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ON THE AVERAGE LOWER BONDAGE NUMBER OF A GRAPH

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Abstract. The domination number is an important subject that it has become one of the most widely studied topics in graph theory, and also is the most often studied property of vulnerability of communication networks. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centers or connection lines until a communication breakdown. Let $G = (V(G), E(G))$ be a simple graph. The bondage number $b(G)$ of a nonempty graph G is the smallest number of edges whose removal from G result in a graph with domination number greater than that of G. If we think a graph as a modeling of network, the average lower bondage number of a graph is a new measure of the graph vulnerability and it is defined by $b_{av}(G) = \frac{1}{|E(G)|}\sum_{e\in E(G)} b_e(G)$, where the lower bondage number, denoted by $b_e(G)$, of the graph G relative to e is the minimum cardinality of bondage set in G that contains the edge e. In this paper, the above mentioned new parameter has been defined and examined. Then upper bounds, lower bounds and exact formulas have been obtained for any graph G. Finally, the exact values have been determined for some well-known graph families.

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1. Introduction

Graph theory has seen an explosive growth due to interaction with areas like computer science, operation research, *etc*. Especially, it has become one of the most powerful mathematical tools in the analysis and study of the architecture of a network. The study of networks has become an important area of multidisciplinary research involving mathematics, informatics, chemistry, social sciences and other theoretical and applied sciences. A network is described as an undirected and unweighted graph in which vertices represent the processing and edges represent the communication channel between them [11, 17, 18].

It is known that communication systems are often exposed to failures and attacks. So robustness of the network topology is a key aspect in the design of computer networks. The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. In the literature, various measures were defined to measure the robustness of network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. Graph vulnerability relates to the study of graph when some of its elements (vertices or edges) are removed. The measures of graph vulnerability are

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usually invariants that measure how a deletion of one or more network elements changes properties of the network. The best known measure of reliability of a graph is its connectivity. The vertex (edge) connectivity is defined to be the minimum number of vertices (edges) whose deletion results in a disconnected or trivial graph [11]. Then toughness [9], integrity [5], domination number [6,14], bondage number [3,4,10,13], *etc.* have been proposed for measuring the vulnerability of networks. Recently, some average vulnerability parameters such as average lower independence number [2,15], average lower domination number [1,15], average connectivity number [7] have been defined. The average parameters have been found to be more useful in some circumstance than the corresponding measures based on worst-case situation [16].

Let $G = (V(G), E(G))$ be a simple undirected graph of order n. We begin by recalling some standard definitions that we need throughout this paper. For any vertex $v \in V(G)$, the *open neighborhood* of v is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of v in G denoted by $d_G(v)$, is the size of its open neighborhood. The *complement* $\overline{G}$ of a graph G has $V(G)$ as its vertex sets, but two vertex are adjacent in $\overline{G}$ if only if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching* in G. If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a *perfect matching* in G. The smallest integer not less than x is denoted by $\lceil x \rceil$. A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G is denoted by $\gamma(G)$ [6,14].

The study of domination in graphs is an important research area, perhaps also the fastest-growing area within graph theory. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real world problems. For instance, dominating sets in graphs are natural models for facility location problems in operations research [14] or domination number is the one of the most important vulnerability parameter for networks [1,14]. When investigating the domination number of a given graph G, one may want to learn the answer of the following question: How does the domination number increases in a graph G? One of the vulnerability parameters known as *bondage number* in a graph G answers this question. The bondage number $b(G)$ was introduced by Fink *et al.* [10] and is defined as follows:

$$b(G) = \min\{|B| : B \subseteq E(G), \gamma(G - B) > \gamma(G)\}.$$

We call such an edge set B that $\gamma(G - B) > \gamma(G)$ the *bondage set* and the minimum one the *minimum bondage set*. If $b(G)$ does not exist, for example empty graphs, then $b(G) = \infty$ is defined.

