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OPTIMAL INVESTMENT WITH TRANSACTION COSTS AND DIVIDENDS FOR AN INSURER

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Abstract. This paper investigates the optimal investment problems for an insurer whose reserve process is approximated by a diffusion model. The insurer is allowed to invest its wealth in the financial market consisting of one risk-free asset (bond) and one risky asset (stock). There are charges which equal to a fixed percentage of the amount transferred between the two assets. Under different criteria, we consider two optimization problems: one is maximizing the expected discounted utility of the dividends; the other is maximizing the insurer's expected utility of the terminal wealth. We obtain that the optimal investment strategies are bang-bang strategies in both of the two problems. Numerical examples are given to illustrate our results.

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1. Introduction

The optimal investment model with transaction costs can be traced back to Magill and Constantinides [14], which introduces the proportional transaction costs to the well known Merton's model (see [15]) and give a fundamental insight that the no-transaction region is a wedge, but the argument is heuristic and no clear prescription as to how to compute the location of the boundaries. Constantinides [2] considers essentially the same problem as that in Magill and Constantinides [14] and proposes an approximate solution based on making certain assumptions on the consumption process. By introducing the singular stochastic control theory, the same optimal portfolio selection problem with transaction costs under some restrictive conditions is solved by Davis and Norman [3]. They provide a precise formulation and analysis, including an algorithm and numerical computations of the optimal investment policy. Shreve and Soner [16] study the same problem as that in Davis and Norman [3], and their main technique is the viscosity solution to Hamilton–Jacobi–Bellman (HJB) equation. Other works about transaction cost can be found in [7, 8, 10–13], and so on. Because it is difficult to solve the corresponding HJB partial differential equations (PDE) of the problem when the transaction costs are included, an explicit formula for the solution is not known in the literatures above. Then the papers above do not provide explicit forms for the no-transaction or target boundaries and they use numerical procedures to solve the HJB PDEs with free boundaries.

Keywords. Optimal investment, transaction costs, partial differential equation, dividend.

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Since the insurance companies (the insurers) have the opportunity to invest in the financial market, the problem of optimal investment in the financial market for an insurer has received more and more attention. The pioneer work about optimal investment problem for an insurer is Browne [1], in which the expected constant absolute risk aversion (CARA) utility from the terminal wealth is maximized. Later, there are some works considering the optimal investment problems under different optimization criteria and in different models, such as [5,9,17] and so on. In the works mentioned above, transaction costs of the market, such as tax and commission, are not taken into consideration. However, in financial practise, transaction costs can not be eliminated.

Inspired by the works above, we add the transaction costs, which are really existing in the real financial market, to the optimal portfolio selection problem for the insurance company. The problems we consider in this paper are much closer to the insurance operational practise, and the optimal portfolio selection strategy we derive is more accessible to the managers in insurance companies. The insurer can invest its wealth into stock and bond in the financial market, and pay dividends to the insured. We consider two optimization problems in this paper. The first optimization problem is maximizing the expected discounted utility of the total dividends by choosing an admissible trading strategy. The second problem is maximizing the expected utility of the terminal wealth corresponding to a fixed terminal time T. Based on the framework of HJB equation, we derive the optimal strategy and value function. Actually, using some intuitive analysis, we can find the optimal bang-bang strategy. The value functions satisfy two two-order PDEs in the two problems. The PDEs are two-dimensional and three-dimensional respectively in the two problems. Using the homogeneous property of the value function, we lower the dimensionality of the PDEs. Then we derive the one-variable ordinary differential equation whose explicit solution is easy to obtain for the first problem and a two-order, two-dimensional PDEs for the second problem. Because it is difficult to obtain the closed form solution of the two-order and two-dimensional PDEs, we give the numerical solutions to illustrate our results.

The organization of this paper is as follows. In Section 2, we formulate the model used in this paper, containing the reserve process of the insurer, the price processes of the risk-free asset and the risky asset, and the resulting wealth process of the insurer after investment with transaction costs and dividends. Section 3 is contributed to solving the optimal investment problem under the criterion of maximizing the expected discounted utility of the dividends. In Section 4, we consider the second optimization problem, *i.e.*, maximizing the expected utility of the terminal wealth corresponding to a fixed terminal time. Some numerical examples are given to illustrate our results in both Sections 3 and 4. At the end of this paper, some concluding remarks are given in Section 5.

2. The model

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ containing all objects defined in the following. We make the following additional notations:

Notation.

