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COMPUTATION AND PROFIT ANALYSIS OF A k-OUT-OF-n:G REPAIRABLE SYSTEM UNDER N-POLICY WITH MULTIPLE VACATIONS AND ONE REPLACEABLE REPAIR FACILITY *

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Abstract. This paper considers a k-out-of-n:G system with N-policy and one repairman who takes multiple vacations, in which the operating times and repair times of components are governed by exponential distributions. Once an operating component breaks down, it is repaired by a repair facility. Moreover, the repair facility is subject to failure during the repair period which results in repair interruptions. Failed repair facility resumes repair after a random period of time. Under such assumptions, applying the quasi-birth-and-death process and the matrix-analytical approach, the system state probabilities are derived. In addition, various steady-state system performance measures such as the availability and the rate of occurrence of failure along with some numerical illustrations are reported. Finally, under a profit structure, we use the direct search method and the parabolic method to search for the optimal system parameters.

Keywords. k-out-of-n:G repairable system, replaceable repair facility, matrix-analytical approach, performance measures, profit analysis.

Mathematics Subject Classification. 60K20, 90B25.

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1. Introduction

In reliability theory, the k-out-of-n:G system as a popular type of redundancy is often encountered in industrial systems. A k-out-of-n:G system consists of n components, in which all n components are active initially even though only k of them are required for the system to be normal. Classical examples of its applications include the power transmission and distribution systems, the communication systems with multiple transmitters, the cables in a suspension bridge, the multi-display system in a cockpit, the multi-pump system in a hydraulic control system and the multi-engine system in an aircraft. Extensive analysis of k-out-of-n:G systems are referred to the monographs by Cao and Cheng [2], and Kuo and Zuo [12]. At an early stage, authors discussed the k-out-of-n:G system under some basic assumptions (see Birnbaum *et al.* [1], Gupta and Sharma [7], and Fawzi and Hawkes [5]). Later, several repair control policies such as T-policy, D-policy, N-policy and their combinations are introduced into k-out-of-n:G systems. For example, Krishnamoorthy and Rekha [9], Krishnamoorthy and Ushakumari [10], Krishnamoorthy *et al.* [11], Ushakumari and Krishnamoorthy [17] as well as the references therein. In [11], authors discussed a k-out-of-n:G system with shut-off rules and N-policy, in which a repairman is activated for repair as soon as the number of broken components accumulates to a predefined value $N(1 \leq N \leq n-k+1)$. Besides, due to the mathematical complexity of the k-out-of-n:G system, Khatab *et al.* [8] proposed an algorithm for automatic construction of the system state transition diagram to analyze the availability of the system with repair priority rule and non-identical components. Subsequently, Moghaddass *et al.* [13] generalized this work to a k-out-of-n:G system with similar or different repair priorities and shut-off rules.

In these studies mentioned above, authors usually assumed that the repairman remains idle until a broken component is present or repair control policies are realized, which will lead to a waste of human resources. In many real life situations, the repairman may perform another assigned job during his/her idle period. The time spent by the repairman to take other secondary tasks is called vacation time. Motivated by vacation queueing theory (see Doshi [4], Tian and Zhang [16]), several authors introduced the "repairman's vacations" into the reliability theory. Yu *et al.* [22] analyzed a phase-type geometric process simple repairable system with spare device procurement lead time and repairman's multiple vacations. Guo *et al.* [6] studied an n-unit series repairable system with a repairman following multiple delayed vacations and a replaceable repair facility. Wu and Ke [20] investigated a machine repair problem under a single vacation policy where the operating times and repair times of machines, the vacation times of repairman all follow exponential distributions. The model is analyzed using the matrix-analytical approach, and the steady-state system probabilities along with some important performance measures, such as system availability and the expected number of failed machines have been calculated. Meanwhile, a total expected cost function per unit time is developed. Yuan [23] considered a k-out-of-n:G system with redundant dependency and repairman's multiple vacations, in which the operating times, the repair times and the vacation times are exponentially distributed random variables.

Traditionally, it is generally supposed that the repair facility (or service station) for the broken components does not fail in repair times. However, in practice, the repair facility may encounter unpredictable failure during the repair period owing to human operational errors, temperature changes and voltage fluctuations. Thus, repairable systems with replaceable repair facility are more general. Whenever the repair facility fails, it should be replaced by a new and identical one. Among some excellent papers in this area are those by Cao and Wu [3], Tang [15], Wang [18], Wang and Kuo [19], Yu *et al.* [21], Zhang and Wu [24]. Therefore, the case of replaceable repair facility is taken into account in this paper.

Based on the above research works, we note that both the "repairman's vacation policy" and "replaceable repair facility" are two important factors in a repairable system, and their influences on the system performance measures can not be neglected. Hence, this inspires us to consider a k-out-of-n:G repairable system with one repairman who takes multiple vacations and a replaceable repair facility. Moreover, in order to make our model more reasonable and flexible, we assume that the repair initiation point is under the control of N-policy. Employing the quasi-birth-and-death process and matrix-analytical approach, the steady-state system performance measures such as the availability, the rate of occurrence of failure (ROCOF), the probability that the system is waiting for repair, the mean waiting time of a broken component and so on are derived. In addition, a profit function per component per unit time is constructed. We then use the direct search method and the parabolic method to determine the optimal values of the number of operating components n and the repair rate μ that will yield the maximum profit.

The rest of this paper is organized as follows. Section 2 provides the assumptions of the system. In Section 3, using the quasi-birth-and-death process and matrix-analytical approach, we obtain the steady-state probabilities. Various system performance measures are given in Section 4. Moreover, Section 5 presents some numerical examples. In Section 6, we develop a profit model to determine the optimal system parameters. Section 7 gives conclusions.

2. Basic model assumptions

We investigate a k-out-of-n:G ($k = 1, 2, \ldots, n$) repairable system under N-policy with a repairman following multiple vacations and one replaceable repair facility by making the following assumptions.

Assumption 2.1. The k-out-of-n:G system consists of n identical and independent components. It functions as long as there are at least k of the n components operate. The system is down until the number of broken components goes up to $n - k + 1$. When the system is down, no other operating components may break down any more.

Assumption 2.2. Initially, a repairable system with n new components and a new repair facility is installed, and the repairman starts to leave for a vacation.

Assumption 2.3. The operating time $X^{\langle 1\rangle}$ of each component has an exponential distribution with parameter $\lambda\,(\lambda > 0)$. Broken components form a single waiting line and receive repair provided by the single repairman in the order of their breakdowns, *i.e.*, FCFS discipline. The repair time χ of every broken component follows an exponential distribution with parameter $\mu\,(\mu > 0)$.