In 2004, Henning introduced the concept of *average domination* and *average independence* [15]. Finding largest dominating sets and independent sets in graphs is the problem which is closely in relation with the concept of average domination and average independence. Also, the average lower domination and average lower independence number are the theoretical vulnerability parameters for a network that is represented by a graph [1,2]. The *average lower domination number* of a graph G, denoted by $\gamma_{av}(G)$, is defined as:

$$\gamma_{av}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \gamma_v(G),$$

where the *lower domination number*, denoted by $\gamma_v(G)$, is the minimum cardinality of a dominating set of the graph G that contains the vertex v [8,15].

Our aim in this paper is to define a new vulnerability parameter, so called average lower bondage number. In Section 2, some well-known basic results are given for bondage number. In Section 3, we define a new parameter namely as average lower bondage number denoted by $b_{av}(G)$. In Section 4, we determine upper bounds, lower bounds and exact solutions of the average lower bondage number for any graph G. Finally, the average lower bondage numbers of the popular well-known graphs are computed in Section 5.

2. Basic results

In this section some well-known basic results are given with regard to bondage number.

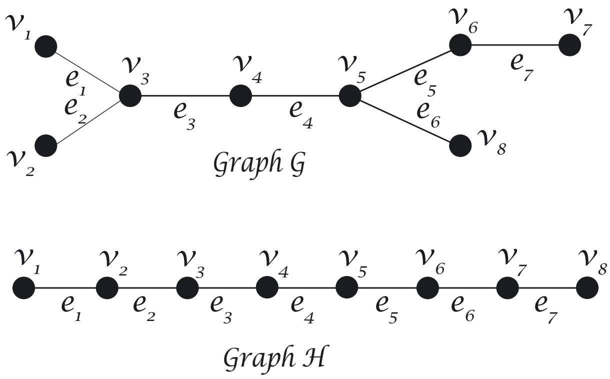

FIGURE 1. Graphs G and H.

Theorem 2.1 ([10]). *For a complete graph K_n of order $n \geq 2$, then $b(K_n) = \lceil \frac{n}{2} \rceil$.*

Theorem 2.2 ([10]). *For a path graph P_n of order $n \geq 2$, then*

$$b(P_n) = \begin{cases} 2 & \text{, if } n \equiv 1 (mod\ 3); \\ 1 & \text{, otherwise.} \end{cases}$$

Theorem 2.3 ([10]). *For a cycle graph C_n of order $n \geq 3$, then*

$$b(C_n) = \begin{cases} 3 & \text{, if } n \equiv 1 (mod\ 3); \\ 2 & \text{, otherwise.} \end{cases}$$

Theorem 2.4 ([10]). *For a star graph $K_{1,n}$ of order $n+1$, where $n \geq 2$. Then,*

$$b(K_{1,n}) = 1.$$

Theorem 2.5 ([19]). *If G is a nonempty graph with a unique minimum dominating set, then $b(G) = 1$.*

3. THE AVERAGE LOWER BONDAGE NUMBER

In this section, we introduce a new graph theoretical parameter: the *average lower bondage number* and it is defined as:

$$b_{av}(G) = \frac{1}{|E(G)|} \sum_{e \in E(G)} b_e(G),$$

where the *lower bondage number*, denoted by $b_e(G)$, of the graph G relative to e is the minimum cardinality of bondage set in the graph G that contains the edge e.

If we think a graph as a modeling of network, the average lower bondage number may be more sensitive than other measures of vulnerability as like connectivity, domination number and bondage number for distinguish two graphs whose number of the vertices and edges are the same. For example, consider two graphs G and H in Figure 1, where $|V(G)| = |V(H)| = 8$ and $|E(G)| = |E(H)| = 7$. They have not only equal connectivity but also equal domination number and bondage number such as $k(G) = k(H) = 1$, $\gamma(G) = \gamma(H) = 3$ and $b(G) = b(H) = 1$. These values could be easily checked by readers. So, how can we distinguish between the graphs G and H?

When the average lower bondage numbers of these two graphs G and H are computed, $b_{av}(G) = \frac{12}{7}$ and $b_{av}(H) = \frac{11}{7}$ are obtained. The results could be checked by readers. Thus, the average lower bondage number may be used for distinguish between these two graphs G and H.

