RAIRO-Oper. Res. 51 (2017) 285–297
DOI: 10.1051/ro/2016022 **RAIRO Operations Research**
www.rairo-ro.org

A NEW APPROACH FOR SOLVING FULLY FUZZY LINEAR FRACTIONAL PROGRAMMING PROBLEMS USING THE MULTI-OBJECTIVE LINEAR PROGRAMMING

Sapan Kumar Das[1], Tarni Mandal[1] and S.A. Edalatpanah[2]

Abstract. This paper deals with developing an efficient algorithm for solving the fully fuzzy linear fractional programming problem. To this end, we construct a new method which is obtained from combination of Charnes−Cooper scheme and the multi-objective linear programming problem. Furthermore, the application of the proposed method in real life problems is presented and this method is compared with some existing methods. The numerical experiments and comparative results presented promising results to find the fuzzy optimal solution.

Mathematics Subject Classification. 90C70, 90C90, 90C05, 90C08.

Received July 20, 2015. Accepted February 27, 2016.

1. Introduction

Consider the following linear fractional programming:

$$\text{Max}\,\frac{c^t x + q}{d^t x + r} = \frac{F(x)}{G(x)} \tag{1.1}$$

s.t.

$$Ax \begin{pmatrix} \leqslant \\ = \\ \geqslant \end{pmatrix} b,$$

$$x \geqslant 0.$$

Where x is the decision variables, A is the coefficients of the technological matrices, $c^t,\ d^t$ are the coefficients associated with the objective function. $q,\ r \in R$. For some values of x, $G(x)$ may be equal to zero. To avoid such cases, one requires that either

$$\left\{ x \geqslant 0,\ Ax \begin{pmatrix} \leqslant \\ = \\ \geqslant \end{pmatrix} b,\ \Rightarrow\ G(x)\ > 0 \right\} \text{ or } \left\{ x \geqslant 0,\ Ax \begin{pmatrix} \leqslant \\ = \\ \geqslant \end{pmatrix} b,\ \Rightarrow\ G(x)\ < 0 \right\}.$$

Keywords. Fully fuzzy linear programming, linear fractional programming, linear programming, multi-objective linear programming; triangular fuzzy number.

[1] Department of Mathematics, National Institute of Technology Jamshedpur, 831014 Jamshedpur India.
cool.sapankumar@gmail.com

[2] Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran.

Article published by EDP Sciences

Nowadays, the problem of linear fractional programming has significant application in different real life areas such as production planning, financial sector, health care and all engineering fields. However, in real world applications, certainty, reliability and precision of data is often illusory. The optimal solution of an LP only depends on a limited number of constraints; therefore much of the collected information has little impact on the solution. It is useful to consider the knowledge of experts about the parameters as fuzzy data.

The fuzzy linear fractional programming problems in which all the parameters and variables are represented by fuzzy numbers are known as fully fuzzy linear fractional programming (FFLFP) problems. In the actual cases the parameters may be uncertain or a vague estimation about the variables is known as those are found in general by some experiment. To overcome the uncertainty and vagueness, one may use the fuzzy numbers in the place of the crisp numbers. Thus the crisp system of linear fractional programming problem becomes a fuzzy system of FLFP problem or FFLFP problem. In the FFLFP problem all the parameters and variables are considered to be fuzzy numbers. It is an important issue to develop mathematical models and numerical techniques which apply in real life circumstances that would appropriately treat the general fuzzy or fully fuzzy linear fractional programming because subtraction and division of fuzzy numbers are not the inverse operations to addition and multiplication respectively. Nowadays, the problem of FLFP problems has significant application in different real life areas such as production planning, financial sector, health care and all engineering fields. For this reason, this is an important area of research in the recent years. In this paper, we consider the FFLFP problem. In recent years, many methods currently exist for solving FFLFP problem under nonnegative fuzzy variables.

The concept of fuzzy set and fuzzy numbers was first introduced [13]. In [7], the authors proposed a method to solve multi-objective linear fractional programming (MOLFP) problem under a fuzzy satisfied. A general concept of Pareto optimal solution and use two types of fuzzy goals (called fuzzy equal and fuzzy min) was introduced in [8]. In [13] the concept of linguistic variation was introduced. The FFLP problem by establishing all the coefficients and variables of a linear program as being fuzzy quantities was introduced in [1].

Recently Pop and Minasian [6], proposed a method for solving fully falsified linear fractional programming problems where all the parameters and variables are triangular fuzzy numbers. In [10, 11], they considered the same problem of [6] for solving fully fuzzy linear fractional programming problem.

In this paper, we modified the methods of [6, 11]. First we transform the FFLFP problem into a FFLP problem with the help of Charnes−Cooper method [3]. Then using a new technique, the FFLP problem will be converted into a multi-objective linear programming (MOLP) problem. We also prove that this solution can be considered as an exact solution of FFLFP problem. Finally, we show that advantages of the proposed method over the existing method [6, 11].

This paper is organized as follows: some basic definitions and notations are presented in Section 2. In Section 3, the general form of linear fractional programming (LFP) problem is presented. In Section 4, the general form of FFLFP problem is presented and a new method is proposed for solving FFLFP problems. Advantages of the proposed method over the existing method with the help of a simple example are provided in Section 5. To show the application of the proposed method a real life problem is solved in Section 6. In Section 7, we discuss the importance of our results. Finally, conclusion has been drawn in the last section.

2. Preliminaries

In this section, we have presented some basic concept of fuzzy triangular number, which was very useful in this paper.

Definition 2.1 ([10])**.** Let X denotes a universal set. Then a fuzzy subset $\widetilde{A}$ of X is defined by its membership function $\mu_{\tilde{A}} : X \to [0,1]$; which assigned a real number $\mu_{\tilde{A}}(X)$ in the interval [0, 1], to each element $x \in X$, where the values of $\mu_{\tilde{A}}(X)x$ shows the grade of membership of x in $\tilde{A}$. A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\tilde{A}}(X)$ and is often written $\tilde{A} = (x, \mu_{\tilde{A}}(x)) : x \in X$ is called a fuzzy set.