Assumption 2.4. The repairman leaves for a vacation whenever there is no broken component in the system. Upon returning from the vacation, if there are at least $N(1 \leq N \leq n-k+1)$ broken components waiting for repair, the repairman begins to repair components but one by one. Otherwise, he/she leaves for another vacation. Vacations are taken repeatedly until at least N broken components are waiting for repair. The vacation time V follows an exponential distribution $V(t) = 1 - \exp(-\theta t), \theta \geq 0, t \geq 0$.

Assumption 2.5. The repair facility may fail during the repair time χ. Once the repair facility fails, it should be replaced by a new and identical one, while the broken component which is being repaired has to wait. The repair facility resumes repair after completion of its replacement. Moreover, the working time $X^{\langle 2\rangle}$ and the replacement time $Y^{\langle 2\rangle}$ of the repair facility are governed by exponential distributions with parameters $\alpha(0 \leq \alpha < \infty)$ and $\beta(0 \leq \beta < \infty)$, respectively.

Assumption 2.6. The random variables $X^{\langle 1\rangle}$, χ, V, $X^{\langle 2\rangle}$, and $Y^{\langle 2\rangle}$ are assumed to be independent of each other.

Remark 2.7. $k = 1$ is a particular case of our system. In this case, the k-out-of-n:G repairable system is reduced to the machine repair problem. That means our system is a generalized repairable system.

3. System analysis

3.1. Steady-state equations

Denote by $L(t)$ the number of broken components (either waiting or being repaired) in the system at time t, $L(t) = i\,(i = 0, 1, \ldots, n-k+1)$. Let $J(t)$ represent the state of the system at time t, and

$$J(t) = \begin{cases} 0, \text{ the repairman is on vacation at time } t, \\ 1, \text{ the repairman is repairing broken components at time } t, \\ 2, \text{ the repair facility is being replaced at time } t. \end{cases}$$

Under the given assumptions, the stochastic process $\{L(t), J(t), t \geq 0\}$ is a quasi-birth-and-death (QBD) process with finite state space

$$\Omega = \{(i,0) : i = 0, 1, \ldots, n-k+1\} \cup \{(i,j) : i = 1, 2, \ldots, n-k+1, j = 1, 2\}.$$

Further, the state transition diagram of the system is shown in Figure 1.

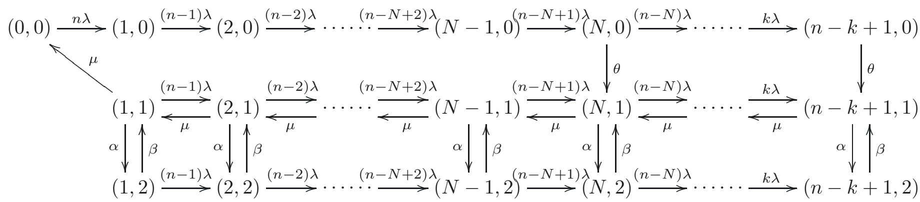

FIGURE 1. State transition diagram of the system.

The description of the states represented by a pair in Figure 1 is provided below.

• $(i, 0)$, $i = 0, 1, \ldots, n-k+1$: there are i broken components in the system, and the repairman is on vacation.

• $(i, 1)$, $i = 1, 2, \ldots, n-k+1$: there are i broken components in the system, and the repairman is busy with a broken component.

• $(i, 2)$, $i = 1, 2, \ldots, n-k+1$: there are i broken components in the system, and the repair facility is being replaced.

Moreover, define the stationary system state probabilities as follows

$$P_{i,j} = \lim_{t\to\infty} P_{i,j}(t) = \lim_{t\to\infty} P\{L(t) = i, J(t) = j\},\ (i, j) \in \Omega.$$

Relating to Figure 1, through a straightforward probability analysis, a set of steady-state equations of the system is given by

$$n\lambda P_{0,0} = \mu P_{1,1}, \tag{3.1}$$

$$(n-i)\lambda P_{i,0} = (n-i-1)\lambda P_{i+1,0},\ i = 0, 1, \ldots, N-2, \tag{3.2}$$

$$(n-i)\lambda P_{i,0} = [\theta + (n-i-1)\lambda]\, P_{i+1,0},\ i = N-1, N, \ldots, n-k-1, \tag{3.3}$$

$$k\lambda P_{n-k,0} = \theta P_{n-k+1,0}, \tag{3.4}$$

$$[(n-i-1)\lambda + \mu + \alpha]\, P_{i+1,1} = (n-i)\lambda P_{i,1} + \mu P_{i+2,1} + \beta P_{i+1,2}, \quad i = 0, 1, \ldots, N-2, \tag{3.5}$$

$$[(n-i-1)\lambda + \mu + \alpha]\, P_{i+1,1} = (n-i)\lambda P_{i,1} + \mu P_{i+2,1} + \beta P_{i+1,2} + \theta P_{i+1,0},\ i = N-1, N, \ldots, n-k-1, \tag{3.6}$$

$$(\mu + \alpha)P_{n-k+1,1} = k\lambda P_{n-k,1} + \beta P_{n-k+1,2} + \theta P_{n-k+1,0}, \tag{3.7}$$

$$[(n-i-1)\lambda + \beta]\, P_{i+1,2} = (n-i)\lambda P_{i,2} + \alpha P_{i+1,1}, i = 0, 1, \ldots, n-k-1, \tag{3.8}$$

$$\beta P_{n-k+1,2} = k\lambda P_{n-k,2} + \alpha P_{n-k+1,1}. \tag{3.9}$$

3.2. MATRIX-ANALYTICAL SOLUTIONS

In this subsection, a matrix-analytical approach is provided to analyze the resulting system of linear equations (3.1)–(3.9). Following the concept by Neuts [14], in order to compute the steady-state equations in a matrix form, the corresponding transition rate matrix $\boldsymbol{Q}$ of this Markov chain can be partitioned as follows

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{C}_0 & & & & & & & \\ \boldsymbol{B}_1 & \boldsymbol{A}_1 & \boldsymbol{C}_1 & & & & & & \\ & \boldsymbol{B}_2 & \boldsymbol{A}_2 & \boldsymbol{C}_2 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & \boldsymbol{B}_N & \boldsymbol{A}_N & \boldsymbol{C}_N & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & \boldsymbol{B}_{n-k} & \boldsymbol{A}_{n-k} & \boldsymbol{C}_{n-k} \\ & & & & & & \boldsymbol{B}_{n-k+1} & \boldsymbol{A}_{n-k+1} \end{bmatrix}.$$

