RAIRO-Oper. Res. 49 (2015) 265–278
DOI: 10.1051/ro/2014036

RAIRO Operations Research
www.rairo-ro.org

A LINEAR FRACTIONAL OPTIMIZATION OVER AN INTEGER EFFICIENT SET

Sara Mahdi[1] and Djamal Chaabane[1]

Abstract. Mathematical optimization problems with a goal function, have many applications in various fields like financial sectors, management sciences and economic applications. Therefore, it is very important to have a powerful tool to solve such problems when the main criterion is not linear, particularly fractional, a ratio of two affine functions. In this paper, we propose an exact algorithm for optimizing a linear fractional function over the efficient set of a Multiple Objective Integer Linear Programming ($MOILP$) problem without having to enumerate all the efficient solutions. We iteratively add some constraints, that eliminate the undesirable (not interested) points and reduce, progressively, the admissible region. At each iteration, the solution is being evaluated at the reduced gradient cost vector and a new direction that improves the objective function is then defined. The algorithm was coded in $MATLAB$ environment and tested over different instances randomly generated.

Keywords. Multiple criteria programming, fractional programming, Integer programming, efficient set.

Mathematics Subject Classification. 90C10, 90C26, 90C32, 90C29.

Received July 21, 2013. Accepted June 6, 2014.

[1] USTHB University, Faculty of Mathematics, Department of Operations Research, Bab-Ezzouar, BP32 El-Alia, 16122 Algiers, Algeria.
`mahdsara@yahoo.fr; dchaabane@usthb.dz`

1. Introduction

Discrete Linear programming problems with multiple objective functions have received considerable attention in many areas of application as well as in academic research papers.

One can read about examples of the rather broad range of applications in [17,20]. For a discussion of some of the basic theoretical properties and other approaches to the problem, see, for example [10, 15].

In addition to vector maximization approach for solving Multi-objective Integer Linear Programming problem (MOILP), recently, an increase interest of researchers and practitioners can be noticed in the approach based on the optimization of functions over the efficient set. In an application of this approach, the decision maker preference function is given explicitly in the form of a linear combination of decision variables, and the problem of finding a most preferred efficient solution can be written as the maximization of this function over the efficient set. This type of optimization problem is not obvious and cannot be solved by a simple mono-objective optimization solver. Consequently, two possibilities are to be considered: either enumerating explicitly all the efficient solutions then evaluate each of them at the preferred DM criterion and choose the best, which is much time consuming, or finding a way that yields to an optimal efficient solution that satisfies the preferences of the DM. The problem has been extensively studied in the linear case, see for instance for continuous case [2–4,8,16,22] and [1,6,7,12,14] when integrity decision variables are imposed.

In this paper, however, we consider a ratio of two affine functions instead of a linear one and we propose a new exact algorithm, based on simple pivoting techniques, taking the advantage of Cambini and Martein idea and bringing together, branch and bound procedure (see Wolsey (1998) [21]) to find integer solutions then reducing progressively the admissible domain by adding more constraints, eliminating all dominated vectors by the current efficient solution until no pivoting operation becomes possible; indicating that the current domain is empty (see [18]).

Consider the following $MOILP$ problem

$$(P) \quad V\max\{Cx,\ x \in D\} \tag{1.1}$$

Where $D = S \cap \mathbb{Z}^n, S = \{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}, A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^{m\times 1}$, $C = (c^i)_{i\in\{1,\dots,p\}} \in \mathbb{Z}^{p\times n}$ are row vectors with $p \geq 2$, $\mathbb{Z}^n$ is the set of integers. We assume throughout the paper that D is not empty and S is a bounded convex polyhedron. The set of all integer efficient solutions of (P) is denoted by $E(P)$.

The main problem is

$$(FP_E) \quad \max\left\{\varphi(x) = \frac{cx+\alpha}{\mathrm{d}x+\beta},\ x \in E(P)\right\}. \tag{1.2}$$

Let the relaxed problem be

$$(FP_R) \qquad \max\left\{\varphi(x) = \frac{cx+\alpha}{\mathrm{d}x+\beta},\ x \in D\right\}, \tag{1.3}$$

Where c, d denotes n-dimensional integer row vectors; α, β are scalars and we assume that $\mathrm{d}x + \beta > 0$ over D.

As in multiple objective linear programming ([20]), the solution to the problem (P) is to find all solutions that are efficient in the sense of the following definition.

