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EXACT TAIL ASYMPTOTICS FOR A TWO-STAGE QUEUE: COMPLETE SOLUTION *VIA* KERNEL METHOD

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Abstract. In this paper, we are interested in tail asymptotics of stationary distributions for a two-stage tandem queue with coupled processors, Poisson arrivals, and exponential service times. The model was motivated by data transfer in cable networks regulated by a reservation procedure, and has been studied in the literature by several researchers. In the present paper, by using the kernel method, we obtain exact tail asymptotics for the stationary distributions. What is the more important is that we give a complete solution of this topic, which means that, given the parameters of the model, exact tail asymptotics for the stationary distributions of this two-stage queue can be obtained based on our results.

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1. Introduction

Since the work of Malyshev [11, 12], stationary distributions of the two-dimensional random walks in the quarter plane have attracted a lot of interest. Two-dimensional random walks are classical queueing models. The study of these models is useful in both theory and applications. For stable queueing models, we are naturally interested in their stationary distributions. However, we can only obtain explicit (closed) solutions of a few components of these stationary distributions. On the other hand, we note that tail asymptotics of stationary distributions are important in applications. For example, we can get performance bounds and approximations from tail asymptotic property. Inspired by the above, we study exact tail asymptotics for a two-stage tandem queue with coupled processors, Poisson arrivals and exponential service times.

There are mainly four alternative methods to study tail asymptotics for stationary distributions. For details, see Miyazawa [13]. In this paper, we apply the kernel method, developed based on the work of Knuth [10] and Banderier *et al.* [2], to obtain tail asymptotics. The original kernel method reads as follows: for a functional equation

$$K(x,y)F(x,y) = A(x,y)G(x) + B(x,y), \tag{1.1}$$

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where $F(x, y)$ and $G(x)$ are unknown functions. In order to study $F(x, y)$, we can find a branch $y = y_0(x)$ such that

$$K(x, y_0(x)) = 0. \tag{1.2}$$

Then, by (1.1) and (1.2),

$$G(x) = -B(x, y_0(x))/A(x, y_0(x)). \tag{1.3}$$

Therefore,

$$F(x, y) = -\frac{A(x, y)B(x, y_0(x))/A(x, y_0(x)) + B(x, y)}{K(x, y)}$$

through analytic continuation. Noting that there is only one unknown function in R.H.S of (1.1), and thus, we cannot apply this method to study tail asymptotics for two-dimensional random walks directly, since there are two unknown functions. More specifically, the analogy to (1.1) is given as follows:

$$h(x, y)\pi(x, y) = h_1(x, y)\pi_1(x) + h_2(x, y)\pi_2(y) + h_0(x, y)\pi_{0,0}, \tag{1.4}$$

where $\pi(x, y)$, $\pi_1(x)$ and $\pi_2(y)$ are unknown generating functions for joint and two boundary probabilities, respectively. Following the idea in the classical kernel method, we find a branch $Y = Y_0(x)$ satisfying the kernel equation $h(x, Y_0(x)) = 0$. However, this time, we can only get a relationship between the two unknown boundary generating functions, *i.e.*,

$$h_1(x, Y_0(x))\pi_1(x) + h_2(x, Y_0(x))\pi_2(Y_0(x)) + h_0(x, Y_0(x))\pi_{0,0} = 0.$$

The generalization of the kernel method is necessary to characterize tail asymptotics in stationary probabilities. Li and Zhao [7] extended the original kernel method systematically to study tail asymptotics for stationary distributions of a generalized two-demand queueing model, and got exact tail asymptotics for boundary probabilities, joint probabilities and marginal distributions. Since then, many scholars have used this method to study exact tail asymptotics for stationary distributions. Later, Dai and Zhao [4] used this method to study exact tail asymptotics for the wireless 3-hop network with stealing. Song, Liu and Dai [17] obtained exact tail asymptotics for a discrete-time preemptive priority queue *via* the kernel method. For more details, readers may refer to Li and Zhao [7, 8], Li, Tavakoli and Zhao [9] and the references therein.

In this paper, we study a two-stage queueing model with coupled processors, Poisson arrivals and exponential service times. The model was motivated by data transfer in cable networks regulated by a reservation procedure. For details, see Resing and Örmeci [16]. The model also has other important applications. See Andradóttir, Ayhan, and Down [1], van Leeuwaarden and Resing [18] and the references therein. Stationary distributions of this model have been studied by many scholars. In particular, Guillemin and van Leeuwaarden [6] studied tail asymptotics for the marginal distributions of this model by solving boundary value problems. In this work, we also focus on exact tail asymptotics for this model. However, compared to [6], we make the following contributions in the present paper:

(1) We systematically study exact tail asymptotic properties for the stationary distributions of this model. In [6], exact tail asymptotics for the marginal (only) stationary probabilities was studied. However, exact tail asymptotics for the marginal stationary distributions cannot directly lead to exact tail asymptotic properties for the joint stationary distributions and for the boundary stationary probabilities. Therefore, further efforts are required. Moreover, further tools are also needed. In the present paper, we provide exact tail asymptotics not only for the marginal distributions, but also for the boundary probabilities and for the joint distributions.

(2) Our results in Theorem 6.4 below refine the results in [6]. In the present paper, exact tail asymptotics of this two-stage queue are presented based on the relationship among the parameters of the model. Based on these results, impact of the parameters can be clearly revealed. However, in [6], exact tail asymptotic properties for the marginal distributions are provided based on the relationship among some functions of the parameters of the model. See Propositions 5 and 6 in [6]. Given the values of the parameters, we can get exact tail asymptotics from our results straightforwardly. However, it is not straightforward to obtain exact tail asymptotic properties for the marginal distributions from [6]. Hence our results have some advantages over the results in [6].

(3) Guillemin and van Leeuwaarden [6] formulated $\pi_1(x)$ and $\pi_2(y)$ in terms of boundary value problems. The solutions to these boundary value problems yield integral expressions of $\pi_1(x)$ and $\pi_2(y)$, see Propositions 3 and 4 in [6], respectively. Based on these expressions, they obtained exact tail asymptotics for the marginal distributions. However, there is no need to express the unknown generating function for the purpose of characterizing tail asymptotics. In the present paper, we apply the kernel method to study tail asymptotics. The kernel method only requires the information about the dominant singularities of the unknown function, including the location and detailed asymptotic property at the dominant singularities. Because of this, the kernel method has a potential advantage over the method used in [6].

The rest of this paper is organized as follows. In Section 2, we describe the two-stage model and present the fundamental equation which plays an important role in our analysis. Section 3 is devoted to studying the kernel equation and branch points. In Section 4, we present some preliminaries for exact tail asymptotics. In Sections 5, 6 and 7, we derive exact tail asymptotics for boundary probabilities, marginal distributions and joint distributions, respectively. In Section 8, we demonstrate our results by numerical examples.

2. Model and analysis steps

2.1. Two-stage model

The model is of independent interest, and has been studied by many scholars, for example, van Leeuwaarden and Resing [18], Guillemin and van Leeuwaarden [6], Resing and Örmeci [16], Denteneer and van Leeuwaarden [5] among others.

We consider a two stage tandem queue, where jobs arrive at queue 1 according to a Poisson rate λ, demanding service at both queues before leaving the system. Each job requires an exponential amount of work with parameter v_j at queue j with $v_1 + v_2 = 1$ (w.l.o.g). The global service rate is set to one. The service rate for one queue is only a fraction (p for queue 1 and $1 - p$ for queue 2) of the global service rate when the other queue is non-empty; when one queue is empty, the other queue has full service rate. Therefore, when both queues are non-empty, the departure rates at queue 1 and 2 are $v_1 p$ and $v_2(1 - p)$, respectively. Let $N_i(t)$ denote the number of jobs at queue i at time t. Then $\{N_1(t), N_2(t)\}$ is a Markov process. This Markov process has a unique stationary distribution if

$$\frac{\lambda}{v_1} + \frac{\lambda}{v_2} < 1. \tag{2.1}$$

The physical meaning of (2.1) can be found in van Leeuwaarden and Resing [18]. In the rest of this paper, we always assume that the condition (2.1) holds.

Denote the joint stationary distribution by $\pi_{m,n}$, *i.e.*, $\pi_{m,n} = \lim_{t\to\infty} \mathbb{P}(N_1(t) = m, N_2(t) = n)$. On the other hand, without loss of generality, we assume that $\max\{\lambda + v_1, \lambda + v_2\} < 1$. Let $\mathbf{Q}$ be the infinitesimal generator of the continuous-time Markov process $\{N_1(t), N_2(t)\}$. Then $\mathbf{P} = I + \mathbf{Q}$ is the transition probability matrix for the uniformized discrete-time Markov chain, where I is the identity matrix. The two Markov chains have the same stationary probability vector $\pi_{m,n}$.

2.2. Analysis steps

In this present paper, we apply the kernel method to study exact tail asymptotics for this tandem queue. In order to reach our aim, we also need the Tauberian-like theorem. Our analysis follows the following steps:

Step 1. We firstly establish the functional equation (1.4), very often referred to as the fundamental form, which plays an important role in our analysis.
Step 2. The interlace between two unknown functions $\pi_1(x)$ and $\pi_2(y)$ plays an important role in the asymptotic analysis of these two functions. Based on the fundamental form, we study the interlace between them, and the corresponding results are presented in Lemma 3.4.
Step 3. In order to apply the Tauberian-like theorem, we need to locate the dominant singularities of the two unknown functions $\pi_1(x)$ and $\pi_2(y)$. Based on Lemma 3.4, we mainly carry out this work in Section 4.
Step 4. To apply the Tauberian-like theorem, we also need asymptotic behavior of the functions $\pi_1(x)$ and $\pi_2(y)$ around its dominant singularities. In Section 5, we carry out the asymptotic analysis of these two unknown functions.
Step 5. In Sections 5, 6 and 7, according to asymptotic analysis of the unknown functions $\pi_1(x)$ and $\pi_2(y)$, we use the Tauberian-like theorem to get exact tail asymptotics of the stationary distributions.

2.3. Fundamental form

In order to apply the kernel method, we first need to establish the fundamental form. Before we present this equation, we introduce the following notation:

$$\pi(x,y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \pi_{i,j} x^{i-1} y^{j-1}, \tag{2.2}$$

$$\pi_1(x) = \sum_{i=1}^{\infty} \pi_{i,0} x^{i-1}, \tag{2.3}$$

$$\pi_2(y) = \sum_{j=1}^{\infty} \pi_{0,j} y^{j-1}. \tag{2.4}$$

Based on the transition probabilities of the uniformized discrete time Markov chain, we get from (1.3.6) in Fayolle, Iasnogorodski and Malyshev [14] that

$$-h(x,y)\pi(x,y) = h_1(x,y)\pi_1(x) + h_2(x,y)\pi_2(y) + h_0(x,y)\pi_{0,0}, \tag{2.5}$$

where

(i)

$$h(x,y) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y) = a(x)y^2 + b(x)y + c(x), \tag{2.6}$$

with $a(x) = pv_1$, $b(x) = -[\lambda + pv_1 + (1-p)v_2]x + \lambda x^2$, $c(x) = (1-p)v_2 x$, and $\tilde{a}(y) = \lambda y$, $\tilde{b}(y) = (1-p)v_2 - \left[\lambda + pv_1 + (1-p)v_2\right]y$, $\tilde{c}(y) = pv_1 y^2$;

(ii)

$$h_1(x,y) = a_1(x)y + b_1(x), \tag{2.7}$$

with $a_1(x) = v_1$ and $b_1(x) = -(\lambda + v_1)x + \lambda x^2$;

(iii)

$$h_2(x,y) = a_2(x)y^2 + b_2(x)y + c_2(x), \tag{2.8}$$

with $a_2(x) = 0$, $b_2(x) = \left[\lambda x - (\lambda + v_2)\right]$ and $c_2(x) = v_2$;

(iv)

$$h_0(x,y) = a_0(x)y + b_0(x), \tag{2.9}$$

with $a_0(x) = 0,$ and $b_0(x) = \lambda x - \lambda.$

3. Kernel equation and their branch points

In this section, we consider the **kernel equation**

$$h(x,y) = 0. \tag{3.1}$$

Specifically, we provide detailed properties of the branch points, and also the branches. We will obtain these properties by using elementary mathematics.

Let

$$D_1(x) = b^2(x) - 4a(x)c(x) \tag{3.2}$$

be the discriminant of the equation (3.1). Therefore, for each x, two solutions to (3.1) are given by

$$Y_\pm(x) = \frac{-b(x) \pm \sqrt{D_1(x)}}{2a(x)}, \tag{3.3}$$

unless $D_1(x) = 0$, for which x is called a branch point of Y.

Symmetrically, let $D_2(y) = \tilde{b}^2(y) - 4\tilde{a}(y)\tilde{c}(y)$. For each fixed y, two solutions to (3.1) are given by

$$X_\pm(y) = \frac{-\tilde{b}(y) \pm \sqrt{D_2(y)}}{2\tilde{a}(y)} \tag{3.4}$$

unless $D_2(y) = 0$, for which y is called a branch point of X. We have the following properties on these branch points.

Lemma 3.1.

(i) *$D_1(x)$ has four zeros satisfying*

$$x_1 = 0 < x_2 < 1 < x_3 < \frac{\lambda + pv_1 + (1-p)v_2}{\lambda} < x_4.$$

Furthermore, $D_1(x) > 0$ in $\big(-\infty,\ x_1\big) \cup \big(x_2,\ x_3\big) \cup \big(x_4,\ \infty\big)$ and $D_1(x) < 0$ in $(x_1, x_2) \cup (x_3, x_4)$.

(ii) *$D_2(y)$ has three zeros satisfying*

$$0 < y_1 < \frac{(1-p)v_2}{\lambda + pv_1 + (1-p)v_2} < y_2 < 1 < y_3.$$

Moreover, $D_2(y) > 0$ in $(-\infty,\ y_1) \cup (y_2,\ y_3)$, and $D_2(y) < 0$ in $(y_1,\ y_2) \cup (y_3,\ \infty)$.

