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TRANSIENT ANALYSIS OF A SINGLE SERVER DISCRETE-TIME QUEUE WITH SYSTEM DISASTER

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Abstract. A discrete-time Geo/Geo/1 queue with system disaster is considered in this paper. The time-dependent and steady state probabilities of number of customers present in the system are obtained in terms of ballot numbers by solving the underlying system of difference equations using the generating function and continued fractions. Further, the busy period distribution is derived in terms of Catalan numbers. For special cases, time-dependent system size probabilities and busy period distribution are verified with the existing results in the literature. Numerical illustrations are provided for different parameter values to see their effect on performance measures and to get more insight of the model behavior.

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1. Introduction

Birth-death processes have a rich history in probabilistic modeling, including applications in ecology, genetics, and evolution. Moreover, populations can suffer dramatic declines from disease or food shortage but, perhaps surprisingly, such populations can survive for long periods of time and, although they may eventually become extinct, they can exhibit an apparently stationary regime.Models based on stochastic processes in the presence of catastrophes have been recently exploited also in another field of mathematical biology, with special reference to the description of the interaction between a myosin head and an actin filament. In particular, meta-population models, epidemics, and migratory flows provide practical examples of populations subject to disasters (*e.g.*, habitat destruction, environmental catastrophes).

Disaster is a special case of so called a negative arrival that removes one customer or a batch of ones of random size from the queueing system (that is, with different types of negative arrivals). A disaster is also called a catastrophe, mass exodus, or queue flushing [6]. Towsley [31] has discussed the presence of disasters in queueing systems for the purpose of analyzing distributed database systems that undergo site failure and later this idea was extended to the M/G/1 queue with disasters [15]. During real queueing system operation, the appearance of disasters is possible which causes the system to loss all customers instantaneously bringing

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system operations to complete halt for a while. Queueing models with disasters seem to be appropriate to model systems like computer networks which are vulnerable to disasters such as massive power outage and denial of service virus attacks for instance.

Birth-death processes with catastrophes have been extensively studied during the past two decades. During the last decade, several authors considered the problem of computing the transient distributions of various processes influenced by Poisson generated total catastrophes/(complete) disasters. The methods used in the specific models vary. Swift [29] used an ordinary differential equation (ODE) technique to solve directly the Chapman−Kolmogorov (C-K) equations for the simple immigration-catastrophe process. Brockwell *et al.* [3] defined several kinds of catastrophes (geometric, binomial, and uniform). Other contributions are due to Kyriakidis [17], Stirzaker [27] and, Economou and Fakinos [8] that extend some previous results on birth-death processes with catastrophes to the more general cases of continuous-time Markov chains with catastrophes resulting from a point process, and of the non-homogeneous Poisson process with total or binomial catastrophes. Economou and Gmez−Corral [10] studied the influence of renewal generated geometric catastrophes on a population of individuals that grows stochastically according to a batch Markovian arrival process (BMAP). These ideas have been further developed and applied in the recent contributions of Gani and Swift [11]. In [9] the authors reviewed the different methodologies for the derivation of the transient distribution and discussed the pros and cons. Chao and Zheng [5] considered an immigration birth and death population process with total catastrophes and studied its transient as well as equilibrium behavior.

In [34], Yechiali discussed the quality of stationary service performance measures such as the rate of customers flushed out of the system due to failures, and the rate of abandonments due to customers impatience for multiple servers queues when the system is down. Sudhesh [28] proposed an M/M/1 queue in the presence of disastrous breakdowns, system repair, and customer impatience and derived an explicit expression for the time-dependent queue size system distributions. A multi-server retrial queue with waiting places in service area along with negative customers and disasters was introduced by Shin [25]. Queues with negative and positive customers have been investigated over the decades ([16, 19, 33]).

Discrete-time queueing models have received considerable growing interest during the last few decades due to their potential applications to a variety of slotted digital computer, communication systems. These queueing models are more accurate and efficient than their continuous-time counterparts to analyze and design digital transmitting systems which are measured in discrete time slots [4]. For comprehensive studies on discrete-time queueing systems, one may refer to Hunter [14], Takagi [30], Park *et al.* [22, 23], and references therein. The early work on negative customers in discrete-time queue can be found in [1] where the authors considered the single-server discrete-time queue with negative arrivals and various killing disciplines caused by negative customers. Wang and Zhang [32] studied a discrete time single server retrial queue with geometrical arrivals of both positive and negative customers in which the server is subject to breakdowns and repairs. Lee and Yang [18] have considered a discrete time Geo/G/1 queue with N policy and disaster and obtained the probability generating functions of the queue length, the sojourn time, and regeneration cycles such as the idle period and busy period.

