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# REPEATED GAMES WITH ASYMMETRIC INFORMATION MODELING FINANCIAL MARKETS WITH TWO RISKY ASSETS *

Victoria Kreps[1] and Victor Domansky[1]

**Abstract.** We consider multistage bidding models where two types of risky assets (shares) are traded between two agents that have different information on the liquidation prices of traded assets. These prices are random integer variables that are determined by the initial chance move according to a probability distribution $\mathbf{p}$ over the two-dimensional integer lattice that is known to both players. Player 1 is informed on the prices of both types of shares, but Player 2 is not. The bids may take any integer values. The model of $n$-stage bidding is reduced to a zero-sum repeated game with lack of information on one side. We show that, if liquidation prices of shares have finite variances, then the sequence of values of n-step games is bounded. This makes it reasonable to consider the bidding of unlimited duration that is reduced to the infinite game $G_\infty(\mathbf{p})$. We give the solutions for these games. Optimal strategies of Player 1 generate random walks of transaction prices. But unlike the case of one-type assets, the symmetry of these random walks is broken at the final stages of the game.

**Keywords.** Multistage bidding model, repeated game, asymmetric information, random walk.

**Mathematics Subject Classification.** 91.

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[1] Institute for Economics and Mathematics RAS, 1, Tchaikovsogo st., 191187, St.Petersburg, Russia. doman@emi.nw.ru

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## 1. Introduction. Modeling financial markets by repeated games

Regular random fluctuations in stock market prices are usually explained by effects from multiple exogenous factors subjected to accidental variations. The work of De Meyer and Saley [4] proposes a different strategic motivation for these phenomena. The authors assert that the Brownian component in the evolution of prices on the stock market may originate from the asymmetric information of stockbrokers on events determining market prices. "Insiders" are not interested in the immediate revelation of their private information. This forces them to randomize their actions and results in the appearance of an oscillatory component in price evolution.

De Meyer and Saley demonstrate this idea on a model of multistage bidding between two agents for risky assets (shares). The liquidation price of a share depends on a random "state of nature". Before the bidding starts a chance move determines the "state of nature" and therefore the liquidation price of a share once and for all. Player 1 is informed on the "state of nature", but Player 2 is not. Both players know the probability of a chance move. Player 2 knows that Player 1 is an insider.

At each subsequent step $t = 1, 2, \ldots, n$ both players simultaneously propose their prices for one share. The maximal bid wins and one share is transacted at this price. If the bids are equal, no transaction occurs. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

In this model Player 2 should use the history of Player 1's moves to update his beliefs about the state of nature. Thus Player 1 must maintain a delicate balance between taking advantage of his private information and concealing it from Player 2.

De Meyer and Saley consider a model where a share's liquidation price takes only two values and players may make arbitrary bids. They reduce this model to a zero-sum repeated game with lack of information on one side, as introduced by Aumann and Maschler [1], but with continual action sets. De Meyer and Saley show that these $n$-stage games have the values (*i.e.* the guaranteed gains of Player 1 are equal to the guaranteed losses of Player 2). They find these values and the optimal strategies of players. As $n$ tends to infinity, the values infinitely grow up with rate $\sqrt{n}$. It is shown that Brownian Motion appears in the asymptotics of transaction prices generated by these strategies.

The same result was demonstrated in De Meyer [2] for models with perfectly general trading mechanisms. The thesis of Gensbittel [9] contains analogous results for a model with two risky assets and with arbitrary bids.

It is more natural to assume that players may assign only discrete bids proportional to a minimal currency unit. De Meyer and Marino [3], Domansky and Kreps [7], Domansky [5] analyze a bidding model with the same mechanism of the game as in the model of De Meyer and Moussa-Saley [4], and where market makers have to post prices within a discrete grid. The $n$-stage games $G_n^m(p)$ are considered with two possible values of liquidation price, 1 with probability $p$ and 0 with probability $1-p$, and with admissible bids being multiples of $1/m$.

The works [3, 5, 7] show that, unlike the model of De Meyer and Saley, the sequence of values $V_n^m(p)$ of the games $G_n^m(p)$ is bounded from above and converges as $n$ tends to $\infty$. The authors calculate its limit $H^m$, that is a continuous, concave, and piecewise linear function with $m$ domains of linearity $[k/m,(k+1)/m]$, $k=0,\dots,m-1$, and the values at peak points $H^m(k/m)=k(m-k)/2m$.

The proof in [3] differs in essential ways from the proof in [5]. The last proof is more concise due to exploiting a "reasonable" strategy of Player 2. In fact, this is his optimal strategy for the game with infinite number of steps.

As the sequence $V_n^m(p)$ is bounded from above, it is reasonable to consider the games $G_\infty^m(p)$ with infinite number of steps. We do this in [7] and in [5]. The games $G_\infty^m(p)$ are infinitely repeated, non-discounted games with non-averaged payoffs that differs from the classical model of Aumann and Maschler [1].

We believe that the model is consistent and tractable with an endogenous random time for information disclosure that happens when a posterior probability takes the value 0 or 1. But the model with infinite number of steps does not allow to determine an exogenous time for information disclosure that is a base for the notion of liquidation value in the works of De Meyer. At time T, each player should be able to sell his shares of the risky asset at this liquidation price.

The infinite game may be reinterpreted in the following way, that allows us to conserve the exogenous time of disclosure $T$. The sequential stages $t_n, n=1,2,\dots$ of the game occur on the interval $[0,T)$ having an accumulation point at the point $T$. This means that transactions become more and more frequent as the disclosure of information approaches. For example, one can take $t_n=T(1-\alpha^n)$ for some $\alpha\in(0,1)$.

Unlike the case of $n<\infty$, the existence of a value for the games $G_\infty^m(p)$ has to be proved. We prove it by constructing explicitly the optimal strategies. We show that the value $V_\infty^m$ is equal to $H^m$, that is the limit of the sequence of values $V_n^m(p)$.

We construct the optimal strategy of Player 1 that provides him the maximal possible expected gain $1/2m$ per step (the fastest optimal strategy). For this strategy the posterior probabilities perform a simple symmetric random walk over the admissible bids $l/m$, $l=0,\dots,m$, with absorbing extreme points 0 and 1. The absorption of posterior probabilities means revealing of the true value of share by Player 2. For the initial probability $k/m$, the expected duration of this random walk before absorption is $k(m-k)$. The bidding terminates almost surely in a finite number of steps, and the expected number of steps is also finite. This random time of absorption is a time for disclosure of information. The game terminates naturally when the posterior expectation of liquidation price coincide with its real value.

The set of all optimal strategies of Player 1 for $G_\infty^m(p)$ consists of the described fastest strategy obtained in [5] and its slower modifications. It can be shown that the constructed fastest optimal strategy of Player 1 for the infinitely repeated game

$G_\infty^m(p)$ is an $\varepsilon$-optimal strategy of Player 1 for any finitely repeated game $G_n^m(p)$ of length $n$, where $\varepsilon = O(\cos^n \pi/m)$. This is not so for slower optimal strategies of Player 1.

