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NETWORK ROBUSTNESS AND RESIDUAL CLOSENESS

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Abstract. A central issue in the analysis of complex networks is the assessment of their robustness and vulnerability. A variety of measures have been proposed in the literature to quantify the robustness of networks and a number of graph-theoretic parameters have been used to derive formulas for calculating network reliability. In this paper, we study the vulnerability of interconnection networks to the failure of individual nodes, using a graph-theoretic concept of residual closeness as a measure of network robustness which provides a much fuller characterization of the network.

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1. Introduction

Networks and complex systems are used for modeling different systems such as chemical systems, neural networks, social systems or the Internet and the World Wide Web. Study of network topology receives great attention in various research fields such as operations research, physical sciences, biological sciences, computer sciences, and mathematics. Graph theory has become one of the most powerful mathematical tools in the analysis and study of the architecture of an interconnection network. It is well known that the underlying topology of an interconnection network is modeled by a graph $G = (V, E)$, where $V(G)$ is the set of processors and $E(G)$ is the set of communication links in the network.

A central issue in the analysis of complex networks is the assessment of their robustness and vulnerability. Robustness of the network topology is a key aspect in the design of computer networks. The study of network robustness receives great attention in all fields of network research. Vulnerability is an important concept in network analysis. The vulnerability of a communication network is defined as the measurement of the global strength of its underlying graph. The design of a good communication network must take into account notions such as reliability and vulnerability. When the network requirements are expressed in terms of graph theoretical parameters, the problem of analysis and design of networks becomes finding a graph G satisfying some specified requirement.

Communication systems are often subjected to failures and attacks. A variety of measures have been proposed in the literature to quantify the robustness of networks and a number of graph-theoretic parameters have been

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used to derive formulas for calculating network reliability. For instance, connectivity of a graph is an important and the earliest measure of robustness of a network [12]. However, the connectivity only partly reflects the ability of graphs to retain connectedness after vertex or edge deletion. Other improved measures were introduced and studied, such as toughness [24], scattering number [11], integrity [2], tenacity [16], *etc.* In contrast to connectivity, these measures consider both the cost to damage a network and how badly the network is damaged. These parameters are appropriate when graph failure means that the graph has become disconnected or trivial. From an algorithmic point of view, it is unfortunate that problem of recognizing these measures of general graphs is NP-complete [8]. This implies that these measures are of no great practical use within the context of complex networks.

The concept of residual closeness is introduced by Dangalchev [4] as a useful measure of graph vulnerability. The aim of residual closeness is to measure the vulnerability even when the removal of vertices does not disconnect the graph. In this model, the vertices fail independently of each other. In [4], it is argued that residual closeness is the most appropriate approach for modeling the robustness of network topologies in the face of possible node or link destruction. Residual closeness can reflect the vulnerability of graphs better than or independent of the other parameters existing in literature. Since residual closeness is considered to be a reasonable measure for the vulnerability of graphs, it is of particular interest to evaluate the residual closeness of different classes of graphs. We studied the closeness and residual closeness in wheels and cycles and related networks [1,32]. Turacı and Aytaç also studied these parameters [22,23].

The aim of this article is to obtain general bounds, general results and efficient formulas for the closeness and residual closeness of graphs.

In this paper, we consider simple finite undirected graphs without loops and multiple edges. Let $G = (V, E)$ be a graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$. The *order* of G is the number of vertices in G. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. If u and v are not connected, then $d(u, v) = \infty$, and for $u = v$, $d(u, v) = 0$. The *eccentricity* of a vertex v in G is the distance from v to a vertex farthest away from v in G. The *diameter*of G, denoted by $diam(G)$, is the largest distance between two vertices in $V(G)$. The *degree* $\deg_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v. The *maximum degree* of G is $\Delta(G) = \max\{\deg_G(v) \mid v \in V(G)\}$. A vertex of degree zero is an *isolated vertex* or an *isolate*. A vertex of degree one is called a *leaf* or an *endvertex*, and its neighbor is called a *support vertex*. For any leaf vertex v and support vertex w, the edge vw is called a *pendant edge* [7,9,10,13].

