PRODUCTION-INVENTORY SYSTEM WITH FINITE PRODUCTION RATE, STOCK-DEPENDENT DEMAND, AND VARIABLE HOLDING COST

Hesham K. Alfares[1]

Abstract. In general, traditional production-inventory systems are based on a number of simplifying – but somewhat unrealistic – assumptions, including constant demand rate, constant holding cost, and instantaneous order replenishment. These assumptions have been individually challenged in numerous variations of production-inventory models. Finite production rate models, such as economic production quantity (EPQ) systems consider gradual order replenishment. Stock-dependent demand models assume the demand rate to be an elastic function of the inventory level. Variable holding cost models assume the holding cost per unit per time period to be a function of the time spent in storage. In this paper, the three simplifying assumptions are simultaneously relaxed in a new production-inventory system with a finite production rate, stock-level dependent demand rate, and variable holding cost. Mathematical models and optimum solution procedures, including nonlinear programming, are presented for two functional forms of holding cost variability. The main contribution of this paper is the formulation and solution of a new production-inventory model that more closely represents real-world situations. The realistic assumptions and efficient solution algorithms should make the model practical and useful for industrial applications.

Keywords. Production-inventory system, stock-dependent demand, variable holding cost, nonlinear programming.

Mathematics Subject Classification. 90B05, 90B30, 90C30.

1. INTRODUCTION

Traditional production-inventory control systems, such as the economic order quantity (EOQ) model, are usually based on the assumptions of constant demand rate, constant inventory carrying cost, and instantaneous order arrival. This paper presents a realistic production-inventory model in which all these assumptions are relaxed. The EOQ assumption of instantaneous order arrival means the whole order quantity (*i.e.*, all units of the purchased lot) is received at once from the supplier. This assumption is not valid if these units are manufactured and received over an extended time interval. The economic production quantity (EPQ) model relaxes this assumption by incorporating a gradual order receipt, *i.e.* a finite production rate. This paper presents a production-inventory control system with a finite production rate, a stock-dependent demand rate, and a variable holding cost.

Demand variability according to item availability and price is a frequently observed phenomenon. In stock-dependent demand models, the demand for a given item increases with higher item availability. A bigger item display tends to attract the attention of more customers, leading to increased sales. Customers may view larger item stock as an indication of the item's popularity and also as a sign of reliable, continuing supply. Furthermore, a larger inventory leads to more on-time delivery and fewer shortages, therefore improving customer satisfaction and stimulating additional demand. Stock-dependent demand is a widely accepted concept in the literature, as illustrated in the Urban [22] survey. Several studies provide strong evidence to support this concept. For example, Koschat [11] empirically confirms the existence of a positive correlation between demand for a given product and its on-shelf inventory. Since higher sales are made on the expense of carrying more stock, firms need to find the optimum balance between the extra holding cost and the additional profit resulting from the induced sales.

In inventory models with variable holding costs, the holding cost is a function of either the storage level or the storage time. Holding cost time variability is used to reflect the fact that longer storage times frequently require higher holding costs. Longer storage periods, especially for perishable products, usually require more expensive specialized storage facilities. Time-dependent holding cost models represent holding cost either as a continuous nonlinear function or as a discontinuous step function of storage time. In continuous nonlinear functions, the unit holding cost gradually increases with longer storage time. However, in discontinuous step functions, the storage time is divided into few intervals, with one unit holding cost rate for each interval. Therefore, the unit holding cost is constant within each interval, but it immediately jumps to the next rate (step) as soon as the storage time extends to the next interval.

The aim of this paper is to present a production-inventory model with a finite production rate, a stock-level dependent demand rate, and a variable holding cost. In this model, the demand rate is an increasing power function of the current inventory level, and the holding cost is an increasing step function of the time spent

in storage. Two types of holding cost step functions are considered: retroactive holding cost, and incremental holding cost.

Subsequent sections of this paper are organized as follows. Relevant literature is reviewed in Section 2. The inventory model is presented and analyzed in Section 3. The model is formulated and the optimal solution process is developed for retroactive and incremental holding costs in Sections 4 and 5, respectively. Finally, results are discussed and conclusions are provided in Section 6.

2. Literature review

Urban [22] provides a comprehensive review of inventory models with stock-dependent demand rates published up to 2004, classifying work in this area into two main types. In the first type, pioneered by Gupta and Vrat [10], the demand rate is a function of the initial inventory. In the second type, pioneered by Baker and Urban [2, 3], the demand rate is a function of the current inventory. A third type could be considered as models in which the demand has two stages, an initial period of level-dependent demand followed by a period of constant demand. In this section, we review several categories of recent production-inventory models with variable holding costs and stock-dependent demand rates.