Here, matrix $\boldsymbol{Q}$ is a square matrix of order $3n-3k+4$, and

$$\boldsymbol{A}_0 = -n\lambda,$$

$$\boldsymbol{A}_i = \begin{bmatrix} -(n-i)\lambda & 0 & 0 \\ 0 & -(\alpha+\mu+(n-i)\lambda) & \alpha \\ 0 & \beta & -(\beta+(n-i)\lambda) \end{bmatrix}, \quad 1 \le i \le N-1,$$

$$\boldsymbol{A}_i = \begin{bmatrix} -(\theta+(n-i)\lambda) & \theta & 0 \\ 0 & -(\alpha+\mu+(n-i)\lambda) & \alpha \\ 0 & \beta & -(\beta+(n-i)\lambda) \end{bmatrix}, \quad N \le i \le n-k,$$

$$\boldsymbol{A}_{n-k+1} = \begin{bmatrix} -\theta & \theta & 0 \\ 0 & -(\alpha+\mu) & \alpha \\ 0 & \beta & -\beta \end{bmatrix},$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 \\ \mu \\ 0 \end{bmatrix}, \qquad \boldsymbol{B}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; 2 \le i \le n-k+1,$$

$$\boldsymbol{C}_0 = (n\lambda, 0, 0), \quad \boldsymbol{C}_i = \begin{bmatrix} (n-i)\lambda & 0 & 0 \\ 0 & (n-i)\lambda & 0 \\ 0 & 0 & (n-i)\lambda \end{bmatrix}, \; 1 \le i \le n-k.$$

Let $\boldsymbol{P}$, partitioned as $\boldsymbol{P} = (\boldsymbol{P}_0, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_{n-k+1})$, denote the steady-state probability vector of $\boldsymbol{Q}$, in which $\boldsymbol{P}_0 = P_{0,0}$ is a nonnegative real number, and $\boldsymbol{P}_i = (P_{i,0}, P_{i,1}, P_{i,2})$, $i = 1, 2, \ldots, n-k+1$ is a row vector of dimension 3. Then, it follows from the steady-state equations $\boldsymbol{PQ} = \boldsymbol{0}$ that

$$\boldsymbol{P}_0\boldsymbol{A}_0 + \boldsymbol{P}_1\boldsymbol{B}_1 = 0, \tag{3.10}$$

$$\boldsymbol{P}_0\boldsymbol{C}_0 + \boldsymbol{P}_1\boldsymbol{A}_1 + \boldsymbol{P}_2\boldsymbol{B}_2 = \boldsymbol{0}, \tag{3.11}$$

$$\boldsymbol{P}_i\boldsymbol{C}_i + \boldsymbol{P}_{i+1}\boldsymbol{A}_{i+1} + \boldsymbol{P}_{i+2}\boldsymbol{B}_{i+2} = \boldsymbol{0}, \; i = 1, 2, \ldots, n-k-1, \tag{3.12}$$

$$\boldsymbol{P}_{n-k}\boldsymbol{C}_{n-k} + \boldsymbol{P}_{n-k+1}\boldsymbol{A}_{n-k+1} = \boldsymbol{0}. \tag{3.13}$$

Moreover, the following normalizing equation should be satisfied

$$\boldsymbol{P}_0 + \sum_{i=1}^{n-k+1} \boldsymbol{P}_i \boldsymbol{e}_3 = 1, \tag{3.14}$$

where $\boldsymbol{e}_3$ is a column vector of dimension 3 with all entries equal to one.

Thus, after some routine manipulation, we have

$$\boldsymbol{P}_i = -\boldsymbol{P}_0 \boldsymbol{C}_0 \left(\boldsymbol{A}_1 + \boldsymbol{\xi}_2 \boldsymbol{B}_2\right)^{-1} \prod_{r=2}^{i} \boldsymbol{\xi}_r, \ i = 2, 3, \ldots, n-k+1, \tag{3.15}$$

$$\boldsymbol{P}_1 = -\boldsymbol{P}_0 \boldsymbol{C}_0 \left(\boldsymbol{A}_1 + \boldsymbol{\xi}_2 \boldsymbol{B}_2\right)^{-1}, \tag{3.16}$$

$$\boldsymbol{P}_0 \left[\boldsymbol{A}_0 - \boldsymbol{C}_0 \left(\boldsymbol{A}_1 + \boldsymbol{\xi}_2 \boldsymbol{B}_2\right)^{-1} \boldsymbol{B}_1\right] = 0, \tag{3.17}$$

where $\boldsymbol{\xi}_i = -\boldsymbol{C}_{i-1} \left(\boldsymbol{A}_i + \boldsymbol{\xi}_{i+1} \boldsymbol{B}_{i+1}\right)^{-1}$, $i = 2, 3, \ldots, n-k$, $\boldsymbol{\xi}_{n-k+1} = -\boldsymbol{C}_{n-k} \boldsymbol{A}_{n-k+1}^{-1}$ are all square matrices of order 3. Once $\boldsymbol{P}_0$ being obtained, then the steady-state probability vector $\boldsymbol{P} = (\boldsymbol{P}_0, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_{n-k+1})$ are determined.

With equations (3.14)–(3.16), we get

$$\boldsymbol{P}_0 \left\{1 - \boldsymbol{C}_0 \left(\boldsymbol{A}_1 + \boldsymbol{\xi}_2 \boldsymbol{B}_2\right)^{-1} \left[\boldsymbol{I}_3 - \sum_{i=2}^{n-k+1} \prod_{r=2}^{i} \boldsymbol{\xi}_i\right] \boldsymbol{e}_3\right\} = 1, \tag{3.18}$$

where $\boldsymbol{I}_3$ is an identity matrix of order 3. Solving equations (3.17) and (3.18) simultaneously would obtain the steady-state solution $\boldsymbol{P}_0$, *i.e.*, $P_{0,0}$ is found.

Further, we get the steady-state probabilities

$$P_{i,j} = \boldsymbol{P}_i \boldsymbol{e}_3(j+1), \ i = 1, 2, \ldots, n-k+1, \ j = 0, 1, 2,$$

where $\boldsymbol{e}_3(j+1)$ denotes the column vector of dimension 3 with 1 in the $(j+1)$th position and 0 elsewhere.

The solution procedure of the steady-state probabilities $\boldsymbol{P}_i, i = 0, 1, \ldots, n-k+1$ is summarized in Table 1.

4. System performance measures

In this section, we list various steady-state system performance measures along with their formulas.

- The steady-state system availability

$$A = \sum_{i=0}^{n-k} P_{i,0} + \sum_{i=1}^{n-k} P_{i,1} + \sum_{i=1}^{n-k} P_{i,2}.$$

- The steady-state rate of occurrence of failure

$$M = k\lambda \left(P_{n-k,0} + P_{n-k,1} + P_{n-k,2}\right).$$

TABLE 1. Computation of the stationary probabilities $\boldsymbol{P}_i$.