$\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$;

$\{W_1(t), t \geq 0\}$ and $\{W_2(t), t \geq 0\}$ are standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Brownian motions, and they are independent;

Y_i is the size of the ith claim and $\{Y_i\}_{i\geq 1}$ are independent and identically distributed random variables with $\mathbb{E}(Y_i) = \mu_1 > 0$ and $\mathbb{E}(Y_i^2) = \mu_2 > 0$;

$\bar{c} = (1+\eta)\lambda\mu_1$ is the premium rate with safety loading $\eta > 0$;

$R_1(t)$ is the classical reserve process of the insurer, and $R_2(t)$ is the diffusion approximation of $R_1(t)$;

$P_0(t)$ is the price process of the bond with interest rate $r > 0$;

$P_1(t)$ is the price process of the stock with appreciation rate $\mu(> r)$ and volatility coefficient σ;

$x(t)$ and $y(t)$ are the amount invested at time t in bond and stock respectively;

U_t and L_t are the total transactions from stock to bond and from bond to stock respectively up to time t;

$c_t = k[x(t) + (1-\lambda_1)y(t)]$ is the insurer's dividend rate with proportionality factor k;

$\lambda_1 \in (0,1)$ and $\frac{1}{1+\lambda_2} \in (0,1)$ (*i.e.*, $\lambda_2 \in (0,\infty)$) are the proportional costs of transactions from stock to bond and from bond to stock respectively;

$\gamma \in (0,1)$ is the parameter in the utility function $u(z) = \frac{z^\gamma}{\gamma}$, and $\rho > 0$ represents the discount rate of the utility;
$a := \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma-1) - \rho$, $b := r - k - \mu + \sigma^2 - \sigma^2\gamma$, $c := \alpha - k + k\lambda_1$, $d := \frac{1}{2}\sigma^2$, $e := \frac{1}{2}\beta^2$;
$\alpha := \eta\lambda\mu_1$, $\beta := \sqrt{\lambda\mu_2}$, $h := \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma-1)$;
$m(x,y) := rx + \alpha - k(x + (1-\lambda_1)y)$, $n(x,y) := \frac{[k(x+(1-\lambda_1)y)]^\gamma}{\gamma}$;
$\Phi(t,x)$ $(J(\cdot,\cdot)$ $,V(\cdot,\cdot))$ is the binary function of t and x;
$\Phi_x(t,x)$ is the first-order partial derivative of $\Phi(t,x)$ with respect to x;
$\Phi_{xx}(t,x)$ is the second-order partial derivative of $\Phi(t,x)$ with respect to x.

The classical reserve process of an insurer is modeled by

$$\mathrm{d}R_1(t) = \bar{c}\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N(t)} Y_i, \tag{2.1}$$

where the constant $\bar{c}$ is the premium rate. Y_i is the size of the ith claim and $\{Y_i\}_{i\geq 1}$ are assumed to be independent and identically distributed (i.i.d) random variables with $\mathbb{E}(Y_i) = \mu_1 > 0$ and $\mathbb{E}(Y_i^2) = \mu_2 > 0$. $\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$, which represents the number of claims occurring in time interval $[0,t]$. Thus the compound Poisson process $\sum_{i=1}^{N(t)} Y_i$ represents the cumulative amount of claims in time interval $[0,t]$. $\{N(t)\}_{t\geq 0}$ and $\{Y_i\}_{i\geq 1}$ are assumed to be independent. The premium rate is assumed to be calculated via the expected value principle, *i.e.*, $\bar{c} = (1+\eta)\lambda\mu_1$ with safety loading $\eta > 0$. This classical reserve model has been studied extensively in the literature, see *e.g.*, [7, 17], and the references therein.

Since there exist Poisson jump processes in the classical model (2.1), it is difficult to solve our optimal portfolio selection problem. The next best way is to look for the approximation of the classical model. According to Grandell [6] (pp. 15–17), we consider the diffusion approximations, *i.e.*, approximating the classical model by a Brownian motion with drift. Mathematically, such approximation is based on the theory of weak convergence of probability measures. One way to express this diffusion approximation is that if the classical risk model is regarded as "large deviation", the diffusion model is related to the "central limit theorem". The diffusion approximation is widely used in the literature on the optimization problems for insurers, such as [1, 6, 9]. In this paper, we consider the following diffusion approximation of the classical model (2.1)

$$\mathrm{d}R_2(t) = (1+\eta)\lambda\mu_1\mathrm{d}t - \lambda\mu_1\mathrm{d}t + \sqrt{\lambda\mu_2}\mathrm{d}W_1(t) := \alpha\mathrm{d}t + \beta\mathrm{d}W_1(t), \tag{2.2}$$

where $\alpha = \eta\lambda\mu_1$, $\beta = \sqrt{\lambda\mu_2}$ and $W_1(t)$ is a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Brownian motion.

The insurer can invest its wealth in a financial market which consists of one risk-free asset (bond) and one risky asset (stock). It is supposed that the two assets can be traded continuously.

The price of the bond is modeled by the following ordinary differential equation

$$\begin{cases} \mathrm{d}P_0(t) = rP_0(t)\mathrm{d}t, t \in [0,T], \\ P_0(0) = p_0, \end{cases} \tag{2.3}$$

where $r(> 0)$ is the constant interest rate of the bond.

The price of the stock is modeled by the following stochastic differential equation

$$\begin{cases} \mathrm{d}P_1(t) = P_1(t)\left[\mu\mathrm{d}t + \sigma\mathrm{d}W_2(t)\right], \\ P_1(0) = p_1, \end{cases} \tag{2.4}$$

where $\mu > r$ is the appreciation rate and σ is the volatility coefficient. $\{W_2(t), t \geq 0\}$ is a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Brownian motion which is independent of $\{W_1(t), t \geq 0\}$.

