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LOST SALES REDUCTION AND QUALITY IMPROVEMENT WITH VARIABLE LEAD TIME AND FUZZY COSTS IN AN IMPERFECT PRODUCTION SYSTEM

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Abstract. This article investigates the effects of lost sales reduction and quality improvement in an imperfect production process under imprecise environment with simultaneously optimizing reorder point, order quantity, and lead time. This study assumes that the demand during lead time follows a mixture of normal distributions and the cost components are imprecise and vague. Under these assumptions, the aim is to study the lost sales reduction and the quality improvement in an uncertainty environment. The objective function in fuzzy sense is defuzzified using Modified Graded Mean Integration Representation Method (MGMIRM). For the defuzzified objective function, theoretical results are developed to establish optimal policies. Finally, some numerical examples and sensitivity analysis are provided to examine the effects of non-stochastic uncertainty.

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1. Introduction

In response to significant practical relevance of Japanese Just-in-Time (JIT) philosophy, substantial research studies have been undertaken on inventory system with controllable lead time and quality improvement. From production inventory management view point, the ultimate goal of JIT is to produce small lot sizes with perfect quality products. The goal of JIT is naturally realized if the sufficient investments are made to shorten the lead time and improve the quality of product. Consequently, many scholars have incorporated these issues for the development of realistic production inventory models. Liao and Shyu [1] first developed an inventory model in which lead time is a unique decision variable and the order quantity is predetermined. Subsequently, many researchers (Ben-Daya and Raouf [2], Ouyang *et al.* [3], Moon and Choi [4], Hariga and Ben-Daya [5], Ouyang and Chuang [6], Chu *et al.* [7], Lee *et al.* [8], and Lin [9]) discussed several optimal inventory policies with lead time reduction under different assumptions. In literature, Porteus [10] and Rosenblatt as well as Lee [11]

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are the first, who explored a significant relationship between quality imperfection and lot size. Later, many scholars, such as Keller and Noori [12], Moon [13], Hong and Hayya [14], Ouyang and Chang [15], further analyzed inventory model considering this issue. Besides, Ouyang *et al.* [16] studied the combined effects of lead time reduction, setup cost reduction, and quality improvement in the lot size reorder point model. Sarkar and Moon [17] extended their work and investigated the relationship between quality improvement, reorder point, and lead time with variable backorder rate in an imperfect production process.

During lead time, the demand of different customers is not identical and thereby it is more reasonable to employ mixture distribution approach to describe the lead time demand than single distribution. This approach has been utilized by Wu and Tsai [18] to extend the model of Ouyang *et al.* [3]. Lee [19] developed an inventory model involving controllable backorder rate and variable lead time demand with mixture of distributions. Later, Lee *et al.* [20] extended the model of Wu and Tsai [18] for variable lead time demand with the mixtures of normal distributions and considered backorder rate as variable. Wu *et al.* [21] relaxed the assumption in Lee *et al.* [20] about the form of the mixture of distribution functions of the lead time demand and considered any mixture of distribution functions of the lead time demand to establish optimal policies using minmax criterion. Cobb [22] presented a mixture distribution procedure for the lead time demand in a continuous review inventory model. Moreover, some authors advocated capital investment to secure more backorders by reducing lost sales rate. In this context, Lin [23] formulated a continuous review inventory model for mixtures of distributions with defective goods and analyzed the effects of increasing investment to reduce the lost sales rate.

The cost parameters in aforesaid studies are assumed to be precisely known. However, in reality, precise values of the cost components are rarely available as they may be vague or imprecise. To cope up it quantitatively with imprecise available information to manage successful inventory, fuzzy set theory has been widely applied in the study of various inventory models. The articles presented by Hseish [24], Kao and Hsu [25], Tutuncu *et al.* [26], Vijayan and Kumaran [27], Handfield *et al.* [28], Shah and Soni [29], and Nezhad *et al.* [30], Kumar *et al.* [31], Kumar and Goswami ([32, 33]) are worth mentioning in this regard. Combining randomness and fuzziness in model formulation, Dey and Chakraborty [34] presented a fuzzy random continuous-review system wherein investment to reduce the setup cost and improvement of process quality have been incorporated. They considered complete backorder case with constant lead time. Kumar and Goswami [35] investigated the impact of a continuous review production-inventory system with finite production rate and stochastic/ fuzzy stochastic demand rate on the reorder level strategy. Moreover, Kumar and Goswami [36] studied a fuzzy random economic production quantity model for imperfect quality items with possibility and necessity constraints. We refer the reader to Wong and Lai [37] in which a survey of the application of fuzzy set theory in production and operations management has been carried out.

This study extends the work of Wu and Tsai [18] and proposes a model to allow for (1) investments to reduce lost sales rate and improve the process quality as well as (2) an imprecise cost parameters to tackle the reality in more effective way. The rest of this paper is organized as follows: Section 2 briefly reviews of basic concepts about fuzzy set. Section 3 describes a fuzzy expected value model along with solution methodology. Section 4 furnishes numerical examples and discusses the results. Section 5 provides conclusions and further extensions.

2. Notation and assumptions

The following notations are used to develop this model.

2.1. Decision variables

Q Order quantity (units).
r Reorder point.
L Replenishment lead time (days).
ρ Fraction of shortages during the stock out period that will be lost, $0 \le \rho \le 1$.
θ Probability of the production process which may goto *out-of-control* state during producing a lot.

2.2. Parameters

D Expected demand per year (units).
$\widetilde{A}$ Ordering cost per order which is TFN represented by $\widetilde{A} = (A - \delta_1^A, A, A + \delta_2^A)$ (\$/order).
μ Mean of the lead time demand.
σ Standard deviation of the lead time demand.
p Weight of the component distributions, $0 \le p \le 1$.
$\widetilde{h}$ Holding cost per unit per year which is TFN represented by $\widetilde{h} = (h - \delta_1^h, h, h + \delta_2^h)$ (\$/unit/unit time).
$\widetilde{s}$ Cost of defective items per unit which is imprecise in nature and characterized by $\widetilde{s} = (s - \delta_1^s, s, s + \delta_2^s)$ (\$/defective item).
$\widetilde{\pi}$ Shortage cost per unit short which is TFN defined by $\widetilde{\pi} = (\pi - \delta_1^\pi, \pi, \pi + \delta_2^\pi)$ (\$/unit shortage).
$\widetilde{\pi_0}$ Marginal profit (*i.e.*, cost of lost demand) per unit which is TFN defined by $\widetilde{\pi_0} = (\pi_0 - \delta_1^{\pi_0}, \pi_0, \pi_0 + \delta_2^{\pi_0})$ (\$/unit).
ρ_0 Original fraction of shortages that will be lost.
θ_0 Initial probability of the production process which may go to *out-of-control* state during producing a lot.
α Annual fractional cost of capital investment per unit per order, $0 < \alpha < 1$ (\$/unit/order).
$I(\rho, \theta)$ Lost sales reduction and capital investment required to reduce the lost sales fraction from ρ_0 to ρ and the *out-of-control* probability from θ_0 to θ.
L_i Length of the lead time with components $i = 1, 2, \ldots, n$ (days).
u_i Component of the lead time with u_i as the minimum duration (days).
v_i Component of the lead time with v_i as normal duration (days).
c_i Component of the lead time with c_i as crashing cost per unit time (\$/unit time).
$E(.)$ Mathematical expectation of (.).
x^+ $\max\{x, 0\}$

The following assumptions are considered to develop this model.