Begin Algorithm
Input: $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $i=0,1,\ldots,n-k+1$, $\boldsymbol{C}_i$, $i=0,1,\ldots,n-k$, $\boldsymbol{e}_3$, $\boldsymbol{I}_3$
Output: $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, …, $\boldsymbol{P}_{n-k+1}$
Step 1 set $\boldsymbol{\xi}_{n-k+1}=-\boldsymbol{C}_{n-k}\boldsymbol{A}_{n-k+1}^{-1}$
Sept 2 for $i=2,3,\ldots,n-k$
Sept 3 set $\boldsymbol{\xi}_i=-\boldsymbol{C}_{i-1}\left(\boldsymbol{A}_i+\boldsymbol{\xi}_{i+1}\boldsymbol{B}_{i+1}\right)^{-1}$
Step 4 end
Sept 5 for $j=2,3,\ldots,n-k+1$
Sept 6 set $\boldsymbol{\Phi}_j=\prod_{r=2}^{j}\boldsymbol{\xi}_j$
Step 7 end
Step 8 Solving $\boldsymbol{P}_0\left[\boldsymbol{A}_0-\boldsymbol{C}_0\left(\boldsymbol{A}_1+\boldsymbol{\Phi}_2\boldsymbol{B}_2\right)^{-1}\boldsymbol{B}_1\right]=0$ and $\boldsymbol{P}_0\left\{1-\boldsymbol{C}_0\left(\boldsymbol{A}_1+\boldsymbol{\xi}_2\boldsymbol{B}_2\right)^{-1}\left[\boldsymbol{I}_3-\sum_{i=2}^{n-k+1}\boldsymbol{\Phi}_i\right]\boldsymbol{e}_3\right\}=1$
Step 9 set $\boldsymbol{P}_1=-\boldsymbol{P}_0\boldsymbol{C}_0\left(\boldsymbol{A}_1+\boldsymbol{\Phi}_2\boldsymbol{B}_2\right)^{-1}$
Step 10 for $i=2,3,\ldots,n-k+1$
Step 11 set $\boldsymbol{P}_i=-\boldsymbol{P}_0\boldsymbol{C}_0\left(\boldsymbol{A}_1+\boldsymbol{\xi}_2\boldsymbol{B}_2\right)^{-1}\boldsymbol{\Phi}_i$
Step 12 end
End Algorithm

- The probability that the system is waiting for repair

$$P_f=P_{n-k+1,0}+P_{n-k+1,2}.$$

- The probability that the repairman is on vacation

$$P_v=\sum_{i=0}^{n-k+1}P_{i,0}.$$

- The probability that the repairman is busy

$$P_b=\sum_{i=1}^{n-k+1}P_{i,1}.$$

- The probability that the repair facility is being replaced

$$P_h=\sum_{i=1}^{n-k+1}P_{i,2}.$$

- The expected number of broken components in the system

$$E\left[N_b\right]=\sum_{i=0}^{n-k+1}iP_{i,0}+\sum_{i=1}^{n-k+1}iP_{i,1}+\sum_{i=1}^{n-k+1}iP_{i,2}.$$

- The expected number of operating components in the system

$$E[N_o] = \sum_{i=0}^{n-k+1} (n-i)P_{i,0} + \sum_{i=1}^{n-k+1} (n-i)P_{i,1} + \sum_{i=1}^{n-k+1} (n-i)P_{i,2}.$$

- The mean waiting time of a broken component (See Appendix A).

$$\begin{aligned}
\overline{W}_q &= \frac{\alpha+\beta}{\mu\beta} \sum_{j=1}^{2}\sum_{i=1}^{n-k} iP^-_{i,j} + \frac{1}{\beta}\sum_{i=1}^{n-k} P^-_{i,2} \\
&+ \sum_{i=0}^{N-2}\sum_{r=0}^{N-i-2} \frac{(-1)^r (n-i-1)!}{(n-N)!\, r!\, (N-i-2-r)!} \frac{1}{n-N+1+r} \\
&\times \left\{ \frac{1}{\theta} + \frac{\theta}{(n-N+1+r)\lambda} + \frac{\theta(\theta-1)}{[(n-N+1+r)\lambda+\theta]^2} \right. \\
&+ \left. \frac{i(\alpha+\beta)[\theta^2+(n-N+1+r)\lambda]}{\mu\beta[(n-N+1+r)\lambda+\theta]} \right\} P^-_{i,0} \\
&+ \sum_{i=N-1}^{n-k} \left[\frac{1}{\theta} + \frac{i(\alpha+\beta)}{\mu\beta} \right] P^-_{i,0},
\end{aligned}$$

where $P^-_{i,0} = \dfrac{(n-i)P_{i,0}}{nP_{0,0} + \sum_{l=1}^{n-k}\sum_{j=0}^{2} (n-l)P_{l,j}}$, $i = 0, 1, \ldots, n-k$,

$$P^-_{i,j} = \frac{(n-i)P_{i,j}}{nP_{0,0} + \sum_{l=1}^{n-k}\sum_{j=0}^{2} (n-l)P_{l,j}}, \; i = 1, 2, \ldots, n-k, \; j = 1, 2.$$

5. Numerical examples

This section presents several numerical examples to show the applicability of the theoretical results.

Example 5.1. In this example, we select $k = 6$, $n = 12$, $N = 3$, $\lambda = 0.6$, $\mu = 4.5$, $\theta = 4.5$, $\alpha = 0.2$, $\beta = 3.0$ to obtain Tables 2–3 which show the steady-state probabilities and various system performance measures. In addition, the graphs of two kinds of reliability measures as functions of N are displayed in Figure 2. From Figure 2, we observe that the system availability goes down with the increasing values of N, while the rate of occurrence of failure increase as N increases. This is because the bigger the threshold value of N is, the more broken components will wait in the system.

Example 5.2. Let $N = 1$, $\theta \to +\infty$, $\alpha = 0$, $\beta \to +\infty$, then our model is reduced to the special case k-out-of-n:G Markov repairable system. Cao and

TABLE 2. The steady-state probabilities of the 6-out-of-12:G system.