2.1. EOQ-type models with stock-dependent demand

In inventory models with variable demand rates, the demand is either a function of time or a function of the stock level. According to Urban [22], demand dependence on the stock level has several functional forms, including: linear, power, and posynomial. Several models have been proposed for deteriorating items with stock-level dependent demand rates. Min and Zhou [12] develop an EOQ model for perishable items with a stock-dependent demand rate. Unsatisfied demands are partially backlogged, and the demand rate during shortages depends on the negative inventory level, *i.e.* the amount of shortage. Sana *et al.* [16] present EOQ and EPQ inventory models in which the demand rate depends three factors: stock level, selling price, and advertising. Both deteriorating and ameliorating items are considered, and the objective is to maximize average profit subject to budget and storage capacity constraints. Yang *et al.* [25] propose an EOQ model for deteriorating items with a stock-dependent demand rate, partial backlogging, inflation, varying replenishment cycles, and varying shortage intervals. Unique optimal solutions are shown to exist, and good heuristic solution procedures are developed.

2.2. EOQ-type models with variable holding cost

An increasing number of EOQ-type inventory models assume variable and nonlinear holding costs. Muhlemann and Valtis-Spanopoulos [14] are the first to introduce variable holding costs into the EOQ model. In their EOQ-type model, the holding cost is assumed to be a function of the average inventory value. Their justification is that the cost of financing increases as the amount of investment (value

of the inventory) increases. Weiss [24] develops deterministic and stochastic EOQ models in which the per-unit holding cost is a non-linear function of the storage duration. According to Weiss [24], this assumption is applicable to inventory systems where the value of stored items decreases non-linearly with storage length. For the deterministic demand case, formulas are developed for the optimal order quantity under both finite and infinite horizons. For the stochastic demand case, a procedure to determine the optimal order quantity is described and computational results are presented. Ferguson *et al.* [4] reexamine the Weiss [24] model in which the cumulative holding cost is a nonlinear function of time. This model is shown to be an approximation to the optimal order quantity for perishable grocery items subject to delivery fees, where discounts are used to increase demand prior to the expiration dates. Given historical data, regression is used to estimate the parameters of the holding function from storage and capital costs, product lifetime, and discount policy.

2.3. EOQ-type models with variable demand and variable holding cost

Goh [9] presents the first inventory model in which the demand is stock dependent and the holding cost varies with storage duration. Goh considers two types of holding cost variation: (a) a nonlinear function of storage time, and (b) a nonlinear function of storage level. For both cases, optimal ordering policies are developed for the deterministic, single-item, and infinite-horizon inventory model. Giri *et al.* [7] present a generalized EOQ-type model for deteriorating items, in which the demand rate, the deterioration rate, the ordering cost, and the holding cost are continuous functions of time. Assuming completely backlogged shortages, the optimal ordering policy is developed for a finite planning horizon. Giri and Chaudhuri [6] develop another EOQ-type model for deteriorating items in which the demand rate is a function of the stock level. The holding cost is assumed to be a continuous nonlinear function of either the stock level or the storage duration, and approximate optimal solutions are derived for the two cases. The assumption of increasing holding cost with longer storage is applicable to deteriorating items that require increasingly better storage facilities to avoid deterioration.

Teng and Yang [20] generalize the EOQ model to consider a fluctuating demand rate, time-varying unit cost, and generalized holding cost. The holding cost is assumed to include both size-related parts (such as the warehouse cost) and value-related parts (such as the insurance cost). The cost function is shown to be convex, and a simple heuristic solution algorithm is proposed. Roy [15] formulates two EOQ models for deteriorating items in which the demand rate is a function of the selling price. Both the deterioration rate and the holding cost are continuous, linearly increasing functions of storage duration. Shortages are allowed and are completely backlogged in the first model, while they are not allowed in the second model.

Gayen and Pal [5] analyze a two-warehouse inventory model for a deteriorating item, in which both the demand rate and the holding cost are assumed to be continuous power functions of the current inventory level. Procedures are provided for approximating the optimal order policy, and results of computational experiments are presented. Mahata and Goswami [13] investigate an EOQ model for deteriorating items, assuming a stock-dependent demand rate and a variable holding cost. The holding cost is considered as a non-linear function of either the length of storage time or the current inventory level. Since real-world problems are characterized by imprecise and/or incomplete cost information, the demand rate, holding cost, ordering cost, and unit purchase cost are considered to be fuzzy numbers. The objective function is shown to be convex, and algorithms are developed for determining the optimal ordering policies.