4. Upper bounds, lower bounds and exact formulas

Theorem 4.1. *Let G be any connected graph of order n. Then,*

$$b(G) \leq b_{av}(G) \leq (b(G)+1) - \frac{b(G)}{|E(G)|}.$$

Proof. Let B be a set including minimum bondage sets. We have two cases depending on the cardinality of B.

Case 1. $|B| = 1$.

It is clear that the minimum bondage set is unique and it is denoted by B^*. Let $e_1^*, e_2^*, \ldots, e_{|B^*|}^*$ be elements of B^*. Then we get $b_{e_i^*}(G) = b(G)$, where $i \in \{1, \ldots, |B^*|\}$. The lower bondage number is $b(G)+1$ for every edge of $E(G) \setminus B^*$. Thus, we have

$$\begin{aligned} b_{av}(G) &= \frac{1}{|E(G)|}\left(\sum_{e_i^* \in B^*} b_{e_i^*}(G) + \sum_{e_i \in E(G)\setminus B^*} b_{e_i}(G)\right) \\ &= \frac{1}{|E(G)|}(|B^*|b(G) + (b(G)+1)(|E(G)| - |B^*|)) \\ &= b(G) + 1 - \frac{|B^*|}{|E(G)|}. \end{aligned}$$

Clearly, $|B^*| = b(G)$. Then we have $b_{av}(G) = b(G) + 1 - \frac{b(G)}{|E(G)|}$. It is an upper bound of $b_{av}(G)$.

Case 2. $|B| > 1$.

If the union of the minimum bondage sets is equal to $E(G)$, then the lower bondage number is $b(G)$ for every edge of $E(G)$. Thus, we get $b_{av}(G) = b(G)$ is also lower bound.

As a result, $b(G) \leq b_{av}(G) \leq (b(G)+1) - \frac{b(G)}{|E(G)|}$ is obtained. Hence the proof is completed. □

Theorem 4.2. *Let G be a connected graph of order n. If the graph G has unique minimum dominating set, then*

$$b_{av}(G) \geq 2 - \frac{\sum_{i=1}^{\gamma(G)} d_G(v_i^*)}{|E(G)|},$$

where minimum dominating set includes v_i^* for $1 \leq i \leq \gamma(G)$.

Proof. Let $S \subseteq V(G)$, and let S be a unique dominating set which includes vertices v_i^*, where $i \in \{1, \ldots, \gamma(G)\}$. Clearly, $|S| = \gamma(G)$. A set which includes edges that are incident to each vertex of S is denoted by B^*. Then let $e_1^*, e_2^*, \ldots, e_{|B^*|}^*$ be elements of B^*, and let $e_1, e_2, \ldots, e_{|E(G)|-|B^*|}$ be elements of $E(G) \setminus B^*$. We have two cases depending on the cardinality of S.

Case 1. $|S| = 1$.

The vertex of S is denoted by v_1^*. We know that $b_{e_i^*}(G) = 1$ for all $e_i^* \in B^*$, where $i \in \{1, \ldots, |B^*|\}$ by the Theorem 2.5. It is not difficult to see that $b_{e_i}(G) = 2$ for all $e_i \in E(G) \setminus B^*$, where $i \in \{1, \ldots, |E(G)| - |B^*|\}$. Thus, we have

$$\begin{aligned} b_{av}(G) &= \frac{1}{|E(G)|}\left(\sum_{e_i^* \in B^*} b_{e_i^*}(G) + \sum_{e_i \in E(G)\setminus B^*} b_{e_i}(G)\right) \\ &= \frac{1}{|E(G)|}(|B^*| + 2(|E(G)| - |B^*|)) \\ &= 2 - \frac{|B^*|}{|E(G)|}. \end{aligned}$$

Clearly, $|B^*| = d_G(v_1^*) = n - 1$. Then $b_{av}(G) = 2 - \frac{n-1}{|E(G)|}$ is obtained.

Case 2. $|S| > 1$.

We have two subcases depending on the intersection of closed neighborhood sets of each pair of vertices of S.