Proof. We only prove part (i) of this lemma. The other part can be proved in the same way. In fact, from (2.6) and (3.2), we have

$$D_1(x) = x\bigg\{ \big[\lambda x - \big[\lambda + pv_1 + (1-p)v_2\big]\big]^2 x - 4p(1-p)v_1v_2 \bigg\}.$$

Set

$$F(x) = \big[\lambda x - \big(\lambda + pv_1 + (1-p)v_2\big)\big]^2 x - 4p(1-p)v_1v_2.$$

We can get

$$F(0) = F\left(\frac{\lambda + pv_1 + (1-p)v_2}{\lambda}\right) = -4p(1-p)v_1v_2, \quad F(1) = \left[pv_1 - (1-p)v_2\right]^2 \geq 0. \tag{3.5}$$

Then, we can get that $D_1(x)$ has four roots x_1, x_2, x_3 and x_4 satisfying¡¡

$$0 = x_1 < x_2 < 1 < x_3 < \frac{\lambda + pv_1 + (1-p)v_2}{\lambda} < x_4. \qquad \square$$

From the complex analysis point of view, the kernel equation defines a two-valued function $Y(x)$ (similarly, $X(y)$). To ensure the continuity of the function, or to avoid the situation in which the function moves from one branch to the other, when x varies, we consider the following cut planes:

- $$\tilde{\mathbb{C}}_x = \mathbb{C} \setminus \big([x_3, x_4] \cup [x_1, x_2]\big) \qquad \text{and} \qquad \tilde{\mathbb{C}}_y = \mathbb{C} \setminus \big([y_1, y_2] \cup [y_3, \infty)\big);$$
- $$\hat{C}_x = \mathbb{C} \setminus [x_3, x_4] \qquad \text{and} \qquad \hat{C}_y = \mathbb{C} \setminus [y_3, \infty),$$

where $\mathbb{C}$ denotes the complex plane.

In the cut plane $\tilde{\mathbb{C}}_y$, define the two branches $X_0(y)$ and $X_1(y)$ of $X(y)$ by

$$X_0(y) = X_-(y), \text{and } X_1(y) = X_+(y) \qquad \text{if} \qquad |X_-(y)| \leq |X_+(y)|,$$

and

$$X_0(y) = X_+(y), \text{ and } X_1(y) = X_-(y) \qquad \text{if} \qquad |X_-(y)| > |X_+(y)|.$$

The two branches $Y_0(x)$ and $Y_1(x)$ of $Y(x)$ can be similarly defined.

Remark 3.2.

(1) The functions $Y_i(x)$, $i = 0, 1$, are meromorphic in the cut plane $\tilde{\mathbb{C}}_x$.
(2) The functions $X_i(y)$, $i = 0, 1$, are meromorphic in the cut plane $\tilde{\mathbb{C}}_y$.

Asymptotic properties for the functions $\pi_1(x)$ and $\pi_2(y)$ are a key for characterizing exact tail asymptotics for the stationary distributions. At the end of this section, we list some properties of the functions $\pi_1(x)$ and $\pi_2(y)$, which play an important role in our analysis.

Before we state these properties, we define the following notation:

$$\Gamma_a = \{x \in \mathbb{C} : |x| = a\}, D_a = \{x : |x| < a\}, \text{ and } B = \{(x, y) : h(x, y) = 0\}.$$

Following Lemma 2.2.1 and Theorem 3.2.3 in Fayolle, Iasnogorodski and Malyshev [14], we have the following lemma.

Lemma 3.3.

(1) $\pi_1(x)$ *is a meromorphic function in the complex cut plane* $\hat{C}_x$. *Similarly,* $\pi_2(y)$ *is a meromorphic function in the complex cut plane* $\hat{C}_y$.
(2) *There exists an* $\epsilon > 0$ *such that the functions* $\pi_1(x)$ *and* $\pi_2(y)$ *can be analytically continued up to the circle* $\Gamma_{1+\epsilon}$ *in their respective complex plane. Moreover, they satisfy the following equation in* $D^2_{1+\epsilon} \cap B$:

$$h_1(x, y)\pi_1(x) + h_2(x, y)\pi_2(y) + h_0(x, y)\pi_{0,0} = 0. \tag{3.6}$$

Under the stability condition (2.1), we can easily verify that the conditions in Lemma 2.2.1 and Theorem 3.2.3 in [14] hold. We naturally conclude the Lemma 3.3.

From the above discussion, we can get that the dominant singularity x_{dom} of $\pi_1(x)$ is in $(1, x_3]$. In the sequel, when we apply the Tauberian-like theorem for getting exact tail asymptotics for the stationary probabilities, we need to standardize the radius of convergence of the function of interest to the unit. For convenience, we introduce the following notation. Assume that $\varPhi(z)$ is analytic at 0 with $R > 0$ the radius of convergence, and let $z_{\text{dom}} > 0$ be the dominant singularity of $\varPhi(z)$. For $0 < \phi < \frac{\pi}{2}$ and $\epsilon > 0$, we define the region $\varDelta(\phi, \epsilon, z_{\text{dom}})$ by

$$\varDelta(\phi, \epsilon, z_{\text{dom}}) = \left\{ z \in \mathbb{C}_z : |\frac{z}{z_{\text{dom}}}| \leq 1 + \epsilon, z/z_{\text{dom}} \neq 1, |\text{Arg}\left(\frac{z}{z_{\text{dom}}} - 1\right)| > \phi \right\}. \tag{3.7}$$

In the sequel, without otherwise stated, the limit of an analytic function $\varPhi(z)$ is always taken in $\varDelta(\phi, \epsilon, z_{\text{dom}})$.

By Li and Zhao [7, 8], we have the following technical lemma.

Lemma 3.4.

(1) *$\pi_1(X_0(y))$ is meromorphic in the cut complex plane $\tilde{C}_y$. If $X_0(y_3)$ is not a pole of $\pi_1(x)$, then the dominant singularity y_{dom} of $\pi_1(X_0(y))$ is y_3. Furthermore, there exist $\epsilon > 0$ and $0 < \phi < \frac{\pi}{2}$ such that*

$$\lim_{y \to y_3} \pi_1(X_0(y)) = \pi_1(X_0(y_3)) \text{ and } \lim_{y \to y_3} \pi_1'(X_0(y)) = \pi_1'(X_0(y_3)).$$

Similarly, $\pi_2(Y_0(x))$ is meromorphic in the cut complex plane $\tilde{C}_x$. If $Y_0(x_3)$ is not a pole of $\pi_2(y)$, then the dominant singularity x_{dom} of $\pi_2(Y_0(x))$ is x_3. Furthermore, there exist $\epsilon > 0$ and $0 < \phi < \frac{\pi}{2}$ such that

$$\lim_{x \to x_3} \pi_2(Y_0(x)) = \pi_2(Y_0(x_3)) \text{ and } \lim_{x \to x_3} \pi_2'(Y_0(x)) = \pi_2'(Y_0(x_3)).$$

(2) *In the cut plane $\tilde{\mathbb{C}}_x$,*

$$\pi_1(x) = \frac{-h_2\big[x, Y_0(x)\big]\pi_2\big[Y_0(x)\big] - h_0\big[x, Y_0(x)\big]\pi_{0,0}}{h_1\big[x, Y_0(x)\big]} \tag{3.8}$$

except at zero of $h_1\big(x, Y_0(x)\big)$, or at a pole of $\pi_1(x)$ or $\pi_2\big(Y_0(x)\big)$.
Similarly, in the cut plane $\tilde{\mathbb{C}}_y$,

$$\pi_2(y) = \frac{-h_1\big[X_0(y), y\big]\pi_1\big[X_0(y)\big] - h_0\big[X_0(y), y\big]\pi_{0,0}}{h_2\big(X_0(y), y\big)} \tag{3.9}$$

except at zero of $h_2\big(X_0(y), y\big)$, or at a pole of $\pi_2(y)$ or $\pi_1\big(X_0(y)\big)$.

4. Preliminaries for exact asymptotics

In this work, our goal is to find exact tail asymptotics for stationary distributions. Following the Tauberian-like theorem (see, for example, Bender [3] and Flajolet and Sedgewick [15]), we first need to locate the dominant singularities of the two unknown functions $\pi_1(x)$ and $\pi_2(y)$, and then study detailed properties of these two functions around these dominant singularities.

In this section, we carry out the first task. It follows from Lemma 3.4 that there are only two possible types of dominant singularities of $\pi_1(x)$: a pole or a branch point. When the dominant singularity is a pole, it is a zero of $h_1(x, y)$ or $h_2(x, y)$. This is stated in the following lemma.

Lemma 4.1. *Let x^* be a pole of $\pi_1(x)$ with the smallest modulus between 1 and x_3. Then either x^* is a zero of $h_1\big[x, Y_0(x)\big]$ or $Y_0(x^*)$ is a zero of $h_2\big[X_0(y), y\big]$. In the latter case, $|Y_0(x^*)| > 1$. Similar results hold for a pole of $\pi_2(y)$ with the smallest modulus in $|y| < y_3$.*

By Li and Zhao [7,8], one can get Lemma 4.1 directly. Here we omit the proof.

For our model, the only possible pole of $\pi_1(x)$ in $(1,\ x_3]$ is detailed in the following lemma.

Lemma 4.2. *If a solution to $h_1\big[x, Y_0(x)\big] = 0$ in $(1,\ x_3]$ exists, then it is*

$$\frac{\lambda + v_1 + v_2 - \sqrt{(\lambda + v_1 + v_2)^2 - 4v_1v_2}}{2\lambda}\cdot$$

Proof. It follows from (2.7) that

$$h_1\big[x, Y_0(x)\big]h_1\big[x, Y_1(x)\big] = v_1^2 Y_0(x)Y_1(x) + [Y_0(x) + Y_1(x)][\lambda x^2 - (\lambda + v_1)x]v_1 + [\lambda x^2 - (\lambda + v_1)x]^2. \tag{4.1}$$

On the other hand, we get that

$$Y_0(x)Y_1(x) = \frac{c(x)}{a(x)}, \quad \text{and} \quad Y_0(x) + Y_1(x) = -\frac{b(x)}{a(x)}\cdot \tag{4.2}$$

Therefore, by (4.1) and (4.2)

$$h_1\big[x, Y_0(x)\big]h_1\big[x, Y_1(x)\big] = (1-p)x\left[v_1v_2 + x\big(\lambda + v_2 - \lambda x\big)\big[\lambda x - (\lambda + v_1)\big]\right]. \tag{4.3}$$

Set

$$Q(x) = v_1v_2 + x\big(\lambda + v_2 - \lambda x\big)\big[\lambda x - (\lambda + v_1)\big].$$

Then,

$$\begin{aligned} Q(x) &= v_1v_2 - (\lambda + v_1)(\lambda + v_2)x + \lambda(2\lambda + v_1 + v_2)x^2 - \lambda^2 x^3 \\ &= (x-1)\Big[-\lambda^2x^2 + \big[\lambda^2 + \lambda(v_1 + v_2)\big]x - v_1v_2\Big]. \end{aligned} \tag{4.4}$$

So, we get that $h_1\big[x, Y_0(x)\big]h_1\big[x, Y_1(x)\big] = 0$ has four roots

$$\begin{aligned} \tilde{x}_0 &= 0, \quad \tilde{x}_4 = \frac{\lambda + v_1 + v_2 + \sqrt{(\lambda + v_1 + v_2)^2 - 4v_1v_2}}{2\lambda}, \\ \tilde{x}_2 &= 1, \quad \tilde{x}_3 = \frac{\lambda + v_1 + v_2 - \sqrt{(\lambda + v_1 + v_2)^2 - 4v_1v_2}}{2\lambda}. \end{aligned}$$

Since $\frac{\lambda}{v_1} + \frac{\lambda}{v_2} < 1$, one can easily confirm that $\tilde{x}_4 > \tilde{x}_3 > 1$.

Next, we study the relationship between $\tilde{x}_3$ and x_3. We will prove

$$\tilde{x}_4 > \frac{\lambda + pv_1 + (1-p)v_2}{\lambda} > x_3 \geq \tilde{x}_3. \tag{4.5}$$

From (4.4), we have that $Q(x) > 0$ for $x \in (\tilde{x}_3,\ \tilde{x}_4)$ and $Q(x) < 0$ for $x \in (\tilde{x}_4,\ \infty)$. Hence, in order to prove (4.5), we only need to show that

$$Q\left(\frac{\lambda + pv_1 + (1-p)v_2}{\lambda}\right) > 0. \tag{4.6}$$

Indeed,

$$Q\left(\frac{\lambda + pv_1 + (1-p)v_2}{\lambda}\right) = \lambda\big[(1-p)v_1 + pv_2\big] + p(1-p)\big[v_1 - v_2\big]^2 > 0. \tag{4.7}$$

By (4.7), we get that

$$\frac{\lambda + pv_1 + (1-p)v_2}{\lambda} < \tilde{x}_4.$$

On the other hand, since $D_1(x) = 0$ at x_3, we get $Y_0(x_3) = Y_1(x_3)$. So we have $Q(x_3) \geq 0$. Hence $\tilde{x}_3 \leq x_3$. We complete the proof of this lemma. □

Remark 4.3. It follows from the proof of Lemma 4.2 that if $\tilde{x}_3$ is a solution to $h_1[x, Y_0(x)] = 0$ between $(1,\ x_3]$, then the solution is unique.

Now, we study the existence of the zero of $h_1\big[x, Y_0(x)\big]$. In order to do this, we first present a technical lemma.

Lemma 4.4. *$\tilde{x}_3$ is a solution to $h_1[x, Y_0(x)] = 0$ between $(1,\ x_3]$ if and only if*

$$g(\tilde{x}_3) \geq 0, \tag{4.8}$$

where

$$g(x) = \Big[\lambda(1-2p) - pv_1 + (1-p)v_2\Big]x + \lambda\big(2p-1\big)x^2.$$

Proof. Noting that

$$b(x) = \lambda x^2 - \big[\lambda + pv_1 + (1-p)v_2\big]x,$$

we have that for any $x \in [0, \frac{\lambda+pv_1+(1-p)v_2}{\lambda}]$,

$$b(x) \leq 0. \tag{4.9}$$

Since $D_1(x) \geq 0$ for any $x \in (x_2, x_3]$, we have

$$Y_0(x) = \frac{-b(x) - \sqrt{D_1(x)}}{2pv_1} \tag{4.10}$$

for any $x \in (1,\ x_3]$. Then

$$\begin{aligned} 2ph_1\big[x, Y_0(x)\big] &= 2pv_1 Y_0(x) + 2p\lambda x^2 - 2p(\lambda + v_1)x \\ &= g(x) - \sqrt{D_1(x)}. \end{aligned} \tag{4.11}$$

We first prove the necessity. In fact, if $\tilde{x}_3$ is the zero of $h_1[x, Y_0(x)]$, then we get $g(\tilde{x}_3) \geq 0$, since $\tilde{x}_3 \leq x_3$ and $D_1(x) \geq 0$ in $(1,\ x_3]$.

Next, we prove the sufficiency. To prove that $\tilde{x}_3$ is a solution of $h_1[x, Y_0(x)] = 0$ between $(1,\ x_3]$, from (4.11), we only need to prove

$$g(\tilde{x}_3) = \sqrt{D_1(\tilde{x}_3)}. \tag{4.12}$$

Since $g(\tilde{x}_3) \geq 0$, (4.12) is equivalent to

$$g^2(\tilde{x}_3) = D_1(\tilde{x}_3). \tag{4.13}$$

By some calculations, one can get that (4.13) is equivalent to

$$Q(\tilde{x}_3) = 0. \tag{4.14}$$

By the proof of Lemma 4.2, (4.14) holds. We complete the proof. □

The next lemma states the condition under which $h_1\big[x, Y_0(x)\big] = 0$ has a solution between $(1,\ x_3]$.