Alfares [1] presents a stock-dependent EOQ-type model with two types of holding cost discontinuous step functions. As the storage time extends to the next time period, the new (higher) holding cost can be applied either retroactively to all storage periods, or incrementally to the new period only. Urban [23] extends Alfares [1] work by allowing non-zero end inventory for each cycle, and shifting to a maximum-profit objective. Singh, Kumar, and Gaur [18] present an EOQ model with time-dependent demand and partial backlogging of unsatisfied demand. They also consider deteriorating items, inflation, and an incremental holding cost function. Approximate optimal solutions are provided to minimize the inflation-adjusted total cost of ordering, holding, purchasing, and backlogging.

2.4. EPQ-type models with variable demand and/or holding cost

Assuming constant holding cost, Goh [8] presents three models of stock-dependent demand inventory systems: (a) an EOQ model with no shortages, (b) an EOQ model with shortages, and (c) an EPQ model. Sarfaraz [17] develops a modified EPQ model in which the holding cost is composed of two components: an investment cost proportional to the dollar value of inventory, and a capacity cost proportional to the maximum inventory level. Two cases are formulated and optimally solved: a deterministic demand model, and stochastic demand model that considers the costs of shortages and safety stocks. As stated earlier, Sana *et al.* [16] analyze an EPQ model with a stock-dependent demand rate and storage capacity limitations.

Tripathy *et al.* [21] formulate an EPQ model for deteriorating items, assuming the demand and the production rates are constant and the holding cost is a non-linear continuous function of storage time. Partially deteriorated items are sold with a discount, while completely deteriorated items are discarded, and shortages are not allowed. Singh, Singh, and Vaish [19] develop an EPQ inventory model where demand is a linear function of time, and the production rate is a linear function of the demand. Two cases of holding costs are considered: (i) holding cost is constant, and (ii) holding cost is a continuous linear function of time. Optimal

solutions are developed to minimize the total cost, including interest charges for payment delays.

The objective of this paper is to develop mathematical models and optimum solution procedures for two types of production-inventory systems with a stock-dependent demand rate, a finite production (replenishment) rate, and time-dependent holding cost functions.

3. MODEL DEVELOPMENT

3.1. NOTATION

The notation below is mostly adopted from Goh [8] and Alfares [1] for the production-inventory model to be developed:

$q(t)$ = quantity on-hand (inventory level) at time t,
D = constant (base) demand rate,
P = production rate during the first phase of the cycle $(0 \leq t \leq t_1)$,
n = number of distinct time periods with different holding cost rates,
t = time from the start of the cycle at $t = 0$,
t_1 = end time of the first (uptime) phase of the cycle,
σ_i = end time of holding-cost interval i, where $i = 1, 2, \ldots n, \sigma_0 = 0$, and $\sigma_n = \infty$,
K = ordering/setup cost per order,
$h(t)$ = holding cost of the item at time t, $h(t) = h_i$ if $\sigma_{i-1} \leq t \leq \sigma_i$,
T = cycle time, *i.e.* time between producing two consecutive orders of size S,
β = demand elasticity rate in relation to the inventory level,
Q = maximum inventory level, corresponding to time $t = t_1$,
S = production lot size $= Pt_1$.

3.2. ASSUMPTION AND LIMITATIONS

1. A single item with an infinite planning horizon is considered.
2. The holding cost is an increasing step function of storage duration $(h_1 < h_2 < \ldots < h_n)$.
3. An order of lot size S is produced gradually in t_1 periods at a constant rate P.
4. The units do not lose value during storage (no item deterioration).
5. Shortages are not allowed.
6. The demand rate R is an increasing power function of the inventory level q, given as:

$$R(q) = Dq^{\beta}, \quad D > 0, \quad 0 < \beta < 1, \quad q \geq 0. \tag{3.1}$$

3.3. THE PRODUCTION-INVENTORY MODEL

As shown in Figure 1, the inventory level variation over time $q(t)$ during a typical cycle is divided into two phases. During the first (uptime) phase of the cycle, a new order is produced at a constant rate P while the inventory is consumed at the
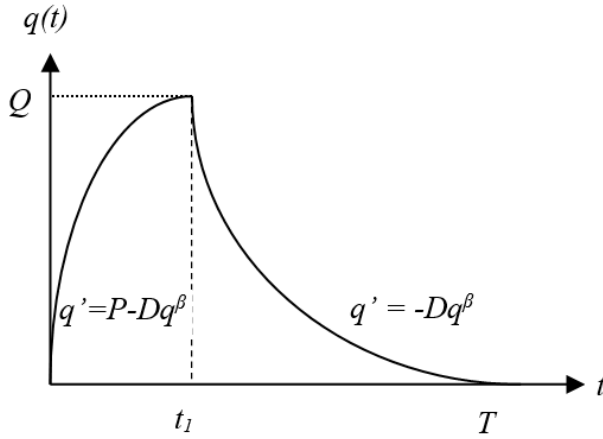
FIGURE 1. The inventory level during the two phases of the cycle.