Definition 2.2 ([10])**.** A fuzzy number $\tilde{A} = (b, c, a)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(X) = \begin{cases} \dfrac{(x-b)}{(c-b)}, & b \leqslant x \leqslant c, \\ \dfrac{(a-x)}{(a-c)}, & c \leqslant x \leqslant a, \\ 0, & \text{else.} \end{cases}$$

Definition 2.3. A triangular fuzzy number $(b,\ c,\ a)$ is said to be non-negative (non-positive) triangular fuzzy number if and only if $b \geqslant 0\,(a \leqslant 0)$.

Definition 2.4 ([10])**.** Two triangular fuzzy number $\tilde{A} = (b, c, a)$ and $\tilde{B} = (e, f, d)$ are said to be equal if and only if $b = e,\ c = f,\ \ a = d$.

Definition 2.5 ([6])**.** A ranking is a function $R : F(R) \to R$ where $F(R)$ is a set of fuzzy number defined on set of real numbers, which maps each fuzzy number into the real line, where a natural order exists. Let $\tilde{A} = (b, c, a)$ is a triangular fuzzy number then $\Re(\tilde{A}) = \frac{b+2c+a}{4}$.

Definition 2.6 ([6])**.** Let$\tilde{A} = (b, c, a)$, $\tilde{B} = (e, f, d)$ be two triangular fuzzy numbers then the fuzzy arithmetic is defined as follows:

(i) $\tilde{A} + \tilde{B} = (b, c, a) + (e, f, d) = (b + e, c + f, a + d)$;
(ii) $-\tilde{A} = (-a,\ -c,\ -b)$;
(iii) $\tilde{A} - \tilde{B} = (b, c, a) - (e, f, d) = (b - d, c - f, a - e)$;
(iv) Let $\tilde{A} = (b, c, a)$ be any triangular fuzzy number and $\tilde{B} = (e, f, d)$ be a non-negative triangular fuzzy number then one may have,

$$\tilde{A} \otimes \tilde{B} = \tilde{A}\tilde{B} = \begin{cases} (be, cf, ad) & \text{if} \quad b \geqslant 0, \\ (bd, cf, ad) & \text{if} \quad b < 0, a \geqslant 0, \\ (bd, cf, cd) & \text{if} \quad c < 0, \end{cases}$$

Definition 2.7. Let $\tilde{A} = (b, c, a)$, $\tilde{B} = (e, f, d)$ be two triangular fuzzy numbers. We say that $\tilde{A}$ is relatively less than $\tilde{B}$, if and only if:

(i) $b <\ e$ or
(ii) $b = e$ and $(b - c) > (e - f)$ or
(iii) $b\ = e,\ \ (b - c) = (e - f)$ and $(a + b) = (d + e)$.

Note. It is clear from Definition 2.7 that $b = e,\ (b - c) = (e - f)$ and $(a + b) = (d + e)$ if and only if $\tilde{A} = \tilde{B}$.

3. Linear fractional programming problem and its fully fuzzy version

In this section the general form of linear fractional programming problem, converting linear fractional programming into the linear programming problem and the general form of multi-objective linear fractional programming problem is discussed. For satisfaction, we assume that (1.1) satisfies the condition that:

$$\left\{ x \geqslant 0,\ Ax \begin{pmatrix} \leqslant \\ = \\ \geqslant \end{pmatrix} b, \Rightarrow G(x)\ > 0 \right\}. \tag{3.1}$$

Now, by the following theorem we show that the linear fractional programming problem can be transform into a LP problem.

Theorem 3.1. *Assume that (y, 0) with $y \geqslant 0$ is feasible for the following LP problem:*

$$\begin{aligned} &\text{Max } c^t y + q\alpha \\ &\text{s.t} \quad d^t y + r\alpha = 1, \\ &Ay - b\alpha = 0, \\ &\alpha \geqslant 0,\ y \geqslant 0. \end{aligned} \tag{3.2}$$

Then following the condition (3.1)*, the LFP* (3.1) *is equivalent into LP* (3.2)*.*

Proof. Let x be the feasible solution of LFP (3.1). Define $q(x) = (y, \alpha)$, with $\alpha = (d^t x + r)^{-1}$ and $y = \alpha x$. Then $y \geqslant 0,\ \alpha > 0$, $Ay - b\alpha = \alpha(Ax - b) = 0,\ d^t y + r\alpha = \alpha(d^t x + r) = 1$, Thus (y, α), is feasible for LP (3.2).

Conversely, if (y, α), is feasible for LP (3.2) and $(y, 0)$ is feasible for LP (3.2), then $\alpha > 0$, and $x = y/t$ satisfies $x >= 0$, $Ax - b = (Ay - b\alpha)/\alpha = 0$. Therefore, the feasible set of LFP problem is one-one onto the feasible set for LP (3.2).

Moreover, the objective functions are related by

$$(c^t x + q)/\,(d^t x + r) = (c^t y + q\alpha)/\,(d^t y + r\alpha) = \ (c^t y + q\alpha)/1.$$

Thus, the theorem is proved. □

Next, we will model the fully fuzzy linear fractional programming problem where all the variables and all the parameters are triangular fuzzy numbers. Let us consider a general format of fully fuzzy linear fractional programming problem as follows:

$$\begin{aligned} &\text{Max } \tilde{Z} = \frac{\tilde{c}^t \tilde{x} + \tilde{q}}{\tilde{d}^t \tilde{x} + \tilde{r}} \\ &\text{s.t.} \quad \tilde{A}\tilde{x} \leqslant \tilde{b} \\ &\tilde{x} \geqslant 0. \end{aligned} \tag{3.3}$$

First, we transform problem (3.3) in to a fully fuzzified LP problem using Charnes−Cooper transformation [3], and obtain the following problem.