Definition 1.1. A feasible solution $x \in D$ is said to be *efficient* of (P), if and only if, there is no feasible solution $\bar{x} \in D$ such that $c^i(\bar{x}) \geq c^i(x)$ for all $i \in \{1, 2, \ldots, p\}$ and $c^i(\bar{x}) > c^i(x)$ for at least one i. The point $Z(x) = Cx$ is then called *non-dominated* vector. Otherwise, x is not efficient and the corresponding vector of criteria $Z(x)$ is said to be dominated.

Throughout this paper we use the following notations:
We consider a linear fractional problem:

$$(FP_k)\begin{cases}\max\ \varphi(x) = \dfrac{cx+\alpha}{\mathrm{d}x+\beta}\\ s.t.\ \ x \in D_k\end{cases} \tag{1.4}$$

Where

- D_k be the current feasible region at iteration k;
- x^k is an optimal integer solution of (FP_k) obtained in D_k;
- $a_{k,j}$ is the activity vector of x_j^k with respect to the current region D_k;
- B_k is the basis associated with solution x^k;
- $I_k = \{i | a_{k,i} \in B_k\}$ is the set of indices of basic variables;
- $N_k = \{j | a_{k,j} \notin B_k\}$ is the set of indices of non-basic variables;
- $y_{k,j} = (y_{k,ij}) = (B_k)^{-1} a_{k,j}$.
- x_{opt} is an optimal solution of (FP_E).
- $\varphi_{\text{opt}} = \varphi(x_{\text{opt}})$ the optimal value of the main criterion $\varphi(x)$.

In the next section some basic definitions and results are presented, followed by a detailed procedure. In Section 4, a numerical illustration is included to explain the proposed algorithm. Section 5 describes details of the implementation and computational experiments. Finally, a general conclusion is given in the last section.

2. Necessary results

The approach adopted to solve the problem (FP_k) at the kth iteration, is the Cambini and Martein's method (see [5]), which is mainly based on the evaluation of the reduced gradient vector $\widehat{\gamma}$ defined by

$$\widehat{\gamma} = \widehat{\beta}\widehat{c} - \widehat{\alpha}\widehat{d}$$

where $\widehat{c}, \widehat{d}, \widehat{\alpha}$ and $\widehat{\beta}$ are the updated values of c, d, α and β respectively. The following theorem allows us to find the optimal solution of (FP_k):

Theorem 2.1 [13]**.** *A feasible solution $\hat{x}$ is an optimal solution of the fractional problem (FP_k) if and only if $\widehat{\gamma}_j \leq 0$ for all non-basic index $j \in N_k$.*

Remark 2.2. Recall that a sufficient condition for the uniqueness of the optimal solution $\hat{x}$ is that the set $J_k = \{j \in N_k / \widehat{\gamma}_j = 0\}$ is empty. Otherwise, there exist another integer feasible solution $\tilde{x} \in D_k$ such that $\varphi(\tilde{x}) = \varphi(\hat{x})$ and we define $\tilde{x}$ as *an alternate optimal solution* to $\hat{x}$.

2.1. Testing efficiency

We may need sometimes along the algorithm process to test the efficiency of a given feasible solution of the problem (P), the following result enables us to perform this task (see [9]).

Theorem 2.3. *Let x^* be an arbitrary element of the admissible region D. $x^* \in E(P)$ if and only if the optimal value of the objective function $\Theta^*(\psi, x)$ is null in the following mixed integer linear programming problem*

$$T_{\text{eff}}(x^*) \begin{cases} \max \Theta = \sum_{i=1}^{p} \psi_i \\ \ s.t. \ \ Cx - I\psi = Cx^* \\ \qquad\quad x \in D, \ \ \psi_i \in \mathbb{R}^+; \forall i = \overline{1, p}, \end{cases} \tag{2.1}$$

Where

- $C = (p \times n)$ matrix, $I = (p \times p)$ identity matrix;
- $\Psi = (\psi_i)_{i=1,\ldots,p}$ are real non negative variables for all i.

2.2. Incident edge

Proposition 2.4. *An edge $E_{j_k}, j_k \in N_k$ incident to a feasible solution x_k is defined as the set*

$$E_{j_k} = \left\{ (x_1, \ldots, x_n) \in D_k \;\middle|\; \begin{array}{l} x_i = x_i^k - \theta_{j_k} y_{k,ij_k}, \ \textit{for } i \in I_k \\ x_{j_k} = \theta_{j_k} \\ x_j = 0, \ \textit{for } j \in N_k \backslash \{j_k\} \end{array} \right\} \tag{2.2}$$

where $0 < \theta_{j_k} \leq \min_{i \in I_k} \{ \frac{x_i^k}{y_{k,ij_k}}; y_{k,ij_k} > 0 \}$, θ_{j_k} is a positive integer and $\theta_{j_k} \times y_{k,ij_k}$ are integers for all $i \in I_k$ if such integer values exist.