(1). We consider that the lead time demand X follows the mixture of normal distributions (Lee [38]) with probability density function given by

$$f(x) = p\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}}\mathrm{e}^{-\frac{1}{2}(\frac{x-\mu_1 L}{\sigma\sqrt{L}})^2} + (1-p)\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}}\mathrm{e}^{-\frac{1}{2}(\frac{x-\mu_2 L}{\sigma\sqrt{L}})^2} \tag{2.1}$$

where $\mu_1 - \mu_2 = \eta\sigma\sqrt{L}$, $\eta > 0$, $x \in \Re$, $0 \le p \le 1$, $\sigma > 0$. However, the mixture of normal distributions is hold for all p if $(\mu_1 - \mu_2)^2) < 27\sigma^2/8L$ (or $0 < \eta < \sqrt{27/8}$) (Everitt and Hand [39]).

(2). The lead time L has n mutually independent components, each having a different crashing cost for reducing lead time. The ith component has a normal duration v_i and the minimum duration u_i with crashing cost per unit time c_i with $c_1 \le c_2 \le c_3 \le \ldots \le c_n$. The lead time demand X follows a mixture of normal distribution.

(3). Let $L_0 = \sum_{j=1}^n v_j$ and L_i be the length of the lead time with components $1, 2, \ldots, i$ crashed to their minimum duration. Then, L_i is assumed as $L_i = L_0 - \sum_{j=1}^i (v_j - u_j)$ and the lead time crashing cost per cycle $R(L)$ is expressed as $R(L) = c_i(L_i - L) + \sum_{j=1}^{i-1} c_j(v_j - u_j)$ for $i = 1, 2, 3, \ldots, n$.

(4). The reorder point r = expected demand during the lead time + safety stock (SS) and SS = k(standard deviation of the lead time demand) in which k is a safety factor, where $r = \mu_* L + k\sigma_*\sqrt{L}$ and $\mu_* = p\mu_1 + (1-p)\mu_2$, $\sigma_* = \sqrt{1 + p(1-p)\eta^2\sigma}$, $\mu_1 = \mu_* + (1-p)\frac{\eta\sigma}{\sqrt{L}}$, $\mu_2 = \mu_* - \frac{p\eta\sigma}{\sqrt{L}}$.

(5). Logarithmic expressions are used for both quality improvement and lost sales reduction.

3. Model formulation

This study extends the work of Wu and Tsai [18] for variable lead time demand with mixture of normal distribution. Based on Wu and Tsai [18], the associated cost of the model is

$$
\begin{aligned}
C(Q,r,L) = {} & \text{ordering cost + holding cost + stock out cost + lost sales cost + crashing cost} \\
= {} & \frac{AD}{Q} + h\left(\frac{Q}{2} + \sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right)\right]\right.\right. \\
& \left.\left. + (1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} + (1-\beta)E(X-r)^+\right) \\
& + \frac{[\pi + \pi_0(1-\beta)]D}{Q}E(X-r)^+ + \frac{D}{Q}R(L)
\end{aligned}
\tag{3.1}
$$

where the expected shortage at the end of the cycle length is given by

$$
E(X-r)^+ = \int_r^\infty (x-r)f(x)\mathrm{d}x = \sigma\sqrt{L}\Psi(r_1, r_2, p) \tag{3.2}
$$

where $\Psi(r_1, r_2, p) = p\{\phi(r_1) - r_1[1-\Phi(r_1)]\} + (1-p)\{\phi(r_2) - r_2[1-\Phi(r_2)]\}$, $r_1 = \frac{r-\mu_1 L}{\sigma\sqrt{L}} = \frac{r-\mu_* L}{\sigma\sqrt{L}} - (1-p)\eta$ and $r_2 = \frac{r-\mu_2 L}{\sigma\sqrt{L}} = \frac{r-\mu_* L}{\sigma\sqrt{L}} + \eta p$; ϕ and Φ denote the standard normal p.d.f and cumulative distribution function (c.d.f), respectively.

After the incorporation of defective item cost, (Sarkar *et al.* [40]), the expected number of defective items in a lot size Q is $\{Q - \frac{\hat{\theta}(1-\hat{\theta}^Q)}{\theta}\}$ (See for instance Appendix A).

Then, the expected annual total cost can be expressed as

$$
M(Q,r,L) = C(Q,r,L) + \frac{sDQ\theta}{2}\cdot \tag{3.3}
$$

For notational convenience, we denote the lost sales rate by $\rho(= 1-\beta)$. Moreover, we consider the following logarithmic investment function for investment in quality improvement which was introduced by Porteus [10]

$$
I(\theta) = b\ \ln\left(\frac{\theta_0}{\theta}\right) \text{ for } 0 < \theta \le \theta_0
$$

and for lost sales reduction

$$
I(\rho) = V\ \ln\left(\frac{\rho_0}{\rho}\right) \text{ for } 0 < \rho \le \rho_0
$$

Hence, the total investment for quality improvement and lost sales reduction becomes as follows:

$$
\begin{gathered}
I(\rho,\theta) = I(\theta) + I(\rho) = U - b\ln\theta - V\ln\rho \\
\text{where } U = b\ \ln(\theta_0) + V\ \ln(\rho_0).
\end{gathered}
$$

When the lost sales rate and probability of *out-of-control* state in (3.3) are considered to be one of the decision variables rather than given, we seek to minimize the sum of investment for quality improvement and lost sales reduction and associated inventory cost as defined in (3.3). That is, for the inventory model associated with investment in lost sales reduction and quality improvement, the cost function for new model can be formulated as

$$\begin{aligned} E(Q,r,\theta,\rho,L) =& \alpha(U - b\ln\theta - V\ln\rho) + (A + R(L))\frac{D}{Q} \\ &+ h\left(\frac{Q}{2} + \sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right)\right]\right.\right. \\ &\left.\left.+(1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} + \rho\sigma\sqrt{L}\Psi(r_1,r_2,p)\right) \\ &+ \frac{[\pi + \pi_0\rho]D}{Q}\sigma\sqrt{L}\Psi(r_1,r_2,p) + \frac{sDQ\theta}{2}\cdot \end{aligned} \tag{3.4}$$

Therefore, crisp optimization problem for the new model can be defined as

$$\begin{aligned} &\text{Minimize } E(Q,r,\theta,\rho,L) \\ &\text{subject to } 0 < \theta \le \theta_0 \text{ and } 0 < \rho \le \rho_0. \end{aligned} \tag{3.5}$$

3.1. Model under fuzzy cost parameters

This article considers cost parameters of the model as imprecise in nature. When parameters A, h, π, π_0, and s treated as triangular fuzzy numbers (as per assumptions), the above cost function defined in (3.4) becomes TFN. Thus, the problem defined in (3.5) can be constructed under fuzzy framework as follows:

$$\begin{aligned} &\text{Minimize } \widetilde{E}(Q,r,\theta,\rho,L) \\ &\text{subject to } 0 < \theta \le \theta_0 \text{ and } 0 < \rho \le \rho_0 \end{aligned} \tag{3.6}$$

where $\widetilde{E}(Q,r,\theta,\rho,L) = (E_1,E_2,E_3)$. Here, E_1, E_2 and E_3 are all positive real valued functions of (Q,r,θ,ρ,L) satisfying the condition

$$E_1(Q,r,\theta,\rho,L) \le E_2(Q,r,\theta,\rho,L) \le E_3(Q,r,\theta,\rho,L)$$

Using Function Principle (see, Chen [41]) the values of $E_i(Q,r,\theta,\rho,L)$ for i = 1,2, and 3 are as follows:

$$\begin{aligned} E_1(Q,r,\theta,\rho,L) =& \alpha(U - b\ln\theta - V\ln\rho) + (A - \delta_1^A + R(L))\frac{D}{Q} \\ &+ (h-\delta_1^h)\left(\frac{Q}{2} + \sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right)\right]\right.\right. \\ &\left.\left.+(1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} + \rho\sigma\sqrt{L}\Psi(r_1,r_2,p)\right) \\ &+ \frac{[\pi - \delta_1^\pi + (\pi_0 - \delta_1^{\pi_0})\rho]D}{Q}\sigma\sqrt{L}\Psi(r_1,r_2,p) + \frac{(s-\delta_1^s)DQ\theta}{2}, \end{aligned} \tag{3.7a}$$

$$\begin{aligned} E_2(Q,r,\theta,\rho,L) =& \alpha(U - b\ln\theta - V\ln\rho) + (A + R(L))\frac{D}{Q} \\ &+ h\left(\frac{Q}{2} + \sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right)\right]\right.\right. \\ &\left.\left.+(1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} + \rho\sigma\sqrt{L}\Psi(r_1,r_2,p)\right) \\ &+ \frac{[\pi + \pi_0\rho]D}{Q}\sigma\sqrt{L}\Psi(r_1,r_2,p) + \frac{sDQ\theta}{2}, \end{aligned} \tag{3.7b}$$

$$
\begin{aligned}
E_3(Q,r,\theta,\rho,L) = {} & \alpha(U - b\ln\theta - V\ln\rho) + (A + \delta_2^A + R(L))\frac{D}{Q} \\
& + (h+\delta_2^h)\left(\frac{Q}{2} + \sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\,\eta\right)\right]\right.\right. \\
& \left.\left. +(1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} + \rho\sigma\sqrt{L}\Psi(r_1,r_2,p)\right) \\
& + \frac{[\pi + \delta_2^\pi + (\pi_0 + \delta_2^{\pi_0})\rho]D}{Q}\sigma\sqrt{L}\Psi(r_1,r_2,p) + \frac{(s+\delta_2^s)DQ\theta}{2}\cdot
\end{aligned}
\tag{3.7c}
$$

3.2. Defuzzification by modified graded mean integration representation

Suppose $\xi = (a,b,c)$ be a triangular fuzzy number. Then, according to Chen and Hsieh [42], the modified graded mean integration representation of ξ is given by

$$P(\xi) = \frac{\lambda a + 2b + (1-\lambda)c}{3} \tag{3.8}$$

where $\lambda \in [0,1]$ is called decision maker's attitude or optimism parameter. The value of λ closer to 0 implies that the decision maker is more pessimistic, whereas the value of λ closer to 1 means that the decision maker is more optimistic.

Hence, the fuzzy cost function with decision maker's λ-preference is represented by $P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))$ and is obtained by the formula (3.8), which is as follows:

$$P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L)) = \frac{\lambda E_1 + 2E_2 + (1-\lambda)E_3}{3}\cdot \tag{3.9}$$

Thus, the optimization problem addressed in this paper is

$$
\begin{gathered}
\text{Minimize } P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L)) \\
\text{subject to } 0 < \theta \le \theta_0 \text{ and } 0 < \rho \le \rho_0.
\end{gathered}
\tag{3.10}
$$

From here, we first ignore the constraints and solve the non-linear program with an analytical method and calculate all the partial derivatives of the $P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))$ with respect to decision variables. After that, we incorporate the restrictions. By taking partial derivatives of $P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))$ with respect to Q, L, r, ρ, θ, we have

$$\frac{\partial P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))}{\partial Q} = -\frac{D\overline{A}}{3Q^2} + \frac{\overline{h}}{6} - \frac{D\sigma\sqrt{L}\Psi(r_1,r_2,p)}{3Q^2}\overline{\pi} + \frac{D\theta\overline{s}}{6} \tag{3.11}$$

$$
\begin{aligned}
\frac{\partial P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))}{\partial r} = {} & \frac{\overline{h}}{3}\left\{p\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right) + (1-p)\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right\} \\
& + \frac{[G_*(r) - 1]}{3}\left[\rho\overline{h} + \frac{D\overline{\pi}}{Q}\right]
\end{aligned}
\tag{3.12}
$$

$$\frac{\partial P_\lambda(\widetilde{E}(Q,r,\theta,\pi_0,L))}{\partial \theta} = -\frac{\alpha b}{\theta} + \frac{DQ\overline{s}}{6} \tag{3.13}$$

$$\frac{\partial P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))}{\partial \rho} = -\frac{\alpha V}{\rho} + \frac{D\sigma\sqrt{L}}{3Q}\Psi(r_1,r_2,p)\overline{\pi_0} + \frac{\sigma\sqrt{L}}{3}\Psi(r_1,r_2,p)\overline{h} \tag{3.14}$$

$$\begin{aligned}\frac{\partial P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))}{\partial L} &= -\frac{c_i D}{Q} + \frac{\sigma\overline{h}}{6\sqrt{L}}\left[p\left\{r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)-\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\right\}\right.\\ &\quad\left.+(1-p)\left\{r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)-\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)\right\}\right]\\ &\quad+\frac{\mu_*\overline{h}}{6}\left[p\Big\{r_1+\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\Big\}\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\right.\\ &\quad\left.+(1-p)\Big\{r_2+\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\Big\}\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)\right]+\frac{\sigma\Psi(r_1,r_2,p)}{6\sqrt{L}}\left[\rho\overline{h}+\frac{D\overline{\pi}}{Q}\right] \end{aligned}\tag{3.15}$$

where $G_*(r) = p\Phi(r_1) + (1-p)\Phi(r_2)$ and [See Appendix for $\overline{A}, \overline{\pi}, \overline{\pi_0}, \overline{h}, \overline{s}$].

It is to be noted that $P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))$ is convex with respect to ρ, keeping other variables fixed, as

$$\frac{\partial^2 P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))}{\partial\rho^2} = \frac{\alpha V}{\rho^2} > 0. \tag{3.16}$$

Also, by calculating the 2nd order sufficient conditions, it can be shown that $P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))$ is not a convex function for L because 2nd order partial order derivative of $P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))$ with respect to L is negative. That is, $P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))$ is concave in $L \in [L_i, L_{i-1}]$ for fixed (Q,r,θ,ρ).

As

$$\begin{aligned}\frac{\partial^2 P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))}{\partial L^2} &= -\left[\frac{\sigma\overline{h}}{12L^{\frac{3}{2}}}\left\{p\left(r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)-\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\right)\right.\right.\\ &\quad\left.+(1-p)\left(r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)-\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)\right)\right\}\\ &\quad+\frac{\mu_* p\overline{h}}{12L}\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\left\{\frac{\mu_*\sqrt{L}}{\sigma}\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\right.\\ &\quad\left.\times\left(r_1+\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)-\left(r_1+\frac{2\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)\right\}\\ &\quad+\frac{\mu_*(1-p)\overline{h}}{12L}\phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)\left\{\frac{\mu_*\sqrt{L}}{\sigma}\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)\right.\\ &\quad\left.\times\left(r_2+\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)-\left(r_2+\frac{2\mu_*\sqrt{L}}{\sigma}-p\eta\right)\right\}\\ &\quad\left.+\frac{\sigma\Psi(r_1,r_2,p)}{12L^{\frac{3}{2}}}\left(\rho\overline{h}+\frac{D\overline{\pi}}{Q}\right)\right]\\ &< 0, \quad \text{if} \quad \frac{\mu_*\sqrt{L}}{\sigma}-p\eta > \sqrt{2}. \end{aligned}\tag{3.17}$$

Thus, the optimal value of $P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))$ occurs at the end points of the interval $[L_i,L_{i-1}]$. On equating first three partial derivatives equal to zero, one can obtain

$$Q=\sqrt{2D}\left[\overline{A}-\sigma\sqrt{L}\Psi(r_1,r_2,p)\overline{\pi}\right]^{\frac{1}{2}}\left[\overline{h}+D\theta\overline{s}\right]^{-\frac{1}{2}} \tag{3.18}$$

$$[1-G_*(r)]=\overline{h}\left\{p\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma}+(1-p)\eta\right)+(1-p)\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma}-p\eta\right)\right\} \times\left[\rho\overline{h}+\frac{D\overline{\pi}}{Q}\right]^{-1} \tag{3.19}$$

$$\theta=\frac{6\alpha b}{DQ\overline{s}}\cdot \tag{3.20}$$

It is to be noted that for fixed L, the convexity of $P_\lambda(\widetilde{E}(Q,r,\theta,\rho,L))$ is not guaranteed for the point (Q,r,θ,ρ). However, for fixed (ρ,L) we can establish following lemma.