$$\begin{aligned} &\text{Max } \tilde{c}^t \tilde{y} + \tilde{q}\tilde{t} \\ &\text{subject to } \tilde{A}\tilde{y} - \tilde{b}\tilde{t} \leqslant \tilde{0}, \\ &\tilde{d}\tilde{y} + \tilde{r}\tilde{t} \leqslant \tilde{1}, \\ &\tilde{y}, \tilde{t} \geqslant 0. \end{aligned} \tag{3.4}$$

Let us assume that all the parameters $\tilde{x}$, $\tilde{c}$, $\tilde{q}$, $\tilde{d}$, $\tilde{r}$, $\tilde{b}$ and $\tilde{z}$ are represented by triangular fuzzy numbers $((x)^p,(x)^q,(x)^r)$, $((c^t)^p,(c^t)^q,(c^t)^r)$, $((q)^p,\ (q)^q,\ (q)^r)$, $((d^t)^p,(d^t)^q,(d^t)^r)$, $((b)^p,\ (b)^q,\ (b)^r)$, $((r)^p,(r)^q,(r)^r)$ and $((a)^p,(a)^q,(a)^r)$, respectively. After computing the fuzzy quantities with respect to Definitions 2.2 and 2.3 we modify the maximization of the objective function and constraints function described by a triangular fuzzy number with the maximization of three values of the fuzzy number. Then we can rewrite the mentioned FFLP as follows:

$$\begin{aligned} &\text{Max } (z_1,z_2,z_3)=((c^ty)^p,\ (c^ty)^q,\ (c^ty)^r)+((q)^p,\ (q)^q,\ (q)^r) \\ &\text{s.t. } ((a)^p,\ (a)^q,\ (a)^r)\otimes((y)^p,(y)^q,(y)^r)-((bt)^p,(bt)^q,(bt)^r)\leqslant\tilde{0}, \\ &((d^ty)^p,\ (d^ty)^q,\ (d^ty)^r+(rt)^p,\ (rt)^q,\ (rt)^r)\leqslant\tilde{1}, \\ &((y)^p,(y)^q,(y)^r)\geqslant 0.\quad ((t)^p,(t)^q,(t)^r)\geqslant 0. \end{aligned} \tag{3.5}$$

Remark 3.2. If $\tilde{y}^*=((y^*)^p,\ (y^*)^q,\ (y^*)^r)$ is said to be an exact optimal solution of problem (3.4) then it will satisfies in the following conditions:

(i) $\tilde{A}\tilde{y}^*=\tilde{b}$;
(ii) $\forall\tilde{y}=((y)^p,(y)^q,(y)^r)$, we have that, $\tilde{c}^t\tilde{y}+\tilde{q}\tilde{t}\leqslant\tilde{c}^t\tilde{y}^*+\tilde{q}\tilde{t}$ (in the case of minimization problem $\tilde{c}^t\tilde{y}+\tilde{q}\tilde{t}\geqslant\tilde{c}^t\tilde{y}^*+\tilde{q}\tilde{t}$).

Remark 3.3. Let $\tilde{y}^*$ be an exact optimal solution of the problem (3.4) then y$'$ is called the alternative exact optimal solution of the problem (3.4) if there exists $\tilde{c}^t\tilde{y}^{'}+\tilde{q}\tilde{t}=\tilde{c}^t\tilde{y}^*+\tilde{q}\tilde{t},\ \exists\,\tilde{y}^{'}\in\tilde{s}$.

The proposed approach for solving FFLFPP can be summarized as follows:

Step 1. With respect to Definitions 2.2 and 2.3 the problem (3.5), can be rewritten as:

$$\text{Max } ((c^ty+qt)^p,\ (c^ty+qt)^q,\ (c^ty+qt)^r)$$

s.t.

$$\begin{aligned} &((ay)^p,\ (ay)^q,\ (ay)^r)-((bt)^p,(bt)^q,(bt)^r)\leqslant\tilde{0}, \\ &((d^ty+rt)^p,\ (d^ty+rt)^q,\ (d^ty+rt)^r)\leqslant\tilde{1}, \\ &(y)^p,\ (y)^q,\ (y)^r\geqslant 0.\quad ((t)^p,(t)^q,(t)^r)\geqslant 0. \end{aligned} \tag{3.6}$$

Step 2. Equivalently, with regard to Definition 2.4 the problem (3.6) may be written as follows:

$$\text{Max}\,((c^ty+qt)^p,\ (c^ty+qt)^q,\ (c^ty+qt)^r)$$

s.t.

$$\begin{aligned} &(ay)^p-(bt)^r\leqslant 0, \\ &(ay)^q-(bt)^q\leqslant 0, \\ &(ay)^r-(bt)^p\leqslant 0, \\ &(d^ty+r)^p+t^p\leqslant 1, \\ &(d^ty+r)^q+t^q\leqslant 1, \\ &(d^ty+r)^r+t^r\leqslant 1, \end{aligned}$$

$$(y)^q-(y)^p\geqslant 0,(y)^r-(y)^q\geqslant 0,(y)^p\geqslant 0,(t)^q-(t)^p\geqslant 0,(t)^r-(t)^q\geqslant 0,(t)^p\geqslant 0. \tag{3.7}$$

Step 3. Regarding Definition 2.6 the problem (3.7) is converted to the MOLP problem with three crisp objective functions and the constraints are changed as follows:

$$\mathrm{Max}\,(c^t y + qt)^p$$

$$\mathrm{Max}\,(c^t y + qt)^p - \ (c^t y + qt)^q$$

$$\mathrm{Max}\,(c^t y + qt)^p + (c^t y + qt)^r$$

s.t

$$(ay)^p - (bt)^r \leqslant 0.$$

$$(ay)^p - (bt)^r - ((ay)^q - (bt)^q) \leqslant 0,$$

$$(ay)^p - (bt)^r + ((ay)^r - (bt)^p) \leqslant 0,$$

$$(d^t y + r)^p + t^p \leqslant 1,$$

$$(d^t y + r)^p + t^p - (d^t y + r)^q - t^q \leqslant 0,$$

$$(d^t y + r)^p + t^p + (d^t y + r)^r + t^r \leqslant 2,$$

$$(y)^q - (y)^p \geqslant 0, (y)^r - (y)^q \geqslant 0, (y)^p \geqslant 0, (t)^q - (t)^p \geqslant 0, (t)^r - (t)^q \geqslant 0, (t)^p \geqslant 0. \tag{3.8}$$