The paper proceeds as follows. In Section 1, the definitions of closeness and residual closeness and known results are given. Sharp upper and lower bounds for the closeness are discussed in terms of order and diameter of a graph in Section 2. Section 3 and 4 include several results on closeness of graphs for which $diam(G)$ is small, and having specific types of vertices. Formulas for the closeness and residual closeness of the graphs obtained via unary and binary graph operations are given in Section 5. Section 6. concludes the paper.

2. Closeness and residual closeness

The *closeness* of a graph is defined as $C = \sum\limits_{i} C(i)$, where $C(i)$ is the closeness of a vertex i and $C(i) = \sum\limits_{j \neq i} 2^{-d(i,j)}$ [4]. This definition can also be used for disconnected graphs.

Let $d_k(i, j)$ be the distance between vertices $i$and j in the graph, received from the original graph where all edges incident to vertex k are deleted. Then the closeness after removing vertex k is defined as $C_k = \sum\limits_{i} \sum\limits_{j \neq i} 2^{-d_k(i,j)}$ [4].

The *vertex residual closeness* (VRC) of the graph is defined as $R = \min\limits_{k} \{C_k\}$ [4].

Theorem 2.1 [4].

The closeness of

(a) *the complete graph K_n with n vertices is $C(K_n) = (n(n-1))/2$;*
(b) *the star graph S_n with n vertices is $C(S_n) = (n-1)(n+2)/4$;*
(c) *the path P_n with vertices is $C(P_n) = 2n - 4 + 2^{2-n}$.*

Theorem 2.2 [1]. *The closeness of the cycle C_n with n vertices is*

$$C(C_n) = \begin{cases} 2n\left(1 - 1/2^{(n-1)/2}\right), & \text{if } n \text{ is odd;} \\ n\left(2 - 3/2^{n/2}\right), & \text{if } n \text{ is even.} \end{cases}$$

Theorem 2.3 [4].

The VRC of

(a) *the complete graph K_n with vertices is $R(K_n) = ((n-1)(n-2))/2$;*
(b) *the star graph S_n with n vertices is $R(S_n) = 0$.*

Theorem 2.4 [4]. *If H is a proper subgraph of graph G, then $R(H) < R(G)$.*

3. Bounds on the closeness of a graph

Theorem 3.1. *For any connected G of order n,*

$$C(G) \leq n(n-1)/2. \tag{3.1}$$

Proof. Consider any vertex v of G. Then, $C(v) \leq (n-1)\left(2^{-1}\right)$. Since this inequality holds for every other vertex of G, combining the inequalities for $\forall v \in V(G)$ yields

$$C(v_1) + C(v_2) + \ldots + C(v_n) = \sum_i C(v_i) = C(G)$$

$$C(G) \leq n\left((n-1)\left(2^{-1}\right)\right)$$

and this leads to the desired bound (3.1). □

Remark 3.2. The result in Theorem 3.1 is best possible. This can be shown by the complete graph K_n of order n having $C(K_n) = n(n-1)/2$. This shows that an equality is obtained in inequality (3.1) if $G \cong K_n$ with $|V(G)| = n$ and consequently, the bound (3.1) is sharp.

Theorem 3.3. *For any connected G of order n,*

$$C(G) \geq n\left(\left(\sum_{i=1}^{diam(G)-1} 1/2^i\right) + (n - diam(G))/2^{diam(G)}\right). \tag{3.2}$$

Proof. Consider any vertex v of G. Then,

$$C(v) \geq (1)\left(2^{-1}\right) + (1)\left(2^{-2}\right) + \ldots + (1)\left(2^{1-diam(G)}\right) + (n - diam(G))\left(2^{-diam(G)}\right).$$

Since this inequality holds for every other vertex of G, combining the inequalities for $\forall v \in V(G)$ yields

$$C(v_1) + C(v_2) + \ldots + C(v_n) = \sum_i C(v_i) = C(G)$$

$$C(G) \geq n\left(\left(2^{-1} + 2^{-2} + \ldots + 2^{1-diam(G)}\right) + (n - diam(G))\left(2^{-diam(G)}\right)\right)$$
$$= n\left(\left(\sum_{i=1}^{diam(G)-1} 1/2^i\right) + (n - diam(G))\left(2^{-diam(G)}\right)\right),$$

and this yields the desired bound (3.2). □

Remark 3.4. The result in Theorem 3.3 is best possible. This can be shown by the complete graph K_n of order n. Since $diam(K_n) = 1$,

$$C(K_n) = n(n-1)/2 \ = n\left(\left(\sum_{i=1}^{diam(K_n)-1} 2^{-i}\right) + (n-1)\left(2^{-1}\right)\right) = n\left((n-1)/2\right).$$

This shows that an equality is obtained in (3.2) if $G \cong K_n$ with $|V(G)| = n$, and thus (3.2) is also sharp.

