperfect production system with repairable items and promotional demand. Taleizadeh *et al.* [52] determined optimal price, replenishment lot size and number of shipments for an EPQ model with rework and multiple shipments. They solved their problem by using a hybrid genetic algorithm. Later, Manna *et al.* [34] introduced a lot–sizing problem for a defective manufacturing system with advertisement dependent demand and manufacturing rate dependent of defective rate. Recently, researcher and scholars have studied the inventory problems with different policies, like Khara *et al.* [28], Bera and Jana [5], Mahata [32], Nobil *et al.* [38] and Benkherouf *et al.* [4]; just to name a few recent works.

One of the reasons why the defective items are produced is that the machines do not work properly. Therefore, the consideration of preventive maintenance (PM) policies in manufacturing systems is very important. Nakagawa [36], Rosenblatt and Lee [47], Lee and Rosenblatt [30], Groenevelt *et al.* [19] and Rahim [43] did first research on production–inventory problems with the consideration of PM policies. Ben–Daya and Makhdoum [2] studied the effects of a wide variety of PM policies on the joint optimization of the EPQ inventory model and the economic design of control chart. The purpose of this research is to determine the optimal production quantity, the optimal PM level and the optimal design of the control chart. Similarly, Rahim and Ben–Daya [44] developed an EPQ inventory model with respect to PM policy for simultaneously determining the optimal production quantity, control chart design and inspection schedule with non–zero inspection time. Then, Ben–Daya [3] considered a faulty production process having a deteriorating distribution with increasing hazard rate and imperfect maintenance to determine EPQ and PM level. Liao and Sheu [31] examined the effect of periodic PM policy on EPQ inventory model. Taleizadeh *et al.* [51] dealt with a problem of single machine production system for defective, scrap and rework, with shortage and temporary break in production process due to PM policy.

One of the important consideration in inventory problems is the inclusion of shortage. This means that the production system faces shortage which is backordered or lost sale. Hadley and Whitin [21], Johnson and Montgomery [27] and Elsayed and Teresi [14] addressed inventory problems with shortage. Cárdenas–Barrón [7] proposed a one algebraic solution procedure without derivatives to find the optimal solution to EPQ inventory model with shortage. Cárdenas–Barrón [10] presented a complete review over several optimizing methods in inventory theory. In this research, he also found the optimal solution to inventory problem with two types of backorder costs, linear and fixed, by geometry and algebraic method. Pentico *et al.* [42] introduced an EPQ inventory model with partial backordering. Then, Taleizadeh *et al.* [53] developed a single machine system for multi items considering defectives and partial backordering. Then, Taleizadeh [50] developed a constrained integrated imperfect manufacturing–inventory system with preventive maintenance and partial–backordering.

In real business conditions, a manufacturing company depends on suppliers of raw materials and consumers of its production. One of the first integrated model in inventory problems is Goyal [16]'s inventory model. He built an integrated inventory model for a single vendor–single buyer EPQ inventory model. Park [39] proposed an integrated procurement–production–inventory (IPPI) model for a perfect production system with a finished product and deteriorating raw materials which are purchased from outside suppliers. They assumed that raw materials are able to decay and finished product cannot decay. In this integrated procurement–production–inventory (IPPI) model shortages are not permitted for raw materials and finished product; all raw materials must be received in the preliminary process of production. Then, with respect to Park's [39] model, Raafat [45] developed his IPPI model with consideration that finished product can decay. Then, Goyal and Gunasekaran [18] presented an integrated procurement–production–inventory (IPPI) model for a perishable product considering pricing and advertising policies. They assumed that the producing system involves a finished product and deteriorating raw materials. They determined the optimal production lot for final item and the optimal order size for raw materials in a multi–stage production system. Kim and Chandra [29] proposed a heuristic procedure that obtains the optimal integrated manufacturing–inventory policies for a single product and raw materials without shortages. Then, Hong and Hayya [25] built an exact algorithm to solve the inventory problem in Kim and Chandra [29]. At the same time, Goyal and Deshmukh [17] presented a review of the literature on integrated procurement–production systems. Roan *et al.* [46] addressed the joint optimization of process mean, production run size and material order quantity for a container–filling process.

Pasandideh and Niaki [41] derived a multi–products EPQ inventory model with discrete delivery orders and constrained space. They optimized their inventory model with a genetic algorithm. Widyadana and Wee [55] revisited Pasandideh and Niaki [41]'s research and extended it to a multi–product EPQ inventory model taking into consideration the well–known just–in–time strategy. They considered a production system that fabricates multi products which are sent to a buyer by batches. They solved the inventory model by Lagrange solution approach and Branch and Bound. Due to complicated calculation of solution to this method and limited time, Cárdenas–Barrón [12] presented a new heuristic algorithm that has precise and optimal solution within of short time and low calculations.

From aforementioned literature, most studies focus on determining the optimal value for production and/or order. Also, some studies focus on integrating procurement system and production system. Therefore, crucial point of organizations is to determine the optimal production lot size of finished item and optimal order size of its raw materials.

This paper presents an integrated procurement–production–inventory (IPPI) model for a finished product and its raw materials. It is supposed that raw materials are turned to finished product by a manufacturing process. In this manufacturing system, it is considered that at the first of process beginning all raw materials are available in producer's stock. Therefore, the producer needs to define the order strategy of raw materials to purcahse from outside suppliers in addition to determine the lot size of fabrication of the product. Likewise, producer have to consider simultaneously the determination of the optmial value of lot size of fabrication for finished product and the optimal value of order size for raw materials for both without and with shortage cases. This paper considers two cases: first one is the consideration of IPPI model without shortage and second one is the consideration of IPPI model with backordering. In the first model, the purpose is to determine simultaneously the EPQ for finished product and the EOQ for each raw material. In the second model, the aim is to determine simultaneously the EPQ and the optimal level of shortage for finished product, and the EOQ for each raw material. In each IPPI model, the objective is to minimize total cost.

The rest of paper continues as follow. Section 2 presents the definition of problem, assumptions and notation. Section 3 formulates the IPPI model without shortage and proposes an exact solution procedure. Section 4 develops the IPPI model with shortage and provides an exact solution procedure. Section 5 solves a numerical example and presents a sensitivity analysis. Finally, Section 7 gives some conclusions and suggestions for future research.

2. Problem definition

This paper derives an integrated procurement–production–inventory (IPPI) system for a single product and its raw materials without/with shortage. The IPPI system considers that the manufacturing process fabricates both perfect and defective finished products. The defective products are considered as scrapped items. The products are fabricated at P rate where β% of these are not useful, so non–defective items are produced with $(1-\beta)P$ rate. In fact, the inventory level of finished product increases with $(1-\beta)\,P-D$ rate, where D is demand rate of items. The finished product needs n type of raw materials to produce it, these are provided from outside suppliers. Thus, producer has to consider cost of ordering and purchasing raw materials in the inventory system costs in addition to cost of producing the item. Here, producer defines the quantity of raw material $j$that needs to be ordered for some producing periods (M_j) and store in his/her stock for beginning of production. The IPPI model is developed without/with shortage for final finished product. For the shortage case is supposed that items in each cycle can have shortage up to Kunits In addition,the following assumptions and notation are considered.

2.1. Assumptions

- There are multiple raw materials and a single finished product in the manufacturing system.
- Shortage is not allowed for all the raw materials.
- Lead time for all the raw materials is zero.
- α_j amount of raw material j required to produce one unit of finished product.
- Finished product and raw materials are imperishable.
- Time horizon is infinite.
- There is not any limitation for ordering.
- Parameters are deterministic and known.