Lemma 4.5.

(i) *If* $p > \frac{1}{2}$, *then* $\tilde{x}_3$ *is a solution to* $h_1[x, Y_0(x)] = 0$ *between* $(1,\ x_3]$ *if and only if*

$$p \le \max\left\{\frac{1}{2}, \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right]\right\}.$$

Furthermore, if $v_2 > v_1$ *and* $p = \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right]$, *then* x_3 *is the solution to* $h_1[x, Y_0(x)] = 0$.

(ii) *If* $p = \frac{1}{2}$, *then* $\tilde{x}_3$ *is a solution to* $h_1[x, Y_0(x)] = 0$ *between* $(1,\ x_3]$ *if and only if* $v_1 \le v_2$. *Furthermore, if* $v_2 = v_1$, *then* x_3 *is the solution to* $h_1[x, Y_0(x)] = 0$.

(iii) *If* $p < \frac{1}{2}$, *then* $\tilde{x}_3$ *is a solution to* $h_1[x, Y_0(x)] = 0$ *between* $(1,\ x_3]$ *if and only if*

$$p \le \min\left\{\frac{1}{2}, \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right]\right\}.$$

Furthermore, if $v_2 < v_1$ *and* $p = \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right]$, *then* x_3 *is the solution to* $h_1[x, Y_0(x)] = 0$.

Proof. We first prove case (i). Since $p > \frac{1}{2}$ and $\tilde{x}_3 > 1$, (4.8) is equivalent to

$$\tilde{x}_3 \ge \hat{x}, \tag{4.15}$$

where

$$\hat{x} = \frac{\lambda(2p-1) + pv_1 - (1-p)v_2}{\lambda(2p-1)}.$$

By some calculations, one can easily get that (4.15) is equivalent to

$$p \le \max\left\{\frac{1}{2}, \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right]\right\}.$$

Furthermore, if $v_2 > v_1$ and $p = \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right]$, then $\tilde{x}_3 = \hat{x}$. Since $g(\hat{x}) = g(\tilde{x}_3) = 0$, then we get $D_1(\hat{x}) = D_1(\tilde{x}_3) = 0$. So $x_3 = \tilde{x}_3$.

Next, we prove case (ii). If $p = \frac{1}{2}$, then $g(x) = \frac{1}{2}(v_2 - v_1)x$. So (4.8) is equivalent to $v_2 \ge v_1$.

On the other hand, if $v_2 = v_1$, then we get $g(x) = 0$ for all x. So the solution is x_3.

Finally, by using the same method as for the proof of case (i), we can prove that case (iii) holds. The proof of this lemma is complete. □

Similar to Lemma 4.2, we have the following.

Lemma 4.6. *If a solution to* $h_2\big[X_0(y), y\big] = 0$ *between* $(1,\ y_3]$ *exists, then it is given by*

$$\frac{v_2}{2\lambda v_1}\left[-\big(\lambda + v_2 - v_1\big) + \sqrt{\big(\lambda + v_2 - v_1\big)^2 + 4\lambda v_1}\right].$$

Proof. It follows from (2.8) that

$$\begin{aligned} h_2\big[X_0(y),y\big]h_2\big[X_1(y),y\big] &= \lambda p v_1 y^3 + p(\lambda+v_2)(v_2-v_1)py^2 + pv_2\big[v_1-2v_2-\lambda\big]y + pv_2^2 \\ &= p(x-1)\big[\lambda v_1 y^2 + (\lambda v_2 + v_2^2 - v_1 v_2)y - v_2^2\big]. \end{aligned}$$

So the solutions to $h_2\big[X_0(y),y\big]h_2\big[X_1(y),y\big]=0$ are

$$\begin{aligned} \tilde{y}_1 &= 1, \\ \tilde{y}_2 &= \frac{v_2}{2\lambda v_1}\left[-\big(\lambda+v_2-v_1\big)+\sqrt{\big(\lambda+v_2-v_1\big)^2+4\lambda v_1}\right], \\ \tilde{y}_3 &= \frac{v_2}{2\lambda v_1}\left[-\big(\lambda+v_2-v_1\big)-\sqrt{\big(\lambda+v_2-v_1\big)^2+4\lambda v_1}\right]. \end{aligned}$$

Set

$$\hat{F}(y) = \lambda v_1 y^2 + (\lambda v_2 + v_2^2 - v_1 v_2)y - v_2^2.$$

It follows from (2.1) that $\hat{F}(1) = \lambda(v_1+v_2) - v_1 v_2 < 0$. So, we get

$$\tilde{y}_2 > 1. \tag{4.16}$$

Using the same method as for the proof of Lemma 4.2, we can prove

$$\tilde{y} \le y_3. \tag{4.17}$$

By (4.16) and (4.17), the lemma holds. □

Similar to Lemma 4.5, we have the following lemma.

Lemma 4.7. *$h_2\big[X_0(y),y\big]=0$ has a solution between $(1,\ y_3]$ if and only if $\lambda < pv_1$.*

Proof: We first prove the necessity. Since

$$\tilde{b}(y) = (1-p)v_2 - \big[\lambda + pv_1 + (1-p)v_2\big]y,$$

we get

$$\tilde{b}(y) < 0 \text{ for } y > 1 > \frac{(1-p)v_2}{\lambda + pv_1 + (1-p)v_2}.$$

Therefore, for $y \in (1,\ y_3]$,

$$X_0(y) = \frac{-\tilde{b}(y) - \sqrt{D_2(y)}}{2\tilde{a}(y)}. \tag{4.18}$$

It follows from (2.8) and (4.18) that, for any $y \in (1,\ y_3]$,

$$2h_2\big[X_0(y),y\big] = \big[pv_1 - \lambda - (1+p)v_2\big]y + (1+p)v_2 - \sqrt{D_2(y)}. \tag{4.19}$$

Set

$$G(y) = \big[pv_1 - \lambda - (1+p)v_2\big]y + (1+p)v_2.$$

We assume $pv_1 \leq \lambda$. Then

$$\frac{(1+p)v_2}{\lambda + (1+p)v_2 - pv_1} \leq 1.$$

So, for any $y \in (1,\ y_3]$,

$$G(y) < 0. \tag{4.20}$$

Noting that $\sqrt{D_2(y)} \geq 0$ for all $y \in (1,\ y_3]$, we get that equation (4.20) contradicts to the fact that there is a solution to $h_1[x_0(y), y] = 0$ between $(1,\ y_3]$. So $pv_1 > \lambda$.

Next, we prove the sufficiency. In fact, from (4.19), we get that

$$2h_2\big[X_0(1), 1\big] = 2(pv_1 - \lambda) > 0, \tag{4.21}$$

and

$$2h_2\big[X_0(y_3), y_3\big] = G(y_3).$$

If $G(y_3) \leq 0$, then it follows from (4.21) that the lemma holds.

Next, we assume $G(y_3) > 0$. If there exists at least one point y_0 between $(1,\ y_3)$ such that $h_2[X_0(y_0), y_0] \leq 0$, then, from (4.21), we can get that the lemma naturally holds. Next we assume that $h_2[X_0(y), y] > 0$ *for all* $y \in (1,\ y_3)$. It follows from (2.8) and (4.18) that

$$h_2\big[X_1(y), y\big] - h_2\big[X_0(y), y\big] = \lambda y\big[X_1(y) - X_0(y)\big] = \sqrt{D_2(y)} > 0,$$

since $D_2(y) > 0$ for all $y \in (1,\ y_3)$. Therefore,

$$h_2\big[X_1(y), y\big]h_2\big[X_0(y), y\big] > 0. \tag{4.22}$$

The equation (4.22) contradicts to the proof of Lemma 4.6. So the lemma holds. □

In order to study the detailed analytic properties of the functions $\pi_1(x)$ and $\pi_2(y)$, we need some technical results for the functions $X_i(y)$, $i = 0, 1$.

Lemma 4.8. *Under the condition that* $pv_1 \geq \lambda$*, we have that*

(1) *if* $pv_1 < (1-p)v_2$*, then* $0 < X_0(y) \leq 1$ *for* $y \in \big(1,\ \frac{(1-p)v_2}{pv_1}\big]$*, and* $X_1(y) \geq 1$ *for* $y \in \big(1,\ \frac{(1-p)v_2}{pv_1}\big]$*;*
(2) *if* $pv_1 \geq (1-p)v_2$*, then* $X_0(y) \geq 1$ *for all* $y \in [1,\ y_3]$*.*

Under the condition that $pv_1 < \lambda$*, we have that*

$X_0(y) \leq 1$ *for* $y \in \big(1,\ \frac{(1-p)v_2}{pv_1}\big]$*;* $X_1(y) \geq 1$ *for* $y \in \big(1,\ \frac{(1-p)v_2}{pv_1}\big]$*.*

Proof. Noting

$$D_2\left(\frac{(1-p)v_2}{pv_1}\right) = \left[\frac{(1-p)v_2}{pv_1}\right]^2 \big[\lambda - (1-p)v_2\big]^2 \geq 0,$$

we get from Lemma 3.1 that

$$y_3 \geq \frac{(1-p)v_2}{pv_1} \geq y_2. \tag{4.23}$$

For $y \in (y_2,\ y_3]$, we have

$$X_0(y) = -\frac{(1-p)v_2 - \big[\lambda + pv_1 + (1-p)v_2\big]y + \sqrt{D_2(y)}}{2\lambda y} \tag{4.24}$$

One can easily get that for $y \in (1,\ y_3]$, $X_0(y) > 0$.

In order to prove the lemma, we need to determine the relationship between $X_0(y)$ and 1. By some calculations, we get that it is equivalent to determine the relationship between $\tilde{F}(y)$ and 1, where

$$\tilde{F}(y) = (1-p)v_2 + \big[\lambda - pv_1 - (1-p)v_2\big]y.$$

In fact, we have that

$$X_0(y) \le 1 \Longleftrightarrow (1-p)v_2 + \big[\lambda - pv_1 - (1-p)v_2\big]y + \sqrt{D_2(y)} \ge 0, \tag{4.25}$$

and

$$X_0(y) \ge 1 \Longleftrightarrow (1-p)v_2 + \big[\lambda - pv_1 - (1-p)v_2\big]y + \sqrt{D_2(y)} \le 0. \tag{4.26}$$

From the condition in (2.1), we can get $\lambda - pv_1 - (1-p)v_2 < 0$. Thus, we get

$$\tilde{F}(y) \ge 0 \text{ for } y \le \frac{(1-p)v_2}{pv_1 + (1-p)v_2 - \lambda}\cdot \tag{4.27}$$

Next, we prove the first part of this lemma. We assume $pv_1 \ge \lambda$. In this case, we get that

$$\frac{(1-p)v_2}{pv_1 + (1-p)v_2 - \lambda} \le 1. \tag{4.28}$$

By (4.27) and (4.28), we get $\tilde{F}(y) \le 0$ for all $y > 1$. Therefore, R.H.S of (4.25) and (4.26) are equivalent to

$$pv_1 y^2 - \big[pv_1 + (1-p)v_2\big]y + (1-p)v_2 \le 0, \tag{4.29}$$

and

$$pv_1 y^2 - \big[pv_1 + (1-p)v_2\big]y + (1-p)v_2 \ge 0,$$

respectively. By some calculations and (4.29), we get that (4.25) and (4.26) are equivalent to

$$\min\left\{1, \frac{(1-p)v_2}{pv_1}\right\} \le y \le \max\left\{1, \frac{(1-p)v_2}{pv_1}\right\},$$

and

$$\min\left\{1, \frac{(1-p)v_2}{pv_1}\right\} \ge y, \text{ or } y \ge \max\left\{1, \frac{(1-p)v_2}{pv_1}\right\}, \tag{4.30}$$

respectively. From the above arguments, we get that

$$X_0(y) \le 1 \text{ for } y \in \left(1,\ \frac{(1-p)v_2}{pv_1}\right], \text{if } pv_1 < (1-p)v_2,$$

and

$$X_0(y) \ge 1 \text{ for } y \in (1,\ y_3], \quad \text{if} \quad (1-p)v_2 \le pv_1.$$

Next, we consider the second part of this lemma. We assume $pv_1 < \lambda$. Under this condition, we get

$$\frac{(1-p)v_2}{pv_1 + (1-p)v_2 - \lambda} > 1. \tag{4.31}$$

On the other hand, from (2.1), we get

$$(1-p)v_2 > \lambda. \tag{4.32}$$

Therefore, we have $pv_1 < (1-p)v_2$. By (4.31) and (4.32), we get that

$$1 < \frac{(1-p)v_2}{pv_1 + (1-p)v_2 - \lambda} < \frac{(1-p)v_2}{pv_1}. \tag{4.33}$$

It follows from (4.27) and (4.33) that

$$\tilde{F}(y) \geq 0 \text{ for } y \in \left(1,\ \frac{(1-p)v_2}{pv_1 + (1-p)v_2 - \lambda}\right], \tag{4.34}$$

and

$$\tilde{F}(y) \leq 0 \text{ for } y \in \left(\frac{(1-p)v_2}{pv_1 + (1-p)v_2 - \lambda},\ y_3\right).$$

It follows from (4.34) that (4.25) holds for $y \in (1,\ \frac{(1-p)v_2}{pv_1+(1-p)v_2-\lambda}]$.

For $y \in (\frac{(1-p)v_2}{pv_1+(1-p)v_2-\lambda}, \frac{(1-p)v_2}{pv_1}]$, by using the same method as in the case $pv_1 \geq \lambda$, we can prove that (4.25) holds. From the above arguments, we can get that for any $y \in (1,\ \frac{(1-p)v_2}{pv_1}]$, $X_0(y) \leq 1$. It follows from (4.30) that for any $y \in [\frac{(1-p)v_2}{pv_1},\ y_3]$, $X_0(y) \geq 1$.

By using the same method as for the proof of results of $X_0(y)$, we can prove that the results for $X_1(y)$ also hold. □

Remark 4.9. From the proof of Lemma 4.8, we get that $\frac{(1-p)v_2}{pv_1}$ is one of branch points if and only if $\lambda = (1-p)v_2$. Furthermore, if it happens, then $pv_1 > \lambda$. Therefore, $pv_1 > (1-p)v_2$. So $y_2 = \frac{(1-p)v_2}{pv_1} < 1$.

Next, we study some properties of $Y_0(x)$ and $Y_1(x)$.

Lemma 4.10.

(i) *If* $\lambda < pv_1$, *then* $0 < Y_0(x) \leq 1$, *and* $Y_1(x) \geq 1$ *for* $x \in (1, \frac{pv_1}{\lambda}]$.
(ii) *If* $\lambda \geq pv_1$, *then* $Y_0(x) \geq 1$ *for* $x \in (1,\ x_3]$.