The results of [5] cannot be extended to a general transaction mechanism introduced by De Meyer [2]. As mentioned in [2], the discretized mechanism does not satisfy axioms of shift- and scale-invariance. Note that in practice a grid of possible bids is not shift- and scale-invariant simultaneously.

A more realistic model is studied in [10]. It is analogous to the model considered in [5], but equipped with a more general transaction mechanism. Namely, the agents fix different stakes for buying and selling a share.

In Domansky and Kreps [8] we consider a model where the share liquidation price may take any integer values according to a probability distribution $\mathbf{p}$. Any integer bids are admissible. This $n$-stage model is reduced to a zero-sum repeated game $\bar{G}_n(p)$ with countable state and action spaces. The games considered in Domansky [5] can be presented as particular cases of these games corresponding to probability distributions with two-point supports and with payoffs rescaling (the payoff for the game $G_n^m(p)$ is multplied by $m$).

We show that if the liquidation price of a share has a finite expectation, then the values of $n$-stage games exist. If its variance is finite, then, as $n$ tends to $\infty$, the sequence of values is bounded from above and converges. The limit $\bar{H}$ is a continuous, concave, piecewise linear function with a countable number of domains of linearity. For distributions with integer mean values the function $\bar{H}$ is equal to the half of the liquidation price variance.

As the sequence of $n$-stage game values is bounded from above, it is reasonable to consider the games $\bar{G}_\infty(\mathbf{p})$ with an infinite number of steps. We show that the value $\bar{V}_\infty(\mathbf{p})$ is equal to $\bar{H}(\mathbf{p})$. We explicitly construct the optimal strategies for these games. To construct the optimal strategies of Player 1 we exploit symmetric representations of univariate probability distributions with given mean values as convex combinations of extreme points of corresponding sets, *i.e.* distributions with the same mean values and with supports containing at most two points.

The insider optimal strategy generates a symmetric random walk of posterior expectations over the one-dimensional integer lattice with absorption. For distributions with integer mean values the expected duration of this random walk is equal to the variance of the liquidation price of a share. The value of infinite game is equal to the expected duration of this random walk multiplied by the constant one-step gain 1/2 of informed Player 1.

In the present paper we consider multistage bidding models where two types of risky assets are traded. We show that, if expectations of share prices are finite, then the values $V_n(\mathbf{p})$ of $n$-stage bidding games $G_n(\mathbf{p})$ exist. The value of such a game does not exceed the sum of values of games modeling the bidding with one-type shares. This means that the simultaneous bidding of two types of risky assets is at most so profitable for the insider as the separate bidding of one-type shares. It is explained by the fact that the simultaneous bidding leads to revealing more insider information, because the bids for shares of each type provide information on shares of the other type.

We show that, if both share prices have finite variances, then the values of $n$-stage bidding games do not exceed the function $H(\mathbf{p})$ that is the smallest piecewise linear function equal to the one half of the sum of share price variances for distributions with integer expectations of both share prices.

This makes it reasonable to consider bidding models of unlimited duration, that are reduced to infinite games $G_\infty(\mathbf{p})$. We get the solutions for games $G_\infty(\mathbf{p})$ with arbitrary probability distributions over a two-dimensional integer lattice having finite component variances. We show that their values $V_\infty(\mathbf{p})$ coincide with $H(\mathbf{p})$, *i.e.* they are equal to the sum of values of corresponding games with one-type risky assets. Thus, the profit that Player 2 gets under simultaneous $n$-step bidding in comparison with separate bidding for each type of shares disappears in a game of unbounded duration.

The optimal strategies of Player 1 generate random walks of transaction prices. But unlike the case of one-type assets, the symmetry of these random walks is broken at the final stages of the game.

## 2. Repeated games modeling multistage bidding with two types of risky assets. Main results

We consider repeated games $G_n(\mathbf{p})$ with incomplete information on one side (see [1]) modeling the bidding with two types of risky assets.

Two players with opposite interests have money and shares of two types. The liquidation prices of both share types may take any integer values, $s^1$ for the first type and $s^2$ for the second one.

At stage 0 a chance move determines the "state of nature" $s = (s^1, s^2)$ and therefore the liquidation prices of shares $s^1$ and $s^2$ for the whole period of bidding $n$ according to the probability distribution $\mathbf{p}$ over the two-dimensional integer lattice known to both Players. Player 1 is informed about the result of chance move $s$, Player 2 is not. Player 2 knows that Player 1 is an insider.

At each subsequent stage $t = 1, \ldots, n$ both Players simultaneously propose their bids, meaning prices for one share of each type, $(i_t^1, i_t^2) \in \mathbb{Z}^2$ for Player 1 and $(j_t^1, j_t^2) \in \mathbb{Z}^2$ for Player 2. The bids are announced to both Players before proceeding to the next stage. The maximal bid wins and one share is transacted at this price. Therefore, if $i_t^e > j_t^e$, Player 1 gets one share of type $e = 1, 2$ from Player 2 and Player 2 receives the sum of money $i_t^e$ from Player 1. If $i_t^e < j_t^e$, Player 2 gets one share of type $e$ from Player 1 and Player 1 receives the sum $j_t^e$ from Player 2. If $i_t^e = j_t^e$, then no transaction of shares of type $e$ occurs. Each player aims to maximize the value of his final portfolio (money plus the liquidation value of obtained shares).

This $n$-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information for Player 2, with countable state space $S = \mathbb{Z}^2$, and with countable action spaces $I = \mathbb{Z}^2$, $J = \mathbb{Z}^2$. The one-step gain $a(s, i, j)$ of Player 1 corresponding to the state $s = (s^1, s^2)$ and the actions $i = (i^1, i^2)$, $j = (j^1, j^2)$ is given with the sum $\sum_{e=1}^{2} a^e(s^e, i^e, j^e)$, where

$$a^e(s^e, i^e, j^e) = \begin{cases} j^e - s^e, & \text{for } i^e < j^e; \\ 0, & \text{for } i^e = j^e; \\ -i^e + s^e, & \text{for } i^e > j^e. \end{cases}$$

At the end of the game Player 2 pays to Player 1 the sum

$$\sum_{t=1}^{n} a(s, i_t, j_t),$$

where $s$ is the result of a chance move. This description is a common knowledge of both Players.

At the step $t$ it is enough for both Players to take into account the sequence $(i_1, \ldots, i_{t-1})$ of Player 1's previous actions only. Thus a mixed behavioral strategy $\sigma$ for Player 1 who is informed on the state is a sequence of moves

$$\sigma = (\sigma_1, \ldots, \sigma_t, \ldots),$$

where the move $\sigma_t = (\sigma_t(s))_{s \in S}$ and $\sigma_t(s) : I^{t-1} \to \Delta(I)$ is the probability distribution used by Player 1 to select his action at stage $t$, given the state $s$ and previous observations. Here $\Delta(\cdot)$ is the set of probability distributions over $(\cdot)$.

A strategy $\tau$ for uninformed Player 2 is a sequence of moves

$$\tau = (\tau_1, \ldots, \tau_t, \ldots),$$

where $\tau_t : I^{t-1} \to \Delta(J)$.