The following notation for the raw material $j=1,2,\ldots,n$ and the finished product is used in this paper:

2.2. Notation

n Number of raw materials
P Production rate for finished product (units/time unit)
D Demand rate for finished product (units/ time unit)
β Proportion of produced scrapped items; $0<\beta<1$
α_j Amount of raw material j required to produce one finished product (amount of raw material j/unit of finished product)
I Maximum on–hand inventory of finished product (units)
A Machine setup cost of producing finished product (\$/setup)
C Production cost of finished product per unit (\$/unit)
d Disposal cost of defective items per unit (\$/unit)
O_j Ordering cost of raw material j per order (\$/order)
R_j Purchasing cost of raw material j per unit (\$/unit)
H Holding cost of finished product per item per unit time (\$/unit/time unit)
h_j Holding cost of raw material j per item per unit time (\$/unit/time unit)
π Backorder cost per item per unit time (\$/unit/time unit)

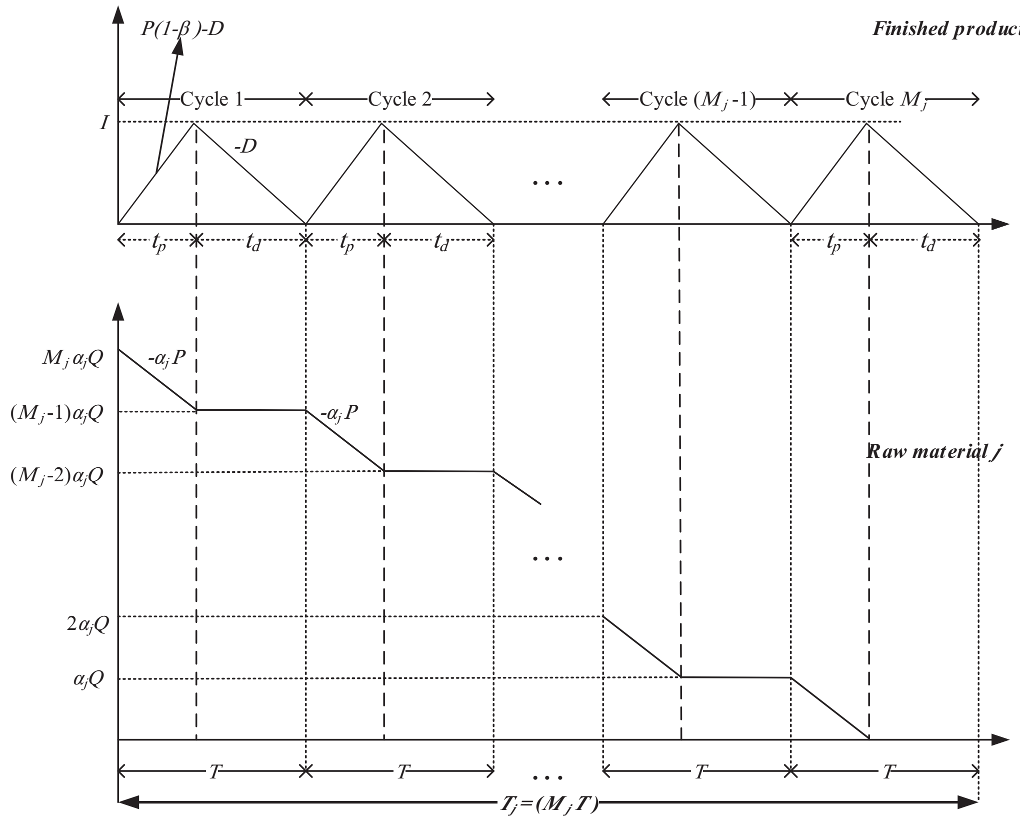

FIGURE 1. The on–hand inventory graph for the IPPI problem without shortage.

Dependent variables:

T Cycle length of the finished product
T_j Cycle length of the raw material j

Decision variables:

Q Production lot size of finished product in a cycle (*a decision variable*)
K Shortage quantity of finished product in a cycle (*a decision variable*)
M_j Number of cycles for raw material j (*decision variables*)

3. The integrated procurement–production–inventory (IPPI) model without shortage

This section describes the IPPI model without shortage and its solution method.

3.1. Mathematical modelling

In this model, it is supposed that shortage is not allowed for finished product and raw materials The on–hand inventory graph of the raw material j and the finished product for the IPPI problem without shortage are shown in Figure 1.

From Figure 1, the following equations are expressed:

$$t_p = \frac{Q}{P} \tag{3.1}$$

$$I = [(1-\beta)P - D]\frac{Q}{P} \tag{3.2}$$

$$t_d = \frac{I}{D} = \left(\frac{(1-\beta)P - D}{DP}\right)(Q) \tag{3.3}$$

$$T = t_p + t_d = \frac{Q}{P} + \left(\frac{(1-\beta)P - D}{DP}\right)(Q) = \frac{(1-\beta)Q}{D} \tag{3.4}$$

then,

$$T_j = M_j T = \frac{(1-\beta)M_j Q}{D} \tag{3.5}$$

The total cost consists of the following costs: production cost, the disposal cost of scrapped items, the setup cost for manufacturing the finished product, holding cost of finished product, the ordering cost of raw materials, the purchasing cost of raw materials and the holding cost for raw materials. Thus, these costs are obtained as follows:

The production cost per cycle and the number of cycles per unit time are equal to CQ and $1/T$ respectively. Therefore, the production cost per unit time is calculated by:

$$\text{Production cost} = \frac{CQ}{T} \tag{3.6}$$

Using equation (3.4) in equation (3.6), hence:

$$\text{Production cost} = \frac{DCQ}{Q(1-\beta)} = \frac{DC}{(1-\beta)} \tag{3.7}$$

The disposal cost per cycle is equal to $d\beta Q$, which β is the proportion of scrapped items, and the number of cycles per unit time is $1/T$. Therefore, the disposal cost per unit time is computed by:

$$\text{Disposal cost} = \frac{d\beta Q}{T} = \frac{Dd\beta Q}{Q(1-\beta)} = \frac{Dd\beta}{(1-\beta)} \tag{3.8}$$

The machine setup cost per cycle is equal to A. Therefore, the setup cost per unit time is determined by:

$$\text{Setup cost} = \frac{A}{T} = \frac{DA}{Q(1-\beta)} = \frac{DA}{(1-\beta)}\left(\frac{1}{Q}\right) \tag{3.9}$$

Based in Figure 1, the holding cost per unit of finished product, thus:

$$\text{Holding cost for finished product} = \frac{H}{2T}(I \times T) \tag{3.10}$$

Substituting I and T from equations (3.2) and (3.4) into equation (3.10), hence:

$$\text{Holding cost for finished product} = \frac{H}{2T}\left[((1-\beta)P - D)\left(\frac{Q}{P}\right) \times \left(\frac{(1-\beta)Q}{D}\right)\right] \tag{3.11}$$

Thus, using equation(3.4) in above relation, thus:

$$\text{Holding cost for finished product} = \frac{H[(1-\beta)P - D]}{2P}Q = \frac{H}{2}\left(1 - \beta - \frac{D}{P}\right)Q \tag{3.12}$$

The ordering cost of the raw material j for M_j cycles and the number of cycles per unit time are equal to O_j and $1/T_j$ respectively. Thus, total ordering cost per unit time is obtained by:

$$\text{Ordering cost} \sum_{j=1}^{n} \frac{O_j}{T_j} = \sum_{j=1}^{n} \frac{DO_j}{(1-\beta)} \left(\frac{1}{M_j Q}\right) \tag{3.13}$$

The purchasing cost of the raw material j for M_j cycles is equal to $M_j R_j \alpha_j Q$ that α_j is proportion of per unit of finished product. Thus, total purchasing cost per unit time is computed by:

$$\text{Purchasing cost} \sum_{j=1}^{n} \frac{M_j R_j \alpha_j Q}{T_j} = \sum_{j=1}^{n} \frac{DR_j \alpha_j}{(1-\beta)} \tag{3.14}$$

From Figure 1, the area of the raw material in this figure is equal to:

$$\begin{aligned} &\left\{\frac{\alpha_j Q t_p}{2} + \frac{3\alpha_j Q t_p}{2} + \frac{5\alpha_j Q t_p}{2} + \cdots + \frac{(2M_j-3)\,\alpha_j Q t_p}{2} + \frac{(2M_j-1)\,\alpha_j Q t_p}{2}\right\} \\ &+ \{\alpha_j Q t_d + 2\alpha_j Q t_d + 3\alpha_j Q t_d + \ldots + (M_j-2)\,\alpha_j Q t_d + (M_j-1)\alpha_j Q t_d\} \end{aligned} \tag{3.15}$$

Based in Appendix A, the area of the raw material j can easily be determined as follows:

$$\text{Area of raw material } j = \left[\frac{\alpha_j}{2P} + \frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right)\right] Q^2 M_j^2 - \frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right) M_j Q^2 \tag{3.16}$$

Thus, total holding cost per unit time for all the raw materials is obtained as follows:

$$\begin{aligned} \text{Holding cost for raw materials} &= \sum_{j=1}^{n} \left\{ \frac{h_j}{T_j} \left(\left[\frac{\alpha_j}{2P} + \frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right)\right] Q^2 M_j^2 \right.\right. \\ &\quad \left.\left. -\frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right) M_j Q^2 \right) \right\} \\ &= \sum_{j=1}^{n} \left\{ \frac{Dh_j}{(1-\beta)} \left[\frac{\alpha_j}{2P} + \frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right)\right] (QM_j) \right. \\ &\quad \left. -\frac{Dh_j \alpha_j}{2\,(1-\beta)}\left(\frac{(1-\beta)\,P-D}{DP}\right) Q \right\} \end{aligned} \tag{3.17}$$