In the literature, analytical results for the transient behavior of queueing models are not as widely available as the steady-state results. It is of interest to practitioners to know how the system will operate in small interval of time [20, 21, 24]. Thus, transient analytical results are pertinent for theory and applications of queueing systems. Continued fractions have been used successfully to find time-dependent probabilities of birth-death queuing models [7, 28].

In this paper, we studied a single server discrete-time queue with system disasters. Suitable generating functions are used to convert the system of difference equations of probabilities into a system of difference equations of generating functions which leads to a continued fraction. We obtained time-dependent and steady state probabilities in terms of ballot numbers which is the generalization of catalan numbers [26]. Also a busy period distribution is derived for this single server disaster queue in terms of Catalan numbers. For special cases, time-dependent system size probabilities and busy period distribution are verified with the existing results in the literature. Numerical illustrations are provided for different parameter values to see their effect on performance measures and to get more insight of the model behavior.

The rest of the paper is organized as follows: Section 2 describes the model under investigation. In Section 3, transient system size probabilities are derived and verified with existing results in the literature for a special case. Transient average number of customers and steady-state probabilities are also derived in this section. In Section 4, the busy period analysis is carried out and verified with the existing results in the literature for a special case. In addition, the average busy period duration is derived. Finally, in Section 5, numerical results of transient system size probabilities and busy period distributions are presented to get more insight of the model under investigation.

2. Model description

We consider a single server discrete-time queueing model where the time axis is divided into equal intervals called slots. All queueing activities such as arrivals, departures and total disasters occur only at the slot boundaries. We use the early arrival system (EAS) in this paper [14, 30]. In EAS, arrivals occur just after the beginning of the slots and departures take place just before the end of the slots. An arriving customer who finds the system empty can get served straight away within the same slot.

The pictorial representation of the system is given as follows (Fig. 1):

Let X_m be a discrete random variable denoting the number of customers in the system at the time epoch m. Then $\{X_m : m = 0, 1, 2, \cdots\}$ is a discrete time Markov chain with state space $\{0, 1, 2, \cdots, \}$.

We assume that customer arrivals occur at the beginning of time epoch $m = m^+, m = 0, 1, 2, \cdots$ with identically distributed (iid) inter-arrival times following a geometric distribution with parameter α $(0 < \alpha < 1)$. Let $\bar{\alpha} = 1-\alpha$. We assume that customer service completions occur at discrete time epoch $m = m^-, m = 1, 2, \cdots$ with iid service times following a geometric distribution with parameter β $(0 < \beta < 1)$. Let $\bar{\beta} = 1 - \beta$. Further, when the system is not empty, total disasters/complete catastrophes occur according to a geometric distribution with parameter ξ $(0 < \xi < 1)$. Let $\bar{\xi} = 1-\xi$. A disaster event will make the system instantly empty. Simultaneously, the system becomes ready to accept new customers. We assume that the probability of more than one arrival, disaster and/or departure events during a given slot is zero and that the events in different slots are independent. Therefore, $\alpha + \beta + \xi < 1$.

As disaster occurs at rate ξ, the behaviour of the process involves:

1. an arrival from state i to state $i + 1$ at rate α for $i = 0, 1, 2, \cdots$;
2. a departure from state i to state $i - 1$ at rate β for $i = 1, 2, 3, \cdots$;
3. a catastrophic jump from any state $i \geq 1, i = 1, 2, 3, \cdots$ to state 0 at rate ξ.

The state transition diagram of our model is given in Figure 2.

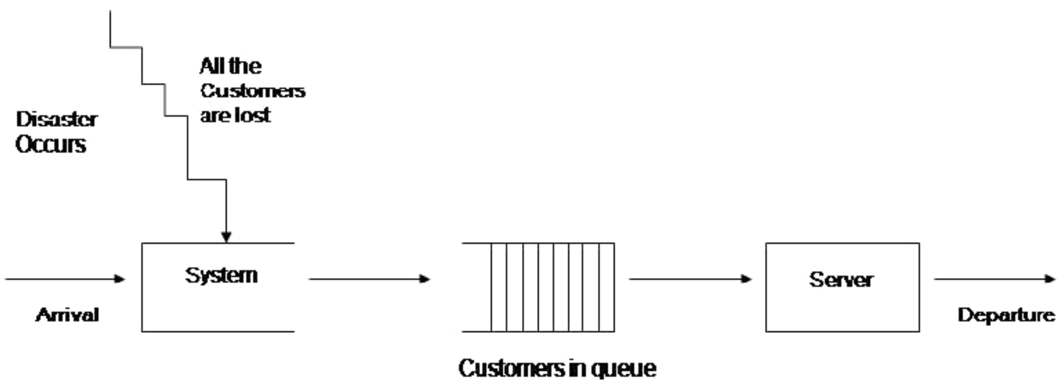

FIGURE 1. Schematic diagram of a queueing model.