(i,j)	$P_{i,j}$	$P_{i,j}^{-}$	(i,j)	$P_{i,j}$	$P_{i,j}^{-}$
(0,0)	0.01588381	0.02939169	(4,1)	0.15353429	0.18940182
(1,0)	0.01732779	0.02939169	(4,2)	0.00667946	0.00823987
(1,1)	0.02541409	0.04310782	(5,0)	0.00370071	0.00399458
(1,2)	0.00052946	0.00089808	(5,1)	0.17804937	0.19218844
(2,0)	0.01906057	0.02939169	(5,2)	0.00939879	0.01014516
(2,1)	0.06346463	0.09786345	(6,0)	0.00191889	0.00177537
(2,2)	0.00179860	0.00277346	(6,1)	0.17840561	0.16506254
(3,0)	0.01155186	0.01603183	(6,2)	0.01138728	0.01053562
(3,1)	0.11243173	0.15603433	(7,0)	0.00153511	–
(3,2)	0.00396166	0.00549804	(7,1)	0.15336942	–
(4,0)	0.00670753	0.00827449	(7,2)	0.02388936	–

TABLE 3. Performance measures of the 6-out-of-12:G system.

$A = 0.82120611$	$P_v = 0.07768625$	$E[N_b] = 4.62101201$
$M = 0.69016239$	$P_b = 0.86466914$	$E[N_o] = 7.37898799$
$P_f = 0.02542447$	$P_h = 0.05764461$	$\overline{W}_q = 0.96094894$

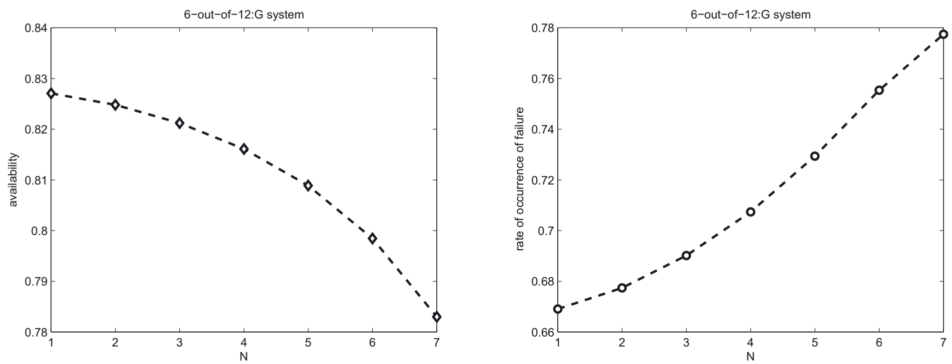

FIGURE 2. Two kinds of reliability measures for different values of N.

Cheng ([2], pp. 224−226) analyzed the Markov repairable system where the operating times and the repair times of components follow exponential distributions with parameters λ and μ, respectively. Applying the Markov analysis method, the stationary reliability measures are given as follows

$$A^a = \frac{\sum\limits_{i=k}^{n} \frac{1}{i!}\left(\frac{\mu}{\lambda}\right)^i}{\sum\limits_{i=k-1}^{n} \frac{1}{i!}\left(\frac{\mu}{\lambda}\right)^i}, \qquad M^a = \frac{\frac{\mu}{(k-1)!}\left(\frac{\mu}{\lambda}\right)^{k-1}}{\sum\limits_{i=k-1}^{n} \frac{1}{i!}\left(\frac{\mu}{\lambda}\right)^i}. \tag{5.1}$$

TABLE 4. The steady-state reliability measures versus (λ, μ).

(λ, μ)	(0.4, 4.5)	(0.5, 4.5)	(0.6, 4.5)	(0.7, 4.5)	(0.8, 4.5)	(0.9, 4.5)
A	0.98527023	0.96663594	0.93939771	0.90525212	0.86668830	0.82610883
A^a	0.98527049	0.96663638	0.93939829	0.90525278	0.86668897	0.82610946
M	0.06628395	0.15013827	0.27271030	0.42636546	0.59990266	0.78251029
M^a	0.06628277	0.15013628	0.27270767	0.42636249	0.59989964	0.78250744
(λ, μ)	(0.75, 2.0)	(0.75, 3.0)	(0.75, 4.0)	(0.75, 5.0)	(0.75, 6.0)	(0.75, 7.0)
A	0.55762254	0.73618059	0.84905792	0.91430371	0.95054097	0.97061984
A^a	0.55762262	0.73618090	0.84905850	0.91430443	0.95054169	0.97062048
M	0.88475491	0.79145824	0.60376834	0.42848146	0.29675418	0.20566112
M^a	0.88475475	0.79145729	0.60376599	0.42847787	0.29674988	0.20565667

To verify the correctness and feasibility of the algorithm provided in this paper, we choose $k = 4$, $n = 8$, $N = 1$, $\theta = 10^5$, $\alpha = 0$, $\beta = 10^5$. The computational results presented in Table 4 indicate that the analytical results derived by Matrix-analytical method exactly match with the dates given by Cao and Cheng [2].

6. Profit analysis

6.1. Profit function

In this subsection, we construct a profit function per component per unit time for the k-out-of-n:G repairable system, in which n and μ are decision variables. Our purpose is to determine the optimal number of operating components, say n^*, and the optimal repair rate, say μ^*, so as to maximum the profit function. First, let us define the following cost elements:

$c_r \equiv$ the revenue per unit time per operating component,
$c_w \equiv$ the cost per unit time per operating component,
$c_\mu \equiv$ the cost per unit time of providing a repair rate for broken components,
$c_f \equiv$ the replacement cost per unit time for the failed repair facility.

Applying the definition of each cost element listed above and its corresponding performance measures, the profit function per component per unit time is

$$F(n, \mu) = \frac{(c_r - c_w)E[N_o] - c_\mu \mu - c_f \beta}{n}. \tag{6.1}$$

As shown in expression (6.1), it would have been an arduous task to develop analytic results for the optimal solution (n^*, μ^*) because the profit function is highly non-linear and complex. Therefore, in the next subsection, we first utilize the direct search method to find the optimal value of the number of operating components, say n^*, when μ is fixed. Then, we fix n^* and use the parabolic method to derive the optimal value of μ, say μ^*.

TABLE 5. The profit function $F(n,\mu)$ for different values of λ.

n	4	5	6	7	8	9	10	11
$\lambda = 0.3$	87.9904	107.2196	119.6053	127.3986	131.9670	134.1493	134.4823	133.3347
$\lambda = 0.4$	82.2875	98.1389	107.5863	112.6212	114.4793	113.9730	111.6912	108.1091
$\lambda = 0.5$	77.4594	90.3356	97.1154	99.6767	99.2361	96.6704	92.6808	87.8557
$\lambda = 0.6$	73.3284	83.6493	88.1424	88.6696	86.5010	82.6089	77.7768	72.6136

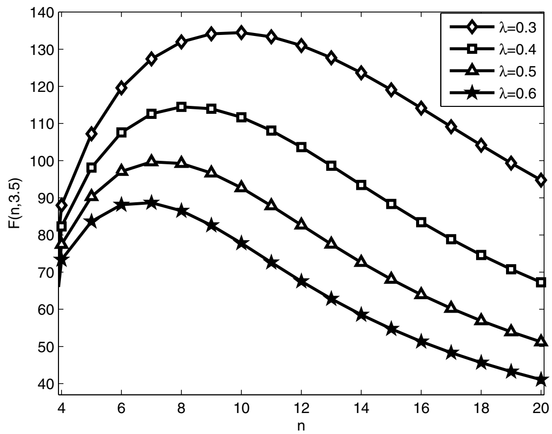

FIGURE 3. The plots of $F(n,\mu)$ against λ.