Subcase 1. If the intersection of closed neighborhood sets of each pair of vertices of S is empty, that is $N_G[v_i^*] \cap N_G[v_j^*] = \emptyset$ for all distinct $i, j \in \{1, 2, \ldots, |S|\}$, then obviously we have $b_{e_i^*}(G) = 1$ for all $e_i^* \in B^*$. Furthermore, we have $b_{e_i}(G) = 2$ for all $e_i \in E(G) \setminus B^*$. Then we know that $b_{av}(G) = 2 - \frac{|B^*|}{|E(G)|}$ by the Case 1. Clearly, $|B^*| = \sum_{i=1}^{\gamma(G)} d_G(v_i^*)$. Thus, $b_{av}(G) = 2 - \frac{\sum_{i=1}^{\gamma(G)} d_G(v_i^*)}{|E(G)|}$ is obtained.

Subcase 2. If at least one intersection of closed neighborhood sets of at least one pair of vertices of S is not empty, then obviously the graph G has either at least one edge between any two vertices of S, or at least one vertex which is adjacent to any two vertices of S. Then $b_{e_i^*}(G) = 2$ is obtained for at least one edge $e_i^* \in B^*$. Thus, we get either $b_{e_i^*}(G) = 1$, or $b_{e_i^*}(G) = 2$ for all $e_i^* \in B^*$. Furthermore, we have $b_{e_i}(G) = 2$ for all $e_i \in E(G) \setminus B^*$. Clearly, $|B^*| \leq \sum_{i=1}^{\gamma(G)} d_G(v_i^*)$. Because of $b_{e_i^*}(G) \leq 2$ for all $e_i^* \in B^*$ and $|B^*| \leq \sum_{i=1}^{\gamma(G)} d_G(v_i^*)$, we get $b_{av}(G) > 2 - \frac{\sum_{i=1}^{\gamma(G)} d_G(v_i^*)}{|E(G)|}$.

As a result, we obtain $b_{av}(G) \geq 2 - \frac{\sum_{i=1}^{\gamma(G)} d_G(v_i^*)}{|E(G)|}$. The proof is completed. □

Theorem 4.3. *Let G be a connected graph of order n and the domination number $\gamma(G) = 1$, and let s be the number of vertices of degree $n-1$, where $s \geq 2$. Then,*

$$b_{av}(G) = \begin{cases} \frac{s+2}{2} - \frac{\binom{s}{2}}{|E(G)|}, & \text{if } s \text{ is even;} \\ \frac{s+3}{2} - \frac{s(n-1)-\binom{s}{2}}{|E(G)|}, & \text{if } s \text{ is odd.} \end{cases}$$

Proof. Let v_i^* be vertices of degree $n-1$, where $i \in \{1, \ldots, s\}$. These vertices form a complete graph, and also it is denoted by K_s. We know that $b(K_s) = \lceil \frac{s}{2} \rceil$ by the Theorem 2.1 and $|E(K_s)| = \binom{s}{2}$. Let x be $\binom{s}{2}$, and let $e_1^*, e_2^*, \ldots, e_x^*$ be elements of $E(K_s)$. We have two cases depending on s.

Case 1. s is even.

Since s is even, $b(K_s) = \frac{s}{2}$ is obtained. The removal of a perfect matching from the sub graph K_s reduces the degree of each vertex to $n-2$ and therefore yields the graph G with $\gamma(G) = 2$. Clearly, $b(G) = \frac{s}{2}$. We know that a perfect matching including any edge e_i^* is removed from the subgraph K_s, the domination number of G increases. Hence we have $b_{e_i^*}(G) = \frac{s}{2}$, where $i \in \{1, \ldots, x\}$. Clearly, the lower bondage number is $\frac{s}{2} + 1$ for every edge of $E(G) \setminus E(K_s)$. Thus, we have

$$\begin{aligned} b_{av}(G) &= \frac{1}{|E(G)|} \left(\sum_{e_i^* \in E(K_s)} b_{e_i^*}(G) + \sum_{e_i \in E(G) \setminus E(K_s)} b_{e_i}(G) \right) \\ &= \frac{1}{|E(G)|} \left(x \left(\frac{s}{2} \right) + (|E(G)| - x) \left(\frac{s}{2} + 1 \right) \right) \\ &= \frac{s+2}{2} - \frac{x}{|E(G)|} \\ &= \frac{s+2}{2} - \frac{\binom{s}{2}}{|E(G)|}. \end{aligned}$$

Case 2. s is odd.