Lemma 3.1. *For fixed ρ and $L\in[L_i,L_{i-1}]$, the Hessian matrix for $P_\lambda(\widetilde{E}(Q,r,\theta,L))$ is always positive definite at the optimal values (Q^*,r^*,θ^*).*

Proof. See Appendix B.

From last equations, it is clear that the value of θ is positive. Based on the restrictions on θ and ρ, we have four conditions for a given $L\in[L_i,L_{i-1}]$ as

(1). If $\theta^*<\theta_0$ and $\rho^*<\rho_0$, then $(Q^*,r^*,\theta^*,\rho^*,L^*)$ is an optimal solution.
(2). If $\theta^*\geq\theta_0$ and $\rho^*<\rho_0$, then it is not profitable to invest in the quality improvement process, *i.e.*, $\theta^*=\theta_0$.
(3). If $\theta^*<\theta_0$ and $\rho^*\geq\rho_0$, then the initial setup cost is an optimal setup cost, *i.e.*, $\rho^*=\rho_0$.
(4). If $\theta^*\geq\theta_0$ and $\rho^*\geq\rho_0$, then we do not consider any investment to reduce setup cost or to improve quality, *i.e.*, and $\rho^*=\rho_0$.

Consequently, we can establish following algorithmic procedure to identify the optimal point $(Q^*,r^*,\theta^*,\rho^*,L^*)$.

Algorithm
Step 1. For each $L_i, i=0,1,\ldots,n$, perform (i) to (ii).

(i) For a given lost sales rate ρ_0, we divide the interval $(0,\rho_0]$ into m equal subintervals, where m is large enough and let $\rho_{j,L_i}=\rho_0-j\rho_0/m$
(ii) For each $\rho_{j,L_i}, j=0,1,\ldots,m$, execute (iii) to (vii).
(iii) Set $r_{j1}=0$.
(iv) Substitute r_{j1} and θ from (3.20) into (3.18) to determine the value of Q_{j1}.
(v) Use Q_{j1} to obtain values of r_{j2} and θ_{j1}.
(vi) Repeat (iv) and (v) until no change occurs in the values of Q_j, r_j and θ_j.
(vii) Set $(Q_{\rho_j,L_i}, r_{\rho_j,L_i}, \theta'_{\rho_j,L_i})=(Q_j,r_j,\theta_j)$.

Step 2. Set $\theta_{\rho_j,L_i}=\min\{\theta_0,\theta'_{\rho_j,L_i}\}$.
Step 3. Compute corresponding $P_\lambda(\widetilde{E}(Q_{\rho_j,L_i},r_{\rho_j,L_i},\theta_{\rho_j,L_i},\rho_j,L_i))$ from (3.9) for $j=0,1,\ldots,m$.
Step 4. Set $P_\lambda(\widetilde{E}(Q_{L_i},r_{L_i},\theta_{L_i},\rho_{L_i},L_i))=\min\limits_{j=0,1,\ldots,m}P_\lambda(\widetilde{E}(Q_{\rho_j,L_i},r_{\rho_j,L_i},\theta_{\rho_j,L_i},\rho_j,L_i))$
Step 5. Compute corresponding $P_\lambda(\widetilde{E}(Q_{L_i},r_{L_i},\theta_{L_i},\rho_{L_i},L_i))$ from (3.9) for $i=0,1,\ldots,n$.
Step 6. Set $P_\lambda(\widetilde{E}(Q^*,r^*,\theta^*,\rho^*,L^*))=\min\limits_{i=0,1,\ldots,n}P_\lambda(\widetilde{E}(Q_{L_i},r_{L_i},\theta_{L_i},\rho_{L_i},L_i))$, then $(Q^*,r^*,\theta^*,\rho^*,L^*)$ is the optimal solution.

4. Numerical examples

Example 4.1. To illustrate the above solution procedure, we consider the following parametric values as:

$D = 600$ units per year, $A = \$200$/order, $\delta_1^A = 20$, $\delta_2^A = 30$, $h = \$20$/unit, $\delta_1^h = 5$, $\delta_2^h = 7$, $\pi = \$50$/unit, $\delta_1^\pi = 7$, $\delta_2^\pi = 5$, $\pi_0 = \$150$/unit, $\delta_1^{\pi_0} = 20$, $\delta_2^{\pi_0} = 12$, $s = \$75$/unit, $\delta_1^s = 12$, $\delta_2^s = 9$, $\sigma = 7$ $\mu = 11$, $\eta = 0.6$, $b = 100$, $V = 125$, $\rho = 1$, $\theta = 0.002$, $\lambda = 0.5$ $\alpha = 0.1$ and the lead time has three components with data as shown in Table 1.

We assume $\rho_{j,L_i} = \rho_0 - j\rho_0/m, j = 0, 1, 2, \ldots, m$ and take $m = 1000$. Applying the procedure of the proposed algorithm for $p = \{0, 0.25, 0.5, 0.75, 1\}$, the optimal solution is listed in Table 2.

From Table 2, the optimal inventory policy can easily be determined by comparing $P_\lambda(\widetilde{E}(Q_{L_i}, r_{L_i}, \theta_{L_i}, \rho_{L_i}, L_i)), i = 0, 1, 2, 3$. Moreover, it is to be noted that the model considers only one kind of customers' demand whenever $p = 0$ or 1, while the model considers two kinds of customers' demand whenever $0 < p < 1$. It implies that $P_\lambda(\widetilde{E}(Q_{L_i}, r_{L_i}, \theta_{L_i}, \rho_{L_i}, L_i))$ for two kinds of customers' demand is higher than that of one kind of customers' demand. The results obtained in Table 2 reflect this feature. That is, as p increases $P_\lambda(\widetilde{E}(Q_{L_i}, r_{L_i}, \theta_{L_i}, \rho_{L_i}, L_i))$ increases and then decreases.

Example 4.2. In this example, we study the effects of investing in lost sales rate reduction. We fix $p = 0.5$ and consider the same set of data as in Example 1. Applying the procedures as proposed in the algorithm, the optimal results are shown in Table 3. Furthermore, we list the optimal results of the no-investment policy (fixed lost sales rate) in the same table to demonstrate the effects of investment in lost sales rate reduction. Based on the computational results shown in Table 3, we can observe that the savings, ranges from 0.87 to 3.62%, can be achieved by controlling the lost sales rate through investment.

Example 4.3. This example studies the effect of decision maker's attitude or optimism parameter, λ. All the parameters are identical to those in Example 2. Computational results are summarized in Table 4 for $\lambda \in \{0, 0.25, 0.50, 0.75, 1\}$.

From Table 4, it can be observed that as λ increases the optimal reorder point (r^*), the optimal order quantity (Q^*), the optimal lost sales rate (ρ^*), and the optimal *out-of-control* probability (θ^*) increase whereas annual total cost $P_\lambda(\widetilde{E}(Q^*, r^*, \theta^*, \rho^*, L^*))$ decreases. These results suggest when the attitude of the decision maker is towards optimistic, *i.e.*, when decision maker takes risks, the total cost incurred by an inventory system reduces.