Remark 2.5. Note that equation (2.2) enables us to compute the integer feasible alternate solutions when the optimal solution obtained by solving (FP_k) is not unique $(J_k \neq 0)$.

3. Description of the procedure

The procedure starts by solving the relaxed problem (FP_R). Obviously, if (FP_R) is infeasible; the problem (FP_E) is also infeasible. If it is not the case, the optimal solution of (FP_R), denoted x^0, is tested of efficiency in order to obtain an initial efficient solution $\hat{x}^0$. This is done by solving the problem $T_{\text{eff}}(x^0)$ (see Eq. (2.1)), then at each iteration k, the main criterion $\varphi(x)$ is optimized on the equivalent efficient solutions of $\hat{x}^k$ by solving the problem (FT_k)

$$(FT_k) : \max \left\{ \varphi(x) = \frac{cx + \alpha}{\mathrm{d}x + \beta} \,\middle|\, Cx = C\hat{x}^k, x \in D \right\}. \tag{3.1}$$

Let $\bar{x}^k$ be an optimal solution of (FT_k), the values of x_{opt} and φ_{opt} are updated, we put $x_{\text{opt}} = \bar{x}^k, \varphi_{\text{opt}} = \varphi(\bar{x}^k)$.

Then, the domain of admissibility is reduced progressively by adding some constraints that eliminate successively all dominated solutions by the current efficient solution $\bar{x}^k$ by solving the problem (FP_k)

$$(FP_k) : \max \left\{ \varphi(x) = \frac{cx + \alpha}{\mathrm{d}x + \beta} \,\middle|\, x \in D_k = D \setminus \bigcup_{s=0}^{k-1} D_s \right\} \tag{3.2}$$

where $D_s = \{x \in \mathbb{Z}^n \mid \; Cx \leq C\bar{x}^s\}$, and $\bar{x}^0, \bar{x}^2, \ldots, \bar{x}^s$ are the efficient solutions obtained before.

The feasible region D_k can be defined by the following constraints:

$$D_k = D_{k-1} \cap \left\{ x \in D \,\middle|\, \begin{array}{ll} c^i(x) \geq (c^i(\bar{x}^k) + 1)y_i^k - M_i(1 - y_i^k) & (*) \\ \sum_{i=1}^{p} y_i^k \geq 1, y_i^k \in \{0, 1\}, i = \overline{1, p} & (**) \end{array} \right\}$$

Where $D_0 = D$ and $-M_i$ is a lower bound to the ith objective function for all $x \in D$.

Note that when $y_i^k = 0$ the constraint $(*)$ is not restrictive and when $y_i^k = 1$ a strict improvement is forced in the ith objective function, the constraint $(**)$ means that at least one criterion is improved.

Let x^k an optimal solution of (FP_k), if it is efficient, the procedure terminates with x^k as an optimal solution of the main problem (FP_E); otherwise, an exploration procedure is applied over the incident edges E_{j_k} of x^k by using the reduced gradient vector $\hat{\gamma}_j$ of the objective function searching for an alternate efficient solution which improves the function $\varphi(x)$. If no such solution can be found, the process continues reducing the domain of admissibility and improving the value of φ until an optimal solution is obtained or the reduced region becomes empty. A technical presentation is given in the following section.

Algorithm 1: Optimizing a Linear Fractional Function over Efficient Set.

Input
$\downarrow A_{(m\times n)}$: matrix of constraints;
$\downarrow b_{(m\times 1)}$: RHS vector;
$\downarrow c_{(1\times n)}, \alpha$: numerator of the main criterion vector;
$\downarrow d_{(1\times n)}, \beta$: denominator of the main criterion vector;
$\downarrow C_{(p\times n)}$: matrix of linear criteria;
Output
$\uparrow x_{\text{opt}}$: optimal solution of the problem (FP_E).
$\uparrow \varphi_{\text{opt}}$: optimal value of the main criterion φ
Initialization