stock-dependent demand rate of Dq^{β}. Hence, the rate of change in the inventory level is expressed as follows:

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = P - Dq(t)^{\beta}, \quad 0 \le t \le t_1, \quad P > D, \quad q(0) = 0 \qquad (3.2)$$

During the second (downtime) phase of the cycle, the rate of change (decrease) in the inventory level is equal to the demand rate, which is given by:

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = -D[q(t)]^{\beta}, \quad t_1 \le t \le T, \quad q(T) = 0 \qquad (3.3)$$

The total cost of the system includes two components: the ordering cost, and the holding cost. Since one order at a cost K is made in each cycle, the ordering cost per cycle is simply K. The holding cost per cycle is obtained by integrating the product of the holding cost $h(t)$ and the inventory level $q(t)$ for the whole cycle. However, the true objective is to minimize the total cost per unit time TC, which is obtained by dividing the total cost per cycle by the cycle time T. Therefore, the total cost per unit time TC is given by:

$$TC = \frac{K}{T} + \frac{1}{T} \int_0^{t_1} h(t)q(t)\mathrm{d}t + \frac{1}{T} \int_{t_1}^{T} h(t)q(t)\mathrm{d}t. \qquad (3.4)$$

During the uptime phase of the cycle ($0 \le t \le t_1$), rearranging and integrating the ordinary differential equation (3.2) results in the following:

$$\int_0^t \frac{1}{1 - \alpha q(t)^{\beta}}\mathrm{d}q = P \int_0^t \mathrm{d}t \qquad (3.5)$$

where

$$\alpha = D/P$$

The left-hand side of (3.5) can be integrated to yield the hyper-geometric function $_2F_1$.

$$\int \frac{1}{1 - \alpha q^\beta} \mathrm{d}q = q \times_2 F_1 \left(1, \frac{1}{\beta}; 1 + \frac{1}{\beta}; \alpha q^\beta\right).$$

After simplification, we obtain:

$$\sum_{n=0}^{\infty} \frac{\alpha^n q(t)^{n\beta+1}}{(n\beta + 1)} |_0^t = Pt. \tag{3.6}$$

Since $q(0) = 0$ while $q(t_1) = Q$, integrating over the range $[0, t_1]$ gives:

$$t_1 = \sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta+1}}{P(n\beta + 1)}. \tag{3.7}$$

In order to obtain an expression for $q(t)$ during the uptime phase, (3.2) is rearranged differently from (3.5) and integrated as follows:

$$\int_a^b q(t)\mathrm{d}t = \frac{1}{P} \int_a^b \frac{q(t)}{1 - \alpha q(t)^\beta} \mathrm{d}q. \tag{3.8}$$

The right-hand side of (3.8) again integrates to the hyper-geometric function $_2F_1$.

After simplification, (3.8) can be written as:

$$\int_a^b q(t)\mathrm{d}t = \sum_{n=0}^{\infty} \frac{\alpha^n \left[q(b)^{n\beta+2} - q(a)^{n\beta+2}\right]}{P(n\beta + 2)}. \tag{3.9}$$

During the downtime phase of the cycle $(t_1 \leq t \leq T)$, Alfares [1] provides the following solution of the ordinary differential equation (3.3):

$$\int_a^b q(t)\mathrm{d}t = \frac{q(a)^{2-\beta} - q(t)^{2-\beta}}{D(2 - \beta)} \tag{3.10}$$

$$T = t_1 + \frac{Q^{1-\beta}}{D(1 - \beta)}. \tag{3.11}$$

Substituting the appropriate terms from (3.7) and (3.9)−(3.11) into (3.4), and setting the derivative with respect to Q equal to 0, we solve for the optimum value of Q. Following the derivation steps described in Appendix A, the minimum total cost TC is obtained by solving the following equation for Q:

$$hQ^{2-\beta} + h(1 - \beta)(2 - \beta)(\alpha Q^2) \sum_{n=0}^{\infty} \frac{(\alpha Q^\beta)^n}{(n\beta + 1)(n\beta + 2)} = KD(1 - \beta)(2 - \beta) \tag{3.12}$$

where

$$\alpha Q^\beta < 1.$$

For convergence of the power series in (3.12), the feasibility condition $(\alpha Q^{\beta} < 1)$ must be satisfied. To solve (3.12), the infinite power series is well approximated with a reasonably large number of terms. Based on extensive numerical experiments, it was found that including $2\,000$ terms is quite sufficient for accurate results, even in the worst case when the value of αQ^{β} is nearly equal to 1.