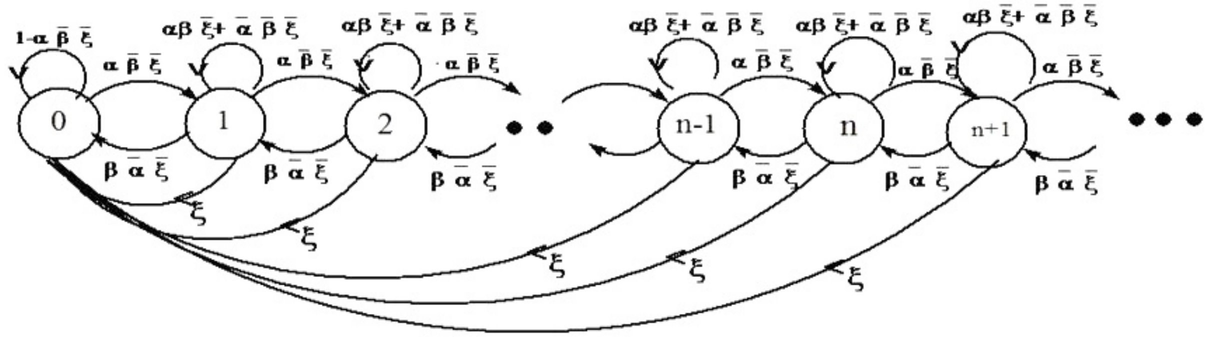

FIGURE 2. Transition rate diagram for the Geo/Geo/1 queue with system disaster.

3. Transient and steady state system probabilities

In this section, we derive the time-dependent and steady state system probabilities in closed form for the Geo/Geo/1 queue described in Section 2.

Let $P_m(n) = P(X_m = n|X_0 = 0),\ m, n = 0, 1, 2, \cdots$ be the probability that there are n customers in the system at time epoch m given that the system is empty at the initial epoch.

Let $P = (P_{ij})$ be the transient probability matrix of the Markov chain.

The (i,j)-component P_{ij} of P is given by

$$P_{ij} = \begin{cases} \alpha\bar{\beta}\bar{\xi} & \text{if } \ j = i+1,\ i \geq 0 \\ \beta\bar{\alpha}\bar{\xi} & \text{if } \ j = i-1,\ i \geq 1 \\ (1-\alpha\bar{\beta}\bar{\xi}) & \text{if } \quad j = 0 \quad i = 0 \\ \alpha\beta\bar{\xi} + \bar{\alpha}\bar{\beta}\bar{\xi} & \text{if } \quad j = i, \quad i \geq 1 \\ \xi & \text{if } \quad j = 0, \quad i \geq 1. \end{cases}$$

Theorem 3.1. *The system size probabilities $P_m(n)$ are given by*

$$P_m(n) = \begin{cases} 0, & \text{if } \ m < n \\ \eta(n, m-n) - \gamma(n, m-n), & \text{if } \ m \geq n \end{cases}, \tag{3.1}$$

where $\eta(n, r)$ and $\gamma(n, r)$ are given by

$$\eta(n, m) = \sum_{k=0}^{[\frac{m}{2}]} b(k, n)\lambda^{n+k}\mu^k \sum_{j=0}^{m-2k} \binom{n+2k-1+j}{j} (1-\lambda-\mu-\xi)^j \tag{3.2}$$

and

$$\gamma(n, m) = \sum_{k=0}^{[\frac{m-1}{2}]} b(k, n+1)\lambda^{n+k+1}\mu^k \sum_{j=0}^{m-2k-1} \binom{n+2k+j}{j} (1-\lambda-\mu-\xi)^j, \tag{3.3}$$

where $[x]$ is the integral part of x, $\eta(0, m) = 1$, for all $\ m \geq 0$, $\gamma(n, 0) = 0$, for all $\ n \geq 0$, $\lambda = \alpha\bar{\beta}\bar{\xi}$, $\mu = \beta\bar{\alpha}\bar{\xi}$, and

$$b(k, n) = \frac{n}{2k+n}\binom{2k+n}{k} \tag{3.4}$$

is a ballot number with $b(0, n) = 1$ for $n \geq 1$, $b(0, 0) = 1$ (see [26]).