Example 4.4. This example assesses the impact of the extent of impreciseness in systems parameters over the decision variables. For this, let us first consider the crisp inventory model with parametric values defined in Example 1 with $p = 0.5$ and setting $\delta_1^A = \delta_2^A = 0$, $\delta_1^h = \delta_2^h = 0$, $\delta_1^\pi = \delta_2^\pi = 0$, $\delta_1^{\pi_0} = \delta_2^{\pi_0} = 0$ and $\delta_1^s = \delta_2^s = 0$. For these input data, the crisp model gives the optimal result as $Q^* = 121.63$, $r^* = 64.24$, $\theta^* = 0.00000365$, $\rho^* = 0.0346$, $L^* = 4$ and associated cost $E(Q^*, r^*, \theta^*, \rho^*, L^*) = 2981.98$.

The sensitivity analysis is carried out by changing the level of impreciseness in each parameter at a time and keeping the remaining parameters as crisp, whose values are defined in the above crisp model. This scenario is suitable, when the decision maker does not find uncertainty in particular system parameter(s) then he/she can treat that parameter(s) as crisp. The different degree of impreciseness in each parameter resulted in −40%, −20%, 0%, 20%, and 40% of its defuzzified values (applying (3.8) for $\lambda = 0.5$) from the corresponding crisp values. The percentage changes from optimal crisp solution are determined using the measurement of $\frac{\Delta r}{r_c}\%$, $\frac{\Delta Q}{Q_c}\%$, $\frac{\Delta \rho}{\rho_c}\%$, $\frac{\Delta \theta}{\theta_c}\%$ and $\frac{\Delta P_\lambda(\widetilde{E}(.))}{P_\lambda(\widetilde{E}(.))_c}\%$. The measurement can be explained as follows: for the measure of $\frac{\Delta Q}{Q_c}\%$, where $\Delta Q = Q_f - Q_c$, Q_f and Q_c denote optimal order quantity in fuzzy and crisp sense, respectively. Hence, $\frac{\Delta Q}{Q_c}\%$ denotes the change in order quantity and can be used to measure this parameter's sensitivity. Similarly, the measures of $\frac{\Delta r}{r_c}\%$, $\frac{\Delta \rho}{\rho_c}\%$, $\frac{\Delta \theta}{\theta_c}\%$, $\frac{\Delta L}{L_c}\%$ and $\frac{\Delta P_\lambda(\widetilde{E}(.))}{P_\lambda(\widetilde{E}(.))_c}\%$ indicate sensitivity on reorder point r, lost sales rate ρ, *out-of-control* probability θ, lead time L, and total annual cost, respectively.

TABLE 1. Lead time data.

Lead time component i	Normal duration v_i (days)	Minimum duration u_i (days)	Unit crashing cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

TABLE 2. Summary of the optimal procedure solution (L_i in weeks).

p	L_i	r_{L_i}	Q_{L_i}	ρ_{L_i}	θ_{L_i}	$P_\lambda(\widetilde{E}(Q_{L_i}, r_{L_i}, \theta_{L_i}, \rho_{L_i}, L_i))$
0.00	8	116.53	117.77	0.0242	0.00000379	3104.26
	6	90.80	117.98	0.0284	0.00000378	3030.75
	4	64.22	120.88	0.0355	0.00000370	2993.70
	3	50.23	128.69	0.0413	0.00000347	3089.17
0.25	8	117.48	118.20	0.0232	0.00000379	3132.90
	6	91.63	118.35	0.0271	0.00000378	3055.60
	4	64.90	121.17	0.0337	0.00000369	3013.75
	3	50.80	128.94	0.0394	0.00000347	3105.83
0.5	8	117.73	118.16	0.0232	0.00000378	3137.17
	6	91.85	118.31	0.0272	0.00000377	3059.26
	4	65.08	121.14	0.0340	0.00000370	3016.52
	3	50.96	128.91	0.0396	0.00000347	3107.77
0.75	8	117.37	117.98	0.0237	0.00000379	3125.85
	6	91.53	118.16	0.0277	0.00000378	3049.41
	4	64.82	121.02	0.0347	0.00000369	3008.31
	3	50.75	128.81	0.0404	0.00000347	3100.38
1	8	116.53	117.77	0.0242	0.00000379	3104.25
	6	90.80	117.98	0.0284	0.00000378	3030.67
	4	64.22	120.88	0.0355	0.00000370	2992.99
	3	50.23	128.69	0.0413	0.00000347	3087.14

TABLE 3. Optimal solution with and without investment in lost sales reduction for different values of ρ.

Model with lost sales reduction investment							Model without lost sales reduction investment					
ρ	L^*	r^*	Q^*	ρ^*	θ^*	$P_\lambda(\widetilde{E}(Q^*, r^*, \theta^*, \rho^*, L^*))$	L	r^*	Q^*	θ^*	$P_\lambda(\widetilde{E}(Q^*, r^*, \theta^*, \rho_0, L^*))$	Saving (%)
0.2	4	65.08	121.15	0.0339	0.00000369	2996.40	4	67.92	120.66	0.00000370	3022.83	0.87
0.4	4	65.08	121.14	0.0340	0.00000370	3005.07	4	70.15	120.32	0.00000371	3061.74	1.85
0.6	4	65.08	121.15	0.0339	0.00000369	3010.14	4	71.75	120.10	0.00000372	3089.87	2.58
0.8	4	65.08	121.15	0.0340	0.00000369	3013.73	4	72.98	119.94	0.00000373	3111.78	3.15
1.0	4	65.08	121.14	0.0340	0.00000370	3016.52	4	73.98	119.81	0.00000373	3129.67	3.62

TABLE 4. Computation results for λ.

λ	L^*	r^*	Q^*	ρ^*	θ^*	$P_\lambda(\widetilde{E}(Q^*, r^*, \theta^*, \rho^*, L^*))$
0.00	4	64.78	118.00	0.0305	0.00000362	3225.57
0.25	4	64.92	119.51	0.0321	0.00000365	3121.33
0.50	4	65.08	121.14	0.0340	0.00000370	3016.52
0.75	4	65.24	122.92	0.0360	0.00000372	2911.07
1.00	4	65.42	124.87	0.0383	0.00000376	2804.91

TABLE 5. Sensitivity analysis when one parameter is imprecise ($p = 0.5$).