From the optimum value of Q, the following quantities can be calculated:

$$\int_0^T q(t)\mathrm{d}t = \frac{Q^{2-\beta}}{D(2-\beta)} + \sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta+2}}{P(n\beta+2)} \tag{3.13}$$

$$S = \sum_0^{\infty} \frac{\alpha^n Q^{n\beta+1}}{n\beta+1}. \tag{3.14}$$

4. RETROACTIVE HOLDING COST

Assuming a retroactive holding cost, only one holding cost is used. That is, the holding cost of the last storage interval is used retroactively for earlier intervals. Assuming that the cycle ends in interval e, $(\sigma_{e-1} \leq T \leq \sigma_e)$, then the rate h_e is applied to holding cost intervals $1, 2, \ldots, e$. Therefore, the total cost per unit time TC is given by:

$$TC = \frac{K}{T} + \frac{h_i}{T} \int_0^T q(t)\mathrm{d}t \qquad \sigma_{i-1} \leq T \leq \sigma_i. \tag{4.1}$$

4.1. SOLUTION PROCEDURE

The optimum solution can be determined by using the following steps:

1. Starting with the lowest holding cost h_1, use (3.12) to determine Q and (3.11) to determine T for each h_i until Q is realizable (*i.e.*, $\sigma_{i1} \leq T \leq \sigma_i$). Call these values T_R and Q_R.
2. Calculate all break-point values of Q, $Q_i = Q(\sigma_i)$, by setting $T = \sigma_i$ in (3.11) and solving for Q_i.
3. For all Q_R and Q_i, use (3.13) to calculate and then substitute the result using the appropriate h_i into (4.1) to calculate the total cost TC.
4. Choose the value of Q that gives the lowest TC, and then use (3.14) to calculate the corresponding production lot size S.

4.2. EXAMPLE 1

Given the following parameters:

$D = 400$ units per year,
$P = 1\,000$ units per year,
$K = \$300$ per order,
$\beta = 0.1$,

$h_1 = \$6/\text{unit/year}$, $h_2 = \$8/\text{unit/year}$, $h_3 = \$10/\text{unit/year}$,
$\sigma_1 = 0.3$ years, $\sigma_2 = 0.6$ years, $\sigma_3 = \infty$.

The solution procedure is implemented in the following steps.

Step 1. Substituting $h_1 = 6$ (for $0 < T \le 0.3$), solving (3.12) gives: $Q = 155$.
Substituting $Q = 155$ into (3.11) gives $T = 0.656$ (not realizable).

Using $h_2 = 8$ (for $0.3 \le T \le 0.6$), solving (3.12) gives: $Q = 135$.
Substituting $Q = 135$ into (3.11) gives $T = 0.567$ (realizable).

Step 2. Setting $T = \sigma_1 = 0.3$, solving (3.11) gives: $Q_1 = 73$.
Setting $T = \sigma_2 = 0.6$, solving (3.11) gives: $Q_2 = 142$.

Step 3. Substituting $Q_R = 135$ and $h_2 = 8$ into (3.13) and (4.1) gives: $TC = \$1,078.09$.
Substituting $Q_1 = 73$ and $h_1 = 6$ into (3.13) and (4.1) gives: $TC = \$1,223.08$.
Substituting $Q_2 = 142$ and $h_2 = 8$ into (3.13) and (4.1) gives: $TC = \$1,079.64$.

Step 4. The minimum cost ($TC = \$1,078.09/\text{year}$) is obtained with $Q = 135$.
Using (3.14), the corresponding lot size is calculated as: $S = 338$. Therefore, the optimum inventory policy is:
Cycle time $T = 0.567$ year.
Lot size $S = 338$.