Note that here we define infinite strategies fitting for games of arbitrary duration. A pair of strategies $(\sigma, \tau)$ creates a probability distribution $\Pi_{(\sigma,\tau)}$ over $(I \times J)^\infty$. The payoff function of the game $G_n(\mathbf{p})$ is

$$K_n(\mathbf{p}, \sigma, \tau) = \sum_{s \in S} \mathbf{p}(s) h_n^s(\sigma, \tau), \tag{2.1}$$

where

$$h_n^s(\sigma, \tau) = \mathbf{E}_{(\sigma,\tau)} \left[ \sum_{t=1}^{n} a(s, i_t, j_t) \right] \tag{2.2}$$

is the $s$-component of the $n$-step vector payoff $h_n(\sigma, \tau)$ for the pair of strategies $(\sigma, \tau)$. Here the expectation is taken with respect to the probability distribution $\Pi_{(\sigma,\tau)}$. Thus we consider n-step games $G_n(\mathbf{p})$ with total (non-averaged) payoffs which differs from the classical model of Aumann and Maschler [1].

We also consider the infinite games $G_\infty(\mathbf{p})$. For certain pairs of strategies $(\sigma, \tau)$, the payoff function $K_\infty(\mathbf{p}, \sigma, \tau)$, given by the infinite series (2.1),(2.2) with $n = \infty$, may be indefinite. If we restrict the set of Player 1's admissible strategies to strategies with positive one-step gains

$$\sum_{s\in S}\mathbf{p}(s)\mathbf{E}_{(\sigma_1(s),j)}a(s,i,j)$$

against any action $j$ of Player 2, then the payoff function of the game $G_\infty(p)$ becomes completely definite (may be infinite). Player 1 has many strategies, ensuring him a positive one-step gain against any action of Player 2. In fact, any reasonable strategy of Player 1 should possess this property.

For the initial probability $\mathbf{p}$, the strategy $\sigma$ ensures the $n$-step payoff

$$w_n(\mathbf{p},\sigma)=\inf_\tau K_n(\mathbf{p},\sigma,\tau).$$

The strategy $\tau$ ensures the $n$-step vector payoff $\mathbf{h}_n(\tau)$ with components

$$h_n^s(\tau)=\sup_{\sigma(s)} h_n^s(\sigma(s),\tau).$$

Now we describe the recursive structure of $G_{n+1}(\mathbf{p})$. A strategy $\sigma$ may be regarded as a pair $(\sigma_1,(\sigma(i))_{i\in I})$, where $\sigma_1(i|s)$ is a probability over $I$ depending on $s$, and $\sigma(i)$ is a strategy depending on the first action $i_1=i$.

Analogously, a strategy $\tau$ may be regarded as a pair $(\tau_1,(\tau(i))_{i\in I})$, where $\tau_1$ is a probability over $J$.

A pair $(\mathbf{p},\sigma_1)$ induces the probability distribution $\pi$ over $S\times I$, $\pi(s,i)=\mathbf{p}(s)\sigma_1(i|s)$. Let

$$\mathbf{q}\in\Delta(I),\quad \mathbf{q}(i)=\sum_S\mathbf{p}(s)\sigma_1(i|s),$$

be the marginal distribution of $\pi$ on $I$ (total probabilities of actions), and let

$$\mathbf{p}(\cdot|i)\in\Delta(S),\quad \mathbf{p}(s|i)=\mathbf{p}(s)\sigma_1(i|s)/\mathbf{q}(i),$$

be the conditional probability on $S$ given $i_1=i$ (a posterior probability).

Conversely, any set of total probabilities of actions $\mathbf{q}\in\Delta(I)$ and posterior probabilities $(\mathbf{p}(\cdot|i)\in\Delta(S))_{i\in I}$, satisfying the equality

$$\sum_{i\in I}\mathbf{q}(i)\mathbf{p}(\cdot|i)=\mathbf{p},$$

define a certain random move of Player 1 for the current probability $\mathbf{p}$. The posterior probabilities contain all information about the previous history of the game, that is essential for Player 1. Thus, to define a strategy of Player 1 it is sufficient to define the random move of Player 1 for any current posterior probability.

The following recursive representation for the payoff function corresponds to the recursive representation of strategies:

$$K_{n+1}(\mathbf{p},\sigma,\tau)=K_1(\mathbf{p},\sigma_1,\tau_1)+\sum_{i\in I}\mathbf{q}(i)K_n(\mathbf{p}(\cdot|i),\sigma(i),\tau(i)).$$

Let, for all $i \in I$, the strategy $\sigma(i)$ ensure the payoff $w_n(\mathbf{p}(\cdot|i), \sigma(i))$ in the game $G_n(\mathbf{p}(\cdot|i))$. Then the strategy $\sigma = (\sigma_1, (\sigma(i))_{i\in I})$ ensures the payoff

$$w_{n+1}(\mathbf{p}, \sigma) = \min_{j\in J} \sum_{i\in I} \left[ \sum_{s\in S} \mathbf{p}(s)\sigma_1(i|s)a(s,i,j) + \mathbf{q}(i) w_n(\mathbf{p}(\cdot|i), \sigma(i)) \right].$$

Let, for all $i \in I$, the strategy $\tau(i)$ ensure the vector payoff $\mathbf{h}_n(\tau(i))$. Then the strategy $\tau = (\tau_1, (\tau^n(i))_{i\in I})$ ensures the vector payoff $\mathbf{h}_{n+1}(\tau)$ with the components

$$h^s_{n+1}(\tau) = \max_{i\in I} \sum_{j\in J} \tau_1(j)(a(s,i,j) + h^s_n(\tau(i)) \quad \forall s \in S.$$

The game $G_n(\mathbf{p})$, where $n \in \mathbb{N} \cup \{\infty\}$, has a value $V_n(\mathbf{p})$ if

$$\inf_\tau \sup_\sigma K_n(\mathbf{p}, \sigma, \tau) = \sup_\sigma \inf_\tau K_n(\mathbf{p}, \sigma, \tau) = V_n(\mathbf{p}).$$

Players have optimal strategies $\sigma^*$ and $\tau^*$ if

$$V_n(\mathbf{p}) = \inf_\tau K_n(\mathbf{p}, \sigma^*, \tau) = \sup_\sigma K_n(\mathbf{p}, \sigma, \tau^*),$$

or, as in the notation introduced above,

$$V_n(\mathbf{p}) = w_n(\mathbf{p}, \sigma^*) = \sum_{s\in S} \mathbf{p}(s) h^s_n(\tau^*).$$

For $n \in \mathbb{N}$ the values $V_n(\mathbf{p})$ should satisfy Bellman optimality equations:

$$V_{n+1}(\mathbf{p}) = \inf_{\tau_1} \sup_{\sigma_1} [K_1(\mathbf{p}, \sigma_1, \tau_1) + \sum_{i\in I} \mathbf{q}(i) V_n(\mathbf{p}(\cdot|i))].)$$

The value $V_\infty(\mathbf{p})$ should satisfy Bellman optimality equation:

$$V_\infty(\mathbf{p}) = \inf_{\tau_1} \sup_{\sigma_1} [K_1(\mathbf{p}, \sigma_1, \tau_1) + \sum_{i\in I} \mathbf{q}(i) V_\infty(\mathbf{p}(\cdot|i))].$$