Step 4. In objective functions the lexicographic method will be used to obtain the optimal solution of the problem (3.8). So, we get:

$$\mathrm{Max}\,(c^t y + qt)^p$$

s.t.

$$(Ay)^p - (bt)^r \leqslant 0,$$

$$(Ay)^p - (bt)^r - ((Ay)^q - (bt)^q) \leqslant 0,$$

$$(Ay)^p - (bt)^r + ((Ay)^r - (bt)^p) \leqslant 0,$$

$$(d^t y + r)^p + t^p \leqslant 1,$$

$$(d^t y + r)^p + t^p - (d^t y + r)^q - t^q \leqslant 0,$$

$$(d^t y + r)^p + t^p + (d^t y + r)^r + t^r \leqslant 2,$$

$$(y)^q - (y)^p \geqslant 0, (y)^r - (y)^q \geqslant 0, (y)^p \geqslant 0, (t)^q - (t)^p \geqslant 0, (t)^r - (t)^q \geqslant 0, \ \ (t)^p \geqslant 0. \tag{3.9}$$

If we get a unique optimal solution then it is an optimal solution of the problem (3.5) and stop. Otherwise go to next step.

Step 5. Solve the following problem over the optimal solutions that are achieved in Step 4 as follows:

$$\mathrm{Max}\,(c^t y + qt)^p - \ (c^t y + qt)^q$$

s.t.

$$(c^t y + qt)^p = l^*$$

$$(Ay)^p - (bt)^r \leqslant 0,$$

$$(Ay)^p - (bt)^r - ((Ay)^q - (bt)^q) \leqslant 0,$$

$$(Ay)^p - (bt)^r + ((Ay)^r - (bt)^p) \leqslant 0,$$

$$(d^t y + r)^p + t^p \leqslant 1,$$

$$(d^ty+r)^p+t^p-(d^ty+r)^q-t^q\leqslant 0,$$
$$(d^ty+r)^p+t^p+(d^ty+r)^r+t^r\leqslant 2,$$

$$(y)^q-(y)^p\geqslant 0,(y)^r-(y)^q\geqslant 0,(y)^p\geqslant 0,(t)^q-(t)^p\geqslant 0,(t)^r-(t)^q\geqslant 0,(t)^p\geqslant 0, \tag{3.10}$$

where l^* is the optimal value of problem (3.9). If the problem (3.9) has a unique optimal solution, then it is an optimal solution of problem (3.5) and stop. Otherwise go to next step.

Step 6. Solve the following problem over the optimal solutions that are achieved in Step 5 as follows:

$$\text{Max } (c^ty+qt)^p+(c^ty+qt)^r$$

s.t.

$$(c^ty+qt)^p-\ (c^ty+qt)^q=k^*$$
$$(c^ty+qt)^p=l^*$$
$$(Ay)^p-(bt)^r\leqslant 0,$$
$$(Ay)^p-(bt)^r-((Ay)^q-(bt)^q)\leqslant 0,$$
$$(Ay)^p-(bt)^r+((Ay)^r-(bt)^p)\leqslant 0,$$
$$(d^ty+r)^p+t^p\leqslant 1,$$
$$(d^ty+r)^p+t^p-(d^ty+r)^q-t^q\leqslant 0,$$
$$(d^ty+r)^p+t^p+(d^ty+r)^r+t^r\leqslant 2,$$

$$(y)^q-(y)^p\geqslant 0,(y)^r-(y)^q\geqslant 0,(y)^p\geqslant 0,(t)^q-(t)^p\geqslant 0,(t)^r-(t)^q\geqslant 0,(t)^p\geqslant 0, \tag{3.11}$$

where k^* is the optimal value of problem (3.10). Hence, the optimal solution of the problem (3.5) is obtained by solving problems (3.11).

Now by theorem we clarify the lexicographic optimal solution of the problem (3.9) can be considered as an exact optimal solution of the problem (3.5).

Theorem 3.4. *If $\tilde{y}^*=(y^*)^p,\ (y^*)^q,\ (y^*)^r$ be an optimal solution of problems (3.9)–(3.11), then it is also an exact optimal solution of the problem (3.10).*

Proof. By the method of contradiction, let $\tilde{y}^*=(y^*)^p,\ (y^*)^q,\ (y^*)^r$ be an optimal solution of (3.9)–(3.11), but it is not the exact optimal solution of the problem (3.5). Let us we consider$\tilde{y}^0=(y^0)^p,\ (y^0)^q,\ (y^0)^r$, such that in the case of maximization:

$$(c^ty^*)^p,\ (c^ty^*)^q,\ (c^ty^*)^r\prec(c^ty^0)^p,\ (c^ty^0)^q,\ (c^ty^0)^r. \qquad \square$$

Based on Definition 2.7, we have three conditions as follows:

Case (i). In case of maximization, we consider $(c^ty^*)^p\prec\ (c^ty^0)^p$. Also, with respect to the assumption we have:

$$(Ay^0)^p=(\tilde{b}_1),$$
$$(Ay^0)^p-(Ay^0)^q=(\tilde{b}_1)-(\tilde{b}_2),$$
$$(Ay^0)^p+(Ay^0)^r=(\tilde{b}_1)+(\tilde{b}_3),$$
$$(y^0)^p-(y^0)^q\geqslant 0,\ (y^0)^r-(y^0)^q\geqslant 0,\ (y^0)^p\geqslant 0.$$

Therefore, $(y^0)^p,\ (y^0)^q,\ (y^0)^r$ is a feasible solution of problem (3.9) in which the objective value in $(y^0)^p,\ \ (y^0)^q,\ \ \ (y^0)^r$ is greater than the objective value in $(y^*)^p,\ \ (y^*)^q,\ \ (y^*)^r$. But it is contradiction.