Let $x(t)$ and $y(t)$ be the amount invested at time t in bond and stock, respectively. We assume that the insurer pay dividend from its bond. c_t is the dividend rate. Simultaneously, it may transfer its stock holdings to the bond and transfer from bond to the stock. U_t, L_t represent the total transactions from stock to bond and from bond to stock respectively up to time t. However, this results in a transaction cost. We assume this cost is linearly proportional in the size of the transaction. Let $\lambda_1 \in (0,1)$, $\frac{1}{1+\lambda_2} \in (0,1)$ (*i.e.*, $\lambda_2 \in (0,\infty)$) be the proportional costs of transactions from stock to bond and from bond to stock respectively. The process U_t and L_t are assumed to be right continuous with left limits, nonnegative, nondecreasing and adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. Using equations (2.2), (2.3) and (2.4), we see that $x(t)$ and $y(t)$ change according to

$$\begin{cases} \mathrm{d}x(t) = rx(t)\mathrm{d}t + \alpha \mathrm{d}t + \beta \mathrm{d}W_1(t) - (1+\lambda_2)\mathrm{d}L_t + (1-\lambda_1)\mathrm{d}U_t - c_t \mathrm{d}t, \\ \mathrm{d}y(t) = y(t)\left[\mu \mathrm{d}t + \sigma \mathrm{d}W_2(t)\right] + \mathrm{d}L_t - \mathrm{d}U_t, \end{cases} \tag{2.5}$$

with $x(0) = x$ and $y(0) = y$.

Moreover, we impose an additional constraint

$$x(t) + (1-\lambda_1)y(t) \geq 0, \ \frac{x(t)}{1+\lambda_2} + y(t) \geq 0, \ \forall t \geq 0,$$

which ensures that the insurer (the investor) has sufficient funds, so that its portfolio can be liquidated to result in a non-negative bond holding and no stock holding at any time if needed. It is more convenient to state the above condition as a state constraint in the following

$$\mathcal{O} = \left\{(x,y) \in \mathbb{R}^2 : x + (1-\lambda_1)y \geq 0, x + (1+\lambda_2)y \geq 0\right\}.$$

Davis and Norman [3] call the cone $\mathcal{O}$ defined above the "solvency region". Then the state (x,y) is required to satisfy $(x(t),y(t)) \in \mathcal{O}$ for all $t \geq 0$.

We assume that the investor is given an initial position $(x(0),y(0)) = (x,y) \in \mathcal{O}$. The investment strategy (L,U) is *admissible* for (x,y) if $(x(t),y(t))$ given by (2.5) is in $\mathcal{O}$ for all $t \geq 0$. We denote by Π the set of all admissible strategies. It is easy to see that there exists at least one admissible strategy for our problem, *i.e.*, the set of all admissible strategies is non-empty.

In this paper, we suppose that the insurer has the hyperbolic absolute risk aversion (HARA) utility function, that is $u(z) = \frac{z^\gamma}{\gamma}$, $\gamma \in (0,1)$. It is assumed that the insurer's dividend rate is a constant fraction of its total asset holding, *i.e.*, $c_t = k[x(t) + (1-\lambda_1)y(t)]$. Then, the goal is to maximize the objective function by choosing $(L_t,U_t) \in \Pi$. In Section 3, the objective function is the expected discounted utility of the total dividends. In Section 4, the objective function is the expected utility of the terminal wealth corresponding to a fixed terminal time T.

3. Maximizing the expected discounted utility of the dividends

In this section, we consider the optimization problem of maximizing the expected discounted utility of the dividends, that is

$$\begin{aligned} &\max \quad \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t}\frac{\{k[x(t)+(1-\lambda_1)y(t)]\}^\gamma}{\gamma}\mathrm{d}t\right] \\ &\text{subject to} \begin{cases} (L,U) \in \Pi \\ (x(\cdot),y(\cdot)) \text{ satisfy } (2.5), \end{cases} \end{aligned} \tag{3.1}$$

where $\rho > 0$ represents the discount rate of the utility. $\rho = r$ means that the utility of the dividends is discounted by the interest rate of the bond.

We define the associated optimal value function by

$$J(x,y) := \max_{(L,U)\in\Pi} \quad \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t}\frac{\{k[x(t)+(1-\lambda_1)y(t)]\}^\gamma}{\gamma}\mathrm{d}t\right].$$

This is a stochastic optimal control problem. In the following, we solve this problem with the help of the HJB equation. According to Fleming and Soner [4], the corresponding HJB equation of problem (2.5)–(3.1) is the following partial differential equation

$$\max_{(L,U)\in\Pi}\Big\{m(x,y)V_x+\frac{1}{2}\beta^2V_{xx}+\mu yV_y+\frac{1}{2}\sigma^2y^2V_{yy}-\rho V+n(x,y)$$
$$+\left[-(1+\lambda_2)V_x+V_y\right]l+\left[(1-\lambda_1)V_x-V_y\right]u\Big\}=0, \tag{3.2}$$

where $m(x,y)=rx+\alpha-k\left[x+(1-\lambda_1)y\right]$ and $n(x,y)=\frac{[k(x+(1-\lambda_1)y)]^\gamma}{\gamma}$.