4. Closeness in graphs with small diameter

Diameter is an important graph theoretic concept. The diameter of a graph corresponds to the maximum number of links over which a message between two nodes must travel. In order to improve or increase the efficiency of message transmission, the diameter of networks are wished to be low.

Theorem 4.1. *For any graph G, if $diam(G) \leq 2$, then*

$$C(G) = (|V(G)|(|V(G)| - 1) + 2|E(G)|)/4.$$

Proof. Consider any vertex v in G. Then, v is adjacent to $\deg(v)$ vertices in G. Since $diam(G) \leq 2$, v is at distance 2 to other remaining vertices of G if there exist some. Hence, we have $C(v) = \deg(v)\left(2^{-1}\right) + (|V(G)| - \deg(v) - 1)\left(2^{-2}\right) = (|V(G)| + \deg(v) - 1)/4$.

Since the value of $C(v)$ holds for every other vertex of G, combining the closeness values for $\forall v \in V(G)$ yields

$$C(v_1) + C(v_2) + \ldots + C\left(v_{|V(G)|}\right) = \sum_i C(v_i) = C(G)$$

$$C(G) = (1/4)\left(|V(G)|^2 + \deg(v_1) + \deg(v_2) + \ldots + \deg\left(v_{|V(G)|}\right) - |V(G)|\right).$$

Furthermore, the sum of the vertex degrees is twice the number of edges in the graph. Therefore,

$$C(G) = (1/4)\left(|V(G)|^2 + 2|E(G)| - |V(G)|\right) = (|V(G)|(|V(G)| - 1) + 2|E(G)|)/4.$$

The theorem is thus proved. □

By Theorem 4.1 , the following corollary is immediate.

Corollary 4.2. *If a vertex v has eccentricity two in G, then $C(v) = (|V(G)| + \deg(v) - 1)/4$.*

5. Closeness in graphs with specific types of vertices

Theorem 5.1. *Let G be a graph and $\{u, v\} \in V(G)$. If u is an endvertex of the support vertex v in G, then $C_v(u) = 0$.*

Proof. Removing the support vertex v from G means removing the pendant edge $e = uv$. Hence, u is an isolate in the remaining graph $G' = G \setminus \{v\}$. Therefore, $d(u, j) = \infty$ for $\forall j \in V(G')$ and $j \neq u$. Then, $C_v(u) = \sum\limits_{j \neq u} 2^{-d(u,j)} = 0$, and the desired result follows. □

Theorem 5.2 [5]**.** *If a vertex k does not belong to any unique geodesic (shortest path) of graph G, then $C(G \setminus k) = C(G) - 2C(k)$.*

Related to Theorem 5.2, the following corollary holds.

Corollary 5.3. *Let G be a graph. Then, for an endvertex u of G, $C_u(G) = C(G) - 2C(u)$.*

Proof. If an endvertex u of G is removed, then in the remaining graph $G' = G \setminus \{u\}$, the distance between each pair of the remaining vertices remains the same as it is in G. Therefore, for $\forall i$,

$$C_u(G) = \sum_{j \neq i} 2^{-d_u(i,j)} = \sum_{j \neq i} 2^{-d(i,j)} - \sum_{j \neq u} 2^{-d(u,j)} - \sum_{j \neq u} 2^{-d(j,u)} = C(G) - 2C(u).$$

This completes the proof. □

6. Graph operations, closeness and residual closeness

6.1. Graph complement

For any graph $G = (V, E)$, the complement $\bar{G} = \left(V, \bar{E}\right)$ is defined to be the graph on the same set of vertices $V(G)$ with $uv \in \bar{E}\left(\bar{G}\right)$ if and only if $uv \notin E(G)$, for all pairs $u \neq v \in V(G)$ [13].