Proof: First, we note that

$$D_1\left(\frac{pv_1}{\lambda}\right) = \frac{p^2v_1^2}{\lambda^2}\frac{\left[\lambda - (1-p)v_2\right]^2}{\lambda} \geq 0. \tag{4.35}$$

Therefore, by Lemma 3.1 and equation (4.35), we get

$$\frac{pv_1}{\lambda} \leq x_3. \tag{4.36}$$

It follows from (4.10) that for any $x \in (1, x_3]$

$$Y_0(x) = \frac{-b(x) - \sqrt{D_1(x)}}{2pv_1}. \tag{4.37}$$

By (4.37), one can easily get

$$Y_0(x) > 0.$$

In order to prove $Y_0(x) \leq 1$, we only need to show that for all $x \in (1,\ x_3]$

$$\big[\lambda + pv_1 + (1-p)v_2\big]x - \lambda x^2 - 2pv_1 \leq \sqrt{D_1(x)}. \tag{4.38}$$

Set

$$\tilde{Q}(x) = \big[\lambda + pv_1 + (1-p)v_2\big]x - \lambda x^2 - 2pv_1.$$

We have that for all $x \in [q_1,\ q_2]$,

$$\tilde{Q}(x) \geq 0, \tag{4.39}$$

where

$$q_1 = \frac{\big[\lambda + pv_1 + (1-p)v_2\big] - \sqrt{\big[\lambda + pv_1 + (1-p)v_2\big]^2 - 8\lambda pv_1}}{2\lambda},$$

and

$$q_2 = \frac{\big[\lambda + pv_1 + (1-p)v_2\big] + \sqrt{\big[\lambda + pv_1 + (1-p)v_2\big]^2 - 8\lambda pv_1}}{2\lambda}.$$

Next, we prove part (i) of this lemma. Based on the relationship between pv_1 and $(1-p)v_2$, we split the proof into two parts, *i.e.*, $pv_1 \geq (1-p)v_2$ and $pv_1 < (1-p)v_2$, respectively.

Part 1. We assume $pv_1 \geq (1-p)v_2$. In this case, we get

$$q_1 \geq 1. \tag{4.40}$$

In order to prove the lemma, we need to consider the relationship between $(1-p)v_2$ and λ. We first assume that $(1-p)v_2 \geq \lambda$. Then one can easily get

$$q_2 \geq \frac{pv_1}{\lambda}\cdot \tag{4.41}$$

It follows from (4.40) and (4.41) that for $x \in (1,\ q_1]$,

$$\tilde{Q}(x) \leq 0, \tag{4.42}$$

and for $x \in [q_1,\ \frac{pv_1}{\lambda}]$,

$$\tilde{Q}(x) \geq 0. \tag{4.43}$$

By Lemma 3.1 and equation (4.42), we get that (4.38) holds for $x \in (1,\ q_1]$.

For $x \in [q_1,\ \frac{pv_1}{\lambda}]$, in order to prove (4.38), we only need to prove

$$\tilde{Q}^2(x) \leq D_1(x). \tag{4.44}$$

One can easily get that (4.44) is equivalent to

$$(\lambda + pv_1)x - \lambda x^2 - pv_1 \geq 0. \tag{4.45}$$

Hence, (4.45) is equivalent to

$$1 \leq x \leq \frac{pv_1}{\lambda}\cdot \tag{4.46}$$

From the above arguments, we get that the lemma holds under the assumption $(1-p)v_2 \geq \lambda$.

Next, we assume $(1-p)v_2 < \lambda$. In this case, we first assume $pv_1 > (1-p)v_2 + \lambda$. Then $q_2 > \frac{pv_1}{\lambda}$. So, by using the same method as above, we can prove that the lemma holds.

Below, we assume $pv_1 \le (1-p)v_2 + \lambda$. Then one can easily get that

$$q_2 < \frac{pv_1}{\lambda}. \tag{4.47}$$

By (4.39), (4.40) and (4.47), we get that

$$\tilde{Q}(x) \le 0, \text{ for } x \in (1,\ q_1] \cup [q_2,\ y_3]. \tag{4.48}$$

By (4.48), we get that (4.38) holds for $x \in (1,\ q_1] \cup [q_2,\ y_3]$.

On the other hand, it follows from (4.39) that if $x \in [q_1,\ q_2]$, then $\tilde{Q}(x) \ge 0$. Hence, we can use a similar method to that for the case $(1-p)v_2 > \lambda$ to prove the lemma.

Part 2. We assume $pv_1 < (1-p)v_2$. Since $(1-p)v_2 > pv_1 > \lambda$, one can easily get that

$$q_2 \ge \frac{pv_1}{\lambda}, \tag{4.49}$$

and

$$q_1 < 1. \tag{4.50}$$

By (4.23), (4.49) and (4.50), for $x \in (1,\ \frac{pv_1}{\lambda}]$, we have

$$\tilde{Q}(x) \ge 0. \tag{4.51}$$

Then, by using the same method as in the case $(1-p)v_2 > \lambda$, the lemma holds.

Next, we prove case (ii) of this lemma. It follows from (2.1) that $\lambda < (1-p)v_2$. Therefore,

$$(1-p)v_2 > pv_1. \tag{4.52}$$

It follows from (4.40) and (4.52) that

$$q_1 \le 1 \le q_2. \tag{4.53}$$

On the other hand, we have

$$D_2(q_2) = pv_1\Big\{\lambda + pv_1 - 3(1-p)v_2 - \sqrt{(\lambda + pv_1 + (1-p)v_2)^2 - 8\lambda pv_1}\Big\}.$$

Since $(1-p)v_2 > pv_1$ and $(1-p)v_2 > \lambda$,

$$D_1(q_2) < 0. \tag{4.54}$$

It follows from Lemma 3.1 and (4.54) that

$$x_3 < q_2. \tag{4.55}$$

By (4.23), (4.53) and (4.55), we get that for all $x \in (1,\ x_3]$,

$$\tilde{Q}(x) \ge 0. \tag{4.56}$$

In order to prove $Y_0(x) \ge 1$, it is sufficient to prove

$$(\lambda + pv_1 + (1-p)v_2)x - \lambda x^2 - 2pv_1 \ge \sqrt{D_1(x)}. \tag{4.57}$$

One can easily get that (4.57) is equivalent to

$$(\lambda + pv_1)x - \lambda x^2 - pv_1 \leq 0. \tag{4.58}$$

(4.58) is equivalent to

$$x \leq \frac{pv_1}{\lambda} \text{ or } x \geq 1, \tag{4.59}$$

since $\frac{pv_1}{\lambda} < 1$. By (4.53), (4.55), (4.57) and (4.59), the lemma holds in this case.

From the above arguments, we proved that the lemma holds for Y_0.

Using similar arguments, we can prove the results for $Y_1(x)$. □

Remark 4.11. From the proof of Lemma 4.10, $x_3 = \frac{pv_1}{\lambda}$ if and only if $\lambda = (1-p)v_2$.

It follows from Lemma 4.1 that the zero of $h_2\Big[X_0\big(Y_0(x)\big), Y_0(x)\Big]$ is important in the study of asymptotics. Next lemma studies it.

Lemma 4.12. *$h_2[X_0\big(Y_0(x)\big), Y_0(x)] = 0$ has no solution between $(1,\ x_3]$ with $Y_0(x) > 1$.*

Proof. By (4.9), we get that for $x \in (1,\ x_3]$, $0 < Y_0(x) \leq Y_1(x)$. We assume that the lemma would not hold, *i.e.*, there were a solution $\tilde{x}$ between $(1, x_3]$ with

$$Y_0(\tilde{x}) > 1. \tag{4.60}$$

Let $\tilde{y} = Y_0(\tilde{x})$. Then $\tilde{x} = X_0(\tilde{y})$, or $\tilde{x} = X_1(\tilde{y})$. Since

$$h_2\Big[X_0\big(\tilde{y}\big), \tilde{y}\Big] = 0, \tag{4.61}$$

we get from Lemma 4.7 that

$$\lambda < pv_1, \tag{4.62}$$

and

$$\tilde{y} = \frac{v_2}{2\lambda v_1}\left[-\big(\lambda + v_2 - v_1\big) + \sqrt{\big(\lambda + v_2 - v_1\big)^2 + 4\lambda v_1}\right]. \tag{4.63}$$

Now, we assume $\tilde{x} = X_1(\tilde{y})$. Let $x^* = X_0(\tilde{y})$. It follows from (4.61) that

$$x^* = \frac{\lambda + v_2}{\lambda} - \frac{v_2}{\lambda \tilde{y}}. \tag{4.64}$$

On the other hand, we have that

$$\tilde{x} + x^* = X_1(\tilde{y}) + X_0(y^*) = -\frac{\tilde{a}(\tilde{y})}{\tilde{b}(\tilde{y})} = \frac{\lambda + pv_1 + (1-p)v_2}{\lambda} - \frac{(1-p)v_2}{\lambda \tilde{y}}. \tag{4.65}$$

By (4.64) and (4.65),

$$\begin{aligned} \tilde{x} &= \frac{\lambda + pv_1 + (1-p)v_2}{\lambda} - \frac{(1-p)v_2}{\lambda \tilde{y}} - X_0(\tilde{y}) \\ &= \frac{p}{\lambda}\left[v_1 - v_2 + \frac{v_2}{\tilde{y}}\right]. \end{aligned} \tag{4.66}$$

Therefore,

$$\tilde{x} = \frac{p}{\lambda}\left[\frac{\lambda + v_2 - v_1 + \sqrt{(\lambda + v_2 - v_1)^2 + 4\lambda v_1}}{2} + v_1 - v_2\right]. \tag{4.67}$$

Since $\lambda(v_1 + v_2) < v_1 v_2$, we get that

$$\tilde{x} = X_1(\tilde{y}) < \frac{pv_1}{\lambda}\cdot \tag{4.68}$$

Similarly, if $\tilde{x} = X_0(\tilde{y})$, then we can also get

$$\tilde{x} < \frac{pv_1}{\lambda}, \tag{4.69}$$

since $0 < X_0(y) \leq X_1(y)$ for $y \in (1,\ y_3]$. It follows from (4.62), (4.68), (4.69) and Lemma 4.10 that

$$Y_0(\tilde{x}) < 1, \tag{4.70}$$

which contradicts to (4.60). Therefore the lemma holds. □

Remark 4.13. In this section, we mainly located the dominant singularities of the two unknown functions $\pi_1(x)$ and $\pi_2(y)$. However, this is not easy. In fact, in order to reach our aim, we first found all possible candidates for the dominant singularities. Next, we studied the relation during these candidates based on the model parameters p, λ, v_1 and v_2. Then, we studied the existence and uniqueness of the dominant singularities. Finally, we determined the locations of these dominant singularities based on the model parameters p, λ, v_1 and v_2.

5. Tail asymptotics for boundary probabilities

In this section, we provide exact tail asymptotic properties in boundary probabilities. The characterization is based on the asymptotic analysis in Section 4 and the Tauberian-like theorem in Flajolet and Sedgewick [15].

Since $\pi_1(x)$ and $\pi_2(y)$ are symmetric, properties for $\pi_2(y)$ can be easily obtained by the counterpart properties for $\pi_1(x)$. Therefore, tail asymptotics for the boundary probabilities $\pi_{0,n}$ can be directly obtained by symmetry. Hence, in this section, we only study tail asymptotics for the boundary probabilities $\pi_{n,0}$.

If $h_1\big[x, Y_0(x)\big]$ has a zero in $(1,\ x_3]$, then such a zero is unique. Recall from Lemma 4.2 that the unique solution x_* is given by

$$x_* = \frac{\lambda + v_1 + v_2 - \sqrt{(\lambda + v_1 + v_2)^2 - 4v_1 v_2}}{2\lambda}$$

or $x_* = x_3$. If $h_1\big[x, Y_0(x)\big]$ does not have a zero in $(1,\ x_3]$, then, for convenience, let $x_* > x_3$.

In order to locate the dominant singularity, we need to study the relationship between x_* and x_3.

Lemma 5.1.

(i) *If* $p < \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1 v_2}}\right]$, *then* $x_* < x_3$.

(ii) *If* $p > \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1 v_2}}\right]$, *then* $x_3 < x_*$.

(ii) *If* $p = \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1 v_2}}\right]$, *then* $x_3 = x_*$.

Proof. We first prove the first part of this lemma. We split the proof into three parts. We first assume $v_2 > v_1$. In such case, we get

$$\frac{1}{2}\left[1+\frac{v_2-v_1}{\lambda+\sqrt{(\lambda+1)^2-4v_1v_2}}\right] > \frac{1}{2}\cdot \tag{5.1}$$

Then it follows from Lemma 4.5 that x_* is the solution to $h_1[x, Y_0(x)] = 0$ and

$$x_* < x_3. \tag{5.2}$$

Next, we assume $v_1 > v_2$. Then

$$\frac{1}{2}\left[1+\frac{v_2-v_1}{\lambda+\sqrt{(\lambda+1)^2-4v_1v_2}}\right] < \frac{1}{2}\cdot$$

It follows from Lemma 4.5 that x_* is the solution to $h_1[x, Y_0(x)] = 0$ and

$$x_* < x_3. \tag{5.3}$$

Finally, we assume $v_1 = v_2$. Then

$$\frac{1}{2}\left[1+\frac{v_2-v_1}{\lambda+\sqrt{(\lambda+1)^2-4v_1v_2}}\right] = \frac{1}{2}\cdot$$

So $p < \frac{1}{2}$. It follows from Lemma 4.5 that x_* is the solution to $h_1[x, Y_0(x)] = 0$ and

$$x_* < x_3. \tag{5.4}$$

By (5.1), (5.3) and (5.4), case (i) holds.

Using the same method as for the proof of case (i), we can prove that cases (ii) and (iii) hold. We complete the proof. □

The following lemma shows the asymptotic behavior of $\pi_1(x)$ at the dominant singularity.

Lemma 5.2. *For the function $\pi_1(x)$, one and only one of the following three types of asymptotics is true.*

(i) *If $p < \frac{1}{2}\left[1+\frac{v_2-v_1}{\lambda+\sqrt{(\lambda+1)^2-4v_1v_2}}\right]$, then*

$$\lim_{x\to x_*}\left(1-\frac{x}{x_*}\right)\pi_1(x) = C_1(x_*), \tag{5.5}$$

where

$$C_1(x) = \frac{\left\{h_2[x, Y_0(x)]\pi_2[Y_0(x)] + h_0[x, Y_0(x)]\pi_{0,0}\right\}h_1[x, Y_1(x)]p}{\lambda^2(1-p)x^2(x-1)(\tilde{x}_4 - x)}.$$

(ii) *If $p > \frac{1}{2}\left[1+\frac{v_2-v_1}{\lambda+\sqrt{(\lambda+1)^2-4v_1v_2}}\right]$, then*

$$\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\pi_1'(x) = C_2(x_3),$$

where

$$C_2(x_3) = \frac{\partial H(x,y)}{\partial y}|_{(x_3,Y_0(x_3))}\frac{q(x_3)}{2\sqrt{x_3}}$$

with $q(x)$ being given by (5.12) below.