5. INCREMENTAL HOLDING COST

An incrementally increasing holding cost means that a different holding cost h_i applies to each storage interval i ($\sigma_{i1} \le t \le \sigma_i$). The objective, which is to minimize the total cost per unit time, is expressed as follows:

$$\text{Minimize } TIC = \frac{K}{T} + \frac{1}{T}\sum_{i=1}^{e} h_i \int_{\tau_{i-1}}^{\tau_i} q(t)dt. \tag{5.1}$$

A given interval $[\sigma_{i-1}, \sigma_i]$ may fall entirely in the first phase (before t_1), entirely in the second phase (after t_1), or straddle both phases. Therefore, depending on the relationship between end points $[\sigma_{i-1}, \sigma_i]$ and t_1, there are three possible procedures for calculating the integrals in (5.1). If the interval falls entirely in the first phase ($\sigma_{i-1} < \sigma_i \le t_1$), then we use (3.9), replacing a and b by σ_{i-1} and σ_i. If the interval falls entirely in the second phase ($t_1 \le \sigma_{i-1} < \sigma_i$), then we use (3.10), replacing a and b by σ_{i-1} and σ_i. However, if the interval straddles both phases ($\sigma_{i-1} \le t_1 < \sigma_i$), then we have to evaluate the integral over $[\sigma_{i-1}, \sigma_i]$ using a combination of (3.9) and (3.10) as follows:

$$\int_{\sigma_{i-1}}^{\sigma_i} q(t)dt = \sum_{n=0}^{\infty} \frac{\alpha^n[Q^{n\beta+2} - q(\sigma_{i-1})^{n\beta+2}]}{P(n\beta+2)} + \frac{Q^{2-\beta} - q(\sigma_i)^{2-\beta}}{D(2-\beta)}, \quad \sigma_{i-1} \le t_1 < \sigma_i. \tag{5.2}$$

The inventory level $q(t)$ is determined from (3.2) during the uptime phase and from (3.3) during the downtime phase. Therefore, the functions $q(\sigma_i)$ and $q(\sigma_{i-1})$ are determined from two different expressions in the two phases.

$$\sum_{n=0}^{\infty} \frac{\alpha^n q(t)^{n\beta+1}}{(n\beta+1)} = Pt, \quad 0 \leq \sigma_i \leq t_1 \tag{5.3}$$

$$q(t) = [Q^{1-\beta} - D(1-\beta)(t-t_1)]^{\frac{1}{1-\beta}}, \quad t_1 \leq \sigma_i \leq T \tag{5.4}$$

If the current interval i is the last interval in the cycle (*i.e.* $i = e$), then the intervals end point $\sigma_i = \sigma_e$ has to be replaced by the cycle time T. Therefore, it should be assumed that $\sigma_i = T$ and $q(\sigma_i) = 0$ for the last interval i in (5.1). Furthermore, the cycle time T must fall in the last interval e.

$$\sigma_{e-1} \leq T \leq \sigma_e \tag{5.5}$$

5.1. SOLUTION PROCEDURE

Since the objective function (5.1) is constrained, direct optimization by differential calculus is not feasible. For the incremental holding cost case, the optimum solution is obtained by nonlinear programming according to the following steps:

1. Substitute the minimum and maximum values of h_i in (3.12) to find the range of values for Q, and then use (3.7) and (3.11) to determine the corresponding range of intervals for t_1 and T.
2. For each possible combination of t_1 and T intervals, formulate a nonlinear programming (NLP) model whose objective function is (5.1), decision variable is Q, and constraints are (3.7), (3.11), and (5.3)−(5.5). In each NLP model, use the applicable terms for each interval $[\sigma_{i-1}, \sigma_i]$ in the objective and constraints.
3. For each combination of t_1 and T intervals, solve the corresponding NLP model to find the optimum solution.
4. Choose the feasible solution with the minimum total cost TC.

5.2. EXAMPLE 2

Resolve Example 1 assuming the holding cost increases incrementally.
The solution procedure is implemented in the following steps.

Step 1. For $h_1 = 6$ ($0 < T \leq 0.3$), (3.12), (3.7), and (3.11) give $Q = 155$, $t_1 = 0.396$, and $T = 0.656$.
For $h_3 = 10$ ($T > 0.6$), (3.12), (3.7), and (3.11) give $Q = 121$, $t_1 = 0.298$, and $T = 0.506$.

Clearly T falls either in the second or the third interval. However, although the results indicate that t_1 may fall either in the first or the second interval, the minimum $t_1 = 0.298$ is nearly in the second interval. Therefore, we will consider t_1 to belong to the second interval only.