Corollary 6.1. *For any graph G, if $diam(G) > 3$, then*

$$C\left(\bar{G}\right) = (|V(G)|(|V(G)| - 1) - |E(G)|)/2.$$

It is not hard to show that if a graph has large diameter, then its complement has small diameter:

Lemma 6.2 [9]**.** *If $diam(G) > 3$, then $diam\left(\bar{G}\right) \leq 2$.*

In order to prove Corollary 6.1, we use Lemma 6.2 above.

Proof. By the definition of graph complement, we have $\left|V\left(\bar{G}\right)\right| = |V(G)| = n$ and $\left|E\left(\bar{G}\right)\right| = |E(K_n)| - |E(G)|$, where K_n is the complete graph of order n. Since $|E(K_n)| = n(n-1)/2$, $\left|E\left(\bar{G}\right)\right| = |V(G)|(|V(G)| - 1)/2 - |E(G)|$. If $diam\left(\bar{G}\right) \leq 2$, then by substituting $|V(G)|$ and $|E(G)|$ in Theorem 4.1 with $\left|V\left(\bar{G}\right)\right|$ and $\left|E\left(\bar{G}\right)\right|$, we receive the desired result of $C\left(\bar{G}\right)$. □

6.2. Graph union

Let G_1 and G_2 be two disjoint graphs. The *union* of G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 \cup G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$ [13].

The following theorems give formulas for closeness and residual closeness of the union of disjoint graphs.

Theorem 6.3. *Let $G_1, G_2, \ldots, G_n$ be disjoint graphs. If $G = \bigcup_{i=1}^{n} G_i$, then $C(G) = \sum\limits_{i=1}^{n} C(G_i)$.*

Proof. The graph $G = G_1 \cup G_2 \cup \ldots \cup G_n$ is a disconnected graph with n components. The components of G are the graphs $G_1, G_2, \ldots, G_n$. Therefore $d(v, w) = \infty$ if $v \in G_i$ and $w \in G_j$, where $i \neq j$ and $v \neq w$. Hence, the result is obvious. □

Theorem 6.4. *Let G_1 and G_2 be two disjoint graphs. Then*

$$R(G_1 \cup G_2) = \min\{R(G_1) + C(G_2), R(G_2) + C(G_1)\}.$$

Proof. Denote the order of G_1 by n and denote the order of G_2 by m. Thus, $|V(G_1 \cup G_2)| = n + m$. Label the vertices of $V(G_1 \cup G_2)$ as $v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{n+m}$, such that $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_2) = \{v_{n+1}, \ldots, v_{n+m}\}$. Then the proof proceeds in the following two cases:

Case 1. If any vertex v_k $(1 \leq k \leq n)$ of G_1 is removed from $G_1 \cup G_2$, then the closeness of the remaining graph is

$$\begin{aligned} C_{v_k} &= \sum_{i \neq k}^{n+m} \sum_{j \neq i, j \neq k}^{n+m} 2^{-d_{v_k}(v_i, v_j)} \\ &= \sum_{i \neq k}^{n} \sum_{j \neq i, j \neq k}^{n} 2^{-d_{v_k}(v_i, v_j)} + \sum_{i=n+1}^{n+m} \sum_{j=n+1, j \neq i}^{n+m} 2^{-d_{v_k}(v_i, v_j)} \\ &= C_{v_k}(G_1) + C(G_2) \end{aligned}$$

Case 2. It is obvious that $G_1 \cup G_2 \cong G_2 \cup G_1$. Thus, if any vertex v_t $(n+1 \leq t \leq n+m)$ of G_2 is removed from $G_1 \cup G_2$, then $C_{v_k} = C_{v_t}(G_2) + C(G_1)$.

By Case 1, Case 2 and the definition of residual closeness, the following result can be derived easily.

For $\forall k, t$,

$$\begin{aligned} R(G_1 \cup G_2) &= \min\{C_{v_k}(G_1) + C(G_2), C_{v_t}(G_2) + C(G_1)\} \\ &= \min\left\{\min_k \{C_{v_k}(G_1)\} + C(G_2), \min_t \{C_{v_t}(G_2)\} + C(G_1)\right\} \\ &= \min\{R(G_1) + C(G_2), R(G_2) + C(G_1)\} \end{aligned}$$

This completes the proof. □

6.3. Graph join

Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 + G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{(u, v) : u \in V(G_1), v \in V(G_2)\}$ [13].