(iii) *If* $p = \frac{1}{2}\Big[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\Big]$, *then*

$$\lim_{x \to x_3} \sqrt{1 - \frac{x}{x_3}}\pi_1(x) = C_3(x_3), \tag{5.6}$$

where

$$C_3(x_3) = \Big[h_2\big[x_3, Y_0(x_3)\big]\pi_2\big[Y_0(x_3)\big] + h_0\big[x_3, Y_0(x_3)\big]\pi_{0,0}\Big]\frac{\sqrt{-X_1''\big[Y_0(x_3)\big]}}{v_1\sqrt{2}}.$$

Proof. First, it follows from (3.8) that

$$\pi_1(x) = -\frac{h_2\big[x, Y_0(x)\big]\pi_2\big[Y_0(x)\big] + h_0\big[x, Y_0(x)\big]\pi_{0,0}p}{h_1\big[x, Y_0(x)\big]}. \tag{5.7}$$

Below, we prove these three cases separately. We first look at case (i).

Case (i). It follows from Lemma 5.1 that

$$x_* < x_3. \tag{5.8}$$

From Lemma 4.1, we get that x_* is a pole of $\pi_1(x)$. By (4.3) and (4.4), we get

$$h_1\big[x, Y_0(x)\big]h_1\big[x, Y_1(x)\big] = \lambda^2(1-p)x(x-1)(x-\tilde{x}_3)(x-\tilde{x}_4). \tag{5.9}$$

By (3.8), (5.7) and (5.9), we get

$$\pi_1(x) = -\frac{\big\{h_2\big[x, Y_0(x)\big]\pi_2\big[Y_0(x)\big] + h_0\big[x, Y_0(x)\big]\pi_{0,0}\big\}\, h_1\big[x, Y_1(x)\big]}{h_1\big[x, Y_0(x)\big]h_1\big[x, Y_1(x)\big]}. \tag{5.10}$$

It follows from (5.7), (5.9) and (5.10) that

$$\lim_{x \to x_*}(1 - \frac{x}{x_*})\pi_1(x) = C_1(x_*).$$

Case (ii). It follows from Lemma 5.1 that $x_3 < x_*$. In order to simplify the discussion, we set

$$H(x, y) = \frac{h_2(x, y)\pi_2(y) + h_0(x, y)\pi_{0,0}}{-h_1(x, y)}.$$

Then, by (3.8), we get that

$$\pi_1'(x) = \frac{\partial H(x, y)}{\partial x} + \frac{\partial H(x, y)}{\partial y}\frac{\partial Y_0(x)}{\partial x}.$$

On the other hand, it follows from Lemma 3.1 that

$$D_1(x) = \lambda^2 x(x - x_2)(x_3 - x)(x_4 - x).$$

So, it follows from (4.10) that for all $x \in (1,\ x_3]$,

$$\begin{aligned} Y_0(x) &= \frac{-b(x)}{2pv_1} - \lambda\frac{\sqrt{x(x-x_2)(x_4-x)}\sqrt{(x_3-x)}}{2pv_1} \\ &= p(x) + q(x)\sqrt{(x_3-x)}, \end{aligned} \tag{5.11}$$

where

$$q(x) = -\frac{\lambda\sqrt{x(x-x_2)(x_4-x)}}{2pv_1} \tag{5.12}$$

and

$$p(x) = \frac{-b(x)}{2pv_1}\cdot$$

Therefore, by (5.11), we get that

$$\frac{\partial Y_0(x)}{\partial x} = p'(x) + q'(x)\sqrt{(x_3-x)} + \frac{q(x)}{2\sqrt{(x_3-x)}}\cdot$$

It is obvious that $p'(x)$ and $q'(x)$ are continuous at x_3. Therefore,

$$\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\frac{\partial Y_0(x)}{\partial x} = \frac{q(x_3)}{2\sqrt{x_3}}\cdot \tag{5.13}$$

Since $\frac{\partial H(x,y)}{\partial y}$ is continuous at the point $(x_3, Y_0(x_3))$, we get that

$$\lim_{x\to x_3}\frac{\partial H(x,y)}{\partial y} = \frac{\partial H(x,y)}{\partial y}|_{(x_3,Y_0(x_3))}.$$

Finally, one can easily get

$$\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\frac{\partial H(x,y)}{\partial x} = 0. \tag{5.14}$$

By (5.13) and (5.14), we get that

$$\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\pi_1'(x) = \frac{\partial H(x,y)}{\partial y}|_{(x_3,Y_0(x_3))}\frac{q(x_3)}{2\sqrt{x_3}}\cdot$$

We notice that $C_2(x_3) = \frac{\partial H(x,y)}{\partial y}|_{(x_3,Y_0(x_3))}\frac{q(x_3)}{2\sqrt{x_3}}$ cannot be zero, since x_3 is a branch point of $Y_0(x)$.

Case (iii). It follows from Lemma 5.1 that $x_* = x_3$. So $h_1\big[x_*, Y_0(x_*)\big] = 0$ and $D_1(x_*) = 0$. Therefore, we have that $h_1\big[x, Y_0(x)\big] = h_1\big[x, Y_0(x)\big] - h_1\big[x_*, Y_0(x_*)\big]$. By (3.8), we get that

$$\sqrt{(1-\frac{x}{x_*})}\pi_1(x) = \frac{\big[h_2\big[x,Y_0(x)\big]\pi_2\big[Y_0(x)\big] + h_0\big[x,Y_0(x)\big]\pi_{0,0}\big]}{\sqrt{x_*}}\frac{\sqrt{x^*-x}}{h_1\big[x_*,Y_0(x_*)\big]-h_1\big[x,Y_0(x)\big]}\cdot$$

Let $\tilde{y} = Y_0(x_3)$. It is obvious that when x is sufficiently close to x_3, we have $X_1\big[Y_0(x)\big] = x$. Since $X_1(y)$ is analytic at $\tilde{y}$ and $X_1'(\tilde{y}) = 0$, by the Taylor expansion, we get that

$$\begin{aligned} X_1(y) &= X_1(\tilde{y}) + X_1''(\tilde{y})(y-\tilde{y})^2 + o(|y-\tilde{y}|^2) \\ &= x_3 + X''(\tilde{y})\big[y - Y_0(x_3)\big]^2 + o(|y-\tilde{y}|^2). \end{aligned} \tag{5.15}$$

Taking $y = Y_0(x)$ in the equation (5.15), where x is sufficiently close to x_3, we get that

$$x = x_3 + X''(\tilde{y})\big[Y_0(x) - Y_0(x_3)\big]^2 + o(|y-\tilde{y}|^2). \tag{5.16}$$

On the other hand, since $X_1(y)$ is concave, we get $X_1''(\tilde{y}) < 0$. It follows from (5.16) that

$$\lim_{x\to x_3}\frac{Y_0(x_3)-Y_0(x)}{\sqrt{x_3-x}} = \frac{\sqrt{2}}{\sqrt{-X_1''\big[Y_0(x_3)\big]}}, \tag{5.17}$$

and

$$\lim_{x \to x_3} Y_0(x) = Y_0(x_3). \tag{5.18}$$

Moreover, it follows from the expression of $h_1(x, y)$ that

$$h_1\big[x_3, Y_0(x_3)\big] - h_1\big[x, Y_0(x)\big] = \big[b(x_3) - b(x)\big] + v_1\Big[Y_0(x_3) - Y_0(x)\Big]. \tag{5.19}$$

Finally, we have that

$$\lim_{x \to x_*} \frac{\big[h_1\big[x_3, Y_0(x_3)\big] - h_1\big[x, Y_0(x)\big]}{\sqrt{x_3 - x}} = \frac{v_1\sqrt{2}}{\sqrt{-X_1''\big[Y_0(x_3)\big]}}. \tag{5.20}$$

By (5.15), (5.19) and (5.20), we get that

$$\lim_{x \to x_3} \sqrt{\left(1 - \frac{x}{x_3}\right)}\pi_1(x) = C_3(x_3). \tag{5.21}$$

□

From Lemma 5.2 and the Tauberian-like theorem, we get the following exact tail asymptotics for the boundary probabilities $\pi_{n,0}$.

Theorem 5.3. *We have the following tail asymptotic properties for the boundary probabilities $\pi_{n,0}$ for large n. In all cases, $C_i(x)$, $i = 1, 2, 3$, are given in Lemma 5.2.*

(i) *If* $p < \frac{1}{2}\Big[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\Big]$, *then*

$$\pi_{n,0} \sim C_1(x_*)\left(\frac{1}{x_*}\right)^{n-1}. \tag{5.22}$$

(ii) *If* $p > \frac{1}{2}\Big[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\Big]$, *then*

$$\pi_{n,0} \sim \frac{C_2(x_3)}{\sqrt{\pi}} n^{-\frac{3}{2}}\left(\frac{1}{x_3}\right)^{n-1}. \tag{5.23}$$

(iii) *If* $p = \frac{1}{2}\Big[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\Big]$, *then*

$$\pi_{n,0} \sim \frac{C_3(x_3)}{\sqrt{\pi}} n^{-\frac{1}{2}}\left(\frac{1}{x_3}\right)^{n-1}. \tag{5.24}$$

6. Tail asymptotics for marginal distributions

In the preceding section, we have seen the asymptotic behavior of the boundary probabilities $\pi_{n,0}$. In this section, we use these results to study tail asymptotics for the marginal distribution $\pi_n^{(1)}$, where

$$\pi_n^{(1)} = \sum_{j=1} \pi_{n,j}.$$

It follows from the fundamental form (2.5) that

$$\pi(x,1) = \frac{h_1(x,1)\pi_1(x) + h_2(x,1)\pi_2(1) + h_0(x,1)\pi_{0,0}}{-h(x,1)}. \tag{6.1}$$

By (2.6) and (3.4), we get that

$$\pi(x,1) = \frac{h_1(x,1)\pi_1(x) + h_2(x,1)\pi_2(1) + h_0(x,1)\pi_{0,0}}{-\lambda\big[x - X_0(1)\big]\big[x - X_1(1)\big]}. \tag{6.2}$$

On the other hand, from the definitions of X_0 and X_1, we have that

(i) if $pv_1 > \lambda$, then

$$X_0(1) = 1, \text{ and } X_1(1) = \frac{pv_1}{\lambda}; \tag{6.3}$$

(ii) if $pv_1 \le \lambda$, then

$$X_0(1) = \frac{pv_1}{\lambda}, \text{ and } X_1(1) = 1. \tag{6.4}$$

Hence, if $pv_1 \le \lambda$, then it follows from (6.2) and (6.4) that $\pi_n^{(1)}$ has the same tail asymptotics as $\pi_{n,0}$. The only difference is the expression for the coefficient, which can be obtained from straightforward calculations.

Next, we consider the case $pv_1 > \lambda$. Before we state our theorem, we need the following lemmas.

Lemma 6.1.

(i)

$$\frac{\lambda + 1 - \sqrt{(\lambda+1)^2 - 4v_1v_2}}{2v_1} > v_2, \tag{6.5}$$

and

$$\frac{v_2 - \lambda}{v_2} > v_2. \tag{6.6}$$

(ii) *If* $v_2 > v_1$, *then*

$$\frac{(\lambda + 1) - \sqrt{(\lambda+1)^2 - 4v_1v_2}}{2v_1} > \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right], \tag{6.7}$$

$$\frac{v_2 - \lambda}{v_2} > \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right], \tag{6.8}$$

and

$$\frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right] > v_2. \tag{6.9}$$

(iii) *If* $v_2 < v_1$, *then*

$$\frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right] < v_2. \tag{6.10}$$

Proof. We first prove (6.5). We note that (6.5) is equivalent to

$$\lambda+1-\sqrt{(\lambda+1)^2-4v_1v_2}>2v_1v_2,$$

i.e.,

$$(\lambda+1-2v_1v_2)^2>(\lambda+1)^2-4v_1v_2. \tag{6.11}$$

By some calculations, we get that (6.11) is equivalent to

$$\lambda<v_1v_2. \tag{6.12}$$

It follows from (2.1) that (6.12) holds.

In order to prove (6.6), we only need to prove

$$v_2-v_2^2>\lambda. \tag{6.13}$$

Since $v_1+v_2=1$, (6.13) is equivalent to

$$v_1v_2>\lambda. \tag{6.14}$$

It follows from (2.1) that (6.14) holds.

Next, we prove (6.7). By using $v_1+v_2=1$, one can get that (6.7) is equivalent to

$$v_2\sqrt{(\lambda+1)^2-4v_1v_2}-(1+v_1)\lambda>v_2^2-v_1v_2. \tag{6.15}$$

In order to prove (6.15), we set

$$f(\lambda)=v_2\sqrt{(\lambda+1)^2-4v_1v_2}-(1+v_1)\lambda.$$

Noting that $0<\lambda<v_1v_2$, we get that

$$f(0)=v_2\sqrt{1-4v_1v_2}=v_2(v_2-v_1), \tag{6.16}$$

and

$$f(v_1v_2)=v_2^2-v_1v_2, \tag{6.17}$$

since $v_2>v_1$.

On the other hand, we have that

$$f'(\lambda)=\frac{v_2}{\sqrt{1-\frac{4v_1v_2}{(\lambda+1)^2}}}-(1+v_1),$$

and

$$f'(v_1v_2)=\frac{(v_1v_2+1)v_2}{1-v_1v_2}-(1+v_1). \tag{6.18}$$

Next, we prove

$$\frac{(v_1v_2+1)v_2}{1-v_1v_2}<(1+v_1). \tag{6.19}$$

Indeed, (6.19) is equivalent to

$$v_2 + 2v_1v_2 < v_1 + 1. \tag{6.20}$$

Since $v_1 + v_2 = 1$, (6.20) is equivalent to $v_2 < 1$. So, $f'(v_1v_2) < 0$. On the other hand, we have

$$f'(0) = \frac{v_2}{v_2 - v_1} - (1 + v_1),$$

since $v_2 > v_1$. One can easily get

$$f'(0) > 0. \tag{6.21}$$

By (6.18), (6.21) and the fact that $f'(\lambda)$ is decreasing in λ, we get that $f'(\lambda) \geq 0$ for $\lambda \in (0, \lambda_0]$ and $f'(\lambda) \leq 0$ for $\lambda \in (\lambda_0,\ v_1v_2)$ with $\lambda_0 \in (0,\ v_1v_2)$. Therefore, $f(\lambda)$ is increasing on $(0, \lambda_0]$ and decreasing on $(\lambda_0,\ v_1v_2)$. So, by (6.16) and (6.17), we get that (6.7) holds.