Case (ii). In case of maximization, let us we consider $(c^t y^*)^p = (c^t y^0)^p$, and $(c^t y^*)^p - (c^t y^*)^q \succ (c^t y^0)^p - (c^t y^0)^q$ also, with respect to the assumption we have:

$$(Ay^0)^p = (\tilde{b}_1),$$

$$(Ay^0)^p - (Ay^0)^q = (\tilde{b}_1) - (\tilde{b}_2),$$
$$(Ay^0)^p + (Ay^0)^r = (\tilde{b}_1) + (\tilde{b}_3),$$
$$(y^0)^p - (y^0)^q \geqslant 0,\ (y^0)^r - (y^0)^q \geqslant 0,\ (y^0)^p \geqslant 0.$$

Therefore, $(y^0)^p$, $(y^0)^q$, $(y^0)^r$ is a feasible solution of problem (3.5) in which the objective value in $(y^0)^p$, $(y^0)^q$, $(y^0)^r$ is greater than the objective value in $(y^*)^p$, $(y^*)^q$, $(y^*)^r$. But it is contradiction.

Case (iii). In case of maximization, let us we consider $(c^t y^*)^p = (c^t y^0)^p$, $(c^t y^*)^p - (c^t y^*)^q = (c^t y^0)^p - (c^t y^0)^q$ and $(c^t y^*)^p + (c^t y^*)^r \prec (c^t y^0)^p + (c^t y^0)^r$. Also, with respect to the assumption we have:

$$(Ay^0)^p = (\tilde{b}_1),$$

$$(Ay^0)^p - (Ay^0)^q = (\tilde{b}_1) - (\tilde{b}_2),$$
$$(Ay^0)^p + (Ay^0)^r = (\tilde{b}_1) + (\tilde{b}_3),$$
$$(y^0)^p - (y^0)^q \geqslant 0,\ (y^0)^r - (y^0)^q \geqslant 0,\ (y^0)^p \geqslant 0.$$

Therefore, $(y^0)^p$, $(y^0)^q$, $(y^0)^r$ is a feasible solution of problem (3.11) in which the objective value in $(y^0)^p$, $(y^0)^q$, $(y^0)^r$ is greater than the objective value in $(y^*)^p$, $(y^*)^q$, $(y^*)^r$. But it is contradiction.

Therefore $\tilde{y}^* = (y^*)^p$, $(y^*)^q$, $(y^*)^r$ is an exact optimal solution of problem (3.5).

4. Numerical example

In order to illustrate the proposed method to solve fully fuzz linear fractional programs, consider the following example:

Example 4.1 ([6])**.** Let us consider the following linear fractional program as follows:

$$\text{Max}\, z = \frac{x_1 - x_2 + 1}{x_1 + x_2 + 2}$$

$$\text{s.t.}\quad x_1 + x_2 \leqslant 2,$$

$$x_1 - x_2 \leqslant 1,$$

$$x_1,\, x_2 \geqslant 0.$$

The optimal solution of the problem is $x_1 = 1$, $x_2 = 0$ and the optimal value of z is $2/3 = 0.66667$.

We attach now to this problem a fully fuzzified problem. Thus the fully fuzzy linear fractional programming problem which we want to solve is as follows:

$$\text{Max}\, z = \frac{(0,1,2)\tilde{x}_1 - (-2,-1,0)_2\tilde{x}_2 + (0,1,2)}{(0,1,2)\tilde{x}_1 + (0,1,2)\tilde{x}_2 + (1,2,3)} \tag{4.1}$$

s.t.

$$(1,2,3)\tilde{x}_1 + (1,2,3)\tilde{x}_2 \leqslant (1,2,3),$$

$$(0,1,2)\tilde{x}_1 - (-2,-1,0)\tilde{x}_2 \leqslant (0,1,2),$$

$$\tilde{x}_1, \tilde{x}_2 \geqslant 0.$$

Now the problem is as follows:

$$\text{Max } \frac{(0,1,2)\otimes((x_1)^p,(x_1)^q,(x_1)^r)-(-2,-1,0)\otimes((x_2)^p,(x_2)^q,(x_2)^r)+(0,1,2)}{(0,1,2)\otimes((x_1)^p,(x_1)^q,(x_1)^r)+(0,1,2)\otimes((x_2)^p,(x_2)^q,(x_2)^r)+(1,2,3)}$$

s.t.

$$\begin{gathered}(0,1,2)\otimes((x_1)^p,(x_1)^q,(x_1)^r)+(0,1,2)\otimes((x_2)^p,(x_2)^q,(x_2)^r)\leqslant(1,2,3),\\ (0,1,2)\otimes((x_1)^p,(x_1)^q,(x_1)^r)-(-2,-1,0)\otimes((x_2)^p,(x_2)^q,(x_2)^r)\leqslant(0,1,2),\\ (x_1)^p,(x_1)^q,(x_1)^r,(x_2)^p,(x_2)^q,(x_2)^r\geqslant 0.\end{gathered}$$

According to Steps 2 and 3, we get the following multiple objective linear programming problem:

$$\text{Max } (-2(y_2)^p,\ (y_1)^q-(y_2)^q+t_2,\ 2(y_1)^r+2t_3)$$

s.t.