6.2. Direct search method

Since n is a discrete variable, the optimal value n^* can be found utilizing direct substitution of successive values of n into the profit function until the maximum value is attained. Numerical results are provided by considering the case of $k = 3$, $N = 2$, $\mu = 3.5$, $\theta = 4.0$, $\alpha = 0.4$, $\beta = 1.5$, $c_r = \$300$, $c_w = \$50$, $c_\mu = \$90$, $c_f = \$60$, and different values of $\lambda = 0.3, 0.4, 0.5, 0.6$.

We observe from Table 5 that a maximum profit per component per unit time (a) of \$134.4823 is achieved at $n^* = 10$ for $\lambda = 0.3$, (b) of \$114.4793 is achieved at $n^* = 8$ for $\lambda = 0.4$, (c) of \$99.6767 is achieved at $n^* = 7$ for $\lambda = 0.5$, (d) of \$88.6696 is achieved at $n^* = 7$ for $\lambda = 0.6$. Figure 3 depicts the different values of λ on the optimal number of operating components.

6.3. Parabolic method

We find n^*, and use the parabolic method to search μ until the maximum value of $F(n^*,\mu)$, say $F(n^*,\mu^*)$, is achieved. The profit maximization problem can be presented mathematically as

$$F(n^*,\mu^*) = \max_{\mu} F(n^*,\mu). \tag{6.2}$$

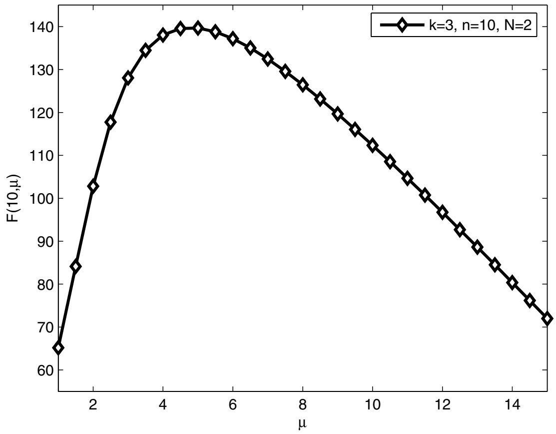

FIGURE 4. μ versus the profit function $F(10, \mu)$.

The steps of the parabolic method are described as follows.

Step 1. Choose starting 3-point pattern $\{x_a, x_b, x_c\}$ along with a stopping tolerance $\varepsilon = 10^{-4}$, and initialize iteration counter $i = 0$.
Step 2. Compute $F(x_a)$, $F(x_b)$, $F(x_c)$, and quadratic fit optimal

$$x_0 = \frac{(x_a^2 - x_b^2)F(x_c) + (x_b^2 - x_c^2)F(x_a) + (x_c^2 - x_a^2)F(x_b)}{2[(x_a - x_b)F(x_c) + (x_b - x_c)F(x_a) + (x_c - x_a)F(x_b)]}, \text{and } F(x_0).$$

Step 3. If $|x_0 - x_b| < \varepsilon$, choose $\max\{F(x_0), F(x_b)\}$ for a maximize, and its according point is the maximize point.
Step 4. When $x_0 \in (x_a, x_b)$, if $F(x_0) < F(x_b)$, then $x_0 \to x_a$; if $F(x_0) = F(x_b)$, then $x_0 \to x_a, x_b \to x_c, \frac{x_a + x_c}{2} \to x_b$; if $F(x_0) > F(x_b)$, then $x_0 \to x_b, x_b \to x_c$, advance $i = i + 1$, and return to step 3.
Step 5. When $x_0 \in (x_b, x_c)$, if $F(x_0) < F(x_b)$, then $x_0 \to x_c$; if $F(x_0) = F(x_b)$, then $x_b \to x_a, x_0 \to x_c, \frac{x_a + x_c}{2} \to x_b$; if $F(x_0) > F(x_b)$, then $x_0 \to x_b, x_b \to x_a$, advance $i = i + 1$, and return to step 3.

Now, we provide a numerical example to illustrate the applicability of the parabolic method. Using the results shown in the first row of Table 5, we know that the optimal number of operating components is n^*=10. With the help of Figure 4, we choose the initial 3-point pattern $\mu_a = 3.5$, $\mu_b = 4.0$, $\mu_c = 5.0$. After five iterations, it appears from Table 6 that the maximum profit converges at the solution $(n^*, \mu^*) = (10, 4.793162)$ with value 139.778316.

7. CONCLUSIONS

This paper studies a k-out-of-n:G repairable system with N-policy, repairman's multiple vacations and one replaceable repair facility. We first established the

TABLE 6. The parabolic method in searching the optimal solution.

Iterations	μ^*	$F(n^*, \mu^*)$	Tolerance	A	M	$E[N_b]$	$\overline{W}_q$
0	4.000000	138.034660	–	0.985292	0.024237	2.678614	0.912251
1	4.726903	139.768123	0.726903	0.990360	0.016219	2.347590	0.762563
2	4.806760	139.777894	0.079856	0.990739	0.015603	2.318451	0.750428
3	4.794496	139.778312	0.012264	0.990683	0.015695	2.322849	0.752249
4	4.793261	139.778316	0.001235	0.990677	0.015704	2.323293	0.752433
5	4.793162	139.778316	0.000099	0.990677	0.015705	2.323329	0.752448

steady-state equations of the system, and then derived the stationary probabilities by using the matrix-analytical approach. Moreover, a variety of system performance measures are discussed. Some numerical examples are provided to illustrate the applicability of the algorithm provided in this paper. Following the construction of the profit function per component per unit time, we employ the direct search method and the parabolic method to find the optimal values of n and μ.

In this work, we assumed that the shut-off rule is suspended animation which implies that there is no additional breakdown occurs when the system is down. Thus, in future work, an interesting extension is introduced the shut-off rules into the k-out-of-n:G repairable system.