Here we have for the sake of simplicity written

$$L_t:=\int_0^t l_s\mathrm{d}s,\quad U_t:=\int_0^t u_s\mathrm{d}s. \tag{3.3}$$

Next we solve the HJB equation (3.2). We give the main result in the following theorem.

Theorem 3.1. *The maximum of the left-hand side of HJB equation (3.2) is attained at (l^*,u^*), where*

$$l^*=\begin{cases}\sup\{l\}, & V_y\geq(1+\lambda_2)V_x,\\ 0, & V_y<(1+\lambda_2)V_x,\end{cases} \tag{3.4}$$

$$u^*=\begin{cases}\sup\{u\}, & V_y\leq(1-\lambda_1)V_x,\\ 0, & V_y>(1-\lambda_1)V_x.\end{cases} \tag{3.5}$$

The classical solution of the HJB equation (3.2) is $V(x,y)=y^\gamma\Psi(\frac{x}{y})$, where

$$\begin{aligned}&\Psi(x)=A\frac{(x+1-\lambda_1)^\gamma}{\gamma},\quad x\leq x_0;\\ &a\Psi(x)+(bx+c)\Psi^{'}(x)+(\mathrm{d}x^2+e)\Psi^{''}(x)+\frac{[k(x+1-\lambda_1)]^\gamma}{\gamma}=0,\ x_0<x<x_1;\\ &\Psi(x)=B\frac{(x+1+\lambda_2)^\gamma}{\gamma},\quad x\geq x_1.\end{aligned} \tag{3.6}$$

Here $a=\mu\gamma+\frac{1}{2}\sigma^2\gamma(\gamma-1)-\rho$, $b=r-k-\mu+\sigma^2-\sigma^2\gamma$, $c=\alpha-k+k\lambda_1$, $d=\frac{1}{2}\sigma^2$, $e=\frac{1}{2}\beta^2$, and constants A, B, x_0 and x_1 can be determined by the smooth condition for $\Psi(x)$.

Proof. Note that both of the derivatives in HJB equation (3.2) must be non-negative since extra wealth provides increased utility. So the maximum of the left-hand side of HJB equation (3.2) is achieved at (3.4) and (3.5). This shows that the optimal transaction strategy is a *bang-bang* strategy (named by Davis and Norman [3]): buying and selling either take place at maximum rate or not at all. And the solvency region is split into three parts: "sell" (S), "no transactions" (NT) and "buy" (B), which correspond to

$$V_y\leq(1-\lambda_1)V_x,$$
$$(1-\lambda_1)V_x<V_y<(1+\lambda_2)V_x$$

and

$$V_y\geq(1+\lambda_2)V_x$$

respectively. At the boundary between the B and NT regions, $V_y=(1+\lambda_2)V_x$ and at the boundary between NT and S, $V_y=(1-\lambda_1)V_x$.

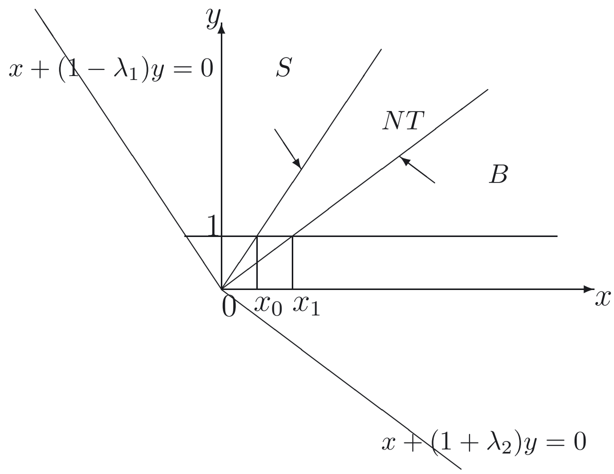

FIGURE 1. The regions S, B and NT in the proof of Theorem 3.1.

Next we show that the boundaries between the transaction and no-transaction (NT) regions are straight lines through the origin. Note that the value function $J(x, y)$ has homogeneous property, so $V(x, y)$ has the same homogeneous property, *i.e.*,

$$V(\theta x, \theta y) = \theta^\gamma V(x, y),$$

which implies

$$V_x(\theta x, \theta y) = \theta^\gamma V_x(x, y)$$

and

$$V_y(\theta x, \theta y) = \theta^\gamma V_y(x, y).$$

It follows that if $V_y = (1 + \lambda_2)V_x$ or $V_y = (1 - \lambda_1)V_x$ for some (x, y), then the same is true at all points along the ray through (x, y). This indicates that the boundaries between the transaction and no-transaction (NT) regions are straight lines. In the transaction region, transactions take place at maximum speed, which implies that the investor makes an instantaneous finite transaction to the boundary of NT. These considerations suggest that the no-transaction region NT is a wedge, the regions above and below it being the sell (S) and buy (B) regions respectively. A finite transaction in the S (B) region moves the portfolio down (up) a line of slope $\frac{-1}{1-\lambda_1}$ $(\frac{-1}{1+\lambda_2})$. After the initial transaction, all further transactions must take place at the boundaries, and this suggests a "local time" type of transaction policy. The analysis above is shown in Figure 1.