Proof. The difference equations governing the model described in Section 2 are given as follows:

$$\begin{aligned} P_{m+1}(0) &= \bar{\alpha}P_m(0) + \alpha\beta\bar{\xi}P_m(0) + \alpha\xi P_m(0) + \beta\bar{\alpha}\bar{\xi}P_m(1) \\ &\quad + \xi[\alpha\beta + \alpha\bar{\beta} + \beta\bar{\alpha} + \bar{\alpha}\bar{\beta}]\sum_{m=1}^{\infty} P_m(n) \\ &= [1 - \alpha\bar{\beta}\bar{\xi} - \xi]P_m(0) + \beta\bar{\alpha}\bar{\xi}P_m(1) + \xi, \qquad (3.5) \\ P_{m+1}(n) &= \alpha\bar{\beta}\bar{\xi}P_m(n-1) + [\alpha\beta\bar{\xi} + \bar{\alpha}\bar{\beta}\bar{\xi}]P_m(n) + \beta\bar{\alpha}\bar{\xi}P_m(n+1) \\ &= \alpha\bar{\beta}\bar{\xi}P_m(n-1) + [1 - \alpha\bar{\beta}\bar{\xi} - \beta\bar{\alpha}\bar{\xi} - \xi]P_m(n) \\ &\quad + \beta\bar{\alpha}\bar{\xi}P_m(n+1), n = 1, 2, 3, \cdots \qquad (3.6) \end{aligned}$$

Let $\lambda = \alpha\bar{\beta}\bar{\xi}$ and $\mu = \beta\bar{\alpha}\bar{\xi}$.

Then the above equations (3.5) and (3.6) can be rewritten as

$$P_{m+1}(0) = (1 - \lambda - \xi)P_m(0) + \mu P_m(1) + \xi, \qquad (3.7)$$

$$P_{m+1}(n) = \lambda P_m(n-1) + (1 - \lambda - \mu - \xi)P_m(n) + \mu P_m(n+1), n = 1, 2, 3, \cdots \qquad (3.8)$$

Let

$$G_z(n) = \sum_{m=0}^{\infty} P_m(n)z^m, \quad |z| \leq 1 \qquad (3.9)$$

be the generating function. □

On applying (3.9) in (3.7) and (3.8), we get

$$G_z(0)\left(\frac{1}{z} - 1 + \lambda + \xi\right) = \frac{P_0(0)}{z} + \mu G_z(1) + \frac{\xi}{1-z}, \qquad (3.10)$$

$$G_z(n)\left(\frac{1}{z} - 1 + \lambda + \mu + \xi\right) - \mu G_z(n+1) - \lambda G_z(n-1) = \frac{P_0(n)}{z}\cdot \qquad (3.11)$$

For the sake of simplicity we set $s = \frac{1}{z} - 1$. This reduces the above system as follows:

$$G_s(0) = \frac{(s+1)(1+\frac{\xi}{s})}{s + \lambda + \xi - \mu\frac{G_s(1)}{G_s(0)}} \qquad (3.12)$$

and

$$\frac{G_s(n)}{G_s(n-1)} = \frac{\lambda}{(s + \lambda + \mu + \xi) - \mu\frac{G_s(n+1)}{G_s(n)}}, \qquad n \geq 1.$$

By iterating, the above equation yields the following continued fraction for $n = 1, 2, 3, \cdots$,

$$\frac{G_s(n)}{G_s(n-1)} = \frac{\lambda}{s + \lambda + \mu + \xi -} \quad \frac{\lambda\mu}{s + \lambda + \mu + \xi -} \quad \frac{\lambda\mu}{s + \lambda + \mu + \xi -} \cdots \qquad (3.13)$$

Using (3.13) in (3.12), we get

$$G_s(0) = \frac{(s+1)(1+\frac{\xi}{s})}{s + \lambda + \xi -} \quad \frac{\lambda\mu}{s + \lambda + \mu + \xi -} \quad \frac{\lambda\mu}{s + \lambda + \mu + \xi -} \cdots, \qquad (3.14)$$

where the notation for the continued fraction is used as

$$\frac{a_1}{b_1-}\quad\frac{a_2}{b_2-}\quad\frac{a_3}{b_3-}\cdots=\cfrac{a_1}{b_1-\cfrac{a_2}{b_2-\cfrac{a_3}{b_3-\cdots}}}.$$

Equation (3.14) can be written as

$$G_s(0)=(s+1)\left(1+\frac{\xi}{s}\right)\left(\frac{1}{s+\lambda+\xi-}\quad\frac{\lambda\mu}{s+\lambda+\mu+\xi-}\quad\frac{\lambda\mu}{s+\lambda+\mu+\xi-}\cdots\right).$$

After some simple algebraic manipulations, the above equation can be rewritten as

$$\begin{aligned}G_s(0)&=\frac{s+1}{s}\left[1-\lambda\left(\frac{s+\lambda+\mu+\xi-\sqrt{(s+\lambda+\mu+\xi)^2-4\lambda\mu}}{2\lambda\mu}\right)\right]\\&=\frac{s+1}{s}\left[1-\lambda\sum_{r=0}^{\infty}C_r\left(\frac{\lambda\mu}{(s+\lambda+\mu+\xi)^2)}\right)^r\frac{1}{s+\lambda+\mu+\xi}\right],\end{aligned}$$

(see [13]) where

$$C_r=\frac{1}{r+1}\binom{2r}{r},\qquad r\geq 0 \tag{3.15}$$

is known as Catalan numbers with $C_0=1$ (see [26]). The relationship between continued fractions and Catalan numbers is given by [13]:

$$1-\quad\frac{x}{1-}\quad\frac{x}{1-}\cdots=\frac{1-\sqrt{1-4x}}{2x}=\sum_{r=0}^{\infty}C_rx^r=C(x), \tag{3.16}$$

$|x|\leq 1/4$, where $C(x)$ is the generating function of Catalan numbers.