- Solve for the lower bound where $\forall i = 1,\dots,p \quad -M_i = \min\{c^i x \mid x \in D\}$,if $c^i_j \geq 0$, $j = 1,\dots,n$ else set $M_i = 0$;
- Let $\varphi_{\text{opt}} \leftarrow -\infty, k = 0$, $End \leftarrow False$;

while *End=False* **do**
  Solve the relaxed problem $(FP^k_R) \equiv (FP_R) \equiv \max\{\varphi(x) = \frac{cx+\alpha}{dx+\beta} \mid x \in D\}$;
  if (FP^k_R) *is unfeasible* **then** (FP_E) is unfeasible, **End** $\leftarrow$ **True**;
  else
    let x^k be an optimal solution of (FP^k_R).
    Solve $T_{\text{eff}}(x^k)$: **Efficiency test**;
    if $x^k \in E(P)$ **then** $x_{\text{opt}} \leftarrow x^k, \varphi_{\text{opt}} \leftarrow \varphi(x^k)$, **End** $\leftarrow$ **True**;
    else
      Let $\hat{x}^k$ be an optimal solution of $T_{\text{eff}}(x^k)$
      Solve $(FT_k) \equiv \max\{\varphi(x) = \frac{cx+\alpha}{dx+\beta} \mid Cx = C\hat{x}^k, x \in D\}$,
      Let $\bar{x}^k$ be an optimal solution of (FT_k),
      if $\varphi(\bar{x}^k) > \varphi_{\text{opt}}$ **then** Let $x_{\text{opt}} \leftarrow \bar{x}^k$ and $\varphi_{\text{opt}} \leftarrow \varphi(\bar{x}^k)$
      $k \leftarrow k+1$, solve the problem (FP_k)
      $(FP_k) \equiv \max\{\varphi(x) \mid x \in D_k = D \setminus \bigcup_{s=0}^{k-1} D_s\}, D_s = \{x \in Z^n \mid Cx \leq C\bar{x}^s\}$
      if (FP_k) *is unfeasible* **then** $(x_{\text{opt}}, \varphi_{\text{opt}}) \equiv SOL(FP_E)$: **End** $\leftarrow$ **True**;
      else
        let x^k be an optimal solution of (FP_k)
        if $\varphi(x^k) \leq \varphi_{\text{opt}}$ **then** $(x_{\text{opt}}, \varphi_{\text{opt}}) \equiv SOL(FP_E)$: **End** $\leftarrow$ **True**;
        Solve $T_{\text{eff}}(x^k)$: **Efficiency test**;
        if $x^k \in E(P)$ **then** $x_{\text{opt}} \leftarrow x^k, \varphi_{\text{opt}} \leftarrow \varphi(x^k)$: **End** $\leftarrow$ **True**;
        else
          let $\hat{x}^k$ be an optimal solution of $T_{\text{eff}}(x^k)$
          Let $J_k = \{j_k \in N_k / \hat{\gamma}_j = 0\}$;
          while $J_k \neq \emptyset$ **do**
            let x^k be an optimal solution of (FP_k)
            select $j_k \in J_k$; $\theta^\star_{j_k} = int\left(\min_{i \in I_k}\{\frac{x^k_i}{y_{k,ij_k}}; y_{k,ij_k} > 0\}\right)$;
            Explore the edge E_{jk} : searching for an alternate integer solution corresponding to $\theta_{j_k}\left(\theta_{j_k} \in \{1,\cdots,\theta^\star_{jk}\}\right)$ starting from $\theta_{jk} = \theta^\star_{jk}$ until $\theta_{jk} = 1$
            if *such efficient point is found* **then** let it be $\tilde{x}$,
            $x_{\text{opt}} \leftarrow \tilde{x}, \varphi_{\text{opt}} \leftarrow \varphi(\tilde{x})$: **End** $\leftarrow$ **True**;

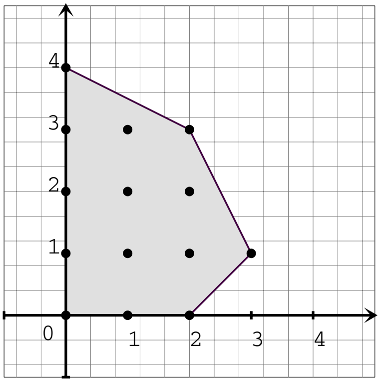

FIGURE 1. The feasible region D.