Now, based in equations (3.7)–(3.9), (3.12)–(3.14) and (3.17), the total cost for the IPPI model without shortage, denoted by TC, is written as follows:

$$\begin{aligned} TC = \sum_{j=1}^{n} &\left\{ \frac{D\,(R_j \alpha_j + d\beta + C)}{(1-\beta)} + \frac{DA}{(1-\beta)}\left(\frac{1}{Q}\right) + \frac{DO_j}{(1-\beta)}\left(\frac{1}{M_j Q}\right) \right. \\ &+ \left[\frac{Dh_j}{(1-\beta)}\left[\frac{\alpha_j}{2P} + \frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right)\right]\right] (QM_j) \\ &\left. + \left[\frac{H}{2}\left(1-\beta-\frac{D}{P}\right) - \frac{Dh_j \alpha_j}{2\,(1-\beta)}\left(\frac{(1-\beta)\,P-D}{DP}\right)\right] (Q) \right\} \end{aligned} \tag{3.18}$$

Therefore, the optimization problem is formulated as follows:

$$\begin{aligned} &TC = \sum_{j=1}^{n} \Delta_1^j + \Delta_2\left(\frac{1}{Q}\right) + \sum_{j=1}^{n} \Delta_3^j \left(\frac{1}{M_j Q}\right) + \sum_{j=1}^{n} \Delta_4^j\,(QM_j) + \sum_{j=1}^{n} \Delta_5^j\,(Q) \\ &\text{s.t. } Q > 0 \\ &\qquad M_j \geq 1 \quad \& \quad \text{integer}; \quad j = 1, 2, \ldots, n \end{aligned} \tag{3.19}$$

where

$$\Delta_1^j = \frac{D\left(R_j\alpha_j + d\beta + C\right)}{(1-\beta)} > 0; \quad j = 1,2,\ldots,n \tag{3.20}$$

$$\Delta_2 = \frac{DA}{(1-\beta)} > 0 \tag{3.21}$$

$$\Delta_3^j = \frac{DO_j}{(1-\beta)} > 0; \quad j = 1,2,\ldots,n \tag{3.22}$$

$$\Delta_4^j = \frac{Dh_j}{(1-\beta)}\left[\frac{\alpha_j}{2P} + \frac{\alpha_j}{2}\left(\frac{(1-\beta)P - D}{DP}\right)\right] > 0; \quad j = 1,2,\ldots,n \tag{3.23}$$

$$\Delta_5^j = \frac{H}{2}\left(1-\beta-\frac{D}{P}\right) - \frac{Dh_j\alpha_j}{2(1-\beta)}\left(\frac{(1-\beta)P - D}{DP}\right); \quad j = 1,2,\ldots,n \tag{3.24}$$

3.2. Solution procedure

It can be shown that the nonlinear objective function in (3.19) is a convex function (see Appendix B). Thus, to determine the optimal values of M_j, the first derivative of equation (3.19) is taken with respect to $M_j$as follows:

$$\frac{\partial TC}{\partial M_j} = \Delta_4^j Q - \frac{\Delta_3^j}{QM_j^2} = 0 \tag{3.25}$$

So,

$$M_j = \sqrt{\frac{\Delta_3^j}{\Delta_4^j Q^2}} \tag{3.26}$$

Also, to determine the optimal values of Q, denoted by Q^*, the first derivative of equation (3.19) are taken with respect to Q , equating it to zero and solving, it follows that:

$$Q^* = \sqrt{\frac{\sum_{j}^{n}\left(\frac{\Delta_3^j}{M_j}\right) + \Delta_2}{\sum_{j=1}^{n}\left(\Delta_5^j + \Delta_4^j M_j\right)}} \tag{3.27}$$

Substituting M_j from equation (3.26), thus:

$$Q^* = \sqrt{\frac{\Delta_2}{\sum_{j=1}^{n}\Delta_5^j}} \tag{3.28}$$

Now, based on the optimal value of Q^*, the optimal value of each M_j, denoted by M_j^*, is obtained as follows:

$$M_j^* = \sqrt{\frac{\Delta_3^j}{\Delta_4^j\left(Q^*\right)^2}} \tag{3.29}$$

where the value of M_j^* must be an integer number. Therefore, the objective function in (3.19) is expressed as follows:

$$TC = \Delta_2\left(\frac{1}{Q}\right) + \sum_{j=1}^{n}\left(\Delta_1^j + \Delta_5^j(Q) + L_j\right) \tag{3.30}$$

where

$$L_j = \Delta_4^j Q\,(M_j) + \frac{\Delta_3^j}{Q}\left(\frac{1}{M_j}\right) \tag{3.31}$$

García–Laguna *et al.* [15] proved that the integer solution to the following mathematical problem:

Min $w_1 U + U/w_2$; where both w_1 and w_2 are positive, $U \geqslant 1$ and integer it follows that:

$$U = \left\lceil -0.5 + \sqrt{0.25 + \frac{w_2}{w_1}} \right\rceil \ \text{ or } \ U = \left\lfloor 0.5 + \sqrt{0.25 + \frac{w_2}{w_1}} \right\rfloor \tag{3.32}$$

in which, $\lceil y \rceil$ is the smallest integer greater than or equal to y and $\lceil y \rceil$ is the largest integer less than or equal to y.

Then based in equation (3.32), the integer solution for each M_j is determined as follows:

$$M_j = \left\lceil -0.5 + \sqrt{0.25 + \frac{\Delta_3^j}{\Delta_4^j\,(Q^*)^2}} \right\rceil \quad \text{or} \quad M_j = \left\lfloor 0.5 + \sqrt{0.25 + \frac{\Delta_3^j}{\Delta_4^j\,(Q^*)^2}} \right\rfloor \tag{3.33}$$

Finally, the following solution procedure is proposed for solving the IPPI model without shortage.

***S*1**. Calculate the coefficients of the objective function, Δ_1^j, Δ_2, $\Delta_3^j \Delta_4^j$ and Δ_5^j by using equations (3.20)–(3.24) and go to $S2$,

***S*2**. If $\sum\limits_{j=1}^{n} \Delta_5^j > 0$, then go to $S3$. Otherwise, there is no feasible solution and go to $S8$,

***S*3**. Compute Q by using equation (3.28),

***S*4**. Given the value of Q, determine y_j for $j = 1, 2, \ldots, n$ by using $y_j = (-0.5 + \sqrt{0.25 + \frac{\Delta_3^j}{\Delta_4^j (Q)^2}})$ and go to $S5$,

***S*5**. If each y_j is not an integer number, then $M_j^* = \lceil y_j \rceil$ is a unique optimal solution for each M_j. Otherwise, both $M_j^* = \lceil y_j \rceil$ and $M_j^* = \lceil y_j \rceil + 1$ are the optimal value of each M_j. Then go to $S6$,

***S*6**. Given the value of each M_j^*, recalculate Q^* by using equation (3.27) and go to $S7$,

***S*7**. Determine TC^* by using equation (3.19) and report the solution.

***S*8**. Stop.

4. The integrated procurement–production–inventory (IPPI) model with shortage

This section derives the IPPI model with shortage and its solution method is proposed.

4.1. Mathematical modelling

In this model, it is supposed that shortage is allowable for finished product and it is fully backordering. The on–hand inventory graph for the raw material j and the finished product for the IPPI model with shortage are shown in Figure 2.