Next, we prove (6.8). In order to simplify the discussion, we set

$$g(\lambda) = \frac{v_2 - \lambda}{v_2} \text{ and } G(\lambda) = \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}}\right].$$

Therefore, we have that

$$g(0) = 1 \text{ and } g(v_1v_2) = \frac{v_2 - v_1v_2}{v_2} = v_2, \tag{6.22}$$

and

$$G(0) = 1 \text{ and } g(v_1v_2) = v_2. \tag{6.23}$$

Since $G(\lambda)$ and $g(\lambda)$ have at most two intersection points, we get from (6.22) and (6.23) that (6.8) holds.

Next, we show (6.10). Note that (6.10) is equivalent to

$$\frac{v_2 - v_1}{\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2}} < v_2 - v_1. \tag{6.24}$$

Since $v_2 < v_1$, (6.24) is equivalent to

$$\lambda + \sqrt{(\lambda+1)^2 - 4v_1v_2} < 1 - \lambda. \tag{6.25}$$

By some calculations, one can get that (6.25) is equivalent to

$$\lambda < v_1v_2. \tag{6.26}$$

It follows from (2.1) that (6.26) holds.

By using a similar proof for (6.10), we can prove that (6.9) holds. We complete the proof. □

To locate the dominant singularity, the next lemma states the relationship among x_*, x_3 and $\frac{pv_1}{\lambda}$.

Lemma 6.2. *Suppose that* $\frac{pv_1}{\lambda} > 1$.

(i) *If* $p = \frac{v_2 - \lambda}{v_2}$, *then*

$$\frac{pv_1}{\lambda} = x_3 < x_*.$$

(ii) *If* $p \neq \frac{v_2-\lambda}{v_2}$, *then*

$$\frac{pv_1}{\lambda} < \min\{x_3, x_*\}.$$

Proof. In order to prove this lemma, we first note the following fact:

$$\frac{pv_1}{\lambda} < \frac{\lambda+1-\sqrt{(\lambda+1)^2-4v_1v_2}}{2\lambda} \Longleftrightarrow p < \frac{(\lambda+1)-\sqrt{(\lambda+1)^2-4v_1v_2}}{2v_1}. \tag{6.27}$$

In order to simplify the discussion, set

$$\hat{x} = \frac{1}{2}\left[1 + \frac{v_2-v_1}{\lambda+\sqrt{(\lambda+1)^2-4v_1v_2}}\right].$$

We split the proof of this lemma into three parts based on the relationship between p and $\frac{1}{2}$.

Case (i). $p > \frac{1}{2}$. Under this assumption, we further split the proof into three parts based on the relationship between v_1 and v_2.

(1) $v_2 < v_1$. In this case, we get $\frac{1}{2} > \hat{x}$. Then, it follows from Lemma 4.5 and (4.36) that

$$x_* > x_3, \tag{6.28}$$

and

$$\frac{pv_1}{\lambda} \leq x_3, \tag{6.29}$$

where the equality holds if and only if $p = \frac{v_2-\lambda}{v_2}$. By (6.28) and (6.29), the lemma holds in this case.

(2) $v_2 > v_1$. In this case, we get $\hat{x} > \frac{1}{2}$. We first assume $p \leq \hat{x}$. It follows from Lemma 4.5 that

$$x_* \leq x_3. \tag{6.30}$$

By (6.8), we get that $p \neq \frac{v_2-\lambda}{v_2}$. It follows from Remark 4.11 that $\frac{pv_1}{\lambda} < x_3$. So, in order to prove the lemma, we only need to show

$$\frac{pv_1}{\lambda} < x_*. \tag{6.31}$$

By (6.7) and (6.27), (6.31) holds. By (6.30) and (6.31), the lemma holds in this case.
Next, we assume $p > \hat{x}$. It follows from Lemma 4.5 that

$$x_* > x_3. \tag{6.32}$$

If $p = \frac{v_2-\lambda}{\lambda}$, then it follows from Remark 4.11 that

$$\frac{pv_1}{\lambda} = x_3. \tag{6.33}$$

So, we have $x_* > \frac{pv_1}{\lambda} = x_3$. If $p \neq \frac{v_2-\lambda}{\lambda}$, then it follows from (6.32) and Remark 4.11 that

$$\frac{pv_1}{\lambda} < \min\{x_3, x_*\}. \tag{6.34}$$

Therefore, by (6.32), (6.33) and (6.34), the lemma holds in this case.

(3) $v_2 = v_1$. We have $\hat{x} = \frac{1}{2}$. So it follows from Lemma 4.5 that

$$x_* > x_3. \tag{6.35}$$

On the other hand, in such a case, we have

$$p \neq \frac{v_2 - \lambda}{v_2}. \tag{6.36}$$

Indeed, if (6.36) would not hold, then we get

$$2\lambda = v_2. \tag{6.37}$$

It follows from (2.1) and (6.37) that $v_1 > 2\lambda$. It contradicts to the assumption $v_1 = v_2$. It follows from Remark 4.11 and (6.36) that

$$\frac{pv_1}{\lambda} < x_3. \tag{6.38}$$

It follows from (6.35) and (6.38) that the lemma holds.

Case (ii). $p = \frac{1}{2}$. We still split the proof into three cases.

(1) $v_1 = v_2$. In this case, it follows from Lemma 4.5 that

$$x_* = x_3. \tag{6.39}$$

It follows from Remark 4.11 and (6.36) that

$$\frac{pv_1}{\lambda} < x_3. \tag{6.40}$$

By (6.7) and (6.27), we get $\frac{pv_1}{\lambda} < x_*$. Therefore, by (6.39) and (6.40), in this case, the lemma holds.

(2) $v_1 < v_2$. It follows from (6.36) and Remark 4.11 that $\frac{pv_1}{\lambda} < x_3$. On the other hand, by Lemma 4.6, we get that $x_* < x_3$. Using a similar method to (6.34), we can show the lemma in this case.

(3) $v_1 > v_2$. It follows from Lemma 4.6 that $x_* > x_3$. By using a similar method in the proof for case (i)−(1), we can show the lemma in this case.

Case (iii). $p < \frac{1}{2}$. We can use a similar method to the proof for the case $p > \frac{1}{2}$ to prove the lemma in this case. □

The following lemma shows the asymptotic behavior of $\pi(x, 1)$ at the dominant singularity.

Lemma 6.3. *For the function $\pi(x, 1)$, we have the following asymptotics, as x approaches to a dominant singularity of $\pi(x, 1)$. In all cases, $C_i(x)$, $i = 1, 2, 3$, are given in Lemma 5.2.*

Case 1. $pv_1 \le \lambda$:

(i) *If* $p < \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\sqrt{(\lambda+1)^2 - 4v_1v_2} + \lambda}\right]$, *then*

$$\lim_{x \to x_*} \left(1 - \frac{x}{x_*}\right) \pi(x, 1) = \frac{h_1(x_*, 1) C_1(x_*)}{\lambda \left(x_* - \frac{pv_1}{\lambda}\right)(1 - x_*)}.$$

(ii) *If* $p > \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\sqrt{(\lambda+1)^2 - 4v_1v_2} + \lambda}\right]$, *then*

$$\lim_{x \to x_3} \sqrt{\left(1 - \frac{x}{x_3}\right)} \pi'(x, 1) = \frac{h_1(x_3, 1) C_2(x_3)}{-\lambda \left(x_3 - \frac{pv_1}{\lambda}\right)(x_3 - 1)}.$$

(iii) *If* $p = \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\sqrt{(\lambda+1)^2 - 4v_1v_2} + \lambda}\right]$, *then*

$$\lim_{x \to x_3} \sqrt{\left(1 - \frac{x}{x_3}\right)}\pi(x,1) = \frac{h_1(x_3,1)C_3(x_3)}{\lambda\left(x_3 - \frac{pv_1}{\lambda}\right)(1 - x_3)}.$$

Case 2. $pv_1 > \lambda$:

(i) *If* $p = \frac{v_2 - \lambda}{v_2}$, *then*

$$\lim_{x \to x_3}(1 - \frac{x}{x_3})\pi(x,1) = \frac{h_1(x_3,1)\pi_1(x_3) + h_2(x_3,1)\pi_2(1) + h_0(x_3,1)\pi_{0,0}}{\lambda\left[x_3 - 1\right]x_3}. \tag{6.41}$$

(ii) *If* $p \neq \frac{v_2 - \lambda}{v_2}$, *then*

$$\lim_{x \to \frac{pv_1}{\lambda}}(1 - \frac{\lambda}{pv_1}x)\pi(x,1) = \frac{h_1(\frac{pv_1}{\lambda},1)\pi_1(\frac{pv_1}{\lambda}) + h_2(\frac{pv_1}{\lambda},1)\pi_2(\frac{pv_1}{\lambda}) + h_0(\frac{pv_1}{\lambda},1)\pi_{0,0}}{\left(\frac{pv_1}{\lambda} - 1\right)pv_1}. \tag{6.42}$$

Proof.
Case 1-(i). In this case, it follows from Lemma 5.1 that $\frac{pv_1}{\lambda} \leq 1 < x_* < x_3$. By (6.4), we get that $h(x,1)$ is continuous at x_*. Therefore, by (5.5) and (6.1), we get that

$$\lim_{x \to x_*}(1 - \frac{x}{x_*})\pi(x,1) = \frac{h_1(x_*,1)C_1(x_*)}{-h(x_*,1)}, \tag{6.43}$$

since $h_i(x,1)$, $i = 0,1,2$, are continuous at x_*.
Case 1-(ii). It follows from Lemma 5.1 that $\frac{pv_1}{\lambda} \leq 1 < x_3 < x_*$. In order to simplify the discussion, let $\tilde{y} = Y_0(x_3)$. Then, we can get that $\pi_2(y)$ is analytic at $\tilde{y}$.

By the Taylor expansion, we get that

$$\pi_2(y) = \pi_2(\tilde{y}) + \pi_2'(\tilde{y})\big(y - \tilde{y}\big) + o(|y - \tilde{y}|). \tag{6.44}$$

It follows from (5.17) that

$$Y_0(x_3) - Y_0(x) = \frac{\sqrt{2}}{\sqrt{-X_1''\left[Y_0(x_3)\right]}}(x_3 - x)^{\frac{1}{2}} + 0\Big(|x_3 - x|^{\frac{1}{2}}\Big). \tag{6.45}$$

By (5.17) and (6.45), we get that

$$\pi_2\left[Y_0(x)\right] = \pi_2\left[Y_0(x_3)\right] - \pi_2'\left[Y_0(x_3)\right]\frac{\sqrt{2}}{\sqrt{-X_1''\left[Y_0(x_3)\right]}}(x_3 - x)^{\frac{1}{2}} + o\Big(|x_3 - x|^{\frac{1}{2}}\Big).$$

On the other hand, we have that

$$
\begin{aligned}
\pi_1(x)-\pi_1(x_3) &= \frac{-h_2\big(x,Y_0(x)\big)\pi_2\big(Y_0(x)\big)-h_0\big(x,Y_0(x)\big)\pi_{0,0}}{h_1\big(x,Y_0(x)\big)}-\pi_1(x_3)\\
&= \frac{-h_2\big[x,Y_0(x)\big]\Big[\pi_2\big(Y_0(x)\big)-\pi_2\big(Y_0(x_3)\big)\Big]}{h_1\big[x,Y_0(x)\big]}\\
&\quad+\frac{\pi_2\big[Y_0(x_3)\big]\Big[h_2\big[x_3,Y_0(x_3)\big]-h_2\big[x,Y_0(x)\big]\Big]}{h_1\big[x,Y_0(x)\big]}\\
&\quad+\frac{\pi_{0,0}\Big[h_0\big[x_3,Y_0(x_3)\big]-h_0\big[x,Y_0(x)\big]\Big]}{h_1\big[x,Y_0(x)\big]}\\
&\quad+\frac{h_2\big[x_3,Y_0(x_3)\big]\pi_2\big[Y_0(x_3)\big]}{h_1\big[x,Y_0(x)\big]h_1\big[x_3,Y_0(x_3)\big]}\Big[h_1\big[x,Y_0(x)\big]-h_1\big[x_3,Y_0(x_3)\big]\Big]\\
&\quad+\frac{h_0\big[x_3,Y_0(x_3)\big]\pi_{0,0}}{h_1\big[x,Y_0(x)\big]h_1\big[x_3,Y_0(x_3)\big]}\Big[h_1\big[x,Y_0(x)\big]-h_1\big[x_3,Y_0(x_3)\big]\Big].
\end{aligned}
\tag{6.46}
$$

Following (2.7)−(2.9) and (4.37),

$$
\begin{aligned}
\lim_{x\to x_3}\frac{h_2\big[x_3,Y_0(x_3)\big]-h_2\big[x,Y_0(x)\big]}{\sqrt{x_3-x}} &= \lim_{x\to x_3}\frac{\lambda\big[x_3Y_0(x_3)-xY_0(x)\big]}{\sqrt{x_3-x}}\\
&\quad+\lim_{x\to x_3}\frac{(\lambda+v_2)\big[Y_0(x_3)-Y_0(x)\big]}{\sqrt{x_3-x}}.
\end{aligned}
\tag{6.47}
$$

Moreover,

$$
x_3Y_0(x_3)-xY_0(x)=x_3\big[Y_0(x_3)-Y_0(x)\big]+[x_3-x]Y_0(x). \tag{6.48}
$$

It follows from (5.18), (6.46)−(6.48) that

$$
\lim_{x\to x_3}\frac{h_2\big[x_3,Y_0(x_3)\big]-h_2\big[x,Y_0(x)\big]}{\sqrt{x_3-x}}=\frac{(\lambda+v_2+\lambda x_3)\sqrt{2}}{\sqrt{-X_1''\big[Y_0(x_3)\big]}}. \tag{6.49}
$$

Furthermore, we have

$$
\lim_{x\to x_3}\frac{h_0\big[x_3,Y_0(x_3)\big]-h_0\big[x,Y_0(x)\big]}{\sqrt{x_3-x}}=0. \tag{6.50}
$$

By Lemma 3.4, (5.17), (5.18), (6.46), (6.49) and (6.50), we get

$$
\lim_{x\to x_3}\pi_1(x)=\pi_1(x_3). \tag{6.51}
$$

Noting that $h_i(x,1)$, $i=0,1,2$, are continuous, we get that

$$
\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\pi'(x,1)=\frac{h_1(x_3,1)C_2(x_3)}{-\lambda(x_3-\frac{pv_1}{\lambda})(x_3-1)}.
$$

Case 1-(iii). It follows from Lemma 5.1 that $\frac{pv_1}{\lambda}\le 1<x_*=x_3$. It follows from (5.6) and (6.4) that

$$
\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\pi(x,1)=\frac{h_1(x_3,1)C_3(x_3)}{-h(x_3,1)}, \tag{6.52}
$$

since $h(x,1)$ and $h_i(x,1)$, $i=0,1,2$, are continuous at x_3.

Case 2-(i). In this case, it follows from Lemma 6.1 that $\frac{pv_1}{\lambda} = x_3 < x_*$. By (6.2) and (6.51), we can get that (6.41) holds.