3.1. Technical description of the method

Proposition 3.1. *The algorithm above converges in a finite number of iterations.*

Proof. Since the feasible region D is a finite bounded set, there are a limited number of efficient solutions ($|E(P)|$ is finite). At each iteration, a new improved efficient solution is generated and the feasible region is being reduced until infeasibility. Thus, the procedure converges to the optimal solution in a finite number of iterations. □

4. Numerical illustration

In this didactic example, we try to highlight different steps of the algorithm. Consider the $MOILP$ problem:

$$(P)\begin{cases} \max\ Z_1 = x_1 - 3x_2 \\ \max\ Z_2 = x_1 + 3x_2 \\ \quad D \quad \begin{cases} x_1 + 2x_2 \leq 8 \\ 2x_1 + x_2 \leq 7 \\ x_1 - x_2 \leq 2 \\ x_1, x_2 \in N. \end{cases} \end{cases}$$

The feasible region D is presented in Figure 1 and the main problem (FP_E) is:

$$(FP_E)\begin{cases} \max\ \varphi(x) = \dfrac{-5x_1 - x_2 - 1}{4x_1 + x_2 + 1} \\ s.t.\ \ x_1, x_2 \in E(P). \end{cases} \tag{4.1}$$

TABLE 1. The optimal table of x^0.

B	x_1	x_2	x_3	x_4	x_5	x_B
x_3	1	2	1	0	0	8
x_4	2	1	0	1	0	7
x_5	1	-1	0	0	1	2
$\hat{c}$	-5	-1	0	0	0	-1
$\hat{d}$	4	1	0	0	0	1
$\widehat{\gamma}$	-1	0	0	0	0	-1

Step 0. Let $\varphi_{\text{opt}} = -\infty, k := 0, D_0 = D$, the optimal solution of the relaxed problem (FP_R) is $x^0 = (0,0)$ and $\varphi(x^0) = -1$.

$$(FP_R)\begin{cases} \max\ \varphi(x) = \dfrac{-5x_1 - x_2 - 1}{4x_1 + x_2 + 1} \\ D \quad \begin{cases} x_1 + 2x_2 \leq 8 \\ 2x_1 + x_2 \leq 7 \\ x_1 - x_2 \leq 2 \\ x_1, x_2 \in N. \end{cases} \end{cases}$$

The corresponding optimal tableau is in Table 1.

Step 1. In order to test the efficiency of x^0 we solve the problem $T_{\text{eff}}(x^0)$

$$T_{\text{eff}}(x^0)\begin{cases} \max\ \Theta = \psi_1 + \psi_2 \\ s.t.\ \ x_1 + 2x_2 \leq 8 \\ \qquad 2x_1 + x_2 \leq 7 \\ \qquad x_1 - x_2 \leq 2 \\ \qquad x_1 - 3x_2 - \psi_1 = 0 \\ \qquad x_1 + 3x_2 - \psi_2 = 0 \\ \qquad x_1, x_2 \in \mathbb{N}, \psi_{i,i=\overline{1,2}} \geq 0. \end{cases}$$

We obtain $x^0 \notin E(P)$ and the optimal solution of $T_{\text{eff}}(x^0)$ is $\hat{x}^0 = (3,1)$. $Z(\hat{x}^0) = (0,6)$.

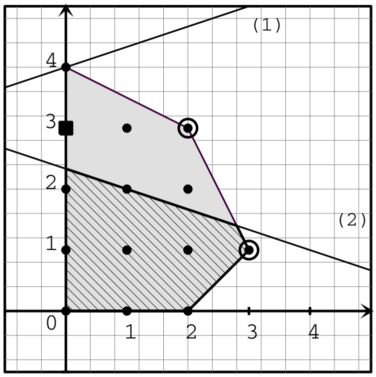

FIGURE 2. The reduced feasible region D_1.

Step 2. We solve the problem (FT_0)

$$(FT_0)\begin{cases} \max\ \varphi(x) = \dfrac{-5x_1 - x_2 - 1}{4x_1 + x_2 + 1} \\ s.t.\ \ x_1 + 2x_2 \leq 8 \\ \qquad 2x_1 + x_2 \leq 7 \\ \qquad x_1 - x_2 \leq 2 \\ \qquad x_1 - 3x_2 = 0 \\ \qquad x_1 + 3x_2 = 6 \\ \qquad x_1, x_2 \in \mathbb{N}. \end{cases}$$

An optimal solution of (FT_0) is $\bar{x}^0 = (3, 1)$ and $\varphi(\bar{x}^0) = -1.21 > \varphi_{\text{opt}}$; we initialize $x_{\text{opt}} = (3, 1)$ and $\varphi_{\text{opt}} = -1.21$.