Since s is odd, $b(K_s) = \frac{s+1}{2}$ is obtained. The removal of a maximum matching from the subgraph K_s leaves the graph G having exactly one vertex of degree $n-1$. Together with the maximum matching, when an edge which is incident to the vertex of degree $n-1$ is removed from the graph G yields the graph G with $\gamma(G) = 2$. Clearly, $b(G) = \frac{s+1}{2}$. Let B^* be a set which includes edges that are adjacent to each edge of the subgraph K_s, and let $e'_i \in E(K_s) \cup B^*$. Clearly, $|E(K_s) \cup B^*| = s(n-1) - x$. We know that a minimum bondage set including any edge e'_i is removed from the graph G, the domination number of G increases. Thus we have $b_{e'_i}(G) = \frac{s+1}{2}$, where $i \in \{1, \ldots, (s(n-1)-x)\}$. Clearly, the lower bondage number is $\frac{s+3}{2}$ for every edge of $E(G) \setminus (E(K_s) \cup B^*)$. Thus, we have

$$\begin{aligned}
b_{av}(G) &= \frac{1}{|E(G)|}\left(\sum_{e'_i \in E(K_s) \cup B^*} b_{e'_i}(G) + \sum_{e_i \in E(G) \setminus (E(K_s) \cup B^*)} b_{e_i}(G)\right) \\
&= \frac{1}{|E(G)|}\left((s(n-1)-x)\left(\frac{s+1}{2}\right) + (|E(G)| - (s(n-1)-x))\left(\frac{s+3}{2}\right)\right) \\
&= \frac{s+3}{2} - \frac{s(n-1)-x}{|E(G)|} \\
&= \frac{s+3}{2} - \frac{s(n-1) - \binom{s}{2}}{|E(G)|} \cdot
\end{aligned}$$

The proof is completed. $\square$

Corollary 4.4. *Let G be a connected graph of order n and the domination number $\gamma(G) = 1$, and let s be the number of vertices of degree $n-1$. If $s = 1$, then*

$$b_{av}(G) = 2 - \frac{n-1}{|E(G)|} \cdot$$

Proof. Since the minimum dominating set is unique, the proof is done as in the Case 1 of Theorem 4.2. $\square$

Definition 4.5 ([12])**.** Let $p_1, p_2, \ldots, p_n$ be a non-negative integers and the graph G be such a graph, where $|V(G)| = n$. The thorn graph of the graph G, with parameters $p_1, p_2, \ldots, p_n$ is obtained by attaching p_i new vertices of degree one to the vertex u_i of the graph G, where $i \in \{1, \ldots, n\}$. The thorn graph of the graph G will be denoted by G^*, or if the respective parameters need to be specified, by $G^*(p_1, p_2, \ldots, p_n)$.

Theorem 4.6. *Let G be any connected graph of order n. If G^* is a thorn graph of G with $p_i \geq 2$. Then,*

$$b_{av}(G^*) = 1 + \frac{|E(G)|}{|E(G^*)|} \cdot$$

Proof. The domination number of thorn graph G^* is $\gamma(G^*) = |V(G)|$. Let $S \subseteq V(G^*)$ and $N_{G^*}[S] = V(G^*)$. Clearly, S is equal to the complement of $V(G^*) \setminus V(G)$ and it is unique dominating set. So, we have $b(G^*) = 1$ by the Theorem 2.5. When an edge which belongs to the set $E(G^*) \setminus E(G)$ removed from G^*, this value is obtained. Let $e^*_1, e^*_2, \ldots, e^*_{|E(G^*) \setminus E(G)|}$ be elements of $E(G^*) \setminus E(G)$. Clearly, we have $b_{e^*_i}(G^*) = 1$, where $i \in \{1, \ldots, |E(G^*) \setminus E(G)|\}$. Furthermore, let $e_1, e_2, \ldots, e_{|E(G)|}$ be elements of $E(G)$. Then we get $b_{e_i}(G^*) = 2$, where $i \in \{1, \ldots, |E(G)|\}$. Thus, we have