Then for $n=1,2,3,\cdots$,

$$\begin{aligned}\frac{G_s(n)}{G_s(n-1)}&=\frac{\lambda}{s+\lambda+\mu+\xi}\left[\sum_{r=0}^{\infty}C_r\left(\frac{\lambda\mu}{(s+\lambda+\mu+\xi)^2}\right)^r\right]\\&=\frac{\lambda C(x)}{s+\lambda+\mu+\xi}=g(s),\ \text{(say)}\end{aligned}$$

where $C(x)$ is the generating function of Catalan numbers with $x=\dfrac{\lambda\mu}{(s+\lambda+\mu+\xi)^2}$.

After some simple algebra we get,

$$G_s(n)=[g(s)]^nG_s(0), \tag{3.17}$$

where

$$g(s)=\frac{\lambda}{s+\lambda+\mu+\xi}C(x).$$

It is known that (see [12], p. 1033):

$$[C(x)]^n=\left(\sum_{r=0}^{\infty}C_rx^r\right)^n=\sum_{k=0}^{\infty}b(k,n)x^k, \tag{3.18}$$

where $b(k,n)$ is the ballot number given in (3.4).

By replacing s by $\frac{1}{z}-1$ in (3.17) and after some simplification we get,

$$G_z(n)=\frac{1}{1-z}[B^n(z)-B^{n+1}(z)],\quad n\geq 1,$$

where

$$B^n(z)=\left[\frac{\lambda z}{(1-z(1-\lambda-\mu-\xi))}\right]^n\left[C\left(\frac{\lambda\mu z^2}{(1-z(1-\lambda-\mu-\xi))^2}\right)\right]^n,\ n\geq 0,$$

which will be obtain from $(g(s))^n$ after changing s by $\frac{1}{z}-1$.

And by using (3.18) for $n=0,1,2,\cdots$, we obtain,

$$\begin{aligned}G_z(n)&=\sum_{m=0}^{\infty}z^m(B^n(z)-B^{n+1}(z))\\&=\sum_{m=0}^{\infty}z^m\left[\frac{\lambda z}{1-z(1-\lambda-\mu-\xi)}\right]^n\sum_{k=0}^{\infty}b(k,n)\left(\frac{\lambda\mu z^2}{(1-z(1-\lambda-\mu-\xi))^2}\right)^k\\&\quad-\sum_{m=0}^{\infty}z^m\left(\frac{\lambda z}{[1-z(1-\lambda-\mu-\xi)]}\right)^{n+1}\sum_{k=0}^{\infty}b(k,n+1)\left(\frac{\lambda\mu z^2}{[1-z(1-\lambda-\mu-\xi)]^2}\right)^k,\end{aligned}$$

where $|\lambda+\mu+\xi|<1$.

The binomial expansion of the terms in the above expression leads to the following for $n\geq 0$:

$$\begin{aligned}G_z(n)&=(\lambda z)^n\sum_{m=0}^{\infty}z^m\sum_{j=0}^{\infty}(1-\lambda-\mu-\xi)^j\sum_{k=0}^{\infty}b(k,n)\binom{n+2k-1+j}{j}z^j(\lambda\mu)^k z^{2k}\\&\quad-\lambda^{n+1}z^n\sum_{m=0}^{\infty}z^{m+1}\sum_{j=0}^{\infty}(1-\lambda-\mu-\xi)^j\sum_{k=0}^{\infty}b(k,n+1)\binom{n+2k+j}{j}z^j(\lambda\mu)^k z^{2k}\\&=(\lambda z)^n\sum_{m=0}^{\infty}z^m\sum_{j=0}^{m}(1-\lambda-\mu-\xi)^j\sum_{k=0}^{\infty}b(k,n)\binom{n+2k-1+j}{j}(\lambda\mu)^k z^{2k}\\&\quad-\lambda^{n+1}z^n\sum_{m=0}^{\infty}z^{m+1}\sum_{j=0}^{m}(1-\lambda-\mu-\xi)^j\sum_{k=0}^{\infty}b(k,n+1)\binom{n+2k+j}{j}(\lambda\mu)^k z^{2k}.\end{aligned}$$