Step 3. Let $k := k + 1 = 1, -M = (-12, 0)$ and solve the problem (FP_1)

$$(FP_1)\begin{cases} \max\ \varphi(x) = \dfrac{-5x_1 - x_2 - 1}{4x_1 + x_2 + 1} \\ D_1 \begin{cases} x_1 + 2x_2 \leq 8 \\ 2x_1 + x_2 \leq 7 \\ x_1 - x_2 \leq 2 \\ -x_1 + 3x_2 + 13y_1^1 \leq 12 & (1) \\ -x_1 - 3x_2 + 7y_2^1 \leq 0 & (2) \\ y_1^1 + y_2^1 \geq 1 \\ (y_1^1, y_2^1) \in \{0, 1\}^2. \end{cases} \end{cases}$$

An optimal solution is $x^1 = (0, 3), Z(x^1) = (-9, 9)$ and $\varphi(x^1) = -1$, see Figure 2.

TABLE 2. The optimal table of x^1.

B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_B
x_5	1	0	0	0	1	0	0	0	0	0	0	0	0	0	2	2
x_6	2	0	0	0	0	1	0	0	0	0	0	0	0	0	1	4
x_7	1	0	0	0	0	0	1	0	0	0	0	0	0	0	−1	5
x_4	0	0	0	1	0	0	0	0	0	−1	0	0	0	−1	0	1
x_2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	−1	3
x_3	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0
x_{11}	0	0	0	0	0	0	0	0	0	0	1	0	0	−1	0	1
x_{12}	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0
x_8	−1	0	0	0	0	0	0	1	0	0	0	0	0	−13	3	3
x_{13}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1	2
x_9	−1	0	0	0	0	0	0	0	1	7	0	0	0	7	−3	2
$\hat{c}$	−5	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	−4
$\hat{d}$	4	0	0	0	0	0	0	0	0	0	0	0	0	0	1	4
$\widehat{\gamma}$	−4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1

Step 4. As $\varphi(x^1) > \varphi_{\text{opt}}$, we test the efficiency of this solution by solving the problem $T_{\text{eff}}(x^1)$

$$T_{\text{eff}}(x^1)\begin{cases} \max \Theta = \psi_1 + \psi_2 \\ s.t.\ \ x_1 + 2x_2 \leq 8 \\ \qquad 2x_1 + x_2 \leq 7 \\ \qquad x_1 - x_2 \leq 2 \\ \qquad x_1 - 3x_2 - \psi_1 = -9 \\ \qquad x_1 + 3x_2 - \psi_2 = 9 \\ \qquad x_1, x_2 \in \mathbb{N}, \psi_{i,i=\overline{1,2}} \geq 0. \end{cases}$$

We obtain $x^1 \notin E(P)$ and $\hat{x}^1 = (2,3)$ is an optimal solution of $T_{\text{eff}}(x^1)$ with $Z(\hat{x}^1) = (-7, 11)$.

Step 5. Using the optimal table of x^1 (see Table 2) we obtain:

$B = \{5,6,7,4,2,3,11,12,8,13,9\}; N = \{1,10,14,15\};$

$\widehat{\gamma}_N = \{-4,0,0,0\}; J_1 = \{j \in N/\widehat{\gamma}_j = 0\} = \{10,14,15\};$

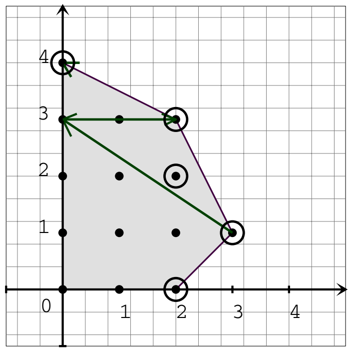

FIGURE 3. The efficient set of (P).

Since the set J_1 is not empty, the incidents edges to x^1 is being explored (see Eq. (2.2)) and we find the only integer solution on the edge E_{15} defined by

$$E_{15} = \begin{cases} x_{15} = \theta_{15} = \min\left\{\dfrac{2}{2}; \dfrac{4}{1}; \dfrac{3}{3}\right\} = 1; \\ x_5 \;\; = 2 - 1(2) = 0; \\ x_6 \;\; = 4 - 1(1) = 3; \\ x_7 \;\; = 5 - 1(-1) = 6; \\ x_4 \;\; = 1 - 1(0) = 1; \\ x_2 \;\; = 3 - 1(-1) = 4; \\ x_3 \;\; = 0 - 1(0) = 0; \\ x_{11} = 1 - 1(0) = 1; \\ x_{12} = 0 - 1(0) = 0; \\ x_8 \;\; = 3 - 1(3) = 0; \\ x_{13} = 2 - 1(-1) = 3; \\ x_9 \;\; = 2 - 1(-3) = 5; \\ x_1 \;\; = x_{10} = x_{14} = 0. \end{cases} \tag{4.2}$$

$\hat{x}^1 = (0, 4)$ is efficient, then $x_{\text{opt}} = (0, 4)$ and $\varphi_{\text{opt}} = -1$ and the procedure terminates.