$$\begin{aligned} b_{av}(G^*) &= \frac{1}{|E(G^*)|}\left(\sum_{e_i^* \in E(G^*) \setminus E(G)} b_{e_i^*}(G^*) + \sum_{e_i \in E(G)} b_{e_i}(G^*)\right) \\ &= \frac{1}{|E(G^*)|}((|E(G^*)| - |E(G)|) + 2(|E(G)|)) \\ &= 1 + \frac{|E(G)|}{|E(G^*)|}. \end{aligned}$$

The proof is completed. □

Corollary 4.7. *Let G be any connected graph of order n. If G^* is a thorn graph of G with $p_i \geq 2$. Then,*

$$b_{av}(G^*) = \frac{\left(\sum_{i=1}^{|V(G)|}(p_i)\right) + 2|E(G)|}{|E(G^*)|}.$$

Proof. Because of the definition of thorn graph G^*, we have $|E(G^*)| - |E(G)| = |V(G^*)| - |V(G)|$. So, $|E(G^*)| = |V(G^*)| - |V(G)| + |E(G)|$ is obtained. Clearly, $|V(G^*)| - |V(G)| = \sum_{i=1}^{|V(G)|}(p_i)$.

Hence we get $b_{av}(G^*) = \frac{(\sum_{i=1}^{|V(G)|}(p_i)) + 2|E(G)|}{|E(G^*)|}$ by the Theorem 4.6. □

5. The average lower bondage number of some well-known graphs

In this section we calculate the average lower bondage number of some well known graphs such as the path graph P_n, the cycle graph C_n, the complete graph K_n, the star graph $K_{1,n}$ and the wheel graph $W_{1,n}$.

Theorem 5.1. *Let P_n be a path graph of order $n \geq 2$. Then,*

$$b_{av}(P_n) = \begin{cases} \frac{4n-6}{3n-3}, & \text{if } n \equiv 0(mod\ 3); \\ 2, & \text{if } n \equiv 1(mod\ 3); \\ \frac{5n-7}{3n-3}, & \text{if } n \equiv 2(mod\ 3). \end{cases}$$

Proof. While we are calculating the average lower bondage number of the path graph P_n, we have three cases according to the number of vertices of P_n.

Case 1. $n \equiv 0(mod\ 3)$.

It is clear that the dominating set of P_n is unique. By the Theorem 4.2, we have

$$b_{av}(P_n) = \frac{4n-6}{3n-3}.$$

Case 2. $n \equiv 1\,(mod\ 3)$.

We know that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ (see [14]). The removal of an edge from P_n leaves a graph H consisting of two paths P_{n_1} and P_{n_2}, where $n_1 + n_2 = n$. Then either $n_1 \equiv 1\,(mod\ 3)$ and $n_2 \equiv 0\,(mod\ 3)$, or $n_1 \equiv n_2 \equiv 2\,(mod\ 3)$. In the former case,

$$\begin{aligned} \gamma(H) = \gamma(P_{n_1}) + \gamma(P_{n_2}) &= \left\lceil \frac{n_1}{3} \right\rceil + \left\lceil \frac{n_2}{3} \right\rceil \\ &= \frac{n_1+2}{3} + \frac{n_2}{3} = \frac{n+2}{3} = \left\lceil \frac{n}{3} \right\rceil = \gamma(P_n). \end{aligned}$$

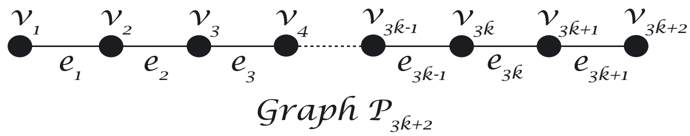

FIGURE 2. The path graph P_n of order $n = 3k + 2$.