Set $\Psi(x) := V(x, 1)$, then $y^\gamma \Psi(\frac{x}{y}) = V(x, y)$. This implies that V can be expressed by using the one-variable function Ψ. So we can treat only the straight line $l = \{(x, y) \in \mathcal{O}, y = 1\}$ instead of the whole solvency region. By ∂_S (the boundary between S and NT) and ∂_B (the boundary between B and NT), the line l is divided into three intervals, whose x-coordinates are defined by $(\lambda_1 - 1, x_0]$, (x_0, x_1), $[x_1, \infty)$, respectively. x_0 and x_1 are as shown in Figure 1.

Using the homogeneous property again, we find that,

$$\begin{aligned} V_x(x, 1) &= \Psi^{'}(x), V_{xx}(x, 1) = \Psi^{''}(x), \\ V_y(x, 1) &= \gamma\Psi(x) - x\Psi^{'}(x), \\ V_{yy}(x, 1) &= \gamma(\gamma - 1)\Psi(x) + 2x(1 - \gamma)\Psi^{'}(x) + x^2\Psi^{''}(x). \end{aligned}$$

According to the optimal policies described above, V has the same value along lines of slope $-\frac{1}{1-\lambda_1}$ in S and along lines of slope $-\frac{1}{1+\lambda_2}$ in B, these and the homogenous property imply the first and third equations in (3.6) for some constants A and B to be determined.

In NT region, the HJB equation takes the following form

$$m(x,y)V_x + \frac{1}{2}\beta^2 V_{xx} + \mu y V_y + \frac{1}{2}\sigma^2 y^2 V_{yy} - \rho V + n(x,y) = 0. \tag{3.7}$$

Denote $a := \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma-1) - \rho$, $b := r - k - \mu + \sigma^2 - \sigma^2\gamma$, $c := \alpha - k + k\lambda_1$, $d := \frac{1}{2}\sigma^2$ and $e := \frac{1}{2}\beta^2$. Set $y = 1$ in equation (3.7), we can derive the following simple ordinary differential equation (ODE) for Ψ

$$a\Psi + (bx + c)\Psi' + \frac{1}{2}(\beta^2 + \sigma^2 x^2)\Psi'' + \frac{[k(x+1-\lambda_1)]^\gamma}{\gamma} = 0, \quad x \in (x_0, x_1) \tag{3.8}$$

with boundary conditions $\Psi(x_0) = A\frac{(x_0+1-\lambda_1)^\gamma}{\gamma}$ and $\Psi(x_1) = B\frac{(x_1+1+\lambda_2)^\gamma}{\gamma}$. Then equation (3.8) becomes the second equation in (3.6).

The unknown constants A, B, x_0, x_1 can be determined by C^2-conditions at the free boundaries. We call these the smooth pasting conditions. Using these conditions, we can derive six algebraic equations with respect to the six unknown constants and these can be solved numerically. Then using $V(x,y) = y^\gamma \Psi(\frac{x}{y})$ we can get the solution $V(x,y)$ of the HJB equation (3.2). □

It remains to *verify* that the optimal control (l^*, u^*) in (3.4) and (3.5) and the optimal function $V(x,y)$ derived in the above theorem are in fact optimal for problem (3.1). Actually, since $V(x,y)$ in the above theorem is twice continuously differentiable, it is clear that all the conditions of the classical verification theorems (see Fleming and Soner [4]) are satisfied, and that these are in fact the optimal strategies and value function, *i.e.*, $V(x,y) = J(x,y)$.

Next we give a numerical example to illustrate our result.

Example 3.2. Set the interest rate $r = 0.07$, the appreciation rate $\mu = 0.12$, the volatility coefficient $\sigma = 0.4$, the parameters of the transaction costs $\lambda_1 = \lambda_2 = 0.01$, the parameter in the utility function $\gamma = 0.5$, the proportional factor in the dividend rate $k = 0.3$, and the discount rate of the utility $\rho = 0.1$, then we have $a = -0.39$, $b = -0.27$, $c = 0.73$, $d = 0.08$ and $e = 0.125$. Using the smooth pasting conditions, we firstly derive $x_0 = 0.4110$ and $x_1 = 1.1009$ numerically with the help of Matlab. Inserting these into boundary conditions, then using the smooth pasting conditions again and plugging them into the second equation in (3.6), we can derive $A = 1.608$ and $B = 2.121$. The numerical solution of the second equation in (3.6) is shown in Figure 2.

From Figure 2, we know $\Psi(x)$ increases as x increases. That is, holding more bond is better for the insurer in the NT region.