Parameter	Fuzzy Numbers	$P(.)$	% Change	$\frac{\Delta L}{L_c}\%$	$\frac{\Delta r}{r_c}\%$	$\frac{\Delta Q}{Q_c}\%$	$\frac{\Delta \rho}{\rho_c}\%$	$\frac{\Delta \theta}{\theta_c}\%$	$\frac{\Delta P_\lambda(\widetilde{E}(.))}{P_\lambda(\widetilde{E}(.))_c}\%$
$\widetilde{A}$	(100, 114, 170)	121.0	−40	0	2.45	−18.89	4.95	23.32	−14.43
	(146, 156, 198)	161.3	−20	0	1.09	−08.73	2.27	09.48	−06.69
	(180, 200, 220)	200.0	0	0	0	0	0	0	0
	(188, 246, 280)	242.0	+20	0	−1.01	08.66	−1.76	−07.91	06.66
	(220, 289, 318)	282.3	+40	0	−1.85	16.35	−3.45	−14.16	12.59
$\widetilde{h}$	(8, 11.3, 20)	12.2	−40	0	3.92	26.00	90.49	−20.75	−23.57
	(12, 15.9, 22)	16.3	−20	0	1.61	10.09	30.21	−09.27	−10.70
	(15, 20, 25)	20.0	0	0	0	0	0	0	0
	(18, 24.6, 30)	24.4	+20	0	−1.53	−08.75	−21.79	09.53	11.61
	(21, 28.7, 35)	28.5	+40	0	−2.73	−14.96	−35.25	17.52	21.62
$\widetilde{\pi}$	(20, 29.7, 40)	29.8	−40	0	−6.75	0.65	−47.06	−0.74	−2.15
	(28, 40.6, 48)	39.7	−20	0	−2.87	0.26	−24.31	−0.31	−0.93
	(45, 50, 55)	50.0	0	0	0	0	0	0	0
	(50, 60.65, 65)	59.6	+20	0	2.09	−0.17	23.74	0.08	0.68
	(56, 70.8, 78)	69.5	+40	0	3.85	−0.32	48.83	0.34	1.25
$\widetilde{\pi_0}$	(70, 86.3, 120)	089.2	−40	0	0	−0.01	65.44	−0.08	−0.21
	(90, 119.65, 145)	118.9	−20	0	0	0	25.44	−0.09	−0.09
	(130, 150, 170)	150.0	0	0	0	0	0	0	0
	(160, 177.6, 200)	178.4	+20	0	0	0.01	−15.44	−0.10	0.07
	(185, 203.45, 250)	208.1	+40	0	0	0.01	−27.24	−0.05	0.13
$\widetilde{s}$	(39, 42.3, 60)	044.7	−40	0	0	0	0.15	67.77	−0.17
	(45, 60.65, 70)	059.6	−20	0	0	0	0.21	25.90	−0.08
	(65, 75, 85)	75.0	0	0	0	0	0	0	0
	(70, 89.1, 110)	089.4	+20	0	0	0	0.13	−16.13	0.06
	(90, 100.2, 135)	104.3	+40	0	0	0	0.14	−28.20	0.11

The results of sensitivity analysis are shown in Tables 5 and 6. A careful study based on the computational results shown in Table 5 reveals the following points:

(1). As the degree of impreciseness of ordering cost (A) varies, the optimal order quantity (Q), the *out-of-control* probability (θ) and associated cost in fuzzy sense are highly sensitive whereas reorder point (r), and lost sales rate (ρ) are moderately sensitive.
(2). The percentage changes in the level of fuzziness in the holding cost (h) causes significant changes in the total cost and θ whereas, the values of Q and ρ show noticeable negative variations in the optimal results.
(3). The variations in the degree of fuzziness in π and π_0 resultes in positive and negative variations in the value of ρ, respectively.
(4). Higher sensitivity has been observed in the value of θ as the degree of impreciseness of s varies.

Moreover, Table 6 shows variations in the optimum decision variables and the optimal cost due to impreciseness in all cost components of the model with absolutely pessimistic ($\lambda = 0$), neutral ($\lambda = 0.5$), and absolutely optimistic ($\lambda = 1$) attitude level.

It can be observed, irrespective the value of λ, that ρ and θ show remarkable negative variations with respect to variations in the level of fuzziness of all the parameters, the optimal cost register noteworthy changes in the same directions with respect to increase in the level of fuzziness of all components. This is a realistic outcome as the fuzzy annual cost considers uncertainty due to lack of information. Hence, if the uncertainties are accounted for in apt manner the fuzzy annual cost increases as compared to the crisp case.

Furthermore, significant difference in the percentage changes from crisp case can be observed for absolute pessimistic and optimistic decision policy. The results under absolute pessimism show that percentage changes

Table 6. The percentage change in optimal policy from the crisp case when all parameters are Fuzzy ($p = 0.5$).

λ	% change in all parameters	$\frac{\Delta L}{L_c}\%$	$\frac{\Delta r}{r_c}\%$	$\frac{\Delta Q}{Q_c}\%$	$\frac{\Delta \rho}{\rho_c}\%$	$\frac{\Delta \theta}{\theta_c}\%$	$\frac{\Delta P_\lambda(\widetilde{E}(.))}{P_\lambda(\widetilde{E}(.))_c}\%$
0	−40	0	−0.17	−1.02	49.53	57.19	−28.75
	−20	0	−0.12	−1.73	13.88	20.01	−11.32
	0	0	−0.70	−2.98	−11.93	−00.91	08.17
	+20	0	−0.79	−2.08	−26.63	−20.38	28.16
	+40	0	−0.71	−2.79	−37.92	−31.12	48.07
0.5	−40	50	41.56	−2.28	40.26	71.68	−36.71
	−20	0	−00.06	0.65	26.27	24.97	−17.65
	0	0	−00.25	−0.40	−01.69	01.25	01.16
	+20	0	−00.37	−1.10	−19.82	−15.26	19.92
	+40	0	−00.46	−1.61	−32.19	−27.04	38.66
1	−40	50	42.35	1.36	71.51	79.61	−44.86
	−20	0	0.04	3.48	42.08	30.70	−24.02
	0	0	0.28	2.66	10.72	02.92	−05.94
	+20	0	0.10	0.04	−11.54	−9.39	11.66
	+40	0	−0.17	−0.24	−25.31	−22.45	29.23

in fuzzy cost is significantly higher than that of crisp case. This result suggests that if the decision maker has less information about system parameters and outcomes, it is worthwhile to select optimistic decision policy ($0.5 < \rho \leq 1$) for decision making.

5. Conclusions

The model extended the work of Wu and Tsai [18] by considering imprecise cost components. This paper considered the logarithmic investment functions for lost sales reduction and quality improvement with two restrictions. The fuzzy cost function was defuzzified using MGMIRM. By analyzing the defuzzified cost function, a lemma was developed to identify the optimum value of decision variables. An efficient algorithm was designed to obtain the numerical results. Finally, some numerical examples were provided to illustrate the proposed model. Numerical results indicated that the saving of total cost were obtained through lostsales reduction. The effect of impreciseness on the optimal solution was examined through numerical example. Higher sensitivity in decision variables and associated cost function has been observed due to impreciseness of ordering and holding cost. Moreover, decision variables and cost function were found highly sensitive whenever all system parameters exhibit fuzziness simultaneously where, optimistic decision strategy was more advantageous than pessimistic decision strategy. The proposed model and results of this model could help the decision maker in seeking business edge in an uncertain environment depending on his risk attitude. While this research generalized the inventory model by Wu and Tsai [18], further investigations may be conducted in a number of directions. For instance, we may extend the proposed model to allow for vendor-buyer, multi-vendor, multi-buyer, and multi-product problems under fuzzy environment. Also, it would be interesting to examine the difference between different defuzzification techniques within these assumptions like quality improvement [43, 55], fuzzy demand [44], delay-in-payments [46], service label constraints [47], imperfect production system [45, 48–54].

5.1. Compliance with ethical standards

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Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Appendix A.