In the latter case,

$$= \frac{n_1+1}{3} + \frac{n_2+1}{3} = \frac{n+2}{3} = \left\lceil \frac{n}{3} \right\rceil = \gamma(P_n).$$

In either case, we get $b(P_n) \geq 2$.

Let H be a graph obtained from the deletion of two adjacent edges of P_n. Then either H may consist of two isolated vertices and a path of order $n-2$, or H may consist of an isolated vertex and two paths P_{n_1} and P_{n_2}, where $n_1 + n_2 = n - 1$. Furthermore, we get either $n_1 \equiv 2(mod\ 3)$ and $n_2 \equiv 1(mod\ 3)$, or $n_1 \equiv n_2 \equiv 0(mod\ 3)$.

If H consists of two isolated vertices and a path of order $n-2$, then we have

$$\begin{aligned}\gamma(H) &= 2 + \gamma(P_{n-2}) = 2 + \left\lceil \frac{n-2}{3} \right\rceil = 2 + \frac{n-1}{3} = 2 + \left(\left\lceil \frac{n}{3} \right\rceil - 1 \right) \\ &= 1 + \left\lceil \frac{n}{3} \right\rceil = 1 + \gamma(P_n),\end{aligned}$$

whence $b(P_n) \leq 2$, and so $b(P_n) = 2$. To calculate the lower bondage number of every edge of P_n, the examining of two subcases is sufficed. These subcases as below:

Subcase 1. If H consists of two isolated vertices and a path of order $n-2$. This case is done above.

Subcase 2. If H consists of an isolated vertex and two paths P_{3m+2} and P_{3s+1}, where $m \geq 0$ and $s \geq 0$. Clearly, $(3m+2) + (3s+1) = n - 1$. So we have $m + s = \frac{n-4}{3}$. Thus,

$$\begin{aligned}\gamma(H) &= 1 + \gamma(P_{3m+2}) + \gamma(P_{3s+1}) = 1 + \left\lceil \frac{3m+2}{3} \right\rceil + \left\lceil \frac{3s+1}{3} \right\rceil = 1 + (m+1) + (s+1) \\ &= 3 + (m+s) = 3 + \frac{n-4}{3} = 3 + \left(\left\lceil \frac{n}{3} \right\rceil - 2 \right) \\ &= 1 + \left\lceil \frac{n}{3} \right\rceil = 1 + \gamma(P_n).\end{aligned}$$

Let e_i be any edges of the graph P_n. As a result, we obtain $b_{e_i}(P_n) = 2$ for all $e_i \in E(P_n)$ by the Subcases 1 and 2. Hence $b_{av}(P_n) = 2$ is obtained.

Case 3. $n \equiv 2(mod\ 3)$.

We know that the graph P_n has $(n-1)$- edges also are labeled by e_i, where $i \in \{1, \ldots, n-1\}$. The graph P_{3k+2} whose vertices and edges are labeled is shown in Figure 2.

We have $b(P_n) = 1$ for $n = 3k + 2$ by the Theorem 2.2. This value is obtained when an edge $\{e_{3k+1}\}$ is removed from the P_n, where $k \in \{0, \ldots, \frac{n-2}{3}\}$. Thus, we have $b_{e_i}(P_n) = 1$ for these edges. Clearly, the lower bondage number of the remaining edges is $b_{e_i}(P_n) = 2$. If we think that the edge set of P_n be $E(P_n) = E_1 \cup E_2$, as follows:

E_1: The set contains edges which are labeled by $\{e_{3k+1}\}$, where $k \in \{0, \ldots, \frac{n-2}{3}\}$.
E_2: The set contains edges of $E(P_n) \setminus E_1$.