For probability distributions $\mathbf{p}$ with finite supports, the games $G_n(\mathbf{p})$, being games with finite state and action spaces, have values $V_n(\mathbf{p})$. The functions $V_n$ are continuous and concave in $\mathbf{p}$. Both players have optimal strategies $\sigma^*_n(\mathbf{p})$ and $\tau^*_n(\mathbf{p})$. The value of such game does not exceed the sum

$$V_n(\mathbf{p}^1) + V_n(\mathbf{p}^2)$$

of values of games modeling the bidding with one-type shares, where $\mathbf{p^1}$ and $\mathbf{p^2}$ are the marginal distributions of the distribution $\mathbf{p}$. This follows from the fact that Player 2 can guarantee himself the loss that does not exceed this sum exploiting the direct combination of optimal strategies $\tau^*_n(\mathbf{p}^1)$ and $\tau^*_n(\mathbf{p}^2)$ for the single asset games $G_n(\mathbf{p}^1)$ and $G_n(\mathbf{p}^2)$ as a strategy for the two asset game $G_n(\mathbf{p})$.

Let $M^1(\mathbb{Z}^2)$ be the set of probability distributions $\mathbf{p}$ over the two-dimension integer lattice $\mathbb{Z}^2$ with finite first moments

$$m_1^1[\mathbf{p}] = \sum_{s\in\mathbb{Z}^2} s^1 \cdot \mathbf{p}(s^1, s^2) < \infty; \quad m_1^2[\mathbf{p}] = \sum_{s\in\mathbb{Z}^2} s^2 \cdot \mathbf{p}(s^1, s^2) < \infty.$$

For $\mathbf{p} \in M^1(\mathbb{Z}^2)$, the liquidation prices of both assets have finite expectations $\mathbf{E_p}[s^1] = m_1^1[\mathbf{p}]$, $\mathbf{E_p}[s^2] = m_1^2[\mathbf{p}]$. The set $M^1$ is a convex subset of the Banach space $L^1(\mathbb{Z}^2, \{1 + |s^1| + |s^2|\})$ of mappings $\mathbf{l} : \mathbb{Z}^2 \to R^1$ with the norm

$$||\mathbf{l}||^1_{\{1+|s^1|+|s^2|\}} = \sum_{s\in\mathbb{Z}^2} \mathbf{l}(s^1, s^2) \cdot (1 + |s^1| + |s^2|).$$

The payoff of the game $G_n(\mathbf{p})$ with $\mathbf{p} \in M^1$ can be approximated using the payoffs of games $G_n(\mathbf{p}_k)$ with probability distributions $\mathbf{p}_k$ having finite support. The next theorem follows immediately from this fact.

**Theorem 2.1.** *If $\mathbf{p} \in M^1$, then the games $G_n(\mathbf{p})$ have values $V_n(\mathbf{p})$. The values $V_n(\mathbf{p})$ are positive and do not decrease, as the number of steps $n$ increases.*

Let $M^2(\mathbb{Z}^2)$ be the set of probability distributions $\mathbf{p}$ over the two-dimension integer lattice $\mathbb{Z}^2$ with finite second moments

$$m_2^1[\mathbf{p}] = \sum_{s\in\mathbb{Z}^2} (s^1)^2 \cdot \mathbf{p}(s^1, s^2) < \infty; \quad m_2^2[\mathbf{p}] = \sum_{s\in\mathbb{Z}^2} (s^2)^2 \cdot \mathbf{p}(s^1, s^2) < \infty.$$

For $\mathbf{p} \in M^2(\mathbb{Z}^2)$, the random variables $s^1$ and $s^2$, determining the prices of shares, belong to $L^2$ and have finite variances

$$\mathbf{D_p}[s^1] = m_2^1[\mathbf{p}] - (m_1^1[\mathbf{p}])^2, \quad \mathbf{D_p}[s^2] = m_2^2[\mathbf{p}] - (m_1^2[\mathbf{p}])^2.$$

We show that, if $\mathbf{p} \in M^2(\mathbb{Z}^2)$, then Player 2 has strategies that guarantee him losses not exceeding the function $H(\mathbf{p})$ in the $n$-stage games $G_n(\mathbf{p})$. Here $H$ is the smallest piecewise linear function equal to $\frac{1}{2}(\mathbf{D_p}[s^1] + \mathbf{D_p}[s^2])$, for distributions with integer expectations of both share prices.

This makes it reasonable to consider infinite games $G_\infty(\mathbf{p})$ where $\mathbf{p} \in M^2(\mathbb{Z}^2)$.

We begin with constructing Player 1's strategies that ensure the gains $H(\mathbf{p})$ for games $G_\infty(\mathbf{p})$ with distributions $\mathbf{p}$ having two- and three-point supports respectively. It follows that these games have the values $V_\infty(\mathbf{p}) = H(\mathbf{p})$ and constructed Player 1's strategies are optimal ones.

For two-point distributions $\mathbf{p}$ the optimal strategies of Player 1 generate asymmetric random walks of posterior probabilities by adjacent points of the lattice formed with those probabilities where at least one of the price expectations has an integer value. The probabilities of jumps provide martingale characteristics of posterior probabilities and with absorption at extreme points.

The martingales of posterior expectations generated by optimal strategies of Player 1 for games with three-point support distributions $\mathbf{p}$ represent symmetric random walks over points of integer lattice lying within the triangle spanned across the support points of distribution. The symmetry is broken at the step when the walk hits the triangle boundary. From this step on the game turns into one of games with distributions having two-point supports.

Using obtained in [6] symmetric decompositions of bivariate probability distributions $\mathbf{p}$ into probability mixtures of distributions with the same mean values and with supports containing at most three points, we build the strategies of Player 1 that ensure the gains $H(\mathbf{p})$, for bidding games $G_\infty(\mathbf{p})$ with distributions $\mathbf{p} \in M^2(\mathbb{Z}^2)$, as convex combinations of his optimal strategies for such games with distributions having two- and three-point supports. Thus we obtain the following result:

**Theorem 2.2.** *For any distribution $\mathbf{p} \in M^2(\mathbb{Z}^2)$, the infinite game $G_\infty(\mathbf{p})$ has the value $V_\infty(\mathbf{p}) = H(\mathbf{p})$.*

## 3. Upper bounds for values $V_n(\mathbf{p})$

The main result of this section is that, for $\mathbf{p} \in M^2(\mathbb{Z}^2)$, the sequence $V_n(\mathbf{p})$ of values remains bounded as $n \to \infty$.

To prove this, we define the set of infinite strategies $\tau^{(k,l)}$ of Player 2, suitable for the games $G_n(\mathbf{p})$ with arbitrary $n$.