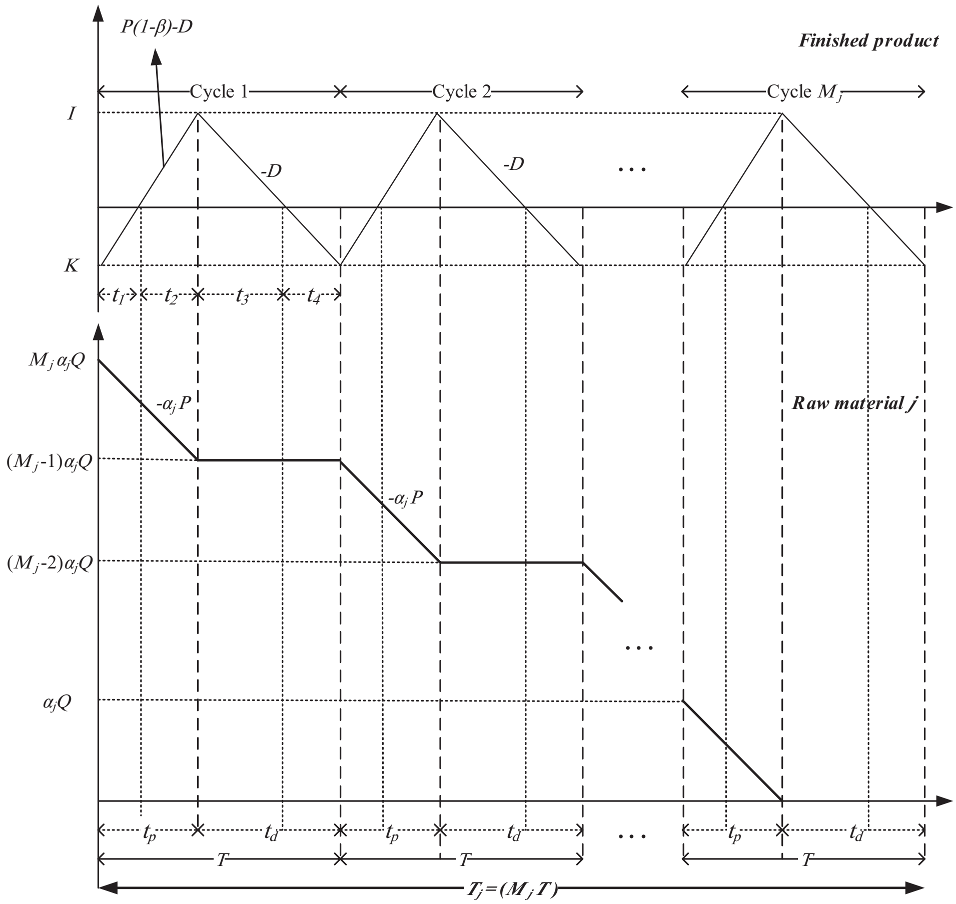

FIGURE 2. The on-hand inventory graph for the IPPI problem with shortage.

From Figure 2, the following equations are deduced.

$$t_1 = \frac{K}{(1-\beta)\,P - D} \tag{4.1}$$

$$I\left[(1-\beta)\,P - D\right]\left(\frac{Q}{P}\right) - K \tag{4.2}$$

$$t_2 = \frac{I}{(1-\beta)\,P - D} = \frac{Q}{P} - \frac{K}{(1-\beta)\,P - D} \tag{4.3}$$

$$t_p = t_1 + t_2 = \frac{Q}{P} \tag{4.4}$$

$$t_3 = \frac{I}{D} = \left(\frac{(1-\beta)\,P - D}{DP}\right)Q - \frac{K}{D} \tag{4.5}$$

$$t_4 = \frac{K}{D} \tag{4.6}$$

$$t_d = t_3 + t_4 = \left(\frac{(1-\beta)\,P - D}{DP}\right)(Q) \tag{4.7}$$

$$T = t_p + t_d = \frac{Q}{P} + \left(\frac{(1-\beta)\,P - D}{DP}\right)(Q) = \frac{(1-\beta)\,Q}{D} \tag{4.8}$$

So,

$$T_j = M_j T = \frac{(1-\beta) M_j Q}{D} \tag{4.9}$$

The total cost is comprised of the following costs: production cost, the disposal cost of scrapped items, the setup cost for producing the finishedo product, the holding cost for finished product, the backorder cost, the ordering cost of raw materials, the purchasing cost of raw materials, and holding cost raw materials. Thus, these costs are obtained as follows:

The production cost per time unit for the finished product is calculated by:

$$\text{Production cost} = \frac{CQ}{T} = \frac{DCQ}{Q(1-\beta)} = \frac{DC}{(1-\beta)} \tag{4.10}$$

The disposal cost per time unit for the scrapped items is determined by:

$$\text{Disposal cost} = \frac{d\beta Q}{T} = \frac{Dd\beta Q}{Q(1-\beta)} = \frac{Dd\beta}{(1-\beta)} \tag{4.11}$$

The setup cost per time unit is obtained with:

$$\text{Setup cost} = \frac{A}{T} = \frac{DA}{Q(1-\beta)} = \frac{DA}{(1-\beta)}\left(\frac{1}{Q}\right) \tag{4.12}$$

Based in Figure 2, the holding cost per unit of finished product is given by

$$\text{Holding cost for finished product} = \frac{H}{2T}\left(I \times (t_2 + t_3)\right) \tag{4.13}$$

Substituting I, t_2, t_3 and T from equations (4.2), (4.3), (4.5) and (4.8) respectively, thus (see Appendix C, for detail calculations)

$$\text{Holding cost for finished product} = \frac{H((1-\beta)P - D)}{2P}Q + \frac{HP}{2((1-\beta)P - D)}\left(\frac{K^2}{Q}\right) - HK \tag{4.14}$$

Based in Figure 2, the backorder cost per unit of finished product is computed by:

$$\text{Bakorder cost} = \frac{\pi}{2T}\left(K \times (t_1 + t_4)\right) \tag{4.15}$$

Substituting t_1 and t_4 from equations (4.1) and (4.6) respectively, hence

$$\text{Bakorder cost} = \frac{\pi}{2T}\left(K \times \left(\frac{K}{(1-\beta)P - D} + \frac{K}{D}\right)\right) \tag{4.16}$$

From equation (4.8),

$$\text{Bakorder cost} = \frac{D\pi}{2(1-\beta)Q}\left(K \times \left(\frac{K}{(1-\beta)P - D} + \frac{K}{D}\right)\right) = \frac{\pi P}{2((1-\beta)P - D)}\left(\frac{K^2}{Q}\right) \tag{4.17}$$

The total ordering cost per for time unit is determined by:

$$\text{Ordering cost} = \sum_{j=1}^{n} \frac{O_j}{T_j} = \sum_{j=1}^{n} \frac{DO_j}{(1-\beta)}\left(\frac{1}{M_j Q}\right) \tag{4.18}$$

The total purchasing cost per time unit is calculated by:

$$\text{Purchasing cost} = \sum_{j=1}^{n} \frac{M_j R_j \alpha_j Q}{T_j} = \sum_{j=1}^{n} \frac{DR_j \alpha_j}{(1-\beta)} \tag{4.19}$$

From Figure 2, the area of the raw material j in this figure is equal to:

$$\left\{\frac{\alpha_j Q t_p}{2}+\frac{3\alpha_j Q t_p}{2}+\frac{5\alpha_j Q t_p}{2}+\ldots+\frac{(2M_j-3)\,\alpha_j Q t_p}{2}+\frac{(2M_j-1)\,\alpha_j Q t_p}{2}\right\}$$
$$+\left\{\alpha_j Q t_d+2\alpha_j Q t_d+3\alpha_j Q t_d+\ldots+(M_j-2)\,\alpha_j Q t_d+(M_j-1)\alpha_j Q t_d\right\} \tag{4.20}$$

Because the uptime period and downtime period, denoted by t_p and t_d, are same as the IPPI model without shortage, in other words, $t_p=\frac{Q}{P}$ and $t_d=\left(\frac{(1-\beta)P-D}{DP}\right)(Q)$. Therefore, the area of the raw material j in Figure 2 is obtained as follows:

$$\text{Area of Raw material } j=\left[\frac{\alpha_j}{2P}+\frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right)\right]Q^2M_j^2-\frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right)M_jQ^2 \tag{4.21}$$

Thus, the total holding cost per time unit is computed as follows:

$$\text{Holding cost for raw materials}=\sum_{j=1}^{n}\left\{\frac{Dh_j}{(1-\beta)}\left[\frac{\alpha_j}{2P}\right.\right.$$
$$\left.\left.+\frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right)\right](QM_j)-\frac{Dh_j\alpha_j}{2\,(1-\beta)}\left(\frac{(1-\beta)\,P-D}{DP}\right)Q\right\} \tag{4.22}$$

Now, based in equations (4.10)–(4.12), (4.14), (4.17)–(4.19) and (4.22), the total cost for the IPPI model with shortage is given by:

$$\begin{aligned}TC=\sum_{j=1}^{n}&\left\{\frac{D\,(R_j\alpha_j+d\beta+C)}{(1-\beta)}+\frac{DA}{(1-\beta)}\left(\frac{1}{Q}\right)+\frac{DO_j}{(1-\beta)}\left(\frac{1}{M_jQ}\right)\right.\\&+\left[\frac{Dh_j}{(1-\beta)}\left[\frac{\alpha_j}{2P}+\frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P-D}{DP}\right)\right]\right](QM_j)\\&+\left[\frac{H}{2}\left(1-\beta-\frac{D}{P}\right)-\frac{Dh_j\alpha_j}{2\,(1-\beta)}\left(\frac{(1-\beta)\,P-D}{DP}\right)\right](Q)-HK\\&\left.+\left[\frac{(H+\pi)\,P}{2\,((1-\beta)\,P-D)}\right]\left(\frac{K^2}{Q}\right)\right\}\end{aligned} \tag{4.23}$$