Case 2-(ii). In this case, it follows from Lemma 6.1 that $\frac{pv_1}{\lambda} < \min\{x_*, x_3\}$. Then, using the same method as for the proof of case (i) in Lemma 5.2, we can prove that (6.42) holds. □

The following theorem is a direct consequence of Lemma 6.3 and the Tauberian-like theorem.

Theorem 6.4. *We have the following exact tail asymptotics for the marginal distribution* $\pi_n^{(1)}$. *In all cases,* $C_i(x)$, $i=1,2,3$, *are given in Lemma 5.2.*

Case 1. $pv_1 \le \lambda$:

(i) *If* $p < \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\sqrt{(\lambda+1)^2 - 4v_1v_2} + \lambda}\right]$, *then*

$$\pi_n^{(1)} \sim \frac{h_1(x_*,1)C_1(x_*)}{\lambda(x_* - \frac{pv_1}{\lambda})(1-x_*)} \left(\frac{1}{x_*}\right)^{n-1}. \tag{6.53}$$

(ii) *If* $p > \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\sqrt{(\lambda+1)^2 - 4v_1v_2} + \lambda}\right]$, *then*

$$\pi_n^{(1)} \sim \frac{h_1(x_3,1)C_2(x_3)}{-\lambda(x_3 - \frac{pv_1}{\lambda})(x_3-1)\sqrt{\pi}} n^{-\frac{3}{2}} \left(\frac{1}{x_3}\right)^{n-1}.$$

(iii) *If* $p = \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\sqrt{(\lambda+1)^2 - 4v_1v_2} + \lambda}\right]$, *then*

$$\pi_n^{(1)} \sim \frac{h_1(x_3,1)C_3(x_3)}{\lambda(x_3 - \frac{pv_1}{\lambda})(1-x_3)\sqrt{\pi}} n^{-\frac{1}{2}} \left(\frac{1}{x_3}\right)^{n-1}. \tag{6.54}$$

Case 2. $pv_1 > \lambda$:

(i) *If* $p = \frac{v_2 - \lambda}{v_2}$, *then*

$$\pi_n^{(1)} \sim \frac{h_1(x_3,1)\pi_1(x_3) + h_2(x_3,1)\pi_2(1) + h_0(x_3,1)\pi_{0,0}}{\lambda\left[x_3 - 1\right]x_3} \left(\frac{1}{x_3}\right)^{n-1}.$$

(ii) *If* $p \neq \frac{v_2 - \lambda}{v_2}$, *then*

$$\pi_n^{(1)} \sim \frac{h_1(\frac{pv_1}{\lambda},1)\pi_1(\frac{pv_1}{\lambda}) + h_2(\frac{pv_1}{\lambda},1)\pi_2\left(\frac{pv_1}{\lambda}\right) + h_0(\frac{pv_1}{\lambda},1)\pi_{0,0}}{\left(\frac{pv_1}{\lambda} - 1\right)pv_1} \left(\frac{\lambda}{pv_1}\right)^{n-1}. \tag{6.55}$$

7. Tail asymptotics for joint probabilities

In the preceding two sections, we have derived exact asymptotic properties for the boundary probabilities and for the marginal distributions. In this section, we provide details for exact tail asymptotic characterization in the joint probabilities $\pi_{m,n}$ for any fixed $n \ge 1$. Parallel results for any fixed $m \ge 1$ can be proved.

In order to simplify the notation, we set for $j \ge 0$

$$\phi_j(x) = \sum_{i=1}^{\infty} \pi_{i,j} x^{i-1}.$$

Using the relevant balance equations of the random walk, we get that

$$c(x)\phi_1(x) + b_1(x)\phi_0(x) = a_0^*(x), \tag{7.1}$$

$$c(x)\phi_2(x) + b(x)\phi_1(x) + a_1(x)\phi_0(x) = a_1^*(x), \tag{7.2}$$

$$c(x)\phi_{j+1}(x) + b(x)\phi_j(x) + a(x)\phi_{j-1}(x) = a_j^*(x), \text{ for } j \geq 2, \tag{7.3}$$

where

$$\begin{aligned} a_0^*(x) &= -c_2(x)\pi_{0,1} - b_0(x)\pi_{0,0}, \\ a_1^*(x) &= -c_2(x)\pi_{0,2} - b_2(x)\pi_{0,1} - a_0(x)\pi_{0,0}, \\ a_j^*(x) &= -c_2(x)\pi_{0,j+1} - b_2(x)\pi_{0,j} - a_2(x)\pi_{0,j-1}. \end{aligned}$$

It follows from (7.3) that

$$\phi_{j+1}(x) = \frac{-b(x)\phi_j(x) - a(x)\phi_{j-1}(x) + a_j^*(x)}{(1-p)v_2 x}. \tag{7.4}$$

By (7.4), we get that $\phi_j(x)$ has the same singularities as $\phi_0(x)$ and

$$\lim_{x\to x_3} \phi_j(x) = \phi_j(x_3). \tag{7.5}$$

Theorem 7.1. *We have the following exact tail asymptotics for the joint probabilities $\pi_{n,j}$. In all cases, $C_i(x)$, $i = 1, 2, 3$, are given in Lemma 5.2.*

(i) *If $p < \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\sqrt{(\lambda+1)^2 - 4v_1v_2} + \lambda}\right]$, then for $j \geq 1$*

$$\pi_{n,j} \sim L_j(x_*)\big(x_*\big)^{1-n}, \tag{7.6}$$

where

$$L_j(x_*) = -\frac{C_1(x_*)b_1(x_*)}{c(x_*)}\Big(Y_1(x_*)\Big)^{1-j}.$$

(ii) *If $p > \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\sqrt{(\lambda+1)^2 - 4v_1v_2} + \lambda}\right]$, then for $j \geq 1$*

$$\pi_{n,j} \sim \hat{L}_j(x_3)n^{-\frac{3}{2}}\big(x_3\big)^{1-n}, \tag{7.7}$$

where

$$\hat{L}_j(x_3) = -C_2(x_3)\frac{b_1(x_3) + (j-1)h_1\big[x_3, Y_0(x_3)\big]}{c(x_3)\sqrt{\pi}}\big[Y_1(x_3)\big]^{1-j}.$$

(iii) *If $p = \frac{1}{2}\left[1 + \frac{v_2 - v_1}{\sqrt{(\lambda+1)^2 - 4v_1v_2} + \lambda}\right]$, then for $j \geq 1$*

$$\pi_{n,j} \sim \tilde{L}_j(x_*)n^{-\frac{1}{2}}\big(x_*\big)^{1-n}, \tag{7.8}$$

where

$$\tilde{L}_j(x_*) = -\frac{C_3(x_*)b_(x_*)}{c(x_*)\sqrt{\pi}}.$$

Proof.
Case (i). In this case, it follows from Lemma 5.2 that

$$\lim_{x\to x_*}(1-\frac{x}{x_*})\pi_1(x)=\lim_{x\to x_*}\left(1-\frac{x}{x_*}\right)\phi_0(x)=C_1(x_*)=L_0(x_*).$$

By induction and equations (7.1) to (7.3), we get that

$$\lim_{x\to x_*}\left(1-\frac{x}{x_*}\right)\phi_j(x)=L_j(x_*)$$

with

$$c(x_*)L_1(x_*)+b_1(x_*)L_0(x_*)=0, \tag{7.9}$$

$$c(x_*)L_2(x_*)+b(x_*)L_1(x_*)+a_1(x_*)L_0(x_*)=0, \tag{7.10}$$

$$c(x_*)L_{j+1}(x_*)+b(x)L_j(x_*)+a(x_*)L_{j-1}(x_*)=0,\ \text{for } j\geq 2.$$

Since $L_j(x_*)$ satisfies the second order recursive relation above, it takes the form of

$$L_j(x_*)=A(x_*)\left(\frac{1}{Y_1(x_*)}\right)^j+B(x_*)\left(\frac{1}{Y_0(x_*)}\right)^j\cdot$$

We can use equations (7.9) and (7.10) to determine $A(x_*)$ and $B(x_*)$. By some calculations, we can get $A(x_*)=0$ and

$$B(x_*)=-\frac{L_0(x_*)b_1(x_*)}{c(x_*)}\cdot$$

It follows from the Tauberian-like theorem in Flajolet and Sedgewick [15] that (7.6) holds.
Case (ii). In this case, we get $x_3>x_*$.

We will prove the lemma by induction. If $j=1$, then it follows from (7.1) that

$$\phi_1'(x)=\frac{[a_0^*(x)]'-b_1'(x)\phi_0(x)-\phi_0'(x)b_1(x)}{c(x)}+c'(x)\frac{a_0^*(x)-b_1(x)\phi_0(x)}{c^2(x)}\cdot$$

From the expression of $b_1(x)$, $c(x)$ and $a_0^*(x)$, and (7.5), we get that

$$\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\phi_1'(x)=-\frac{C_2(x_3)b_1(x_3)}{c(x_3)}\cdot$$

Now we assume that for $j\leq k$,

$$\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\phi_j'(x)=\hat{L}_j(x_3).$$

It follows from (7.4) that

$$\phi_{k+1}'(x)=\frac{[a_k^*(x)]'-b'(x)\phi_k(x)-b(x)\phi_k'(x)-a'(x)\phi_{k-1}(x)-a(x)\phi_{k-1}'(x)}{c(x)}$$
$$+\frac{a_k^*(x)-b(x)\phi_k(x)-a(x)\phi_{k-1}(x)}{c^2(x)}c'(x).$$

Thus, we get that

$$\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\phi'_{k+1}(x)=\frac{-b(x_3)\hat{L}_k(x_3)-a(x_3)\hat{L}_{k-1}(x_3)}{c(x_3)}=\hat{L}_{k+1}(x_3). \tag{7.11}$$

By (7.11), we get that

$$\begin{aligned} c(x_3)\hat{L}_1(x_3)+\hat{L}_0(x_3)b_1(x_3) &= 0,\\ c(x_3)\hat{L}_2(x_3)+\hat{L}_1(x_3)b(x_3)+\hat{L}_0(x_3)a_1(x_3) &= 0,\\ c(x_3)\hat{L}_{j+1}(x_3)+\hat{L}_j(x_3)b(x_3)+\hat{L}_{j-1}(x_3)a(x_3) &= 0, \text{ for } j\geq 2.\end{aligned}$$

Using a similar method to the proof for case (i), we can get that (7.7) holds.

Case (iii). In this case, we get $x_3 = x_*$. By Lemma 5.2, we get

$$\lim_{x\to x_3}\sqrt{1-\frac{x}{x_3}}\pi_1(x)=C_3(x_3)=\tilde{L}_0(x_3).$$

Similar to the proof for case (i), we can get that (7.8) holds. □

Remark 7.2. In Sections 5 and 6, we applied the Tauberian-like theorem to obtain exact tail asymptotics for the boundary and marginal stationary probabilities. In order to apply the Tauberian-like theorem, we used the kernel method to locate the dominant singularities of $\pi_1(x)$ and $\pi(x,1)$, and study the asymptotic behavior of $\pi_1(x)$ and $\pi(x,1)$ around their dominant singularities, respectively. However, for the joint stationary distributions, this method is not available. In this section, based on the balance equation, we obtained the asymptotic properties for the joint stationary distributions by induction.

8. Numerical examples

We shall compare the asymptotic estimates in Theorems 5.3, 6.4 and 7.1 against results by numerical calculations. Here, all numerical results presented were obtained by using the block rectangle-iterative (BRI) algorithm introduced by Zhang [19]. On the other hand, since exact tail asymptotics for the marginal stationary distributions have been demonstrated by numerical examples in [6], we only demonstrate those for the boundary and joint stationary distributions.

For a first scenario, we take $\lambda = 0.1$, $v_1 = 0.3$, $v_2 = 0.7$ and $p = 0.2$. In this case, $x_* = 2.458619$ and $x_3 = 5.6589$. Hence, we have regime (5.22), (6.53) and (7.6) for queue 1. Since (6.53) has been demonstrated in [6], we only focus on (5.22) and (7.6). Here we first compare (5.22) against results obtained by numerical calculations. It follows from (5.22) that as $n\to\infty$

$$\frac{\pi_{n+1,0}}{\pi_{n,0}}\to\frac{1}{x_*}, \tag{8.1}$$

and

$$\pi_{n,0}x_*^{n-1}\to\ C_1(x_*). \tag{8.2}$$

Moreover, (8.2) can be rewritten as the following:

$$\log\left(\pi_{n,0}x_*^n\right)\to\log C_1(x_*)+\log(x_*). \tag{8.3}$$

Conversely, we also can get from Zhang [19] that (8.1) and (8.3) imply (5.22). Hence, it suffices to compare (8.1) and (8.3) against numerical results. These results are plotted in Figures 1 and 2, respectively. Moreover, we

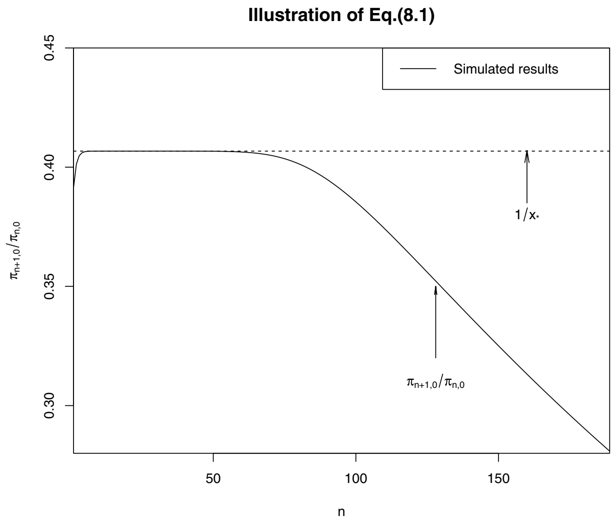

FIGURE 1.

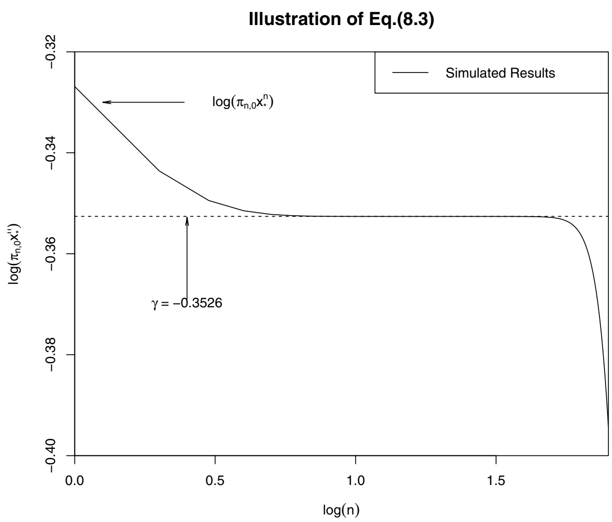

FIGURE 2.

find that $C_1(x_*) = 0.1805936$. We see from Figures 1 and 2 that (8.1) and (8.3) converge fast to $\frac{1}{x_*}$ and γ, respectively. On the other hand, Figure 2 also confirms the correctness of $C_1(x_*)$.