$$\begin{aligned} -3t_3&\leqslant 0,\\ (y_1)^q+(y_2)^q-2t_2&\leqslant 0,\\ 2(y_1)^r+2(y_2)^r-t_1&\leqslant 0,\\ -2(y_2)^r-2t_3&\leqslant 0,\\ (y_1)^q-(y_2)^q-t_2&\leqslant 0,\\ 2(y_1)^r&\leqslant 0,\\ t_1&\leqslant 1,\\ (y_1)^q+(y_2)^q+2t_2&\leqslant 1,\\ 2(y_1)^r+2(y_2)^r+3t_3&\leqslant 1,\\ (y)^q-(y)^p\geqslant 0,\quad (y)^r-(y)^q\geqslant 0,\quad (y)^p&\geqslant 0,\quad t_1,t_2,t_3\geqslant 0.\end{aligned}$$

Using Steps 3, 4, 5 and Step 6, the optimal solution of the problem is:

$$\begin{aligned}&\tilde{y}^*=\{\tilde{y}_1^*=((y_1^*)^p,(y_1^*)^q,(y_1^*)^r)=(0,\ 0.2,\ 0.2),\\ &\tilde{y}^*=\{\tilde{y}_2^*=((y_2^*)^p,(y_2^*)^q,(y_2^*)^r)=(0,\ 0,\ 0),\\ &t^*=\{(t_1,\ t_2,t_3)=(0.1,\ 0.2,\ 0.2).\end{aligned}$$

Then the solution of the problem is:

$$\tilde{x}^*=\{\tilde{x}_1^*=((x_1^*)^p,(x_1^*)^q,(x_1^*)^r)=(0,\ 1,\ 1),$$

$$\tilde{x}^*=\{\tilde{x}_2^*=((x_2^*)^p,(x_2^*)^q,(x_2^*)^r)=(0,\ 0,\ 0).$$

The triangular fuzzy number $\tilde{z}^*=(0,\ 0.66667,\ 4)$

The obtained result is exactly the optimal value of the problem which start with the original problem.

By comparing proposed method results with existing method [6, 11], based on Definition 2.7, we conclude that our result is more efficient than other existing method.

$$0=(z_1^*)_{\text{proposed method}}=(z_1^*)_{\text{method of [3]}}>(z_1^*)_{\text{method of [2]}}=-0.21$$

$$0.66667=(z_1^*+z_2^*)_{\text{proposed method}}=(z_1^*+z_2^*)_{\text{method of [2]}}=0.66667>(z_1^*+z_2^*)_{\text{method of [3]}}=0.55$$

$$(0,0.66667,4)=(z^*)_{\text{proposed method}}>(z^*)_{\text{method of [2]}}=(-0.21,0.66667,5.822)>(z_1^*)_{\text{method of [3]}}=(0,0.55,1.09).$$

5. Application of the proposed method in real life problems

In the existing method [6,11] it is difficult to solve for the fully fuzzy transportation problem in real life case. In the existing methods [6, 11], problem reduced to the deterministic MOLFP problem subject to conjunctive system of disjunctive non-linear constraints. So, it is very difficult to solve a transportation problem. So, we applied in our proposed method for solving fully fuzzy transportation problem.

In this section, to show the application of proposed method the real life problem is solved by the proposed method and it is concluded that the proposed method applied any real life problems.

Example 5.1 (Production planning in Taiwan)**.**

Dali Company is the leading producer of soft drinks and low-temperature foods in Taiwan. Currently, Dali plans to develop the South-East Asian market and broaden the visibility of Dali products in the Chinese market. Notably, following the entry of Taiwan to the World Trade Organization, Dali plans to seek strategic alliance with prominent international companies and introduced international bread to lighten the embedded future impact. In the domestic soft drinks market, Dali produces tea beverages to meet demand from four distribution centers in Taichung, Chiayi, Kaohsiung, and Taipei, with production being based at three plants in Changhua, Touliu, and Hsinchu. According to the preliminary environmental information, Table 1 summarizes the potential supply available from the given three plants. The forecast demand from the four distribution centers as is shown Table 2. The profit of the company gained by each route is presented in Table 3. Table 4 summarizes the unit shipping cost for each route for the upcoming season. The environmental coefficient and related parameters generally are imprecise numbers with triangular possibility distributions over the planning horizon due to incomplete or unobtainable information. For example, the unavailable supply of the Changhua plant is (7.2, 8, 8.8) thousand dozen bottles, the forecast demand of the Taichung distribution center is (6.2, 7, 7.8) thousand dozen bottles, profit per dozen bottles for Changhua to Taichung is (8, 10, 10.8) dollars and the transportation cost per dozen bottles for Changhua to Taichung is (8, 10, 10.8)dollars. The management of Dali is initiating a study to maximize the profit as much as possible.

Table 1. Supply of the plants.

Source	Changhua	Touliu	Hsinchu
Supply (Thousand dozen bottles)	(7.2, 8, 8.8)	(12, 14, 16)	(10.2, 12, 13.8)

Table 2. Demand of the destinations.

Destination	Taichung	Chiayi	Kaohsiung	Taipei
Demand (thousand dozen bottles)	(6.2, 7, 7.8)	8.9, 10, 11.1)	(6.5, 8, 9.5)	(7.8, 9, 10.2)

Table 3. Profit of the company.

Source	Destination			
	Taichung	Chiayi	Kaohsiung	Taipei
Changhua	(8, 10, 10.8)	(20.4, 22, 24)	(8, 10, 10.6)	(18.8, 20, 22)
Touliu	(14, 15, 16)	(18.2, 20, 22)	(10, 12, 13)	(6, 8, 8.8)
Hsinchu	(18.4, 20, 21)	(9.6, 12, 13)	(7.8, 10, 10.8)	(14, 15, 16)

TABLE 4. Shipping costs.