Step 2 (a). Assuming both t_1 and T are in the second interval, then $e = 2$. In the objective function, (3.9) is used for the first interval $(0 < t \leq 0.3)$ which is in the uptime phase, while (5.2) is used for the second interval $(0.3 < t \leq T)$ which straddles the uptime and downtime phases. Therefore, the NLP model is formulated as follows:

$$\text{Minimize } TIC = \frac{300}{T} + \frac{6}{T} \sum_{n=0}^{\infty} \frac{0.4^n q (0.3)^{0.1n+2}}{1000(0.1n+2)}$$
$$+ \frac{8}{T} \left(\sum_{n=0}^{\infty} \frac{0.4^n [Q^{0.1n+2} - q(0.3)^{0.1n+2}]}{1000(0.1n+2)} \right. \tag{5.6}$$
$$\left. + \frac{Q^{2-0.1}}{400(2-0.1)} \right).$$

Subject to:

$$\sum_{n=0}^{\infty} \frac{0.4^n q (0.3)^{0.1n+1}}{(0.1n+1)} = 1000(0.3) \quad 0 < \sigma_i < t_1 \tag{5.7}$$

$$t_1 = \sum_{n=0}^{\infty} \frac{0.4^n Q^{0.1n+1}}{1000(0.1n+1)} \tag{5.8}$$

$$T = t_1 + \frac{Q^{1-0.1}}{400(1-0.1)} \tag{5.9}$$

$$0.3 \leq T \leq 0.6. \tag{5.10}$$

Step 2 (b). Assuming t_1 is in the second interval and T is in the third interval, then $e = 3$. The NLP model is formulated as follows:

$$\text{Minimize } TIC = \frac{300}{T} + \frac{6}{T} \sum_{n=0}^{\infty} \frac{0.4^n q (0.3)^{0.1n+2}}{1000(0.1n+2)}$$
$$+ \frac{8}{T} \left(\sum_{n=0}^{\infty} \frac{0.4^n [Q^{0.1n+2} - q(0.3)^{0.1n+2}]}{1000(0.1n+2)} \right.$$
$$\left. + \frac{Q^{2-0.1} - q(0.6)^{2-0.1}}{400(2-0.1)} \right) \tag{5.11}$$
$$+ \frac{10}{T} \left(\frac{q(0.6)^{2-0.1}}{400(2-0.1)} \right).$$

This function is minimized subject to constraints (5.7)−(5.9) above, in addition to the two following constraints:

$$q(0.6) = [Q^{1-0.1} - 400(1-0.1)(0.6 - t_1)]^{\frac{1}{1-0.1}} \tag{5.12}$$

$$T > 0.6. \tag{5.13}$$

Step 3. The optimum solution of the NLP in step 2(a) is:

$$Q = 126,\ t_1 = 0.312,\ T = 0.528,\ S = 312,\ TC = \$1,007.01.$$

The optimum solution of the NLP in step 2(b) is:

$$Q = 143,\ t_1 = 0.361,\ T = 0.603,\ S = 361,\ TC = \$1,015.62.$$

Step 4. The optimum solution of Example 2 for incremental holding cost is:

$$Q = 126,\ t_1 = 0.312,\ T = 0.528,\ S = 312,\ TC = \$1,007.01.$$

6. Discussion and conclusions

The two algorithms presented in sections 4 and 5 are quite efficient. In the retroactive case, the number of algorithm iterations is proportional to the number of holding cost rates, *i.e.*, number of holding cost intervals, n. Therefore, the complexity of the retroactive holding cost algorithm is simply $O(n)$. In the incremental case, the number of algorithm iterations is proportional to the number of feasible combinations of t_1 and T intervals. Let u and e respectively denote holding costs intervals corresponding to t_1 and T. Since t_1 cannot exceed T, then u cannot exceed e ($u = 1, \ldots, e$), and since e cannot exceed n, then the number of feasible combinations of u and e cannot exceed $\sum_{i=1}^{n} i = n(n+1)/2$. Therefore, the complexity of the incremental holding cost algorithm is $O(n^2)$.

To effectively use the models and algorithms in real-life applications, concerned mangers need to validate that the models assumptions are applicable in their organizations. First, production managers have to confirm that the demand for the given product is indeed an increasing power function of the stock level. Subsequently, the parameters of this function (D and β) need to be estimated by least-squares regression from historical inventory and demand data. In addition, the parameters of the holding cost function (σ_i and h_i) and its particular form (retroactive or incremental) have to be determined. For the solution of the equations and nonlinear programs, managers may conveniently use Microsoft Excel with Solver add-in.