As $\hat{\theta} = 1 - \theta$ is approximately 1, we use the Taylor series expansion of $\hat{\theta}^Q$ and obtain

$$\hat{\theta}^Q = \mathrm{e}^{(\ln\hat{\theta})Q} \cong 1 + (\ln\hat{\theta})Q + \frac{[(\ln\hat{\theta})Q]^2}{2}$$

Hence, we have the number of defective items

$$\begin{aligned}
&= Q - \frac{\hat{\theta}(1-\hat{\theta}^Q)}{\theta}\\
&= Q - \frac{1 - 1 - (\ln\hat{\theta})Q - \frac{(\ln\hat{\theta})^2 Q^2}{2}}{\theta}\\
&= Q - \frac{\frac{\theta}{\hat{\theta}}Q - \frac{\theta^2}{2\hat{\theta}^2}Q^2}{\theta}\\
&= Q - \frac{\theta Q - \frac{\theta^2 Q^2}{2}}{\theta}\\
&= \frac{\theta Q^2}{2}.
\end{aligned}$$

The expected annual defective cost is

$$\begin{aligned}
&= sD - \frac{sD\hat{\theta}(1-\hat{\theta}^Q)}{\theta Q}\\
&= sD - \frac{sD\left(1 - 1 - (\ln\hat{\theta})Q - \frac{(\ln\hat{\theta})^2 Q^2}{2}\right)}{\theta Q}\\
&\quad \times \left[\text{since } \hat{\theta} = 1 - \theta \cong 1 \text{ and } \hat{\theta}^Q = \mathrm{e}^{(\ln\hat{\theta})Q} \cong 1 + (\ln\hat{\theta})Q + \frac{[(\ln\hat{\theta})Q]^2}{2}\right]\\
&= sD - \frac{sD\left(\frac{\theta}{\hat{\theta}}Q - \frac{\theta^2}{2\hat{\theta}^2}Q^2\right)}{\theta Q}\\
&= sD - \frac{sD\left(\theta Q - \frac{\theta^2 Q^2}{2}\right)}{\theta Q}\\
&= sD - sD\left(1 - \frac{\theta Q}{2}\right)\\
&= \frac{sDQ\theta}{2}.
\end{aligned}$$

Appendix B.

Proof of Lemma 1

We compute the Hessian matrix at the optimal values for a given $L \in [L_i, L_{i-1}]$ and fixed ρ, as follows:

$$H_{33} = \det \begin{pmatrix} \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial {Q^*}^2} & \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial Q^* \partial r^*} & \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial Q^* \partial \theta^*} \\ \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial r^* \partial Q^*} & \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial {r^*}^2} & \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial r^* \partial \theta^*} \\ \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial \theta^* \partial Q^*} & \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial \theta^* \partial r^*} & \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial {\theta^*}^2} \end{pmatrix}$$

where $P_\lambda \widetilde{E^*}(\cdot) = P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, L)$. The second order partial derivatives at the optimal values are

$$\begin{aligned}
\frac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, L)}{\partial {Q^*}^2} &= \left[\frac{2D}{3{Q^*}^3}\right]\left[\overline{A} + \sigma\sqrt{L}\Psi(r_1, r_2, p)\overline{\pi}\right] \\
\frac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, L)}{\partial {r^*}^2} &= \frac{1}{3\sigma\sqrt{L}}\left[\rho\overline{h} + \frac{D\overline{\pi}}{Q^*}\right]\left[p\phi(r_1) + (1-p)\phi(r_2)\right] \\
\frac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, L)}{\partial {\theta^*}^2} &= \frac{\alpha b}{{\theta^*}^2} \\
\frac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, L)}{\partial Q^* \partial r^*} &= \frac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, \rho^*, L)}{\partial r^* \partial Q^*} = \frac{[1 - G_*(r)]D\overline{\pi}}{3Q^{*2}} \\
\frac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, L)}{\partial Q^* \partial \theta^*} &= \frac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, \rho^*, L)}{\partial \theta^* \partial Q^*} = \frac{D\overline{s}}{6} \\
\frac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, L)}{\partial r^* \partial \theta^*} &= \frac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, L)}{\partial \theta^* \partial r^*} = 0
\end{aligned}$$

where

$$\begin{aligned}
\overline{A} &= \lambda(A - \delta_1^A + R(L)) + 2(A + R(L)) + (1-\lambda)(A + \delta_2^A + R(L)) \\
\overline{\pi} &= \lambda(\pi - \delta_1^\pi + (\pi_0 - \delta_1^{\pi_0})\rho) + 2(\pi + \pi_0\rho) + (1-\lambda)(\pi + \delta_2^\pi + (\pi_0 + \delta_2^{\pi_0})\rho) \\
\overline{h} &= \lambda(h - \delta_1^h) + 2h + (1-\lambda)(h + \delta_2^h) \\
\overline{\pi_0} &= \lambda(\pi_0 - \delta_1^{\pi_0}) + 2\pi_0 + (1-\lambda)(\pi_0 + \delta_2^{\pi_0}) \\
\overline{s} &= \lambda(s - \delta_1^s) + 2s + (1-\lambda)(s + \delta_2^s)
\end{aligned}$$

The first principal minor at the optimal values are

$$\begin{aligned}
\det(H_{11}) &= \det\left(\tfrac{\partial^2 P_\lambda \widetilde{E^*}(Q^*, r^*, \theta^*, \rho^*, L)}{\partial {Q^*}^2}\right) \\
&= \frac{2D}{3Q^{*3}}\left[\overline{A} + \sigma\sqrt{L}\Psi(r_1, r_2, p)\overline{\pi}\right] > 0
\end{aligned}$$

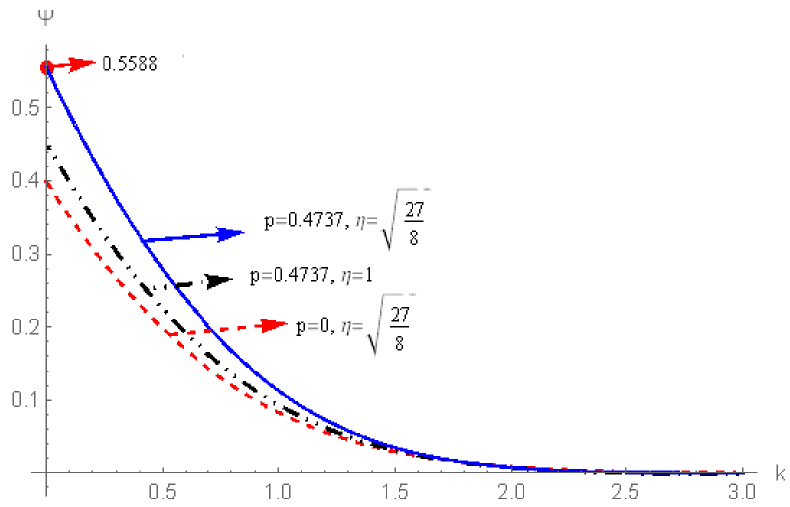

FIGURE 1. The behaviour of $\Psi(k,p,\eta)$ for different values of p and η.

This first principal minor is greater than zero because all terms are positive.

$$\begin{aligned}\det(H_{22}) &= \det\begin{pmatrix} \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial {Q^*}^2} & \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial Q^* \partial r^*} \\ \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial r^* \partial Q^*} & \frac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial {r^*}^2} \end{pmatrix} \\ &= \omega\tau - \nu^2\end{aligned}$$

where

$$\begin{aligned}\omega &= \frac{2D}{3{Q^*}^3}\left[\overline{A} + \sigma\sqrt{L}\Psi(r_1,r_2,p)\overline{\pi}\right] \\ \tau &= \frac{1}{3\sigma\sqrt{L}}\left[\rho\overline{h} + \frac{D\overline{\pi}}{Q^*}\right]\left[p\phi(r_1) + (1-p)\phi(r_2)\right] \\ \nu &= \left[\frac{D\overline{\pi}\{1 - G_*(r)\}}{3{Q^*}^2}\right].\end{aligned}$$

After some simplifications, we obtain

$$\begin{aligned}\det(H_{22}) = &\frac{2D\overline{A}}{9\sigma\sqrt{L}{Q^*}^3}\left[\rho\overline{h} + \frac{D\overline{\pi}}{Q^*}\right]\left[p\phi(r_1) + (1-p)\phi(r_2)\right] \\ &+ \frac{2\rho D\overline{\pi}\overline{h}}{9{Q^*}^3}\Psi(r_1,r_2,p)\left[p\phi(r_1) + (1-p)\phi(r_2)\right] \\ &+ \frac{D^2\overline{\pi}^2}{9{Q^*}^4}\left\{2\Psi(r_1,r_2,p)\left[p\phi(r_1) + (1-p)\phi(r_2)\right] - \left[G_*(r) - 1\right]^2\right\}.\end{aligned}$$