4. Maximizing the expected utility of the terminal wealth

In this section, we consider another optimization problem, that is, the problem of maximizing the insurer's expected utility of the terminal wealth at the terminal time T. We also set the utility function $u(z) = \frac{z^\gamma}{\gamma}$, $\gamma \in (0,1)$ and dividends rate $c(t) = k[x(t) + (1-\lambda_1)y(t)]$. Then the goal is to maximize the insurer's expected utility of the terminal wealth $\mathbb{E}[u[x(T) + (1-\lambda_1)y(T)]$ by choosing $(L_t, U_t) \in \Pi$. That is

$$\max \quad \mathbb{E}\left\{\frac{[x(T) + (1-\lambda_1)y(T)]^\gamma}{\gamma}\right\}$$
$$\text{subject to} \begin{cases} (L_t, U_t) \in \Pi \\ (x(\cdot), y(\cdot)) \text{ satisfy (2.5).} \end{cases} \tag{4.1}$$

The insurer's value function is defined as

$$\begin{aligned} \bar{J}(t,x,y) &= \max_{(L,U)\in\Pi} \mathbb{E}\left\{u\left[x(T) + (1-\lambda_1)y(T)\middle| x(t) = x, y(t) = y\right]\right\} \\ &= \max_{(L,U)\in\Pi} \mathbb{E}\left\{\frac{[x(T) + (1-\lambda_1)y(T)]^\gamma}{\gamma}\middle| x(t) = x, y(t) = y\right\} \end{aligned} \tag{4.2}$$

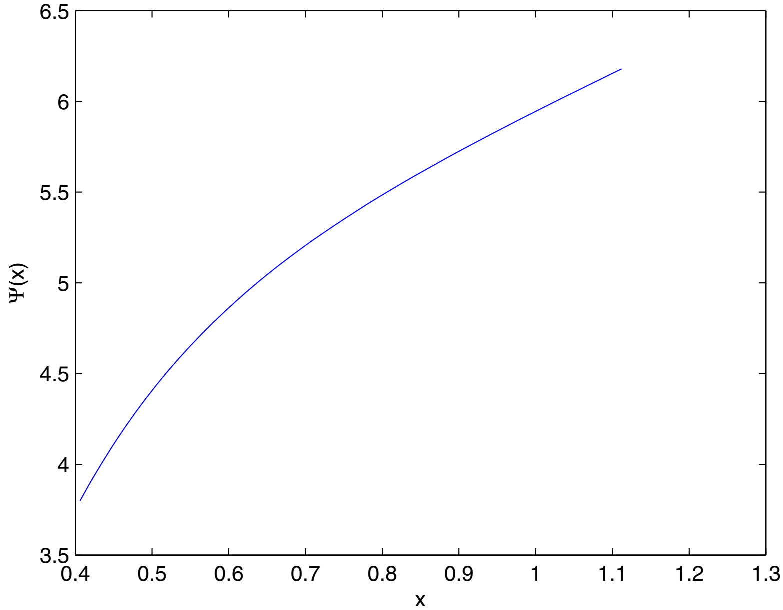

FIGURE 2. The function $\Psi(x)$ in Example 3.2.

with the terminal time condition

$$\bar{J}(T, x, y) = \frac{[x + (1-\lambda_1)y]^\gamma}{\gamma}. \tag{4.3}$$

Following the similar method to that in Section 3, we solve this problem with the help of the HJB equation. The corresponding HJB equation of problem (2.5)–(4.1) is the following partial differential equation

$$\begin{aligned}\max_{(L,U)\in\Pi} \Big\{ & V_t + [rx + \alpha - k(x + (1-\lambda_1)y)]V_x + \frac{1}{2}\beta^2 V_{xx} + \mu y V_y \\ & + \frac{1}{2}\sigma^2 y^2 V_{yy} + [-(1+\lambda_2)V_x + V_y]l + [(1-\lambda_1)V_x - V_y]u \Big\} = 0. \end{aligned} \tag{4.4}$$

where $l,\ u$ are defined in (3.3).

Next we give the solution to the HJB equation (4.4) in the following theorem.

Theorem 4.1. *The optimal transaction strategies to the HJB equation (4.4) are also bang-bang. The classical solution of the HJB equation (4.4) is* $V(t,x,y) = y^\gamma \Phi(t, \frac{x}{y})$ *, where*

$$\Phi(t,x) = f(t)\frac{(x+1-\lambda_1)^\gamma}{\gamma}, \quad x \le x_0,\ t \in [0,T]; \tag{4.5}$$

$$\begin{cases} \Phi_t + h\Phi + (bx + c)\Phi_x + (\mathrm{d}x^2 + e)\Phi_{xx} = 0, x \in (x_0, x_1), t \in [0,T] \\ \Phi(T,x) = \frac{(x+1-\lambda_1)^\gamma}{\gamma}, x \in [x_0, x_1]; \end{cases} \tag{4.6}$$

$$\Phi(t,x) = g(t)\frac{(x+1+\lambda_2)^\gamma}{\gamma}, \quad x \ge x_1,\ t \in [0,T]. \tag{4.7}$$