**Definition 3.1.** The first move $\tau_1^{(k,l)}$ is the action $(k, l)$. For $t > 1$, the $e$th component of the move $\tau_t^{(k,l)}$, $e = 1, 2$, depends on the last observed pair of $e$th components of actions $(i_{t-1}^e, j_{t-1}^e)$ for both players:

$$j_t^e = \begin{cases} j_{t-1}^e - 1, & \text{if } i_{t-1}^e < j_{t-1}^e\ ; \\ j_{t-1}^e, & \text{if } i_{t-1}^e = j_{t-1}^e; \\ j_{t-1}^e + 1, & \text{if } i_{t-1}^e > j_{t-1}^e. \end{cases}$$

**Proposition 3.2.** *For the state $s = (u, v) \in \mathbb{Z}^2$ the strategy $\tau^{(k,l)}$ ensures the payoff*

$$\max_\sigma h_n^{u,v}(\sigma, \tau^{(k,l)}) \leq (u - k)(u - k - 1)/2 + (v - l)(v - l - 1)/2. \tag{3.1}$$

*Proof.* According to the strategy $\tau^{(k,l)}$ Player 2 operates separately with each of the assets. Hence Player 1 can do the same. Therefore the assertion follows from Proposition 1 of Domansky and Kreps [8]. This proves Proposition 3.2. □

Set

$$H(\mathbf{p}) = 1/2 \cdot (\mathbf{D_p}[u] + \mathbf{D_p}[v] - \alpha(\mathbf{p})(1 - \alpha(\mathbf{p})) - \beta(\mathbf{p})(1 - \beta(\mathbf{p}))) \tag{3.2}$$

where $\alpha(\mathbf{p}) = \mathbf{E_p}[u] - ent[\mathbf{E_p}[u]]$, $\beta(\mathbf{p}) = \mathbf{E_p}[v] - ent[\mathbf{E_p}[v]]$ and $ent[x]$, $x \in R^1$ is the integer part of $x$.

$H(\mathbf{p})$ is a continuous, concave, and piecewise linear function over $M^2(\mathbb{Z}^2)$. The domains of linearity of function $H(\mathbf{p})$ are

$$L(k,l) = \{\mathbf{p} : \mathbf{E}_\mathbf{p}[u] \in [k,k+1], \mathbf{E}_\mathbf{p}[v] \in [l,l+1]\}, \quad (k,l) \in \mathbb{Z}^2.$$

Its peak points are

$$\Theta(k,l) = \{\mathbf{p} : \mathbf{E}_\mathbf{p}[u] = k, \mathbf{E}_\mathbf{p}[v] = l\}.$$

**Theorem 3.3.** *For $\mathbf{p} \in M^2(\mathbb{Z}^2)$, the values $V_n(\mathbf{p})$ are bounded from above by the function $H(\mathbf{p})$.*

*For $\mathbf{p} \in L(k,l)$ the upper bound $H$ is ensured with the strategy $\tau^{(k,l)}$. For $\mathbf{p} \in \Theta(k,l)$ the upper bound $H$ is ensured with the strategies $\tau^{(k,l)}$, $\tau^{(k-1,l)}$, $\tau^{(k,l-1)}$, and $\tau^{(k-1,l-1)}$.*

*Proof.* It follows from Proposition 3.2 that there is the following not depending on $n$ upper bound for $V_n(\mathbf{p})$:

$$V_n(\mathbf{p}) \le \min_{(k,l)} \frac{1}{2} \sum_{u,v=-\infty}^{\infty} ((u-k)(u-k-1) + (v-l)(v-l-1)) \cdot p(u,v). \quad (3.3)$$

Observe that, if $\mathbf{E}_\mathbf{p}[u] - k = \alpha$, $\mathbf{E}_\mathbf{p}[v] - l = \beta$, then

$$\frac{1}{2} \sum_{u,v=-\infty}^{\infty} ((u-k)(u-k-1) + (v-l)(v-l-1)) \cdot p(u,v)$$

$$= \frac{1}{2}(\mathbf{D}_\mathbf{p}[u] + \mathbf{D}_\mathbf{p}[v] - \alpha(\mathbf{p})(1-\alpha(\mathbf{p})) - \beta(\mathbf{p})(1-\beta(\mathbf{p}))).$$

Consequently, for $\mathbf{p} \in L(k,l)$ the minimum in formula (3.3) is attained on $(k,l)$, and the equality (3.2) holds. In particular, for $\mathbf{p} \in \Theta(k,l)$, this minimum is attained on $(k,l)$, $(k-1,l)$, $(k,l-1)$, and $(k-1,l-1)$. □

**Corollary 3.4.** *The strategies $\tau^{k,l}$ guarantee the same upper bound $H(\mathbf{p})$ for the upper value of the infinite game $G_\infty(\mathbf{p})$.*

## 4. Solutions for games $G_\infty(\mathbf{p})$ with two states

In this section we show that, for games $G_\infty(\mathbf{p})$ with supports of distributions $\mathbf{p}$ containing two states $z_1, z_2 \in \mathbb{Z}^2$, the values $V_\infty(\mathbf{p})$ are equal to $H(\mathbf{p})$. Observe that later on we use the notation $z = (x,y) \in \mathbb{Z}^2$ instead of $s = (s^1, s^2) \in \mathbb{Z}^2$.

A distribution with the support $z_1 = (x_1,y_1), z_2 = (x_2,y_2)$ is uniquely determined with expectations for coordinates. For any point $w = (u,v) = p_1 z_1 + p_2 z_2$, $p_i \in [0,1]$, $p_1 + p_2 = 1$, the distribution $\mathbf{p}^w_{z_1,z_2}$ such that $\mathbf{E}_{\mathbf{p}^w_{z_1,z_2}}[x] = u$, $\mathbf{E}_{\mathbf{p}^w_{z_1,z_2}}[y] = v$, is given with probabilities $\mathbf{p}^w_{z_1,z_2}(z_i) = p_i$.

Without loss of generality we assume that one of these points is $(0,0)$. Thus there are two states $0 = (0,0)$ and $z = (x,y)$, where $x$ and $y$ are integers and

$x > 0$. The distribution $\mathbf{p}_{z,0}^{pz}$ can be depicted with a scalar parameter $p \in [0, 1]$ – the probability of state $z$. For definiteness set $y > 0$.

Observe that the function $H(p)$ is equal to the sum of values

$$H(p) = V_\infty^x(p) + V_\infty^y(p) \tag{4.1}$$

of one asset games $G_\infty^x(p)$ and $G_\infty^y(p)$ considered in Domansky [5].

The function $V_\infty^m(p)$ is a piecewise linear continuous concave function of $p \in [0, 1]$. The set of its break points is the regular lattice $\{k/m, k = 0, \ldots, m\}$ with values $V_\infty^m(k/m) = k(m-k)/2$. Therefore, for $p \in [k/m, (k+1)/m]$,

$$\begin{aligned} V_\infty^m(p) &= (pm-k)(k+1)(m-k-1)/2 + (1-pm+k)k(m-k)/2 \\ &= k(m-k)/2 + (pm-k)(m-2k-1)/2. \end{aligned} \tag{4.2}$$

For $p \in [(k-1)/m, k/m]$,

$$V_\infty^m(p) = k(m-k)/2 - (k-pm)(m-2k+1)/2. \tag{4.3}$$

Thus the function $H(p)$ is a piecewise linear continuous concave function of $p \in [0, 1]$. The set of its break points is the irregular lattice $D(x, y) \subset [0, 1]$:

$$D(x, y) = \{k/x, k = 0, \ldots, x\} \cup \{l/y, l = 0, \ldots, y\}.$$

Further we enumerate the points of the lattice $D(x, y)$ in ascending order $D(x, y) = \{p_i\}$, $i = 0, 1, \ldots, I$, $p_0 = 0$, $p_I = 1$, $p_i < p_{i+1}$.