Thus, the optimization problem is stated as follows:

$$\begin{aligned}&TC=\sum_{j=1}^{n}\left\{\varDelta_1^j+\varDelta_2\left(\frac{1}{Q}\right)+\varDelta_3^j\left(\frac{1}{M_jQ}\right)+\varDelta_4^j\,(QM_j)+\varDelta_5^j\,(Q)-HK+\varDelta_6\left(\frac{K^2}{Q}\right)\right\}\\&\text{s.t. } QK>0\\&\qquad M_j\geq 1\&\text{integer};\qquad j=1,2,\ldots,n\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}
\Delta_1^j &= \frac{D\left(R_j\alpha_j + d\beta + C\right)}{(1-\beta)} > 0\\
\Delta_2 &= \frac{DA}{(1-\beta)} > 0\\
\Delta_3^j &= \frac{DO_j}{(1-\beta)} > 0\\
\Delta_4^j &= \frac{Dh_j}{(1-\beta)}\left[\frac{\alpha_j}{2P} + \frac{\alpha_j}{2}\left(\frac{(1-\beta)P - D}{DP}\right)\right] > 0\\
\Delta_5^j &= \frac{H}{2}\left(1-\beta-\frac{D}{P}\right) - \frac{Dh_j\alpha_j}{2(1-\beta)}\left(\frac{(1-\beta)P-D}{DP}\right)\\
\Delta_6 &= \frac{(H+\pi)P}{2\left((1-\beta)P - D\right)} > 0
\end{aligned} \tag{4.25}$$

4.2. Solution procedure

It can be shown that the nonlinear objective function in (4.24) is a convex function (see Appendix D). Thus, to determine the optimal values of each M_j and K, the partial derivatives of equation (4.24) are taken with respect to M_j and K as follows:

$$\begin{aligned}
M_j &= \sqrt{\frac{\Delta_3^j}{\Delta_4^j Q^2}}\\
K &= \frac{HQ}{2\Delta_6}
\end{aligned} \tag{4.26}$$

Also, to determine the optimal values of Q, denoted by Q^*, the first derivative of equation (4.24) is taken with respect to Q as follows:

$$Q = \sqrt{\frac{\sum_j^n \left(\frac{\Delta_3^j}{M_j}\right) + \Delta_2 + \Delta_6 K^2}{\sum_j^n \left(\Delta_5^j + \Delta_4^j M_j\right)}} \tag{4.27}$$

Substituting K from equation (4.26), thus:

$$Q = \sqrt{\frac{\sum_j^n \left(\frac{\Delta_3^j}{M_j}\right) + \Delta_2}{\sum_j^n \left(\Delta_5^j + \Delta_4^j M_j\right) - \frac{H^2}{4\Delta_6}}} \tag{4.28}$$

Then, substituting M_j from equation (4.26):

$$Q^* = \sqrt{\frac{\Delta_2}{\sum_j^n \left(\Delta_5^j\right) - \frac{H^2}{4\Delta_6}}} \tag{4.29}$$

Now, given the optimal value of Q^*, the optimal values of each M_j and K, denoted by M_j^* and K^*, are obtained by using equation (4.26). But, the value of each M_j^* must be an integer number. So, based on García–Laguna *et al.* [15] 's study, the following solution procedure for the IPPI model with shortage is proposed.

TABLE 1. Data for raw materials.

Raw material	α_j	h_j (\$/unit/year)	R_j (\$/unit)	O_j (\$/ordering)
1	0.5	2	50	400
2	0.2	3	40	500
3	0.4	4	30	300

TABLE 2. Values of the coefficient of the objective function for the IPPI model without shortage.

Raw material	Δ_1^j	Δ_2	Δ_3^j	Δ_4^j	Δ_5^j
1	46 122.4489795918		408 163.265306123	0.5	2.54755102040816
2	28 775.5102040816	204081.632653061	510 204.081632653	0.5	2.69653061224490
3	32857.1428571429		306 122.448979592	0.8	2.32408163265306

***S*1**. Calculate the coefficients of the objective function, Δ_1^j, Δ_2, Δ_3^j, Δ_4^j, Δ_5^j and Δ_6, by using equation (4.25) and go to $S2$,

***S*2**. If $\sum_j^n \left(\Delta_5^j\right) - \frac{H^2}{4\Delta_6} > 0$, then go to $S3$. Otherwise, there is no feasible solution and go to $S9$,

***S*3**. Compute Q by using equation (4.29) and go to $S4$,

***S*4**. Given the value of Q, determine y_j for $j = 1, 2, \ldots, n$ by using $y_j = (-0.5 + \sqrt{0.25 + \frac{\Delta_3^j}{\Delta_4^j (Q)^2}})$ and go to $S5$,

***S*5**. If each y_j is not an integer number, then $M_j^* = \lceil y_j \rceil$ is a unique optimal solution for M_j. Otherwise, both $M_j^* = \lceil y_j \rceil$ and $M_j^* = \lceil y_j \rceil + 1$ are the optimal value of M. Then go to $S6$,

***S*6**. Given the value of each M_j^*, recalculate Q^* by using equation (4.28) and go to $S7$,

***S*7**. Given the value of Q^*, recalculate K^* by using equation (4.26) and go to $S8$,

***S*8**. Determine TC^* by using equation (4.24) and report the solution.

***S*9**. Stop.

5. EXAMPLE AND SENSITIVITY ANALYSIS

In order to illustrate the applicability of the IPPI models without/with shortages a numerical example is solved. This numerical example consists a production system with a product and three raw materials. The following parameters for both models are: $P = 4000$ units/year, $D = 1000$ units/year, $\beta = 0.02$, $C = \$20$ per unit of finished product, $d = \$10$ per defective unit of finished product, $A = \$200$/setup, $H = \$\,8$/unit/year and $\pi = \$8$/unit/year. Thus, $P(1-\beta) - D = 2920$. The rest of data which correspond to raw materials are given in Table 1.

5.1. The IPPI model without shortage

The proposed solution procedure for solving the IPPI model without shortage is exposed below:

***S*1**. Calculated the coefficients of the objective function by using equations (3.20)–(3.24). The coefficients values are shown in Table 2. Then go to $S2$,

***S*2**. Since $\left(\sum_j^n \left(\Delta_5^j\right) = 7.568\right) > 0$, then go to $S3$.

TABLE 3. Values of the coefficients of the objective function for the IPPI model with shortage.

Raw material	Δ_1^j	Δ_2	Δ_3^j	Δ_4^j	Δ_5^j	Δ_6
1	46 122.44897959		408 163.2653061	0.5	2.547551020408	
2	28 775.51020408	20 4081.632653	510 204.0816326	0.3	2.696530612244	10.9589041095
3	32 857.14285714		30 6122.4489795	0.8	2.324081632653	

***S*3**. Compute Q by using equation (3.28), thus $Q = \sqrt{\frac{\Delta_2}{\sum_j^n (\Delta_5^j)}} = 164.212689388708$, then go to $S4$.

***S*4**. Given the value of Q, determine y_j:

$$y_1 = \left(-0.5 + \sqrt{0.25 + \frac{\Delta_3^1}{\Delta_4^1 (Q)^2}}\right) = 5.50205898380093;$$

$$y_2 = \left(-0.5 + \sqrt{0.25 + \frac{\Delta_3^2}{\Delta_4^2 (Q)^2}}\right) = 7.94153808848666;$$

$y_3 = \left(-0.5 + \sqrt{0.25 + \frac{\Delta_3^3}{\Delta_4^3 (Q)^2}}\right) = 3.76700227268965$, then go to $S5$.

***S*5**. Since y_1, y_2 and y_3 are not the integer numbers, so $M_1^* = \lceil 5.502 \rceil = 6$, $M_2^* = \lceil 7.941 \rceil = 8$ and $M_3^* = \lceil 3.767 \rceil = 4$. Then go to $S6$.

***S*6**. Given the value of each M_j^*, recalculate Q^* by using equation (3.27).

$Q^* = \sqrt{\frac{\sum_j^n \left(\frac{\Delta_3^j}{M_j}\right) + \Delta_2}{\sum_j^n (\Delta_5^j + \Delta_4^j M_j)}} = 159.711748914276$ and go to $S7$,

***S*7**. Determine TC^* by using equation (3.19) and report solution. The total cost TC^* is equal to \$ 112 919.593304484, $Q^* = 159.711748914276$, $M_1^* = 6$, $M_2^* = 8$ and $M_3^* = 4$.