By rearranging the powers of z in the above equation we get,

$$\begin{aligned}G_z(n)&=\sum_{k=0}^{\infty}b(k,n)\lambda^{n+k}\mu^k z^{n+2k}\sum_{m=0}^{\infty}z^m\sum_{j=0}^{m}\binom{n+2k-1+j}{j}(1-\lambda-\mu-\xi)^j\\&\quad-\sum_{k=0}^{\infty}b(k,n+1)\lambda^{n+k+1}\mu^k z^{n+2k}\sum_{m=1}^{\infty}z^m\sum_{j=0}^{m-1}\binom{n+2k+j}{j}(1-\lambda-\mu-\xi)^j\end{aligned}$$

which yields

$$G_z(n) = z^n \sum_{m=0}^{\infty} \eta(n,m) z^m - z^n \sum_{m=1}^{\infty} \gamma(n,m) z^m, \tag{3.19}$$

where $\eta(n,r)$ and $\gamma(n,r)$ are given by (3.2) and (3.3), respectively.

By comparing the coefficients of z^m on both sides of (3.19), we obtain the system size probabilities as in (3.1). Hence the theorem.

Remark 3.2. The average number of customers $E[X_m]$ in the system at time m is given by

$$E[X_m] = \sum_{n=0}^{\infty} n P_m(n), \quad m = 0,1,2\cdots \tag{3.20}$$

Remark 3.3. Since $s = \frac{1}{z} - 1$, using the limit theorem of z-transforms, we can obtain the steady state probabilities π_n for $n = 0,1,2,\cdots$ as

$$\begin{aligned}\pi_n &= \lim_{m\to\infty} P_m(n) = \lim_{s\to 0} sG_s(n)\\ &= \left(\frac{\lambda}{\lambda+\mu+\xi}\right)^n \sum_{k=0}^{\infty} \left[b(k,n) - \frac{\lambda}{\lambda+\mu+\xi} b(k,n+1)\right] \left(\frac{\lambda\mu}{(\lambda+\mu+\xi)^2}\right)^k.\end{aligned}$$

4. Busy period analysis

Busy period of a single server queue is the first passage time to state zero starting from state one. In busy period analysis, we modify the actual model in Section 2 by making state 0 an absorbing state. Let T be a random variable denoting the time until the process reaches the absorbing state 0 starting from state 1 initially. That is, the variable T denotes the duration of a busy period. Let $B(m) = P(T = m)$, $m = 1,2,3,\cdots$, the probability mass function of T.

Theorem 4.1. *The busy period probability $B(m)$ is given by*

$$B(m) = \mu q_{m-1}(1) + \xi[1 - q_{m-1}(0)], \quad m = 1,2,3,\cdots, \tag{4.1}$$

where

$$q_m(n) = P\left(X_m = n, \mid X_0 = 1\right) \tag{4.2}$$

with X_m denoting the number of customers in the system at time m. In particular,

$$\begin{aligned} q_{m+1}(0) = \xi \sum_{k=0}^{m} (1-\xi)^{m-k} + \mu \sum_{k=0}^{[\frac{m}{2}]} C_k (\lambda\mu)^k \\ \times \sum_{j=0}^{m-2k} \binom{2k+j}{j} (1-\lambda-\mu-\xi)^j (1-\xi)^{m-j-2k}, \end{aligned} \tag{4.3}$$

and

$$q_m(1) = \sum_{k=0}^{[\frac{m}{2}]} C_k \binom{m}{2k} (\lambda\mu)^k (1-\lambda-\mu-\xi)^{m-2k}, \tag{4.4}$$

where $\lambda = \alpha\bar{\beta}\bar{\xi}$, $\mu = \beta\bar{\alpha}\bar{\xi}$, and C_k is the Catalan number given by (3.15).

Proof. We note that the function $q_m(n)$ satisfies the following difference equations:

$$q_{m+1}(0) = (1-\xi)q_m(0) + \mu q_m(1) + \xi, \quad n = 0, \tag{4.5}$$

$$q_{m+1}(1) = (1-\lambda-\mu-\xi)q_m(1) + \mu q_m(2), \quad n = 1, \tag{4.6}$$

$$q_{m+1}(n) = \lambda q_m(n-1) + (1-\lambda-\mu-\xi)q_m(n) + \mu q_m(n+1), \; n = 2, 3 \cdots \tag{4.7}$$