The set of all efficient solutions of the problem (P) is

$E(P) = \{(2, 0), (3, 1), (2, 2), (2, 3), (0, 4)\}$. Whereas, the proposed algorithm optimizes the linear fractional function $\varphi(x)$ without having to pass by all these solutions but only by $\{(3, 1), (2, 3), (0, 4)\}$, see Figure 3.

TABLE 3. Results of samples randomly generated.

p	$p=3$		$p=5$		$p=8$	
$n\times\ m$	cpu (s)	# iter	cpu (s)	# iter	cpu (s)	# iter
$5\times\ 5$	0.39	3.5	0.6	3	0.67	3
	[0.02; 1.56]	[1; 6]	[0.03; 23.5]	[1; 8]	[0.13; 38]	[1; 7]
$10\times\ 5$	1.09	3.5	2.26	3.5	3.25	3
	[0.08; 15.64]	[1; 7]	[0.08; 34.54]	[1; 9]	[0.13; 75.35]	[1; 10]
$15\times\ 5$	3.21	4.5	5.18	3.5	8.56	4
	[0.14; 96.88]	[1; 11]	[0.15; 48.69]	[1; 6]	[0.18; 91.29]	[1; 8]
$20\times\ 5$	4.39	3.5	6.14	3	8.02	4
	[0.11; 39]	[1; 6]	[1.35; 129.6]	[1; 7]	[0.28; 141]	[1; 8]
$20\times\ 10$	6.56	3.5	7.42	4	14.5	4.5
	[0.23; 101]	[1; 6]	[0.27; 148]	[1; 10]	[0.35; 165]	[1; 7]
$30\times\ 10$	18.04	3.5	18.11	4	20.5	3.5
	[0.27; 153]	[2; 8]	[0.51; 213]	[1; 8]	[3.48; 195]	[1; 6]
$35\times\ 15$	22.27	4	25.79	4	24.69	3.5
	[1.51; 296]	[2; 7]	[1.50; 312]	[1; 8]	[2.57; 325.27]	[1; 7]
$40\times\ 15$	32.27	3.5	38.42	3.5	48.93	4.5
	[3.07; 387]	[1; 6]	[11.89; 398.36]	[2; 7]	[7.32; 430]	[1; 8]
$50\times\ 15$	75	4	84.47	4.5	69.75	4.5
	[8; 370]	[2; 7]	[3.14; 714.88]	[1; 7]	[11.19; 516.59]	[2; 6]
$60\times\ 20$	97.35	3.5	88.7	3.5	102	3.5
	[12.75; 500.25]	[2; 6]	[12.37; 506]	[2; 6]	[14.3; 1014]	[2; 8]
$70\times\ 20$	93.5	3.5	100	3	125	3
	[23.35; 892]	[2; 6]	[29; 1355]	[2; 7]	[33.7; 1804]	[2; 7]
$80\times\ 20$	119	3	131	4	126.3	3.5
	[29; 725]	[2; 5]	[32.53; 1874]	[2; 7]	[15.45; 1488]	[1; 6]

5. COMPUTATIONAL RESULTS

The presented procedure was implemented in the $MATLAB$ environment and run on a PC Intel(R)Core(TM) i3 CPU 2.13 GHZ, the performance is evaluated using 360 instances randomly generated from discrete uniform distribution; $A \in U([1,30])$, $b \in U([25,100])$ and $C \in U([-15,15])$. The vectors c, d, and α are generated in the same way as C and the constant β is generated such that $dx+\beta > 0$.

The problems were grouped according to the number of variables, constraints and objective functions into 36 categories, we consider sets of 3,5 and 8 objective functions $p = 3, 5, 8$. For each category of problems, 10 instances were solved.

The results reported in Table 3 − median cpu time (in seconds), required iterations, the lower and upper bounds for each measure − show that the proposed algorithm for small and relatively medium dimensions works efficiently in terms of number of iterations (# iter) and execution time (cpu (s)). In all these problems, the number of the efficient solutions generated (which is equal to the number of the performed iterations) does not exceed 11 solutions. As regards larger dimensions, the resolution of such problems becomes difficult due to the factors, such as multiple objective and discrete nature of the research area.