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Appendix A. Computation of the waiting time of a broken component

Let $L^-(t) = i(i = 0, 1, \ldots, n-k)$ be the number of broken components at an instant just prior to time t which is a broken component arrival epoch. Denote

$$P^-_{i,0} = P\{L^- = i, J = 0\} = \lim_{t\to\infty} P\{L^-(t) = i, J(t) = 0\}, i = 0, 1, \ldots, n-k,$$
$$P^-_{i,1} = P\{L^- = i, J = 1\} = \lim_{t\to\infty} P\{L^-(t) = i, J(t) = 1\}, i = 1, 2, \ldots, n-k,$$
$$P^-_{i,2} = P\{L^- = i, J = 2\} = \lim_{t\to\infty} P\{L^-(t) = i, J(t) = 2\}, i = 1, 2, \ldots, n-k.$$

Denote by $A(t, t+\Delta t)$ the event that one of the operating components breaks down during the time interval $(t, t+\Delta t]$. We know that

$$\begin{aligned} P^-_{i,0} &= \lim_{t\to\infty}\lim_{\Delta t\to 0} P\{L(t) = i, J(t) = 0 | A(t, t+\Delta t)\} \\ &= \lim_{t\to\infty}\lim_{\Delta t\to 0} \frac{P\{A(t,t+\Delta t)|L(t) = i, J(t) = 0\}P_{i,0}(t)}{n\lambda\Delta t P_{0,0}(t) + \sum_{l=1}^{n-k}\sum_{j=0}^{2} P\{A(t,t+\Delta t)|L(t) = l, J(t) = j\}P_{l,j}(t)} \end{aligned}$$

$$= \lim_{t\to\infty}\lim_{\Delta t\to 0}\frac{(n-i)\lambda\Delta t P_{i,0}(t)+o(\Delta t)}{n\lambda\Delta t P_{0,0}(t)+\sum\limits_{l=1}^{n-k}\sum\limits_{j=0}^{2}(n-l)\lambda\Delta t P_{l,j}(t)+o(\Delta t)}$$

$$= \frac{(n-i)P_{i,0}}{nP_{0,0}+\sum\limits_{l=1}^{n-k}\sum\limits_{j=0}^{2}(n-l)P_{l,j}},\ i=0,1,\ldots,n-k.$$

Similarly, we get

$$P_{i,j}^{-}=\frac{(n-i)P_{i,j}}{nP_{0,0}+\sum\limits_{l=1}^{n-k}\sum\limits_{j=0}^{2}(n-l)P_{l,j}},\ i=1,2,\ldots,n-k,\ j=1,2.$$

Let $\widetilde{\chi}$ be the "generalized repair time" of the broken component, that is, the length of time since the broken component starts to be repaired until the repair is finished, where $\widetilde{\chi}$ contains some replacement times of the repair facility due to its failures. Moreover, let $\widetilde{G}(t)=P\{\widetilde{\chi}\le t\},\ t\ge 0$, it follows from [15] that

$$\begin{aligned}\widetilde{G}(t)&=\sum_{l=0}^{\infty}P\left\{\chi+\sum_{r=1}^{l}Y_r^{\langle 2\rangle}\le t,\sum_{r=1}^{l}X_r^{\langle 2\rangle}\le\chi<\sum_{r=1}^{l+1}X_r^{\langle 2\rangle}\right\}\\&=\sum_{l=0}^{\infty}\int_0^t Y^{(l)}(t-x)\frac{(\alpha x)^l}{l!}\mathrm{e}^{-\alpha x}\mu\mathrm{e}^{-\mu x}\,\mathrm{d}x,\ t\ge 0,\end{aligned}$$

where $Y(t)=P\{Y^{\langle 2\rangle}\le t\}$, and $Y^{(l)}(t)$ is the l-fold convolution of $Y(t)$, $l\ge 1$, with $Y^{(0)}(t)=1$, $t\ge 0$. The Laplace−Stieltjes transform of $\widetilde{G}(t)$ is

$$\widetilde{g}(s)=\int_0^{\infty}\mathrm{e}^{-st}\,\mathrm{d}\widetilde{G}(t)=\frac{s\mu+\mu\beta}{s^2+s(\alpha+\beta+\mu)+\mu\beta},\ \Re(s)>0,$$

where $\Re(s)$ is the real part of the complex number s.

Denote by $\widetilde{\widehat{\chi}}^1$ the "generalized remaining repair time" of the broken component, *i.e.*, the time interval from the broken component is being repaired until the repair is completed, where $\widetilde{\widehat{\chi}}^1$ includes replacement times of the repair facility which fails during the remaining repair period. Setting $\widetilde{\widehat{G}}_1(t)=P\{\widetilde{\widehat{\chi}}^1\le t\},t\ge 0$, we have

$$\begin{aligned}\widetilde{\widehat{G}}_1(t)&=\sum_{l=0}^{\infty}P\left\{\widehat{\chi}^1+\sum_{r=1}^{l}Y_r^{\langle 2\rangle}\le t,\widehat{X}_1^{\langle 2\rangle}+\sum_{r=2}^{l}X_r^{\langle 2\rangle}\le\widehat{\chi}^1<\widehat{X}_1^{\langle 2\rangle}+\sum_{r=2}^{l+1}X_r^{\langle 2\rangle}\right\}\\&=\sum_{l=0}^{\infty}\int_0^t Y^{(l)}(t-x)\frac{(\alpha x)^l}{l!}\mathrm{e}^{-\alpha x}\mu\mathrm{e}^{-\mu x}\,\mathrm{d}x,\ t\ge 0,\end{aligned}$$

where $\widehat{\chi}^1$ is the actual remaining repair time. Its Laplace−Stieltjes transform is

$$\widetilde{\widehat{g}}_1(s)=\widetilde{g}(s)=\frac{s\mu+\mu\beta}{s^2+s(\alpha+\beta+\mu)+\mu\beta},\ \Re(s)>0.$$

The "generalized remaining repair time" of the broken component $\widetilde{\widehat{\chi}}^2$ denote the time interval since the repair facility is being replaced (*i.e.*, the broken component is waiting for repair) until the repair is completed. Similarly, let $\widetilde{\widehat{G}}_2(t) = P\{\widetilde{\widehat{\chi}}^2 \leq t\}, t \geq 0$, we obtain that

$$\begin{aligned}
\widetilde{\widehat{G}}_2(t) &= \sum_{l=0}^{\infty} P\left\{\widehat{\chi}^2 + \widehat{Y}_1^{\langle 2\rangle} + \sum_{r=2}^{l+1} Y_r^{\langle 2\rangle} \leq t, \sum_{r=1}^{l} X_r^{\langle 2\rangle} \leq \widehat{\chi}^2 < \sum_{r=1}^{l+1} X_r^{\langle 2\rangle}\right\} \\
&= \sum_{l=0}^{\infty} \int_0^t Y^{(l+1)}(t-x) \frac{(\alpha x)^l}{l!} \mathrm{e}^{-\alpha x} \mu \mathrm{e}^{-\mu x}\, \mathrm{d}x,\ t \geq 0,
\end{aligned}$$

where $\widehat{\chi}^2$ is the actual remaining repair time of the broken component, $\widehat{Y}^{\langle 2\rangle}$ is the remaining replacement time of the failed repair facility. Moreover, its Laplace−Stieltjes transform is

$$\widetilde{\widehat{g}}_2(s) = \frac{\mu\beta}{s^2 + s(\alpha+\beta+\mu) + \mu\beta},\ \Re(s) > 0.$$