By definition of $q_m(n)$, the system starts with one customer at the initial epoch of time. If $G_z(n) = \sum_{m=0}^{\infty} q_m(n) z^m, \quad |z| \leq 1$, then the above equations lead to,

$$G_z(1) = \frac{\frac{1}{z}}{\frac{1}{z} - 1 + \lambda + \mu + \xi -} \quad \frac{\lambda\mu}{\frac{1}{z} - 1 + \lambda + \mu + \xi -} \quad \frac{\lambda\mu}{\frac{1}{z} - 1 + \lambda + \mu + \xi -} \cdot$$

and

$$G_z(0) = \frac{1}{1 - z(1-\xi)} \left[\frac{\xi z}{1-z} + \left(\frac{\mu}{\frac{1}{z} - 1 + \lambda + \mu + \xi -} \quad \frac{\lambda\mu}{\frac{1}{z} - 1 + \lambda + \mu + \xi -} \cdots \right) \right].$$

After some algebra, we get

$$G_z(0) = \frac{1}{1 - z(1-\xi)} \left[\frac{\xi z}{1-z} + \mu \sum_{k=0}^{\infty} C_k \frac{(\lambda\mu)^k z^{2k+1}}{[1 - z(1-\lambda-\mu-\xi)]^{2k+1}} \right] \tag{4.8}$$

and

$$G_z(1) = \sum_{k=0}^{\infty} C_k \frac{(\lambda\mu)^k z^{2k}}{[1 - z(1-\lambda-\mu-\xi)]^{2k+1}} \cdot \tag{4.9}$$

On comparing the coefficients of z^m in (4.8) and (4.9), we get the probabilities $q_{m+1}(0)$ and $q_m(1)$ as in (4.3) and (4.4), respectively. For $m = 1, 2, 3, \cdots$,

$$B(m) = P(T = m) = P(T \leq m) - P(T \leq m-1) \tag{4.10}$$

$$= q_m(0) - q_{m-1}(0), \quad m = 1, 2, 3, \cdots, \tag{4.11}$$

$$= \mu q_{m-1}(1) + \xi[1 - q_{m-1}(0)]. \tag{4.12}$$

The last equation follows from (4.5). Hence the theorem. □

Remark 4.2. If $\xi = 0$, then $B(m)$ agrees with Böhm ([2], p. 10).

Remark 4.3. The average busy period duration $E[T]$ is given by

$$E[T] = \sum_{m=1}^{\infty} m B(m). \tag{4.13}$$

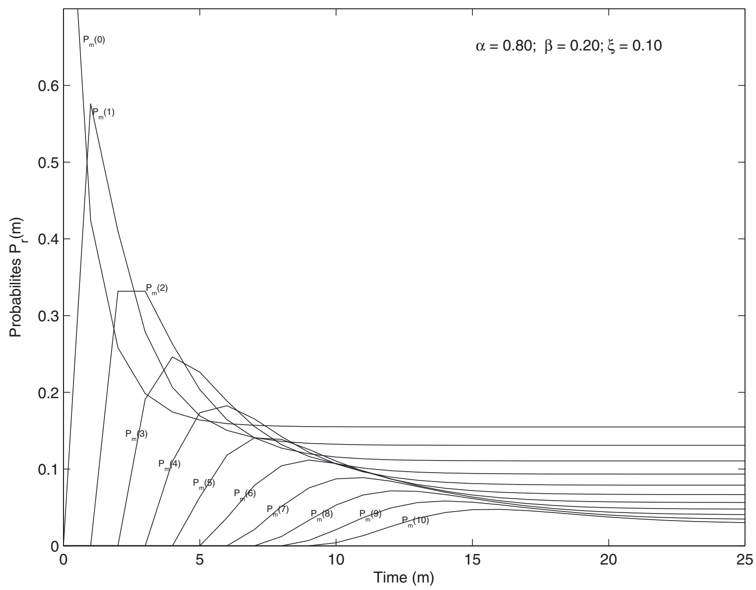

FIGURE 3. System size probabilities $P_m(n)$ for $\alpha = 0.6$, $\beta = 0.2$ and $\xi = 0.1$.

TABLE 1. Expected busy period for different parameter values.

$\beta = 0.2; \xi = 0.2$		$\alpha = 0.2; \xi = 0.2$		$\alpha = 0.2; \beta = 0.2$	
α	$E[T]$	β	$E[T]$	ξ	$E[T]$
0.1	3.168	0.1	4.185	0.1	5.648
0.2	3.517	0.2	3.517	0.2	3.517
0.3	3.821	0.3	2.979	0.3	2.604
0.4	4.080	0.4	2.547	0.4	2.091
0.5	4.299	0.5	2.199	0.5	1.757

5. Numerical results

In this section, we provide some numerical results of the system size probabilities $P_m(n)$, busy period probabilities $B(m)$, expected system size $E[X(m)]$ and busy period $E(T)$ for specific values of α, β and ξ.