Here $h = \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma-1) = a + \rho$*, a, b, c, d, e, x_0 and x_1 are shown in Theorem 3.1 and $f(t)$, $g(t)$ is determined by*

$$\begin{aligned} f^{'}(t) + \Big\{ \left[\mu\gamma + \tfrac{1}{2}\sigma^2\gamma(\gamma-1)\right] + \tfrac{\gamma}{x_0+1-\lambda_1}\left[x_0(r-k-\mu+\sigma^2-\sigma^2\gamma) + \alpha - k + k\lambda_1\right] \\ + \tfrac{1}{2}(\beta^2+\sigma^2 x_0^2)\tfrac{\gamma(\gamma-1)}{(x_0+1-\lambda_1)^2} \Big\} f(t) = 0 \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} g^{'}(t) + \Big\{ \left[\mu\gamma + \tfrac{1}{2}\sigma^2\gamma(\gamma-1)\right] + \tfrac{\gamma}{x_1+1+\lambda_2}\left[x_1(r-k-\mu+\sigma^2-\sigma^2\gamma) + \alpha - k + k\lambda_1\right] \\ + \tfrac{1}{2}(\beta^2+\sigma^2 x_1^2)\tfrac{\gamma(\gamma-1)}{(x_1+1+\lambda_1)^2} \Big\} g(t) = 0, \end{aligned} \tag{4.9}$$

with $f(T) = 1$ *and* $g(T) = \left(\frac{x_1+1-\lambda_1}{x_1+1+\lambda_2}\right)^{\gamma}$.

Proof. Using the similar analysis to that in the proof of Theorem 3.1, we can conclude that the optimal transaction policies are also bang-bang.

It is easy to show that the $V(t,x,y)$ satisfies $V(t,\rho x,\rho y) = \rho^{\gamma}V(t,x,y)$. Set $\Phi(t,x) \triangleq V(t,x,1)$, then $y^{\gamma}\Phi(t,\frac{x}{y}) = V(t,x,y)$. This implies that V can be expressed by using the two-variable function Φ. So we can treat only the straight line $l = \{(x,y) \in \mathcal{O}, y = 1\}$ instead of the whole solvency region for any given $t \geq 0$. By ∂_S (the boundary between S and NT) and ∂_B (the boundary between B and NT), the line l is divided into three intervals $(\lambda_1 - 1, x_0]$, (x_0, x_1) and $[x_1, \infty)$.

Using $V(t,x,y) = y^{\gamma}\Phi(t,\frac{x}{y})$ again, we find that

$$\begin{aligned} V_x(t,x,1) &= \Phi_x(t,x), V_{xx}(t,x,1) = \Phi_{xx}(t,x), \\ V_y(t,x,1) &= \gamma\Phi(t,x) - x\Phi_x(t,x), V_t(t,x,1) = \Phi_t(t,x), \\ V_{yy}(t,x,1) &= \gamma(\gamma-1)\Phi(t,x) + 2x(1-\gamma)\Phi_x(t,x) + x^2\Phi_{xx}(t,x). \end{aligned}$$

According to the optimal strategies described above, V has the same value along lines of slope $\frac{-1}{1-\lambda_1}$ in S and along lines of slope $\frac{-1}{1+\lambda_2}$ in B, and this implies (4.5) and (4.7) for some functions $f(t)$ and $g(t)$ to be determined.

In NT region, the HJB equation becomes

$$V_t + [rx + \alpha - k(x + (1-\lambda_1)y)]V_x + \frac{1}{2}\beta^2 V_{xx} + \mu y V_y + \frac{1}{2}\sigma^2 y^2 V_{yy} = 0. \tag{4.10}$$

Set $y = 1$ in equation (4.10), we can obtain the following PDE about $\Phi(t,x)$

$$\begin{aligned} \Phi_t + \left[\mu\gamma + \tfrac{1}{2}\sigma^2\gamma(\gamma-1)\right]\Phi + \left[x(r-k-\mu+\sigma^2-\sigma^2\gamma) + \alpha - k + k\lambda_1\right]\Phi_x \\ + \tfrac{1}{2}(\beta^2+\sigma^2x^2)\Phi_{xx} = 0, \quad x \in (x_0, x_1), \end{aligned} \tag{4.11}$$

with the boundary conditions $\Phi(t,x_0) = f(t)\frac{(x_0+1-\lambda_1)^{\gamma}}{\gamma}, 0 \leq t \leq T$, $\Phi(t,x_1) = g(t)\frac{(x_1+1+\lambda_2)^{\gamma}}{\gamma}$, $0 \leq t \leq T$ and the terminal time condition $\Phi(T,x) = V(T,x,1) = \frac{(x+1-\lambda_1)^{\gamma}}{\gamma}$.

Set $h = \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma-1) = a + \rho$, then equation (4.11) becomes (4.6). Using the terminal time condition and the boundary conditions, we obtain $f(T) = 1$ and $g(T) = \left(\frac{x_1+1-\lambda_1}{x_1+1+\lambda_2}\right)^{\gamma}$. Using the smooth pasting condition, we get (4.8) and (4.9). This completes the proof. □

Since $V(t,x,y)$ in the above theorem is twice continuously differentiable, all the conditions of the classical verification theorems are satisfied. So the optimal control and optimal function are in fact the optimal strategies and value function, *i.e.*, $V(t,x,y) = \bar{J}(t,x,y)$.

The closed form solution of the PDE (4.6) is not easy to obtain, so we give a numerical example in the following.