***S*8**. Stop.

5.2. The IPPI model with shortage

The proposed solution procedure for solving the IPPI model with shortage is as follows:

***S*1**. Calculate the coefficients of the objective function, Δ_1^j, Δ_2, Δ_3^j, Δ_4^j, Δ_5^j and Δ_6, by using equation (4.25). The values of coefficients are given in Table 2. Then go to $S2$,

***S*2**. Since $\sum_j^n \left(\Delta_5^j\right) - \frac{H^2}{4\Delta_6} = 6.10816326530612 > 0$, then go to $S3$.

***S*3**. Compute Q by using equation (4.29). Thus, $Q = \sqrt{\frac{\Delta_2}{\sum_j^n (\Delta_5^j) - \frac{H^2}{4\Delta_6}}} = 182.787562533778$, then go to $S4$.

***S*4**. Given the value of Q, determine y_j:

$$y_1 = \left(-0.5 + \sqrt{0.25 + \frac{\Delta_3^1}{\Delta_4^1 (Q)^2}}\right) = 4.94293971855054;$$

TABLE 4. Sensitivity analysis of P, D, β and α parameters.

Parameter	% Changes	% Changes					
		M_1^*	M_2^*	M_3^*	K^*	Q^*	TC^*
P	–25	0	0	0	–8.961064351	2.77075725307	–0.11759921654
	+25	0	0	0	5.183944610	–1.55861594150	0.06906164345
	+50	0	14.2	33.3	0.479604982	–9.81543651168	0.11122391838
D	–25	0	0	33.3	–12.91235753	–19.7804681350	–24.4191785123
	+25	0	0	0	4.333603204	14.1026671752	24.3261069465
	+50	–20.0	–14.2	0	13.30926803	36.7202738323	48.5809456921
β	–25	0	0	0	0.245960654	–0.43598465637	–0.62377067369
	+25	0	0	0	–0.248153023	0.43979074909	0.63016958197
	+50	0	0	0	–0.498527635	0.88343725888	1.26683704121
α_j	–25	20.0	14.2	33.3	–4.456390154	–4.45639015461	–10.4632334833
	+25	–20.0	–14.2	0	1.734126510	1.73412651047	10.4173895742
	+50	–20.0	–14.2	0	–2.660450592	–2.66045059208	20.8133577158

$$y_2 = \left(-0.5 + \sqrt{0.25 + \frac{\varDelta_3^2}{\varDelta_4^2\,(Q)^2}}\right) = 7.13451894273312;$$

$y_3 = (-0.5 + \sqrt{0.25 + \frac{\varDelta_3^3}{\varDelta_4^3(Q)^2}}) = 3.38419948029796$, then go to $S5$.

***S*5**. Since y_1, y_2 and $y_3$are not the integer numbers, so $M_1^* = \lceil 4.942 \rceil = 5$, $M_2^* = \lceil 7.134 \rceil = 8$ and $M_3^* = \lceil 3.384 \rceil = 4$. Then go to $S6$.

***S*6**. Given the value of each M_j^*, recalculate Q^* by using equation (4.28). Thus, $Q^* = \sqrt{\frac{\sum_j^n \left(\frac{\varDelta_3^j}{M_j}\right) + \varDelta_2}{\sum_j^n \left(\varDelta_5^j + \varDelta_4^j M_j\right) - \frac{H^2}{4\varDelta_6}}} =$ 187.460841606887 and go to $S7$,

***S*7**. Given the value of Q^*, recalculate K^* by using equation (4.26). Thus, $K^* = \frac{HQ^*}{2\varDelta_6} = 68.4232071865137$ and go to $S8$.

***S*8**. Determine TC^* by using equation (4.24) and report solution. The total cost TC^* is equal to \$ 112 669.636676086, $Q^* = 187.460841606887$, $K^* = 68.4232071865137$, $M_1^* = 5$, $M_2^* = 8$ and $M_3^* = 4$.

***S*9**. Stop.

5.3. Sensitivity analysis

This subsection presents a sensitivity analysis on problem's parameters. Notice that the total cost for the IPPI model with shortage is less than or equal to the total cost for the IPPI model without shortage. In the IPPI model with shortage, if it considered the value of shortage $K$is zero then the IPPI model without shortage is obtained. Because of that in this subsection, it is considered IPPI model with shortage for the sensitivity analysis. So in order to analyze the sensitivity of parameters, it takes into consideration the numerical example given in the Section 5. The sensitivity analysis results are shown in the Tables 4 and 5.

Finally, from results of Tables 4 and 5 the following observations are made:

- The optimal value of each M_j is insensitive to changes in the value of production cost, disposal cost, purchasing cost, shortage cost and proportion of produced scrapped items.

TABLE 5. Sensitivity analysis of cost parameters.

Parameter	% Changes	% Changes					
		M_1^*	M_2^*	M_3^*	K^*	Q^*	TC^*
	–25	0	0	0	0	0	–13.5849576696
C	+25	0	0	0	0	0	13.5849576696
	+50	0	0	0	0	0	27.1699153392
	–25	0	0	0	0	0	–0.13584957669
d	+25	0	0	0	0	0	0.13584957669
	+50	0	0	0	0	0	0.27169915339
	–25	20.0	14.2	33.3	–16.38176030	–16.3817603054	–0.26935617638
A	+25	–20.0	–14.2	0	12.170624929	12.1706249290	0.23226131648
	+50	–20.0	–.2	0	17.310847536	17.3108475368	0.44278881980
	–25	–20.0	–14.2	0	–1.555717044	14.8516634484	–0.29563072916
H	+25	20.0	14.2	33.3	–5.807587970	–15.2268291736	0.24478711993
	+50	20.0	28.5	33.3	–4.523106082	–20.4359217350	0.46205384600
	–25	0	0	0	15.205943122	0.80520023230	–0.03484146928
π	+25	0	0	0	–11.65608746	–0.61309840030	0.02690769755
	+50	0	0	0	–20.87643200	–1.09554000338	0.04831565488
	–25	–20.0	–14.2	0	–1.35121244	–1.351212445	–0.321551014
O_j	+25	20.0	14.2	33.3	–5.20132207	–5.201322077	0.2778499270
	+50	20.0	28.5	33.3	–2.05540229	–2.055402292	0.5295979247
	–25	0	0	0	0	0	–10.18871825
R_j	+25	0	0	0	0	0	10.18871825
	+50	0	0	0	0	0	20.3774365044
	–25	20.0	14.2	33.3	–4.45639015	–4.456390154	–0.274515231
h_j	+25	–20.0	–14.2	0	1.734126510	1.7341265104	0.2286713219
	+50	–20.0	–14.2	0	–2.66045059	–2.660450592	0.4359212114

- The optimal value of shortage quantity (K) and the optimal value of production quantity (Q) are insensitive to changes in the value of production cost, disposal cost and purchasing cost.
- The optimal value of objective function is highly sensitive to changes in the value of production cost, purchasing cost, demand rate and α_j. It is slightly sensitive to changes in the value of shortage cost and proportion of produced scrapped items. Also, it is moderately sensitive to changes in the value of rest of parameters.
- The optimal value of production quantity (Q) is highly sensitive to changes in the value of holding cost for finished product, setup cost and demand rate. It is slightly sensitive to changes in the value of shortage cost and proportion of produced scrapped items. Also, it is moderately sensitive to changes in the value of rest of parameters.
- The optimal value of shortage quantity (K) is highly sensitive to changes in the value of shortage cost, setup cost and demand rate. It is slightly sensitive to changes in the value proportion of produced scrapped items. Also, it is moderately sensitive to changes in the value of rest of parameters.

6. Conclusion

This paper presents IPPI models without/with shortage. In the IPPI models, one type of finished product and several raw materials are considered Here, the raw materials after a manufacturing process are turned to the finished product. Therefore, the producer faces to determine the order policy of raw materials from outside suppliers in addition to define the lot size of fabrication of the product. Consequently, producer must consider simultaneously the determination of the economic production quantity for finished product and the economic order quantity for raw materials for both without and with shortage cases. It was suggested two algorithms with simple approach to solve IPPI models without/with shortage. A numerical example was solved in order to show the solution approach of algorithms Also, a sensitivity analysis of parameters was done For future study, there are some suggestions. For example, consider the case of multi–objective functions. Besides, the proposed IPPI models without/with shortage can be developed taking into account that the product and/or raw materials deteriorate over time. Furthermore, the IPPI models without/with shortage can be formulated for several products. Additionally, the discount in purchasing price of raw materials can be included too. Finally, limited warehouse space, investment and service level can be stated as constraints of the IPPI models without/with shortage.