Finally, we give formulas for closeness and residual closeness of the join of two disjoint graphs.

Note that throughout Corollary 6.5 and Corollary 6.6, $|V(G_1)| = v_1$, $|V(G_2)| = v_2$, $|E(G_1)| = e_1$, $|E(G_2)| = e_2$.

Corollary 6.5. *Let G_1 and G_2 be two disjoint graphs, then*

$$C(G_1 + G_2) = (e_1 + e_2 + v_1 v_2)/2 + (v_1 + v_2)(v_1 + v_2 - 1)/4.$$

Proof. The demonstration of the validity of Corollary 6.5 is elementary. First, note that any two vertices of $G_1 + G_2$ are either adjacent or at distance two. The distance-two pairs are those corresponding to nonadjacent vertices in either G_1 or G_2. Therefore, obviously $diam(G_1 + G_2) \leq 2$. Since $|V(G_1 + G_2)| = v_1 + v_2$ and $|E(G_1 + G_2)| = e_1 + e_2 + v_1 v_2$, by substituting $|V(G_1 + G_2)|$ and $|E(G_1 + G_2)|$ with Theorem 4.1, we arrive at result. □

Corollary 6.6. *Let G_1 and G_2 be two disjoint graphs with $v_1 > 1$ and $v_2 > 1$, then*

$$R(G_1+G_2) = \begin{cases} C(G_1+G_2) - (v_2+\Delta(G_2)-1)/2 - v_1, & \text{if } \Delta(G_1)-\Delta(G_2) \le v_1 - v_2; \\ C(G_1+G_2) - (v_1+\Delta(G_1)-1)/2 - v_2, & \text{otherwise.} \end{cases}$$

Proof. We distinguish two cases:

Case 1. Removing a vertex v of G_1 from G_1+G_2:
Let the remaining subgraph be $G' = (G_1+G_2) \setminus \{v\}$. It is easily seen that G' is a graph which is obtained by the join of two disjoint graphs: $(G_1 \setminus \{v\}) + G_2$. Since $v_1 > 1$, we have $diam\,(G') \le 2$. Being $|V(G')| = (v_1-1)+v_2$ and $|E(G')| = \left(e_1 - \deg_{G_1}(v)\right) + e_2 + v_2(v_1-1)$, by substituting $|V(G')|$ and $|E(G')|$ with Theorem 4.1, we arrive the result that for $\forall v \in V(G_1)$,

$$C_v = \left((v_1+v_2)^2 + 2\left(v_1v_2 - v_1 - 2v_2 + e_1 + e_2 + 1 - \deg_{G_1}(v)\right) - v_1 - v_2\right)/4.$$

Case 2. Removing a vertex u of G_2 from G_1+G_2:
It is obvious that $G_1+G_2 \cong G_2+G_1$. Then, the remaining subgraph is $G' = G_1+(G_2 \setminus \{u\})$. Since, $v_2 > 1$, we have $diam\,(G') \le 2$. Being $|V(G')| = v_1+(v_2-1)$ and $|E(G')| = e_1+e_2-\deg_{G_2}(u)+v_1(v_2-1)$, by substituting $|V(G')|$ and $|E(G')|$ with Theorem 4.1, we arrive the result that for $\forall u \in V(G_2)$,

$$C_u = \left((v_1+v_2)^2 + 2\left(v_1v_2 - 2v_1 - v_2 + e_1 + e_2 + 1 - \deg_{G_2}(u)\right) - v_1 - v_2\right)/4.$$

By the definition of the vertex residual closeness (VRC) of a graph, we have $R(G_1+G_2) = \min\{C_v, C_u\}$, where $v \in V(G_1)$ and $u \in V(G_2)$.