According to Corollary 3.4 the optimal strategy $\tau^*$ guarantees to Player 2 the loss not exceeding the function $H(p)$. Therefore it is sufficient to show that there is an optimal strategy $\sigma^*$ for Player 1 that guarantees him this gain at the break points of function $H(p)$, *i.e.* for the initial probability $p$ belonging to the lattice $D(x, y)$.

Now we present a definition of first moves for the strategy $\sigma^*$ for $p_i \in D(x, y)$.

**Definition 4.1.** For any initial probability $p_i$ the first move of the strategy $\sigma^*$ makes use of two actions $a_i^-$ and $a_i^+$.
For $p_i = k/x \neq l/y$, these actions are $a_i^- = (k-1, l)$ and $a_i^+ = (k, l)$, where $l = \mathrm{ent}(yp_i)$ and $\mathrm{ent}(z)$ is the integral part of $z$.
For $p_i = l/y \neq k/x$, these actions are $a_i^- = (k, l-1)$ and $a_i^+ = (k, l)$, where $k = \mathrm{ent}(xp_i)$.
For $p_i = k/x = l/y$, these actions are $a_i^- = (k-1, l-1)$ and $a_i^+ = (k, l)$.
The posterior probabilities $p(z|a_i^-)$ and $p(z|a_i^+)$ are the left and right adjacent points $p_{i-1}$ and $p_{i+1}$ of the lattice $D(x, y)$ correspondingly.
Consequently the total probabilities of actions are

$$q(a_i^-) = \frac{p_{i+1} - p_i}{p_{i+1} - p_{i-1}}, \quad q(a_i^+) = \frac{p_i - p_{i-1}}{p_{i+1} - p_{i-1}}.$$

This first move is realized with the following conditional probabilities of action $a_i^+$:

$$f^*(a_i^+|z) = \frac{(p_i - p_{i-1})p_{i+1}}{(p_{i+1} - p_{i-1})p_i}, \quad f^*(a_i^+|0) = \frac{(p_i - p_{i-1})(1 - p_{i+1})}{(p_{i+1} - p_{i-1})(1 - p_i)}.$$

As the posterior probabilities also belong to the lattice $D(x, y)$ this set of moves defines the infinite strategy $\sigma^*$. The defined strategy $\sigma^*$ of Player 1 generates the asymmetric random walk of posterior probabilities for state $z$ by adjacent points of the irregular lattice $D(x, y)$ with the probabilities of jumps that provide the martingale characteristics for posterior probabilities and with absorption at the extreme points $p_0 = 0$ and $p_I = 1$.

**Theorem 4.2.** *The value $V_\infty(p)$ of the game $G_\infty(p)$ with two states 0 and $z = (x, y)$, and with the probability $p$ of the state $z$ is equal to the function $H(p)$. Both players have optimal strategies.*
*For the initial probability $p_i \in D(x, y)$, one of optimal strategies of Player 1 is the strategy $\sigma^*$ of Definition 4.1.*
*For the initial probability $p \in (k/x, (k+1)/x) \cap (l/y, (l+1)/y)$ a unique optimal strategy of Player 2 is the strategy $\tau^* = \tau^{k,l}$, defined in Definition 3.1. Any optimal strategy for adjacent intervals is also optimal for points of the lattice $D(x, y)$.*

*Proof.* At first we show that Bellman optimality equations are satisfied with the one-step gain of Player 1 corresponding to the first move $\sigma_1^*$ combined with the optimal gain $H$ at the points of posterior probabilities generated by this move and weighted by total probabilities of actions.

For $p_i = k/x \neq l/y$, the one-step gain of Player 1 corresponding to the first move $\sigma_1^*$ in the game $G_\infty(0, z, p)$ is equal to his gain in the one-asset game $G_\infty^x(p)$

$$\min_{(k',l')} K_1(\sigma_1^*, (k', l')|0, z, p_i) = \min_{k'} K_1^{m_1}(\sigma_1^*, k'|p_i)$$

$$= \frac{x(p_{i+1} - p_i)(p_i - p_{i-1})}{p_{i+1} - p_{i-1}}. \tag{4.4}$$

Here the minimum in the left part is attained at $(k', l') = (k-1, l)$ and $(k, l)$, and the minimum in the right part is attained at $k' = k - 1$ è $k$.

For this move, taking into account (4.2), (4.3), (4.4), we get

$$\min_{k'} K_1^x(\sigma_1^{x,y}, k'|p_i) + q(k-1)V_\infty^x(p_{i-1}) + q(k)V_\infty^x(p_{i+1})$$

$$= \frac{x(p_{i+1} - p_i)(p_i - p_{i-1})}{p_{i+1} - p_{i-1}}$$

$$+\frac{p_{i+1} - p_i}{p_{i+1} - p_{i-1}}(k(x-k)/2 - x(p_i - p_{i-1})(x - 2k + 1)/2)$$

$$+\frac{p_i - p_{i-1}}{p_{i+1} - p_{i-1}}(k(x-k)/2 + x(p_{i+1} - p_i)(x - 2k - 1)/2)$$

$$= k(x-k)/2 = V_\infty^x(p_i). \tag{4.5}$$

Thus the Bellman optimality equation is fulfilled for a one-asset game. On the other hand, three points $p_{i-1}$, $p_i$ and $p_{i+1}$ are situated on the same linearity interval of function $V_\infty^y(p)$, *i.e.*

$$q(k-1)V_\infty^y(p_{i-1}) + q(k)V_\infty^y(p_{i+1}) = V_\infty^y(p_i). \tag{4.6}$$

Summing (4.5) and (4.6), and also taking into account (4.1) we obtain

$$\min_{(k',l')} K_1(\sigma_1^*,(k',l')|0,z,p_i)+q(k-1,l)H(p_{i-1})+q(k,l)H(p_{i+1})=H(p_i),$$

*i.e.*, for $p_i = k/x \neq l/y$ and for the move $\sigma_1^*$ in the game $G_\infty(0,z,p)$, function $H$ satisfies the Bellman optimality equation.

For $p_i = l/y \neq k/x$, the proof of this fact is analogous with replacement of $x$ and $y$.

For $p_i = k/x = l/y$, the Bellman optimality equations (4.5) are fulfilled for both one-asset games $G_\infty^x(p)$ and $G_\infty^y(p)$. Summarizing these optimality equations we obtain the optimality equation for the two-asset game $G_\infty(p)$.