A. Determination of the area of the raw material j in Figure 1

From Figure 1,

$$\text{Area of raw material } j = \left\{\frac{\alpha_j Q t_p}{2} + \frac{3\alpha_j Q t_p}{2} + \frac{5\alpha_j Q t_p}{2} + \cdots + \frac{(2M_j - 3)\,\alpha_j Q t_p}{2} + \frac{(2M_j - 1)\,\alpha_j Q t_p}{2}\right\} + \{\alpha_j Q t_d + 2\alpha_j Q t_d + 3\alpha_j Q t_d + \ldots + (M_j - 2)\,\alpha_j Q t_d + (M_j - 1)\alpha_j Q t_d\} \quad (A.1)$$

From equations (3.1) and (3.2), the area of the Figure 1 is calculated as follows:

$$\text{Area of raw material } j = \left\{\frac{\alpha_j Q^2}{2P} + \frac{3\alpha_j Q^2}{2P} + \frac{5\alpha_j Q^2}{2P} + \cdots + \frac{(2M_j - 3)\,\alpha_j Q^2}{2P} + \frac{(2M_j - 1)\,\alpha_j Q^2}{2P}\right\} + \left\{\alpha_j\left(\frac{(1-\beta)\,P - D}{DP}\right)Q^2 + 2\alpha_j\left(\frac{(1-\beta)\,P - D}{DP}\right)Q^2 + 3\alpha_j\left(\frac{(1-\beta)\,P - D}{DP}\right)Q^2 + \ldots + (M_j - 2)\,\alpha_j\left(\frac{(1-\beta)\,P - D}{DP}\right)Q^2 + (M_j - 1)\alpha_j\left(\frac{(1-\beta)\,P - D}{DP}\right)Q^2\right\} \quad (A.2)$$

So, it can be expressed as follows:

$$\text{Area of raw material } j = \frac{\alpha_j Q^2}{2P}\sum_{i=1}^{M_j}(2i-1) + \alpha_j\left(\frac{(1-\beta)\,P - D}{DP}\right)Q^2\sum_{i=1}^{M_j - 1}(i) \quad (A.3)$$

Knowing that $\sum_{i=1}^{M_j}(2i-1)$ and $\sum_{i=1}^{M_j-1}(i)$ are calculated as follows:

$$\sum_{i=1}^{M_j}(2i-1) = 1+3+5+7+\ldots+(2M_j-3)+(2M_j-1) = M_j^2 \quad (A.4)$$

$$\sum_{i=1}^{M_j-1}(i) = 1+2+3+4+\ldots+(M_j-2)+(M_j-1) = \frac{(M_j-1)\,M_j}{2} \quad (A.5)$$

Using (A.4) and (A.5) in (A.3), thus

$$\text{Area of raw material } j = \frac{\alpha_j Q^2}{2P}M_j^2 + \alpha_j\left(\frac{(1-\beta)\,P - D}{DP}\right)Q^2\left(\frac{(M_j-1)\,M_j}{2}\right) \quad (A.6)$$

$$= \left[\frac{\alpha_j}{2P} + \frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P - D}{DP}\right)\right]Q^2 M_j^2 - \frac{\alpha_j}{2}\left(\frac{(1-\beta)\,P - D}{DP}\right)M_j Q^2 \quad (A.7)$$

B. Proof of the convexity of the IPPI model without shortage in equation (3.19)

Based on the objective function in (3.19):

$$TC=\sum_{j=1}^{n}\Delta_1^j+\Delta_2\left(\frac{1}{Q}\right)+\sum_{j=1}^{n}\Delta_3^j\left(\frac{1}{M_jQ}\right)+\sum_{j=1}^{n}\Delta_4^j\left(QM_j\right)+\sum_{j=1}^{n}\Delta_5^j\left(Q\right) \tag{B.1}$$

So,

$$\frac{\partial TC}{\partial Q}=\sum_{j=1}^{n}\Delta_5^j+\sum_{j=1}^{n}\Delta_4^jM_j-\frac{\Delta_2}{Q^2}-\sum_{j=1}^{n}\frac{\Delta_3^j}{M_jQ^2} \tag{B.2}$$

$$\frac{\partial^2 TC}{\partial^2 Q}=\frac{2\Delta_2}{Q^3}+\sum_{j=1}^{n}\frac{2\Delta_3^j}{M_jQ^3} \tag{B.3}$$

$$\frac{\partial TC}{\partial M_j}=\Delta_4^jQ-\frac{\Delta_3^j}{QM_j^2} \tag{B.4}$$

$$\frac{\partial^2 TC}{\partial^2 M_j}=\frac{2\Delta_3^j}{QM_j^3} \tag{B.5}$$

$$\frac{\partial^2 TIC}{\partial M_j\partial Q}=\frac{\partial^2 TIC}{\partial Q\partial M_j}=\Delta_4^j+\frac{\Delta_3^j}{Q^2M_j^2} \tag{B.6}$$

$$\frac{\partial^2 TIC}{\partial M_j\partial M_r;\, j\neq r}=0 \tag{B.7}$$

Thus, the Hessian matrix is:

$$\text{Hessian}=\begin{bmatrix}\frac{\partial^2 TC}{\partial^2 Q} & \frac{\partial^2 TC}{\partial Q\partial M_1} & \frac{\partial^2 TC}{\partial Q\partial M_2} & \cdots & \frac{\partial^2 TC}{\partial Q\partial M_n}\\ \frac{\partial^2 TC}{\partial M_1\partial Q} & \frac{\partial^2 TC}{\partial^2 M_1} & \frac{\partial^2 TC}{\partial M_1\partial M_2} & \cdots & \frac{\partial^2 TC}{\partial M_1\partial M_n}\\ \frac{\partial^2 TC}{\partial M_2\partial Q} & \frac{\partial^2 TC}{\partial M_2\partial M_1} & \frac{\partial^2 TC}{\partial^2 M_2} & \cdots & \frac{\partial^2 TC}{\partial M_2\partial M_n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \frac{\partial^2 TC}{\partial M_n\partial Q} & \frac{\partial^2 TC}{\partial M_n\partial M_1} & \frac{\partial^2 TC}{\partial M_n\partial M_2} & \cdots & \frac{\partial^2 TC}{\partial^2 M_n}\end{bmatrix} \tag{B.8}$$

From (B.3), (B.5), (B.6) and (B.7),

$$\text{Hessian}=\begin{bmatrix}\frac{2\Delta_2}{Q^3}+2\sum_{j=1}^{n}\frac{\Delta_3^j}{M_jQ^3} & \Delta_4^1+\frac{\Delta_3^1}{Q^2M_1^2} & \Delta_4^2+\frac{\Delta_3^2}{Q^2M_2^2} & \cdots & \Delta_4^n+\frac{\Delta_3^n}{Q^2M_n^2}\\ \Delta_4^1+\frac{\Delta_3^1}{Q^2M_1^2} & \frac{2\Delta_3^1}{QM_1^3} & 0 & \cdots & 0\\ \Delta_4^2+\frac{\Delta_3^2}{Q^2M_2^2} & 0 & \frac{2\Delta_3^2}{QM_2^3} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \Delta_4^n+\frac{\Delta_3^n}{Q^2M_n^2} & 0 & 0 & \cdots & \frac{2\Delta_3^n}{QM_n^3}\end{bmatrix} \tag{B.9}$$

Therefore,

$$\mu = X^T\,(\text{Hessian})\,X \tag{B.10}$$

From (B.9)

$$\mu = [Q, M_1, M_2, \ldots, M_n]\,(\text{Hessian})\begin{bmatrix} Q \\ M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix} = \frac{2\Delta_2}{Q} + \sum_{j=1}^{n}\frac{6\Delta_3^j}{M_j Q} + \sum_{j=1}^{n} 2\Delta_4^j M_j Q \tag{B.11}$$

Since Q, $M_j\Delta_2\Delta_3^j\Delta_4^j > 0$, therefore:

$$\mu = \frac{2\Delta_2}{Q} + \sum_{j=1}^{n}\frac{6\Delta_3^j}{M_j Q} + \sum_{j=1}^{n} 2\Delta_4^j M_j Q \geqslant 0 \tag{B.12}$$

So, the Hessian matrix in (B.9) is positive semi–definite and this means that the nonlinear form of objective function in (B.1) is a convex function

C. Simplifying holding cost of finished product for the IPPI model with shortage

Based on equation (4.13), the holding cost of the proposed model with shortage is equal to:

$$\text{Holding cost for finished product} = \frac{H}{2T}\left(I \times (t_2 + t_3)\right) \tag{C.1}$$