This paper presented a production-inventory system with a finite production rate in which both the demand rate and the holding cost are variables. The demand rate is stock dependent, varying as an increasing power function of the instantaneous inventory level. The holding cost is time dependent, varying either retroactively or incrementally as an increasing step function of the storage duration. Optimization models and solution procedures have been developed, and examples have been solved. Several steps of nonlinear programming are used to solve problems with incremental holding cost srtucture. For future research, the work presented here can be extended in several directions, including variable production rates, variable ordering costs, and price-dependent demands.

Appendix A. Derivation of equation (3.12)

Equation (3.4) can be written as follows:

$$TC = \frac{K}{T} + \frac{h}{T} \int_0^{t_1} q(t)\mathrm{d}t + \frac{h}{T} \int_{t_1}^{T} q(t)\mathrm{d}t. \qquad (A.1)$$

From (3.9), the uptime phase integral is:

$$\int_0^{t_1} q(t)dt = \frac{Q^2}{P} \sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta}}{(2+n\beta)}. \qquad (A.2)$$

From (3.10), the downtime phase integral is:

$$\int_{t_1}^{T} q(t)\mathrm{d}t = \frac{Q^{2-\beta}}{D(2-\beta)}. \qquad (A.3)$$

Substituting (A.2) and (A.3) and the values of t_1 and T from (3.7) and (3.11) into (A.1) gives:

$$TC = \frac{K + h_i \left(\frac{Q^2}{P} \sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta}}{(2+n\beta)} + \frac{Q^{2-\beta}}{D(2-\beta)} \right)}{\sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta+1}}{P(1+n\beta)} + \frac{Q^{1-\beta}}{D(1-\beta)}}$$

$$= \frac{K + \sum_{n=0}^{\infty} \frac{h_i \alpha^n Q^{n\beta+2}}{P(2+n\beta)} + \frac{h_i Q^{2-\beta}}{D(2-\beta)}}{\sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta+1}}{P(1+n\beta)} + \frac{Q^{1-\beta}}{D(1-\beta)}}.$$

Setting the derivative of TC with respect $Q = 0$:

$$TC' = \frac{\left(\sum_{n=0}^{\infty} \frac{h_i \alpha^n Q^{n\beta+1}}{P} + \frac{h_i Q^{1-\beta}}{D} \right) \left(\sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta+1}}{P(1+n\beta)} + \frac{Q^{1-\beta}}{D(1-\beta)} \right)}{\left(\sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta+1}}{P(1+n\beta)} + \frac{Q^{1-\beta}}{D(1-\beta)} \right)^2}$$

$$- \frac{\left(K + \sum_{n=0}^{\infty} \frac{h_i \alpha^n Q^{n\beta+2}}{P(2+n\beta)} + \frac{h_i Q^{2-\beta}}{D(2-\beta)} \right) \left(\sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta}}{P} + \frac{Q^{-\beta}}{D} \right)}{\left(\sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta+1}}{P(1+n\beta)} + \frac{Q^{1-\beta}}{D(1-\beta)} \right)^2}$$

$$= 0.$$

Since the numerator $= 0$, then:

$$\left(\sum_{n=0}^{\infty} \frac{h_i \alpha^n Q^{n\beta+1}}{P} + \frac{h_i Q^{1-\beta}}{D} \right) \left(\sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta+1}}{P(1+n\beta)} + \frac{Q^{1-\beta}}{D(1-\beta)} \right)$$

$$= \left(K + \sum_{n=1}^{\infty} \frac{h_i \alpha^n Q^{n\beta+2}}{P(2+n\beta)} + \frac{h_i Q^{2-\beta}}{D(2-\beta)} \right) \left(\sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta}}{P} + \frac{Q^{-\beta}}{D} \right).$$

Dividing both sides by $\left(\sum_{n=0}^{\infty} \frac{\alpha^n Q^{n\beta}}{P} + \frac{Q^{-\beta}}{D}\right)$ and simplifying gives:

$$\sum_{n=0}^{\infty} \frac{h\alpha^n Q^{n\beta+2}}{P(1+n\beta)(2+n\beta)} + \frac{hQ^{2-\beta}}{D(1-\beta)(2-\beta)} = K.$$

Multiplying both sides by $\frac{D(1-\beta)(2-\beta)}{h}$ gives:

$$(1-\beta)(2-\beta) \sum_{n=0}^{\infty} \frac{D\alpha^n Q^{n\beta+2}}{P(1+n\beta)(2+n\beta)} + Q^{2-\beta} = \frac{KD(1-\beta)(2-\beta)}{h}.$$

Finally, substituting $\alpha = D/P$ in the above equation and rearranging leads to equation (3.12).

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