Thus function $H$ satisfies the Bellman optimality equation for all initial probabilities $p_i \in D(x,y)$. Iterating this optimality equation and taking into account the fact that a random walk of posterior probabilities generated by the strategy $\sigma^*$ terminates in a finite mean number of steps, we see that, for the initial probability $p_i \in D(x,y)$, the strategy $\sigma^*$ guarantees Player 1 the gain of $H(p_i)$. □

**Remark 4.3.** Observe all first moves of Player 1 for the one-asset game $G_\infty^m(p)$ that satisfy Bellman optimality equation (4.5). For the initial probability $p = k/m$ these are the moves $m(k,a,b)$ that use only two actions, $k-1$ and $k$ with posterior probabilities $a = p(\cdot|k-1) \in [(k-1)/m, k/m]$ and $b = p(\cdot|k) \in [k/m,(k+1)/m]$, and only these moves.

For the initial probability $p = \delta\cdot k/m+(1-\delta)\cdot(k+1)/m$, $0<\delta<1$, these are any moves $\delta\cdot m(k,a_1,b_1)+(1-\delta)\cdot m(k+1,a_2,b_2)$ that use three actions, $k-1$, $k$ and $k+1$, and jumps which stay in an area on which the function is linear. These jumps are of two types: the first type use the actions $k-1$ and $k$ with posterior probabilities $p(\cdot|k-1) = k/m$ and $p(\cdot|k) \in [p,(k+1)/m]$; the second type use the actions $k$ and $k+1$ with posterior probabilities $p(\cdot|k) \in [k/m,p]$ and $p(\cdot|k+1) = (k+1)/m$. In both cases a boundary point of the interval $[k/m,(k+1)/m]$ appears among the posteriors.

A jump with two loose end points is impossible in the univariate case, but it is possible for one coordinate process in the bivariate case, because this jump is realized by means of two different bids for the other asset, but with the same bids for the asset under consideration.

For the games with two-type risky assets it follows that, if the integer high price of one asset is a multiple of the integer high price of the other asset, then the constructed optimal strategy of Player 1 provides the fastest possible convergence of $n$-step gains for Player 1 to the value $V_\infty(p)$ among all optimal strategies of Player 1 for the two-asset game $G_\infty(p)$.

## 5. Solutions for games $G_\infty(\mathbf{p})$ with three states

In this section we show that, for games $G_\infty(\mathbf{p})$ with the support of distribution $\mathbf{p}$ containing three states $z_1,z_2,z_3 \in \mathbb{Z}^2$, the value $V_\infty(\mathbf{p})$ coincides with $H(\mathbf{p})$.

We assume that three points

$$z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2), \quad z_3 = (x_3, y_3), \quad z_1, z_2, z_3 \in \mathbb{Z}^2$$

are enumerated counterclockwise. It follows that, for $w \in \triangle(z_1, z_2, z_3)$, $\det[z_i - w, z_{i+1} - w] \geq 0$, where $\det[z_i, z_{i+1}] = x_i \cdot y_{i+1} - y_i \cdot x_{i+1}$. Notice that arithmetical operations with subscripts are fulfilled modulo 3.

A distribution with the support $z_1, z_2, z_3$ is uniquely determined with expectations of coordinates. For any point $w = (u, v) \in \triangle(z_1, z_2, z_3)$ the distribution $\mathbf{p}^w_{z_1,z_2,z_3}$ such that

$$\mathbf{E}_{\mathbf{p}^w_{z_1,z_2,z_3}}[x] = u, \quad \mathbf{E}_{\mathbf{p}^w_{z_1,z_2,z_3}}[y] = v,$$

is given with probabilities

$$\mathbf{p}^w_{z_1,z_2,z_3}(z_i) = \frac{\det[z_{i+1} - w, z_{i+2} - w]}{\sum_{j=1}^{3} \det[z_j - w, z_{j+1} - w]}. \tag{5.1}$$

Observe that $\sum_{j=1}^{3} \det[z_j - w, z_{j+1} - w] = \det[z_1 - z_3, z_2 - z_3]$ does not depend on $w$.

By Corollary 3.4 the optimal strategy $\tau^*$ guarantees Player 2 the loss not exceeding $H(\mathbf{p})$. It follows from Theorem 3.2 that, for $\mathbf{p}^w_{z_1,z_2,z_3}$ with $w = (u, v)$ belonging to the boundary of the triangle $\triangle(z_1, z_2, z_3)$, the equality $V_\infty(\mathbf{p}^w_{z_1,z_2,z_3}) = H(\mathbf{p}^w_{z_1,z_2,z_3})$ holds. For other points $w = (u, v) \in \triangle(z_1, z_2, z_3)$, the function $H(\mathbf{p}^w_{z_1,z_2,z_3})$ is the least concave majorant of its values at the points $\mathbf{p}^w_{z_1,z_2,z_3}$ with $w = (u, v) \in \mathbb{Z}^2$ and at the boundary of $\triangle(z_1, z_2, z_3)$. Therefore this is sufficient to show that there is a strategy $\sigma^*$ of Player 1 that guarantees him $H(\mathbf{p}^w_{z_1,z_2,z_3})$, for $w = (u, v) \in \mathbb{Z}^2$.

For the point $w = (u, v) \in \mathbb{Z}^2$ that belongs to the triangle $\triangle(z_1, z_2, z_3)$

$$H(\mathbf{p}^w_{z_1,z_2,z_3}) = \frac{1}{2}\left(\sum_{i=1}^{3}(x_i^2 + y_i^2)\mathbf{p}^w_{z_1,z_2,z_3}(z_i) - (u^2 + v^2)\right). \tag{5.2}$$

For $\mathbf{p}^w_{z_1,z_2,z_3}$ with $w = (u, v) \in \mathbb{Z}^2$, the first step of strategy $\sigma^*$ may efficiently use the actions $(u-1, v-1)$, $(u, v-1)$, $(u-1, v)$ and $(u, v)$. With the help of these actions Player 1 can perform moves such that the modulus of difference between posterior expectations of each coordinate and its initial expectation is not more than one.

There are several types of optimal first moves of Player 1, in particular, the first moves $\sigma_1^{NE-SW}$ (north-east – south-west), $\sigma_1^{NW-SE}$, and their probabilistic mixtures. Denote $e = (1, 1), \bar{e} = (1, -1)$. The first move $\sigma_1^{NE-SW}$ exploits only two actions $w - e$ and $w$ with posterior expectations $w - b \cdot e$ and $w + a \cdot e$. The first move $\sigma_1^{NW-SE}$ makes use of actions $(u-1, v)$ and $(u, v-1)$ with posterior expectations $w - b\bar{e}$ and $w + a\bar{e}$.

Further we define the first move $\sigma_1^{NE-SW}$ both in terms of posterior expectations and in terms of conditional probabilities of actions. We assume w.l.o.g. that

$w = 0 \in \triangle(z_1, z_2, z_3)$. The span of this move is defined with a mutual disposition of the points $-e$, $e$ and the triangle $\triangle(z_1, z_2, z_3)$. If $z_i = k \cdot e$ for some $i = 1, 2, 3$, $k > 0$, then put $a = 1$. If $z_i = k \cdot -e$ for some $i = 1, 2, 3$, $k > 0$, then put $b = 1$.