Denote by I_m^n the length of time that there are m of the n operating components break down, and $I_m^n(t) = P\{I_m^n \leq t\}$, $t \geq 0$. It follows that

$$\begin{aligned}
I_m^n(t) &= \binom{n}{m-1}\binom{n-m+1}{1} P\Big\{X_{(1)}^{\langle 1\rangle} \leq X_{(m)}^{\langle 1\rangle}, \ldots, X_{(m-1)}^{\langle 1\rangle} \leq X_{(m)}^{\langle 1\rangle}, \\
&\qquad X_{(m)}^{\langle 1\rangle} \leq t, X_{(m+1)}^{\langle 1\rangle} > X_{(m)}^{\langle 1\rangle}, \ldots, X_{(n)}^{\langle 1\rangle} > X_{(m)}^{\langle 1\rangle}\Big\} \\
&= \binom{n}{m} \sum_{r=0}^{m-1} (-1)^r \binom{m-1}{r} \frac{m}{n-m+1+r}\left(1 - \mathrm{e}^{(n-m+1+r)\lambda t}\right),
\end{aligned}$$

where $X_{(1)}^{\langle 1\rangle}$, $X_{(2)}^{\langle 1\rangle}$, …, $X_{(n)}^{\langle 1\rangle}$ is the order statistics of $X_1^{\langle 1\rangle}$, $X_2^{\langle 1\rangle}$, …, $X_n^{\langle 1\rangle}$.

Let W_q be the waiting time of a broken component under steady-state condition, and $W_q(t) = P\{W_q \leq t\}$, $t \geq 0$, we then know that

$$\begin{aligned}
W_q(t) &= \sum_{j=1}^{2} \sum_{i=1}^{n-k} P\left\{\widetilde{\widehat{\chi}}^j + \widetilde{\chi}_2 + \ldots + \widetilde{\chi}_i \leq t\right\} P_{i,j}^- \\
&+ \sum_{i=0}^{N-2} P\left\{\widehat{V} + \widetilde{\chi}_1 + \widetilde{\chi}_2 + \ldots + \widetilde{\chi}_i \leq t, I_{N-i-1}^{n-i-1} \leq \widehat{V}\right\} P_{i,0}^- \\
&+ \sum_{i=0}^{N-2} \sum_{m=1}^{\infty} P\Big\{I_{N-i-1}^{n-i-1} + \widehat{V}_m + \widetilde{\chi}_1 + \widetilde{\chi}_2 + \ldots + \widetilde{\chi}_i \leq t, \\
&\qquad \widehat{V} + V_1 + \ldots + V_{m-1} \leq I_{N-i-1}^{n-i-1} < \widehat{V} + V_1 + \ldots + V_m\Big\} P_{i,0}^-
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=N-1}^{n-k} P\left\{\widehat{V}+\widetilde{\chi}_1+\widetilde{\chi}_2+\ldots+\widetilde{\chi}_i \leq t\right\} P_{i,0}^{-} \\
&=\sum_{i=1}^{n-k}\left[\widetilde{\widehat{G}}_1(t) * \widetilde{G}^{(i-1)}(t)\right] P_{i,1}^{-}+\sum_{i=1}^{n-k}\left[\widetilde{\widehat{G}}_2(t) * \widetilde{G}^{(i-1)}(t)\right] P_{i,2}^{-} \\
&+\sum_{i=0}^{N-2}\binom{n-i-1}{N-i-1} \sum_{r=0}^{N-i-2}(-1)^r\binom{N-i-2}{r} \frac{(N-i-1)\theta}{n-N+1+r} \\
&\times \int_0^t \widetilde{G}^{(i)}(t-x) \mathrm{e}^{-\theta x}\left[1-\mathrm{e}^{-(n-N+1+r)\lambda x}\right] \mathrm{d}x P_{i,0}^{-}+\sum_{i=0}^{N-2} \sum_{m=1}^{\infty} \sum_{r=0}^{N-i-2} \\
&\times \frac{(-1)^r(n-i-1)!}{(n-N)!\, r!\,(N-i-2-r)!} \int_0^t \int_0^{t-x} \widetilde{G}^{(i)}(t-x-y) \frac{(\theta x)^m}{m!} \mathrm{e}^{-\theta x}\theta \\
&\times \mathrm{e}^{-\theta y} \mathrm{e}^{-(n-N+1+r)\lambda x}\, \mathrm{d}y\, \mathrm{d}x P_{i,0}^{-}+\sum_{i=N-1}^{n-k}\left[V(t) * \widetilde{G}^{(i)}(t)\right] P_{i,0}^{-},
\end{aligned}
$$

where $\widehat{V}$ is the remaining vacation time. Its Laplace–Stieltjes transform is

$$
\begin{aligned}
w_q(s)=&\sum_{i=1}^{n-k} \widetilde{\widehat{g}}_1(s) \widetilde{g}^{i-1}(s) P_{i,1}^{-}+\sum_{i=1}^{n-k} \widetilde{\widehat{g}}_2(s) \widetilde{g}^{i-1}(s) P_{i,2}^{-} \\
&+\sum_{i=0}^{N-2} \sum_{r=0}^{N-i-2} \frac{(-1)^r(n-i-1)!}{(n-N)!\, r!\,(N-i-2-r)!}\left\{\frac{\theta}{(n-N+1+r)(s+\theta)}\right. \\
&+\frac{\theta}{s+(n-N+1+r)\lambda+\theta}\left[\frac{\lambda\theta}{(s+\theta)(s+(n-N+1+r)\lambda)}\right. \\
&\left.\left.-\frac{1}{n-N+1+r}\right]\right\} \widetilde{g}^i(s) P_{i,0}^{-}+\sum_{i=N-1}^{n-k} \frac{\theta}{s+\theta} \widetilde{g}^i(s) P_{i,0}^{-}.
\end{aligned}
$$

Noting that $\overline{W}_q=-\frac{\mathrm{d}}{\mathrm{d}s} w_q(s)|_{s=0}$, we obtain the desired result.

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