Example 5.1. Consider an $Geo/Geo/1$ with arrival, service completion, and disaster probabilities $\alpha = 0.6, \beta = 0.2$ and $\xi = 0.1$, respectively. Using (3.1), the system size probabilities $P_m(n)$ for $n = 0, 1, 2, \cdots, 10$ are calculated and are plotted in Figure 3 for different values of m. For clarity sake, the x- and y-axes are adjusted to show times from 0 to 25 and probabilities from 0 to 0.7, respectively. Because the system starts with 0 customers at time 0, the graph of $P_m(0)$ decreases from 1 while the graphs of $P_m(n)$ for $n \geq 1$ increase from 0, and reach the steady-state around $m = 20$ time units. Note that for $n \geq 1$, the graphs of $P_m(n)$ remain zero until $m < n$ as we expected.

Example 5.2. Consider an $Geo/Geo/1$ with arrival and service completion probabilities $\alpha = 0.2$ and $\beta = 0.2$, respectively. Using (3.20) and (4.1), we computed the average number of customers $E[X_m]$ in the system at time m and busy period probabilities $B(m)$ for different disaster probabilities $\xi = 0.1, 0.2, 0.3, 0.4$ and 0.5, and are plotted in Figures 4 and 5 for different values of m. The expected behavior that the $E[X_m]$ decreases as ξ

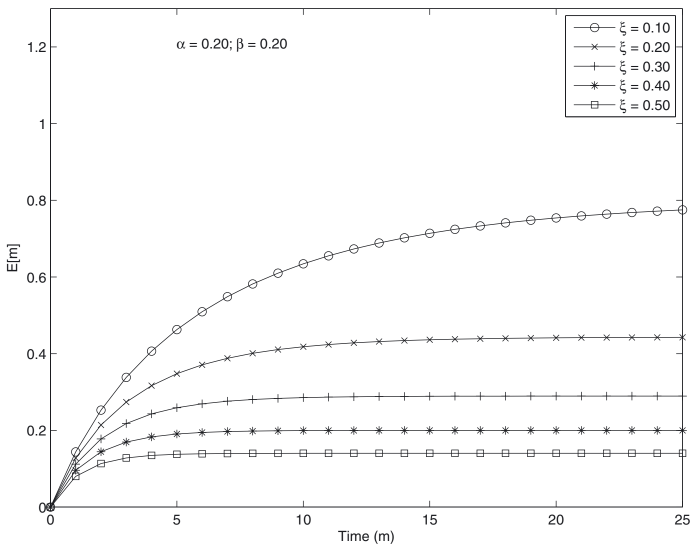

FIGURE 4. $E[X_m]$ for $\alpha = 0.2, \beta = 0.1, 0.2, 0.3, 0.4, 0.5$ and $\xi = 0.2$.

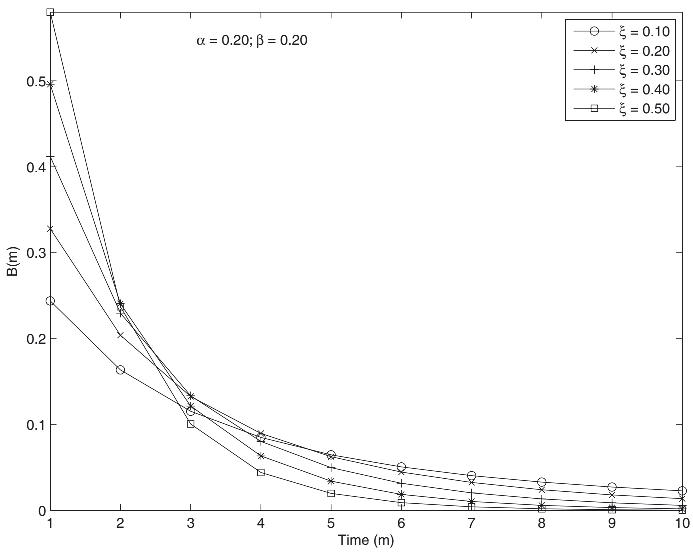

FIGURE 5. $B(m)$ for $\alpha = 0.2, \beta = 0.1, 0.2, 0.3, 0.4, 0.5$ and $\xi = 0.2$.

increases is clear from Figure 4 (more disasters result in less number of customers in the system). The graph of $E[X_m]$ for $\xi = 0.1$ (less disastrous) takes longer time to reach steady-state when compared to the case when $\xi = 0.5$ (more disastrous). The $B(m)$ values in Figure 5 are smaller for larger values of ξ (because large ξ means more disasters which in turn result in shorter busy periods). This behavior is noticed when $m \geq 3$. Further investigation is needed to understand the behavior of graphs of $B(m)$ when $m < 3$. Note that the chance for a busy period to last longer than 10 time units is less than 10%.

Using (4.13), the expected busy period $E[T]$ are computed and tabulated for different parameter values in Table 1. The $E[T]$ increases as α increases and decreases as β and ξ increase.

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