Let us show how to deduce $\min\{C_v, C_u\}$:

$$\min\{C_v, C_u\} = \min\left\{\min_v \{C_v\}, \min_u \{C_u\}\right\} (\forall v, u)$$

$$\min_v \{C_v\} = \left((v_1+v_2)^2 + 2\left(v_1v_2 - v_1 - 2v_2 + e_1 + e_2 + 1 - \max_v \deg_{G_1}(v)\right) - v_1 - v_2\right)/4.$$

$$\min_u \{C_u\} = \left((v_1+v_2)^2 + 2\left(v_1v_2 - 2v_1 - v_2 + e_1 + e_2 + 1 - \max_u \deg_{G_2}(u)\right) - v_1 - v_2\right)/4.$$

We deduce that if $\Delta(G_1) - \Delta(G_2) \le |V(G_1)| - |V(G_2)|$, then $\min_v \{C_v\} \ge \min_u \{C_u\}$. Hence, we receive

$$R(G_1+G_2) = \begin{cases} \min_u \{C_u\}, & \text{if } \Delta(G_1)-\Delta(G_2) \le |V(G_1)| - |V(G_2)|; \\ \min_v \{C_v\}, & \text{otherwise.} \end{cases}$$

The theorem is thus proved. ∎

Corollary 6.7. *For any graph G,*

$$R(G+K_1) = \min\{C(G), C(G+K_1) - (|V(G)| + \Delta(G)+1)/2\}.$$

Proof. We distinguish two cases:

Case 1. Removing the vertex v of K_1 from $G+K_1$: the remaining subgraph is G. Therefore, obviously $C_v = C(G)$.

Case 2. Removing a vertex u of G from $G+K_1$: the survival subgraph is $G' = (G \setminus \{u\}) + K_1$. Since $diam$ $(G') \le 2$, being $|V(G')| = (|V(G)|-1)+1$ and $|E(G')| = |E(G)| - \deg_G(u) + |V(G)| - 1$, by substituting $|V(G')|$ and $|E(G')|$ with Theorem 4.1, we arrive the result below for $\forall u \in V(G)$.

$$C_u = (|V(G)|(|V(G)|+1) + 2|E(G)| - 2\deg_G(u) - 2)/4.$$

By the definition of the vertex residual closeness (VRC) of a graph, we have $R(G+K_1)=\min\{C_v,C_u\}$, where $v\in V(K_1)$ and $u\in V(G_2)$.

Let us show how to deduce $\min\{C_v,C_u\}$:

$$\min\{C_v,C_u\}=\min\left\{C(G),\min_u C_u\right\}(\forall u)$$

$$\min_u C_u=\Big(|V(G)|(|V(G)|+1)+2|E(G)|-2\max_u\{\deg_G(u)\}-2\Big)/4.$$

By writing the value of $\min\limits_u C_u$ in terms of the closeness of G and K_1, finally we conclude that

$$R(G+K_1)=\min\{C(G),C(G+K_1)-(|V(G)|+\Delta(G)+1)/2\}.$$

This completes the proof. □

7. Conclusion

Operations research handles designing complex and interdependent systems. Complex network is widely used to model the structure of many complex systems in nature and society [19, 20]. Network models are used to analyze social and practical problems [14], such as information, communication and computer networks, facility location problems [17], supply chain management [26], transportation networks [25], power grid networks [15,18,27,28], water distribution networks [29], network optimization [6,30,31], game theory all in operations research [33,34]. Military, government, commercial, and civilian operations depend on the security and availability of computer systems and networks [3]. Networks are known to be prone to node failures. A central issue in the analysis of networks is the assessment of their stability and reliability. The main aim is to understand, predict, and possibly even control the behavior of a networked system under attacks or disfunctions of any type. A central concept that is used to assess stability and robustness of the performance of a network under failures is that of vulnerability [21]. Different approaches to characterize network vulnerability been proposed. Vulnerability measures for graphs are essential to guide the designer in choosing an appropriate topology. Residual closeness is a new graph vulnerability measure, a graph-based approach to network vulnerability analysis, and more sensitive than some other existing vulnerability measures [4]. In this paper, sharp upper and lower bounds for closeness, closeness values for complex networks with small diameter , and for specific types of vertices in networks are investigated. Those obtained bounds and values are of practical importance for measuring the residual closeness of a handling complex network. Also, the vulnerability of complex networks gained by graph operations are evaluated via residual closeness, and the calculated closeness values of related networks are used for evaluation. Calculation of closeness and residual closeness for simple graph types is important because if one can break a more complex network into smaller networks, then under some conditions the solutions for the optimization problem on the smaller networks can be combined to a solution for the optimization problem on the larger network and by calculating the residual closeness for some real networks very good practical results can be achieved.

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