6. Conclusion

In this paper, we have introduced a new exact method that optimizes a linear fractional function over the integer efficient set of $MOILP$ problem, the difficulties arise are mainly due to the non convexity of the efficient set. The algorithmic complexity is also well known from the integer linear programming problems, the NP-hard so-called problems. Nevertheless, the quality of the solution remains our main interest.

The proposed method solves the problem avoiding the explicit enumeration of all efficient solutions bringing together Sylva and Crema's cuts for eliminating the efficient solutions previously finding and a process of exploring the incident edges by using the reduced gradient vector of the current solution in order to find an alternate efficient solution that improves the value of the objective function.

The algorithm was coded using the $MATLAB$ environment and it was tested for several problems, randomly generated from a discrete uniform distribution. As all exact cutting plane algorithms, our method gives an exact optimal solution for relatively medium dimensions in reasonable CPU execution time, but for higher dimensions, we suggest for future research work, cooperation of such exact technique with meta-heuristics techniques to take into account both quality and CPU execution time together.

Acknowledgements. The authors would like to thank the referees for their many constructive and helpful suggestions.

References

[1] M. Abbas and D. Chaabane, Optimizing a Linear Function over an Integer Efficient Set. *Eur. J. Oper. Res.* **174** (2006) 1140–1161.
[2] H.P. Benson, Optimization over the Efficient Set. *J. Math. Anal. Appl.* **98** (1984) 562–580.
[3] H.P. Benson, A finite Non adjacent Extreme Point Search Algorithm over the Efficient Set. *J. Opt. Theor. Appl.* **73** (1992) 47–64.
[4] H.P. Benson and S. Sayin, Optimizing over the Efficient Set: Four Special Cases. *J. Optim. Theor. Appl.* **80** (1994) 3–18.
[5] A. Cambini and L. Martein, A modified version of Martos's algorithm for the linear fractional problem. *Math. Oper. Res.* **53** (1986) 33–44.
[6] D. Chaabane and M. Pirlot, A method for optimizing over the integer efficient set. *J. Ind. Manag. Optim.* **6** (2010) 811–823.
[7] D. Chaabane, B. Brahmi and Z. Ramdani, The augmented weighted Tchebychev norm for optimizing a linear function over an integer efficient set of a multicriteria linear program. *Int. Trans. Oper. Res.* **00** (2012) 1–15.

[8] J.G. Ecker and J.H. Song, Optimizing a Linear Function over an Efficient Set. *J. Optim. Theor. Appl.* **83** (1994) 541–563.
[9] J.G. Ecker and I.A. Kouada, Finding Efficient Points for Multi-objectve Linear Programs. *Math. Program.* **8** (1975) 375–377.
[10] A.M. Geoffrion, Proper efficiency and the theory of vector maximization. *J. Math. Anal. Appl.* **22** (1968) 618–630.
[11] R. Gupta and R. Malhotra, Multi-criteria integer linear programming problem. *Cahiers du CERO* **34** (1992) 51–68.
[12] J.M. Jorge, An algorithm for optimizing a linear function over an integer efficient set. *Eur. J. Oper. Res.* **195** (2009) 98–103.
[13] B. Martos, Hyperbolic Programming. *Naval Res. Logist. Quart.* **11** (1964) 135–155.
[14] N.C. Nguyen, An Algorithm for Optimizing a Linear Function over the Integer Efficient Set. *Konrad-Zuse-zentrum fur Informationstechnik*, Berlin (1992).
[15] J. Philip, Algorithms for the vector maximization problem. *Math. Program.* **2** (1972) 207–229.
[16] S. Sayin, Optimizing over the Efficient Set using a Top-Down Search of Faces. *Oper. Res.* **48** (2000) 65–72.
[17] R.E. Steuer, Multiple Criteria Optimization : Theory, Computation and Application. John Wiley & Sons, New York (1986).
[18] J. Sylva, A. Crema, A method for finding the set of non-dominated vectors for multiple objective integer linear programs. *Eur. J. Oper. Res.* **158** (2004) 46–55.
[19] J. Sylva and A. Crema, A method for finding well-dispersed subsets of non-dominated vectors for multiple objective mixed integer linear programs. *Eur. J. Oper. Res.* **180** (2007) 1011–1027.
[20] J. Teghem and P. Kunsch, A survey of techniques for finding efficient solutions to multi-objective integer linear programming. *Asia Pac. J. Oper. Res.* **3** (1986) 95–108.
[21] L.A. Wolsey, Integer Programming. Wiley-Interscience publication, New York (1998).
[22] Y. Yamamoto, Optimization over the Efficient Set: overview. *J. Global Optim.* **22** (2002) 285–317.