Source	Destination			
	Taichung	Chiayi	Kaohsiung	Taipei
Taichung	(1.5, 2, 2.5)	(4, 5, 6)	(1.3, 2, 2.5)	(3, 4, 5)
Touliu	(2.5, 3, 4)	(2, 3, 4)	(2.3, 3, 4)	(1.5, 2, 2.5)
Hsinchu	(3, 4, 5)	(2, 3, 4)	(1.5, 2, 2.7)	(2, 3, 4)

The real world problem can be modeled to the following FFLFP problem:

$$\operatorname{Max} Z = \frac{\left\{\begin{array}{l}(8,\, 10,\, 10.8)\tilde{x}_{11} + (20.4,\, 22,\, 24)\tilde{x}_{12} + (8,\, 10,\, 10.6)\tilde{x}_{13} + (18.8,\, 20,\, 22)\tilde{x}_{14} + \\ (14,\, 15,\, 16)\tilde{x}_{21} + (18.2,\, 20,\, 22)\tilde{x}_{22} + (10,\, 12,\, 13)\tilde{x}_{23} + (6,\, 8,\, 8.8)\tilde{x}_{24} + \\ (18.4,\, 20,\, 21)\tilde{x}_{31} + (9.6,\, 12,\, 13)\tilde{x}_{32} + (7.8,\, 10,\, 10.8)\tilde{x}_{33} + (14,\, 15,\, 16)\tilde{x}_{34}\end{array}\right\}}{\left\{\begin{array}{l}(1.5,\, 2,\, 2.5)\tilde{x}_{11} + (4,\, 5,\, 6)\tilde{x}_{12} + (1.3,\, 2,\, 2.5)\tilde{x}_{13} + (3,\, 4,\, 5)\tilde{x}_{14} + \\ (2.5,\, 3,\, 4)\tilde{x}_{21} + (2,\, 3,\, 4)\tilde{x}_{22} + (2.3,\, 3,\, 4)\tilde{x}_{23} + (1.5,\, 2,\, 2.5)\tilde{x}_{24} + \\ (3,\, 4,\, 5)\tilde{x}_{31} + (2,\, 3,\, 4)\tilde{x}_{32} + (1.5,\, 2,\, 2.7)\tilde{x}_{33} + (2,\, 3,\, 4)\tilde{x}_{34}\end{array}\right\}}$$

s.t.

$$\begin{aligned}
\tilde{x}_{11} + \tilde{x}_{12} + \tilde{x}_{13} + \tilde{x}_{14} &\leqslant (7.2,\, 8,\, 8.8)\\
\tilde{x}_{21} + \tilde{x}_{22} + \tilde{x}_{23} + \tilde{x}_{24} &\leqslant (12,\, 14,\, 16)\\
\tilde{x}_{31} + \tilde{x}_{32} + \tilde{x}_{33} + \tilde{x}_{34} &\leqslant (10.2,\, 12,\, 13.8)\\
\tilde{x}_{11} + \tilde{x}_{21} + \tilde{x}_{31} &\geqslant (6.2,\, 7,\, 7.8)\\
\tilde{x}_{12} + \tilde{x}_{22} + \tilde{x}_{32} &\geqslant (8.9,\, 10,\, 11.1)\\
\tilde{x}_{13} + \tilde{x}_{23} + \tilde{x}_{33} &\geqslant (6.5,\, 8,\, 9.5)\\
\tilde{x}_{14} + \tilde{x}_{24} + \tilde{x}_{34} &\geqslant (7.8,\, 9,\, 10.2)\\
\tilde{x}_{ij} \geqslant 0, \quad \text{where} \quad \tilde{x}_{ij} &= (x_{ij}^1,\, x_{ij}^2,\, x_{ij}^3), i = 1,2,3; j = 1,2,3,4.
\end{aligned}$$

After solving this problem by using Steps 1, 2, 3, 4, 5 and 6, we have

$$\tilde{x} = \begin{bmatrix}(0,\, 0,\, 0),\\ (0,\, 0,\, 0.25),\\ (0,\, 0,\, 0.625),\\ (7.64,\, 8,\, 8),\\ (0,\, 0,\, 0),\\ (0,\, 0,\, 0),\\ (0,\, 0, 1.45\,),\\ (12, 14,\, 14),\\ (5.8, 7, 7.5),\\ (8.8, 10, 11.25),\\ (6.4, 8, 8),\\ (5.8, 10, 12.5).\end{bmatrix}.$$

Now the optimal value of the problem may be written as:
$(\tilde{Z})$ proposed method $= ((Z_1),\ (Z_2),\ (Z_3)) = (2.26, 4.64, 9.48)$.

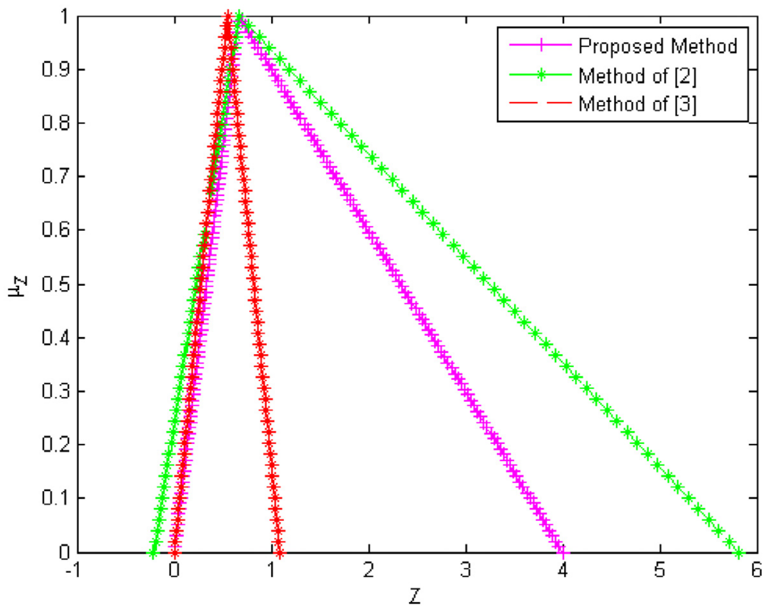

FIGURE 1. Membership functions of the optimal solution by the present method and existing method [6, 11].