Here, we should point out that the tail of the curve in Figure 1 comes down since numerical calculations stop after finite iteration. The numerical experiments also led to the interesting insight that as we increase the number of iteration, the location where the tail begins to come down moves forward along the horizontal line $\frac{1}{x_*}$. See Figure A below.

Next, we focus on (7.6). Here, we only take $j = 5$. Instead of (7.6), we demonstrate the following two equations by numerical results:

$$\lim_{n\to\infty} \frac{\pi_{n+1,5}}{\pi_{n,5}} = \frac{1}{x_*}, \tag{8.4}$$

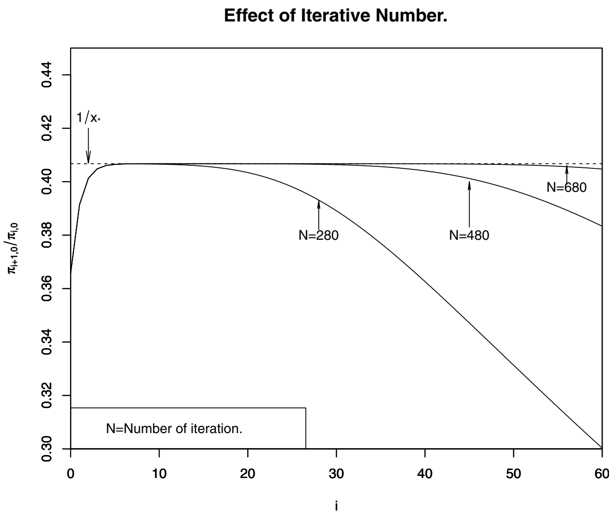

FIGURE A.

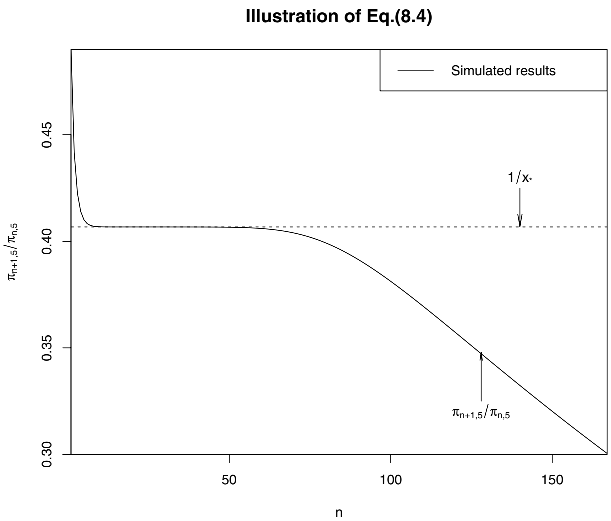

FIGURE 3.

and

$$\lim_{n\to\infty} \log(\pi_{n,5} x_*^n) = \log L_5(x_*) + \log x_*. \tag{8.5}$$

These results are plotted in Figures 3 and 4, respectively.

For a second scenario, we take $\lambda = 0.05$, $v_1 = 0.5$, $v_2 = 0.5$ and $p = 0.5$. From Lemma 5.1, we have $x_3 = x_*$. Indeed, $x_* = x_3 = 7.298438$. Hence, we have regime (5.24), (6.54) and (7.8) for queue 1. Instead of (5.24), we demonstrate the following equations by numerical results:

$$\lim_{n\to\infty} \frac{(n+1)^{\frac{1}{2}} \pi_{n+1,0}}{n^{\frac{1}{2}} \pi_{n,0}} = \frac{1}{x_*}, \tag{8.6}$$

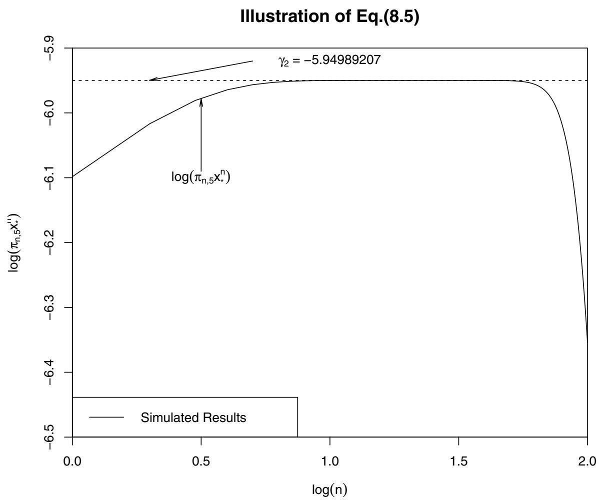

FIGURE 4.

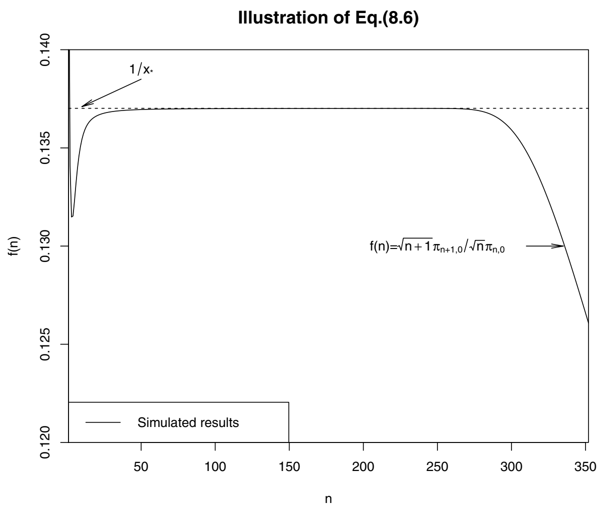

FIGURE 5.

and as $n \to \infty$

$$\log(\pi_{n,0}x_*^n) \sim -\frac{1}{2}\log n + \log x_* + \log\frac{C_3(x_*)}{\sqrt{\pi}}. \tag{8.7}$$

See Figures 5 and 6, respectively.

Instead of (7.8), we demonstrate the following equations by numerical results. Here, we take j=6.

$$\lim_{n\to\infty}\frac{(n+1)^{\frac{1}{2}}\pi_{n+1,6}}{(n)^{\frac{1}{2}}\pi_{n,6}} = \frac{1}{x_*}, \tag{8.8}$$

and as $n \to \infty$

$$\log(\pi_{n,6}x_*^n) \sim -\frac{1}{2}\log n + \log \tilde{L}_6(x_*) + \log x_*. \tag{8.9}$$

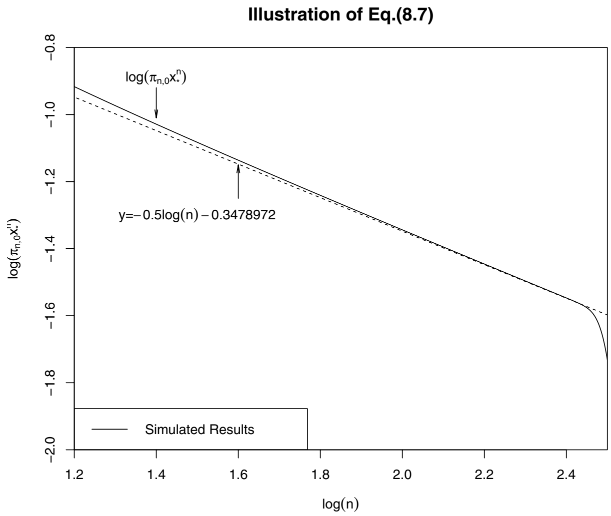

FIGURE 6.

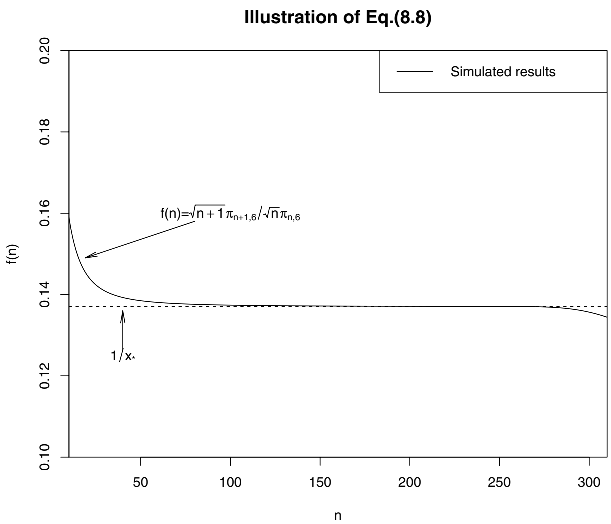

FIGURE 7.

See Figures 7 and 8, respectively.

For a third scenario, we take $\lambda = 0.1$, $v_1 = 0.4$, $v_2 = 0.6$ and $p = 0.8$. In this scenario, we have regime (5.23), (6.55) and (7.7) for queue 1. Here, we get $x_3 = 3.213828$. Instead of (5.23), we demonstrate the following equations by numerical results:

$$\lim_{n\to\infty} \frac{(n+1)^{\frac{3}{2}}\pi_{n+1,0}}{n^{\frac{3}{2}}\pi_{n,0}} = \frac{1}{x_3}, \tag{8.10}$$

and as $n \to \infty$

$$\log(\pi_{n,0}x_3^n) \sim -\frac{3}{2}\log n + \log x_3 + \log\frac{C_2(x_3)}{\sqrt{\pi}}. \tag{8.11}$$

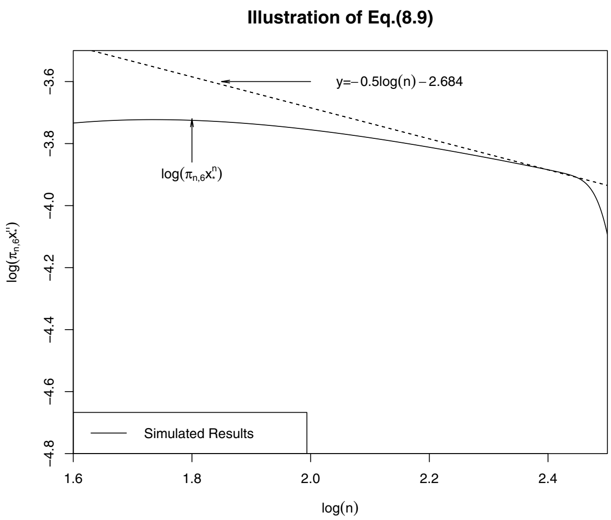

FIGURE 8.

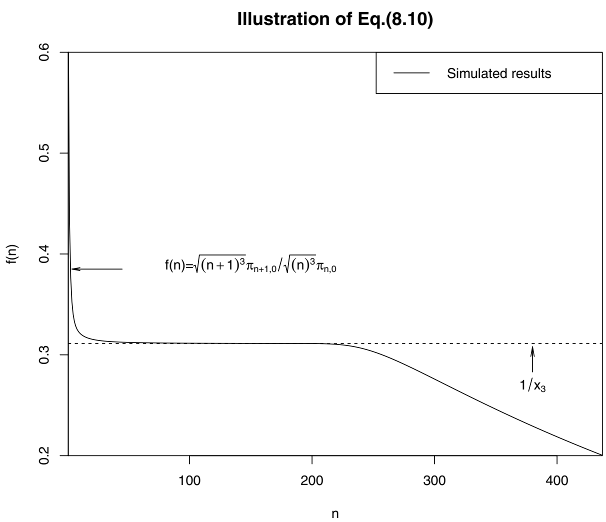

FIGURE 9.

See Figures 9 and 10, respectively.

Analogously, we focus on the following two equations instead of (7.7). Here, we still take $j = 6$.

$$\lim_{n\to\infty} \frac{(n+1)^{\frac{3}{2}}\pi_{n+1,6}}{n^{\frac{3}{2}}\pi_{n,6}} = \frac{1}{x_3}, \tag{8.12}$$

and as $n \to \infty$

$$\log(\pi_{n,6}x_3^n) \sim -\frac{3}{2}\log n + \log x_3 + \log \hat{L}_6(x_3). \tag{8.13}$$

These results are plotted in Figures 11 and 12, respectively.

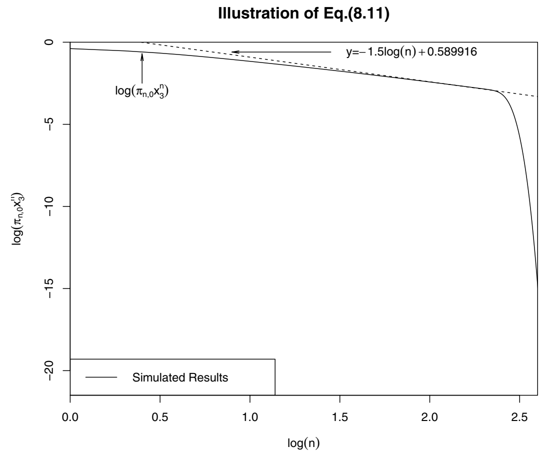

FIGURE 10.

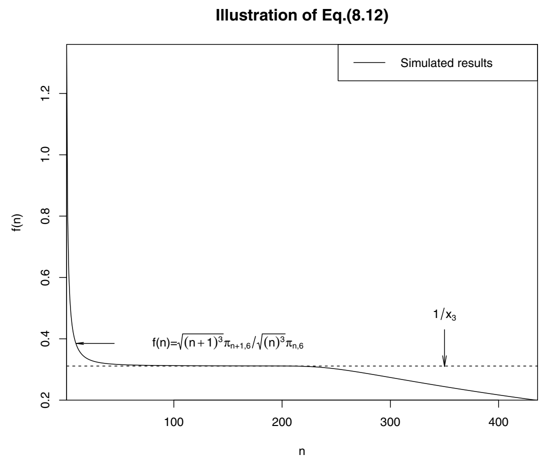

FIGURE 11.

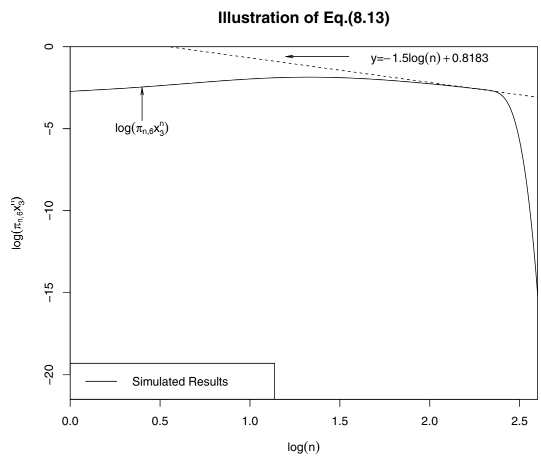

FIGURE 12.

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