Clearly, $|E_1| = \frac{n+1}{3}$ and $|E_2| = \frac{2n-4}{3}$. Thus, we have

$$\begin{aligned} b_{av}(P_n) &= \frac{1}{|E(P_n)|}\left(\sum_{e_i \in E_1} b_{e_i}(P_n) + \sum_{e_i \in E_2} b_{e_i}(P_n)\right) \\ &= \frac{1}{n-1}\left(\left(\frac{n+1}{3}\right) + 2\left(\frac{2n-4}{3}\right)\right) \\ &= \frac{5n-7}{3n-3}. \end{aligned}$$

The proof is completed. □

Theorem 5.2. *Let C_n be a cycle graph of order $n \geq 3$. Then,*

$$b_{av}(C_n) = \begin{cases} 3, & \text{if } n \equiv 1 (mod\ 3); \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Let e_i be any edges of the graph C_n. We know that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ (see [14]). We have two cases depending on n.

Case 1. $n \equiv 0, 2\, (mod\ 3)$.

Due to $\gamma(C_n) = \gamma(P_n)$, we have $b(C_n) \geq 2$. The removal of two adjacent edges from the graph C_n leaves a graph H consisting of an isolated vertex and a path of order $n-1$. Thus,

$$\gamma(H) = 1 + \gamma(P_{n-1}) = 1 + \left\lceil \frac{n-1}{3} \right\rceil = 1 + \left\lceil \frac{n}{3} \right\rceil = 1 + \gamma(C_n),$$

so that $b(C_n) \leq 2$. Thus, $b(C_n) = 2$. Since $b(C_n) = 2$ is obtained when any two adjacent edges are removed from the graph C_n, we get $b_{e_i}(C_n) = 2$ for all $e_i \in E(C_n)$. Hence $b_{av}(C_n) = 2$ is obtained.

Case 2. $n \equiv 1 (mod\ 3)$.

We know that $b(C_n) \geq 3$ for $n = 3k+1$ by the definition of C_n and the Case 2 of Theorem 5.1. Furthermore, we know that the domination number of C_n increases when any three consecutive edges of C_n are removed by the Theorem 2.3. Due to $b(C_n) \leq 3$, we have $b(C_n) = 3$. Clearly, we get $b_{e_i}(C_n) = 3$ for all $e_i \in E(C_n)$. Hence $b_{av}(C_n) = 3$ is obtained.

The proof is completed. □

Theorem 5.3. *Let K_n be a complete graph of order $n \geq 2$. Then,*

$$b_{av}(K_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Theorem 4.3 is ensured for the graph K_n. We have two cases in the proof according to the parity of the number of vertices of K_n. Therefore, we know that $|V(K_n)| = n$, $|E(K_n)| = \binom{n}{2}$, and also the graph K_n has n-vertices of degree $n-1$. By the Theorem 4.3, the average lower bondage number is $\frac{n}{2}$ and $\frac{n+1}{2}$ for n is even number and odd number, respectively. Thus, $b_{av}(K_n) = \lceil \frac{n}{2} \rceil$ is obtained. □

Theorem 5.4. *Let $K_{1,n}$ be a star graph of order $n+1$, where $n \geq 2$. Then,*

$$b_{av}(K_{1,n}) = 1.$$

Proof. Since dominating set is unique, $|V(K_{1,n})| = n+1$ and $|E(K_{1,n})| = n$, we have $b_{av}(K_{1,n}) = 1$ by the Case 1 of Theorem 4.2. □

Theorem 5.5. *Let $W_{1,n}$ be a wheel graph of order $n+1$, where $n \geq 3$. Then,*

$$b_{av}(W_{1,n}) = \frac{3}{2}.$$

Proof. Since dominating set is unique, $|V(W_{1,n})| = n+1$ and $|E(W_{1,n})| = 2n$, we have $b_{av}(W_{1,n}) = \frac{3}{2}$ by the Case 1 of Theorem 4.2. $\square$

6. Conclusion

In this study, a new graph theoretical parameter namely the average lower bondage number has been presented for the network vulnerability. The present parameter has been constructed by summing of the lower bondage number of every edge of a graph divided by the number of edges of the graph. Additionally, the stability of popular interconnection networks has been studied and their average lower bondage numbers have been computed. These networks have been modeled with the complete graphs, the path graphs, the cycle graphs, the star graphs and the wheel graphs. Then upper bounds, lower bounds and exact formulas of the average lower bondage number have been obtained for any given graph G. As a further study, exact formulas or bounds may be obtained for graph operations and trees.

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