By comparing proposed method results with existing method [6, 11], based on Definition 2.7, we conclude that our result is more efficient than other existing method.

$$2.26 = (z_1^*)_{\text{proposed method}} = (z_1^*)_{\text{method of [3]}} > (z_1^*)_{\text{method of [2]}} = 2$$

$$4.64 = (z_1^* + z_2^*)_{\text{proposed method}} = (z_1^* + z_2^*)_{\text{method of [2]}} = 4.64 > (z_1^* + z_2^*)_{\text{method of [3]}} = 4.2$$

$$(2.26, 4.64, 9.48) = (z^*)_{\text{proposed method}} > (z^*)_{\text{method of [2]}} = (2, 4.64, 9) > (z_1^*)_{\text{method of [3]}} = (2, 4.2, 11).$$

6. Results and discussion

The FFLFP problem, chosen in Example 4.1 and one real life transportation problem can be solved by using the proposed method. In the existing method [6, 11] it is difficult to solve for fully fuzzy transportation problem in real life case. Obtained result by the present method has been compared with the results of existing method [6, 11]. It is worth mentioning that one may check that the results obtained by the existing method may not be satisfied the constraints properly where the results obtained by the present method satisfied those constraints exactly. In the proposed methodology the FFLFP problem turns into a crisp LP problem and that problem is solved by using LINGO Version 11.0. According optimal solution by both the methods is depicted in Figures 1 and 2.

7. Conclusions

In this paper, a new solving procedure has been suggested to solve the FFLFP problem. Using Charnes−Cooper transformation method, we transformed the fully fuzzy linear fractional programming problem into fully fuzzy linear programming problem. After that, the fully fuzzy linear programming problem is converted into its equivalent MOLP problem. It is our belief that the proposed method for solution of FFLFP

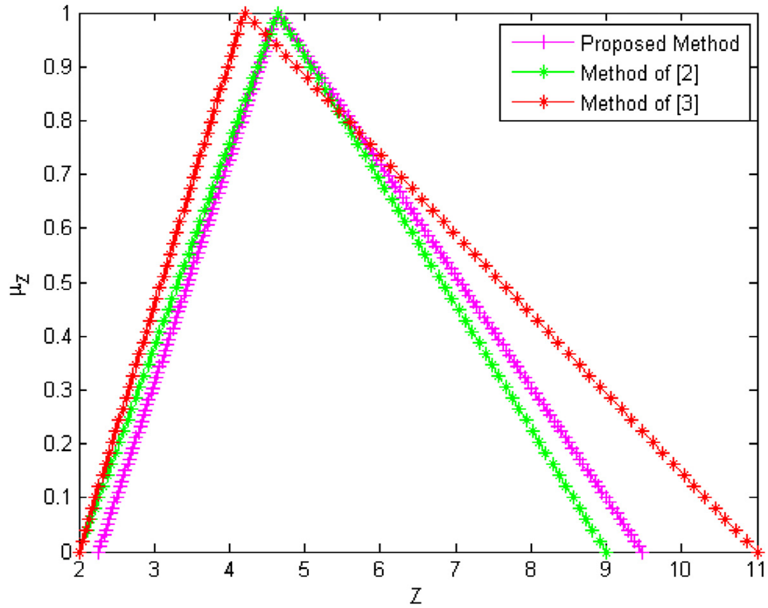

FIGURE 2. Membership functions of the optimal solution by the present method and existing method [6, 11].

problem in real life problem as well as simple problem may be of considerable interest for mathematician working in this field.

REFERENCES

[1] J.J. Buckley and T. Feuring, Evolutionary algorithm solution to fuzzy problems: fuzzy linear programming. *Fuzzy Sets Syst.* **109** (2000) 35–53.

[2] M. Chakraborty and S. Gupta, Fuzzy mathematical programming for multi objective linear fractional programming problem. *Fuzzy Sets Syst.* **125** (2002) 335–342.

[3] A. Charnes and W.W. Cooper, Programming with linear fractional functionals. *Nav. Res. Logist. Q.* **9** (1962) 181–186.

[4] D. Dutta, R.N. Tiwari and J.R. Rao, Multiple objective linear fractional programming- A fuzzy set theoretic approach. *Fuzzy Sets Syst.* **52** (1992) 39–45.

[5] R. Ezzati, E. Khorram and R. Enayati, A new algorithm to solve fully fuzzy linear programming problems using MOLP problem. *Appl. Math. Model.* **39** (2015) 3183–3193.

[6] B. Pop and I.M. Stancu-Minasian, A method of solving fully fuzzified linear fractional programming problems. *J. Appl. Math. Comput.* **27** (2008) 227–242.

[7] M. Sakawa and H. Yano, An interactive fuzzy satisficing method for multiobjective linear fractional programming problems. *Fuzzy Sets Syst.* **28** (1988) 129–144.

[8] M. Sakawa, H. Yano and J. Takahashi, Pareto optimality for multiobjective linear fractional programming problems with fuzzy parameters. *Inf. Sci.* **63** (1992) 33–53.

[9] I.M. Stancu-Minasian and B. Pop, On a fuzzy set approach to solving multiple objective linear fractional programming problem. *Fuzzy Sets Syst.* **134** (2003) 397–405.

[10] B. Stanojevic and I.M. Stancu-Minasian, On solving fuzzified linear fractional programs. *Adv. Model. Optim.* **11** (2009) 503–523.

[11] B. Stanojevic and I.M. Stancu-Minasian, Evaluating fuzzy inequalities and solving fully fuzzified linear fractional programs. *Yugoslav J. Oper. Res.* **1** (2012) 41–50.

[12] M.D. Toksari, Taylor series approach to fuzzy multi objective linear fractional programming. *Inf. Sci.* **178** (2008) 1189–1204.

[13] L.A. Zadeh, Fuzzy Sets. *Inf. Control* **8** (1965) 338–353.