Using equations (4.2)–(4.4), hence

$$\begin{aligned}
\text{Holding cost for finished product} &= \frac{H}{2T}\left[\left((1-\beta)\,P - D\right)\left(\frac{Q}{P}\right) - K\right] \\
&\quad \times \left[\frac{Q}{P} - \frac{K}{(1-\beta)\,P - D} + \left(\frac{(1-\beta)\,P - D}{DP}\right)Q - \frac{K}{D}\right] \\
&= \frac{H}{2T}\left[\left((1-\beta)\,P - D\right)\left(\frac{Q}{P}\right) - K\right] \times \left[\frac{(1-\beta)\,Q}{D} - \frac{(1-\beta)\,PK}{D\left((1-\beta)\,P - D\right)}\right] \\
&= \frac{H}{2T}\left[\frac{\left((1-\beta)\,P - D\right)(1-\beta)\,Q^2}{PD} + \frac{(1-\beta)\,PK^2}{D\left((1-\beta)\,P - D\right)} - \frac{2\,(1-\beta)\,KQ}{D}\right]
\end{aligned} \tag{C.2}$$

From equation (4.8), so,

$$\begin{aligned}\text{Holding cost for finished product} &= \frac{DH}{2(1-\beta)Q}\left[\frac{((1-\beta)P-D)(1-\beta)Q^2}{PD} + \frac{(1-\beta)PK^2}{D((1-\beta)P-D)} - \frac{2(1-\beta)KQ}{D}\right] \\ &= \frac{H((1-\beta)P-D)}{2P}Q + \frac{HP}{2((1-\beta)P-D)}\left(\frac{K^2}{Q}\right) - HK \qquad \text{(C.3)}\end{aligned}$$

D. Proof of the convexity of the IPPI model with shortage in equation (4.24)

Based on the objective function in (4.24):

$$TC = \sum_{j=1}^{n}\left\{\Delta_1^j + \Delta_2\left(\frac{1}{Q}\right) + \Delta_3^j\left(\frac{1}{M_jQ}\right) + \Delta_4^j(QM_j) + \Delta_5^j(Q) - HK + \Delta_6\left(\frac{K^2}{Q}\right)\right\} \qquad \text{(D.1)}$$

So,

$$\frac{\partial TC}{\partial Q} = \sum_{j=1}^{n}\Delta_5^j + \sum_{j=1}^{n}\Delta_4^jM_j - \frac{\Delta_2}{Q^2} - \sum_{j=1}^{n}\frac{\Delta_3^j}{M_jQ^2} - \frac{\Delta_6K^2}{Q^2} \qquad \text{(D.2)}$$

$$\frac{\partial^2 TC}{\partial^2 Q} = \frac{2\Delta_2 + 2\Delta_6K^2}{Q^3} + \sum_{j=1}^{n}\frac{2\Delta_3^j}{M_jQ^3} \qquad \text{(D.3)}$$

$$\frac{\partial TC}{\partial M_j} = \Delta_4^jQ - \frac{\Delta_3^j}{QM_j^2} \qquad \text{(D.4)}$$

$$\frac{\partial^2 TC}{\partial^2 M_j} = \frac{2\Delta_3^j}{QM_j^3} \qquad \text{(D.5)}$$

$$\frac{\partial^2 TIC}{\partial M_j\partial Q} = \frac{\partial^2 TIC}{\partial Q\partial M_j} = \Delta_4^j + \frac{\Delta_3^j}{Q^2M_j^2} \qquad \text{(D.6)}$$

$$\frac{\partial TC}{\partial K} = -H + \frac{2\Delta_6K}{Q} \qquad \text{(D.7)}$$

$$\frac{\partial^2 TC}{\partial^2 K} = \frac{2\Delta_6}{Q} \qquad \text{(D.8)}$$

$$\frac{\partial^2 TIC}{\partial K\partial Q} = \frac{\partial^2 TIC}{\partial Q\partial K} = -\frac{2\Delta_6K}{Q^2} \qquad \text{(D.9)}$$

$$\frac{\partial^2 TIC}{\partial K\partial M_j} = \frac{\partial^2 TIC}{\partial M_j\partial K} = 0 \qquad \text{(D.10)}$$

$$\frac{\partial^2 TIC}{\partial M_j\partial M_r; j\neq r} = 0 \qquad \text{(D.11)}$$

Thus, the Hessian matrix is:

$$\text{Hessian} = \begin{bmatrix} \frac{\partial^2 TC}{\partial^2 Q} & \frac{\partial^2 TC}{\partial Q \partial K} & \frac{\partial^2 TC}{\partial Q \partial M_1} & \frac{\partial^2 TC}{\partial Q \partial M_2} & \cdots & \frac{\partial^2 TC}{\partial Q \partial M_n} \\ \frac{\partial^2 TC}{\partial K \partial Q} & \frac{\partial^2 TC}{\partial^2 K} & \frac{\partial^2 TC}{\partial K \partial M_1} & \frac{\partial^2 TC}{\partial K \partial M_2} & \cdots & \frac{\partial^2 TC}{\partial K \partial M_n} \\ \frac{\partial^2 TC}{\partial M_1 \partial Q} & \frac{\partial^2 TC}{\partial M_1 \partial K} & \frac{\partial^2 TC}{\partial^2 M_1} & \frac{\partial^2 TC}{\partial M_1 \partial M_2} & \cdots & \frac{\partial^2 TC}{\partial M_1 \partial M_n} \\ \frac{\partial^2 TC}{\partial M_2 \partial Q} & \frac{\partial^2 TC}{\partial M_2 \partial K} & \frac{\partial^2 TC}{\partial M_2 \partial M_1} & \frac{\partial^2 TC}{\partial^2 M_2} & \cdots & \frac{\partial^2 TC}{\partial M_2 \partial M_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 TC}{\partial M_n \partial Q} & \frac{\partial^2 TC}{\partial M_n \partial K} & \frac{\partial^2 TC}{\partial M_n \partial M_1} & \frac{\partial^2 TC}{\partial M_n \partial M_2} & \cdots & \frac{\partial^2 TC}{\partial^2 M_n} \end{bmatrix} \tag{D.12}$$

From (D.3), (D.5), (D.6), (D.8), (D.9), (D.10) and (D.11),

$$\text{Hessian} = \begin{bmatrix} \frac{2\Delta_2 + 2\Delta_6 K^2}{Q^3} + 2\sum_{j=1}^{n} \frac{\Delta_3^j}{M_j Q^3} & \frac{-2\Delta_6 K}{Q^2} & \Delta_4^1 + \frac{\Delta_3^1}{Q^2 M_1^2} & \Delta_4^2 + \frac{\Delta_3^2}{Q^2 M_2^2} & \cdots & \Delta_4^n + \frac{\Delta_3^n}{Q^2 M_n^2} \\ \frac{-2\Delta_6 K}{Q^2} & \frac{2\Delta_6}{Q} & 0 & 0 & \cdots & 0 \\ \Delta_4^1 + \frac{\Delta_3^1}{Q^2 M_1^2} & 0 & \frac{2\Delta_3^1}{Q M_1^3} & 0 & \cdots & 0 \\ \Delta_4^2 + \frac{\Delta_3^2}{Q^2 M_2^2} & 0 & 0 & \frac{2\Delta_3^2}{Q M_2^3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_4^n + \frac{\Delta_3^n}{Q^2 M_n^2} & 0 & 0 & 0 & \cdots & \frac{2\Delta_3^n}{Q M_n^3} \end{bmatrix} \tag{D.13}$$

Therefore,

$$\mu = X^T \,(\text{Hessian})\, X \tag{D.14}$$

From (D.13)

$$\mu = [Q, K, M_1, M_2, \cdots, M_n]\,(\text{Hessian}) \begin{bmatrix} Q \\ K \\ M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix} = \left[\frac{2\Delta_2}{Q^2} + \sum_{j=1}^{n} \Delta_4^j M_j + \sum_{j=1}^{n} \frac{3\Delta_3^j}{M_j Q^2}, 0, \Delta_4^1 Q \right.$$

$$\left. + \frac{3\Delta_3^1}{M_1^2 Q}, \Delta_4^2 Q + \frac{3\Delta_3^2}{M_2^2 Q}, \cdots, \Delta_4^n Q + \frac{3\Delta_3^n}{M_n^2 Q} \right] \begin{bmatrix} Q \\ K \\ M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix} = \frac{2\Delta_2}{Q} + \sum_{j=1}^{n} \frac{6\Delta_3^j}{M_j Q} + \sum_{j=1}^{n} 2\Delta_4^j M_j Q \tag{D.15}$$

Since Q, $M_j \Delta_2 \Delta_3^j \Delta_4^j > 0$, therefore:

$$\mu = \frac{2\Delta_2}{Q} + \sum_{j=1}^{n} \frac{6\Delta_3^j}{M_j Q} + \sum_{j=1}^{n} 2\Delta_4^j M_j Q \geqslant 0 \tag{D.16}$$

So, the Hessian matrix in (D.13) is positive semi-definite and this means that the nonlinear form of objective function in (D.1) is a convex function

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