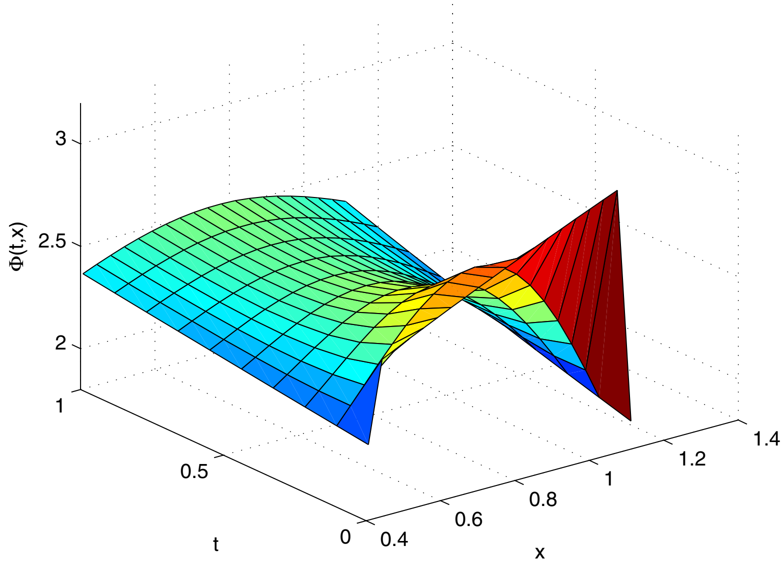

FIGURE 3. The function $\Phi(t,x)$ in Example 4.2.

TABLE 1. The numerical solution of $\Phi(t,x)$ in Example 4.2.

$x \diagdown t$	0	0.1111	0.2222	0.3333	0.4444	0.5556	0.6667	0.7778	0.8889	1.0000
0.4432	2.5679	2.2700	2.2710	2.2796	2.2925	2.3086	2.3270	2.3470	2.3682	2.3902
0.5175	2.6336	2.4355	2.3941	2.3744	2.3684	2.3724	2.3834	2.3992	2.4182	2.4395
0.5918	2.6978	2.5876	2.5073	2.4586	2.4338	2.4258	2.4297	2.4414	2.4586	2.4796
0.6661	2.7604	2.7054	2.5928	2.5195	2.4793	2.4620	2.4605	2.4696	2.4860	2.5073
0.7404	2.8217	2.7725	2.6342	2.5456	2.4969	2.4750	2.4716	2.4804	2.4976	2.5206
0.8147	2.8816	2.7736	2.6195	2.5292	2.4813	2.4614	2.4604	2.4719	2.4921	2.5183
0.8891	2.9404	2.6937	2.5441	2.4680	2.4313	2.4203	2.4264	2.4440	2.4694	2.5004
0.9634	2.9979	2.5255	2.4115	2.3652	2.3497	2.3540	2.3716	2.3981	2.4308	2.4680
1.0377	3.0544	2.2786	2.2330	2.2291	2.2425	2.2672	2.2996	2.3372	2.3787	2.4232
1.1120	1.9370	1.9808	2.0256	2.0714	2.1182	2.1661	2.2150	2.2651	2.3163	2.3687

Example 4.2. Taking the same example as that in Example 3.2, and setting $T = 1$, we get $h = -0.29$, $x_0 = 0.443$, $x_1 = 1.112$, $f(t) = \mathrm{e}^{-0.0855(1-t)}$ and $g(t) = 0.9953\mathrm{e}^{-0.2012(1-t)}$. Then equation (4.6) becomes

$$\begin{cases} \Phi_t + (-0.29)\Phi + (-0.27x + 0.73)\Phi_x + (0.08x^2 + 0.125)\Phi_{xx} = 0, x \in (0.443, 1.112), t \in [0,1] \\ \Phi(t, 0.443) = 2.3630\mathrm{e}^{-0.0855(1-t)}, t \in [0,1] \\ \Phi(t, 1.112) = 2.3687\mathrm{e}^{-0.2012(1-t)}, t \in [0,1] \\ \Phi(1,x) = 2(x + 0.99)^{0.5}, x \in (0.443, 1.112). \end{cases}$$

The numerical solution of equation (4.6) is shown in Table 1 and Figure 3.

5. CONCLUDING REMARKS

In this paper, we add the transaction costs into the optimal portfolio selection problems for the insurance company (the insurer) to make the analysis much closer to the insurance companies' operation reality. We consider two optimal portfolio selection problems for the insurance company with proportional transaction costs. One is maximizing the expected discounted utility of the total dividends, and the other is maximizing the insurer's expected utility of the terminal wealth. We give the bang-bang optimal strategies for these optimization problems. Using the homogeneous property, we reduce the dimensionality of the value function and give the numerical solutions to illustrate our results.

There are some possible extensions of this paper. Firstly, the parameters (such as the interest rate r, the appreciation rate μ and the volatility coefficient σ of the stock) in our model can be changed from constants to general stochastic processes. Secondly, one can extend the model to the case of $m > 1$ risky assets, but m risky assets imply 3^m possible transaction regions. This may be difficult to deal with. Finally, one can consider the case when the reserve process of the insurer is the classical Cramér-Lundberg model, which is very popular in the risk theory in actuarial science.

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