As $r_1 = k\sqrt{1 + p(1-p)\eta^2} - (1-p)\eta$ and $r_2 = k\sqrt{1 + p(1-p)\eta^2} + p\eta$, we can write $\Psi(r_1,r_2,p) \equiv \Psi(k,p,\eta)$. The behaviour of $\Psi(k,p,\eta)$ is shown in Figure 1. It is to be noted from the Figure 1 that $\Psi(k,p,\eta)$ is decreasing function of k and increasing function of η for fixed p. Moreover, $\Psi(k,p,\eta) > 0$ (by examining standard normal table) for all $k \in [0,\infty), p \in [0,1]$ and $\eta \in \left(0, \sqrt{\frac{27}{8}}\right)$ Besides the behaviour of $\xi_1(k,p,\eta) \equiv$

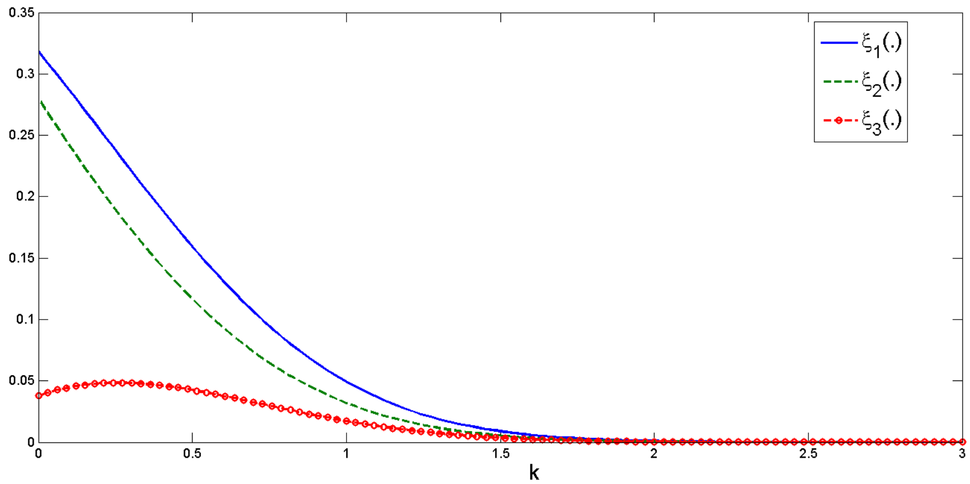

FIGURE 2. The behaviour of $\xi_1(.), \xi_2(.)$ and $\xi_3(.)$

$2\Psi(k,p,\eta)[p\phi(r_1) + (1-p)\phi(r_2)]$, $\xi_2(k,p,\eta) \equiv [p\Phi(r_1)+(1-p)\Phi(r_2)-1]^2$ and $\xi_3(k,p,\eta) \equiv \xi_1(k,p,\eta)-\xi_2(k,p,\eta)$ is depicted in Figure 2 for fixed p. Note that Figure 2 shows that though the values of $\xi_1(k,p,\eta), \xi_2(k,p,\eta)$ and $\xi_3(k,p,\eta)$ are small (< 0.40), they are positive, and as $k \to 0$, $\xi_1(k,p,\eta), \xi_2(k,p,\eta)$ and $\xi_3(k,p,\eta)$ approaches to 0^+. Hence, $\det(H_{22}) > 0$.

$$\begin{aligned}
\det(H_{33}) = \det &\begin{pmatrix} \dfrac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial {Q^*}^2} & \dfrac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial Q^* \partial r^*} & \dfrac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial Q^* \partial \theta^*} \\ \dfrac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial r^* \partial Q^*} & \dfrac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial {r^*}^2} & \dfrac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial r^* \partial \theta^*} \\ \dfrac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial \theta^* \partial Q^*} & \dfrac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial \theta^* \partial r^*} & \dfrac{\partial^2 P_\lambda \widetilde{E^*}(\cdot)}{\partial {\theta^*}^2} \end{pmatrix} \\
=&\frac{2D}{3{Q^*}^3}\left[\overline{A} + \sigma\sqrt{L}\Psi(r_1,r_2,p)\overline{\pi}\right] \\
&\times \frac{\alpha b}{3\sigma\sqrt{L}{\theta^*}^2}\left[\rho\overline{h} + \frac{D\overline{\pi}}{Q^*}\right]\left[p\phi(r_1) + (1-p)\phi(r_2)\right] \\
&- \frac{\alpha b}{{\theta^*}^2}\left[\frac{D\overline{\pi}\{1-G_*(r)\}}{3{Q^*}^2}\right]^2 - \frac{D^2\overline{s}^2}{36} \\
&\times \frac{1}{3\sigma\sqrt{L}}\left[\rho\overline{h} + \frac{D\overline{\pi}}{Q^*}\right]\left[p\phi(r_1) + (1-p)\phi(r_2)\right] \\
=&\left(\frac{\alpha b}{{\theta^*}^2}\omega - \frac{D^2\overline{s}^2}{36}\right)\tau - \frac{\alpha b}{{\theta^*}^2}\nu^2.
\end{aligned}$$

Therefore, it is enough to show

$$\left(\frac{\alpha b}{{\theta^*}^2}\omega - \frac{D^2\overline{s}^2}{36}\right)\tau > \frac{\alpha b}{{\theta^*}^2}\nu^2.$$

Now

$$\begin{aligned}\left(\frac{\alpha b}{\theta^{*2}}\omega-\frac{D^2\overline{s}^2}{36}\right)\tau &= \left[\frac{\alpha b}{\theta^{*2}}\left\{\frac{2D}{3Q^{*3}}\left(\overline{A}+\sigma\sqrt{L}\Psi(r_1,r_2,p)\overline{\pi}\right)\right\}-\frac{D^2\overline{s}^2}{36}\right]\tau\\ &> \frac{\alpha b}{\theta^{*2}}\left[\frac{2D}{3Q^{*3}}\left(\overline{A}+\sigma\sqrt{L}\Psi(r_1,r_2,p)\overline{\pi}\right)-\frac{\alpha b}{Q^{*3}}\right]\tau\\ &> \frac{\alpha b}{\theta^{*2}}\left[\frac{2D}{3Q^{*3}}\sigma\sqrt{L}\Psi(r_1,r_2,p)\overline{\pi}-\frac{\alpha b}{Q^{*3}}\right]\tau\\ &> \frac{\alpha b}{\theta^{*2}}\frac{D^2\overline{\pi}^2}{9Q^{*4}}2\Psi(r_1,r_2,p)\Big[p\phi(r_1)+(1-p)\phi(r_2)\Big]\quad\text{if}\quad(2D\overline{A}>3\alpha b)\\ &> \frac{\alpha b}{\theta^{*2}}\frac{D^2\overline{\pi}^2}{9Q^{*4}}\left[1-G_*(r)\right]^2\quad\text{since}\;\; 2\Psi(r_1,r_2,p)\Big[p\phi(r_1)+(1-p)\phi(r_2)\Big]>\left[1-G_*(r)\right]^2\\ &> \frac{\alpha b}{\theta^{*2}}\nu^2\end{aligned}$$

Hence $P_\lambda\widetilde{E^*}(Q^*,r^*,\theta^*,L)$ is a convex function for a given $L\in[L_i,L_{i-1}]$ and fixed ρ.

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