If $z_i \neq k \cdot e$, $i = 1, 2, 3$, $k > 0$, then there is a unique $i = i^+$ such that the half-line starting at 0 and passing through $e$ crosses the side $z_{i^+}, z_{i^++1}$ of the triangle $\triangle(z_1, z_2, z_3)$. If $z_i \neq k \cdot -e$, $i = 1, 2, 3$, $k > 0$, then there is a unique $i = i^- \neq i^+$ such that the half-line starting at 0 and passing through $-e$ crosses the side $z_{i^-}, z_{i^-+1}$. Put

$$a = \min\left(\frac{\det[z_{i^+}, z_{i^++1}]}{\det[e, z_{i^++1} - z_{i^+}]}, 1\right), \quad b = \min\left(\frac{\det[z_{i^-}, z_{i^-+1}]}{\det[-e, z_{i^-+1} - z_{i^-}]}, 1\right).$$

**Definition 5.1.** The first move $\sigma_1^{NE-SW}$ for the game $G_\infty(\mathbf{p}^0_{z_1,z_2,z_3})$ makes use of actions $-e$ and 0. The posterior expectations are

$$\mathbf{E}_p[z|-e] = -b \cdot e, \quad \mathbf{E}_p[z|0] = a \cdot e.$$

The total probabilities of actions are

$$q(e) = a/(b+a), \quad q(0) = b/(b+a).$$

This move is realized with the conditional probabilities of actions:

$$f^*(-e|z_i) = \frac{a \det[z_{i+1} + b \cdot e, z_{i+2} + b \cdot e]}{(b+a)\det[z_{i+1}, z_{i+2}]}, \quad i = 1, 2, 3;$$

$$f^*(0|z_i) = \frac{b \det[z_{i+1} - a \cdot e, z_{i+2} - a \cdot e]}{(b+a)\det[z_{i+1}, z_{i+2}]}, \quad i = 1, 2, 3.$$

**Remark 5.2.** The martingale of posterior expectations generated by the optimal strategy of Player 1 is a symmetric random walk over the adjacent points of the lattice $\mathbb{Z}^2$ disposed inside the triangle $\triangle(z_1, z_2, z_3)$. The symmetry of this random walk is broken at the moment when it hits the triangle boundary. Beginning from this moment the game degenerates into one of two-point games with the distribution support being either $z_{i^+}, z_{i^++1}$, or $z_{i^-}, z_{i^-+1}$.

If $a < 1$, then after observing the action 0 the next game is $G_\infty(\mathbf{p}^{ae}_{z_{i^+}, z_{i^++1}})$ with the probabilities of states

$$p(z_{i^+}) = \frac{\det[e, z_{i^++1}]}{\det[e, z_{i^++1} - z_{i^+}]}, \quad p(z_{i^++1}) = \frac{\det[z_{i^+}, e]}{\det[e, z_{i^++1} - z_{i^+}]}.$$

If $b < 1$, then after observing the action $-e$ the next game is $G_\infty(\mathbf{p}^{-be}_{z_{i^-}, z_{i^-+1}})$ with the probabilities of states

$$p(z_{i^-}) = \frac{\det[e, z_{i^-+1}]}{\det[e, z_{i^-+1} - z_{i^-}]}, \quad p(z_{i^-+1}) = \frac{\det[z_{i^-}, e]}{\det[e, z_{i^-+1} - z_{i^-}]}.$$

**Theorem 5.3.** *The value $V_\infty(\mathbf{p}^0_{z_1,z_2,z_3})$ of the game $G_\infty(\mathbf{p}^0_{z_1,z_2,z_3})$ is equal to the function $H(\mathbf{p})$ given by (5.2). Both players have optimal strategies.*
*The optimal strategy of Player 2 is given by Definition 3.1.*
*For $w = (u, v) \in \mathbb{Z}^2$, one of optimal strategies of Player 1 is the strategy $\sigma^*$ of Definition 5.1.*

*Proof.* Taking into account Corollary 3.4 and Theorem 4.2 this is sufficient to show that the one-step gain corresponding to the first move $\sigma_1^{NE-SW}$ of optimal strategy of Player 1 combined with the gain $H(\mathbf{p})$ at the points of posterior probabilities generated by this move and weighted by the total probabilities of actions satisfies Bellman optimality equations.

The best replies of Player 2 to the first move $\sigma_1^{NE-SW}$ are actions $0$, $-e$, $(-1, 0)$, and $(0, -1)$. Corresponding one-step gain of Player 1 is equal to $2ab/(b + a)$. In fact,

$$K_1(\sigma_1^{NE-SW}, 0|\mathbf{p}^0_{z_1,z_2,z_3}) = -q(-e)\mathbf{E}_p[x + y| - e] = 2ab/(b + a);$$

$$K_1(\sigma_1^{NE-SW}, -e|\mathbf{p}^0_{z_1,z_2,z_3}) = q(0)\mathbf{E}_p[x + y|0] = 2ab/(b + a).$$

For actions $(0, -1)$ and $(-1, 0)$ of Player 2 the proof is analogous.

It follows from (5.1) and (5.2) that

$$H(\mathbf{p}^0_{z_1,z_2,z_3}) = \frac{\sum_{i=1}^3 (x_i^2 + y_i^2) \det[z_{i+1}, z_{i+2}]}{2 \det[z_1 - z_3, z_2 - z_3]};$$

$$H(\mathbf{p}^{ae}_{z_1,z_2,z_3}) = H(\mathbf{p}^0_{z_1,z_2,z_3}) - a\frac{\sum_{i=1}^3 (x_i^2 + y_i^2) \det[z_{i+1} - z_{i+2}, e]}{2 \det[z_1 - z_3, z_2 - z_3]} - a;$$

$$H(\mathbf{p}^{-be}_{z_1,z_2,z_3}) = H(\mathbf{p}^0_{z_1,z_2,z_3}) + b\frac{\sum_{i=1}^3 (x_i^2 + y_i^2) \det[z_{i+1} - z_{i+2}, e]}{2 \det[z_1 - z_3, z_2 - z_3]} - b.$$

We get

$$2ab/(b + a) + q(-e)H(\mathbf{p}^{-be}_{z_1,z_2,z_3}) + q(0)H(\mathbf{p}^{ae}_{z_1,z_2,z_3}) = H(\mathbf{p}^0_{z_1,z_2,z_3}),$$

*i.e.*, for $\mathbf{p}^0_{z_1,z_2,z_3}$ and for the move $\sigma_1^{NE-SW}$ in the game $G_\infty(\mathbf{p}^0_{z_1,z_2,z_3})$, function $H$ satisfies Bellman optimality equation. □

## 6. Decomposition of bivariate distributions

Further we consider games $G_\infty(\mathbf{p})$ with prices given by arbitrary probability distributions $\mathbf{p} \in \Delta(\mathbb{Z}^2)$. We get solutions for the games $G_\infty(\mathbf{p})$ as combinations of the solutions of games with two and three states that were obtained in sections 4 and 5. To realize this idea we use symmetric representations of distributions over $\mathbb{R}^2$ with given mean values as convex combinations of distributions with the same mean values and with supports containing at most three points. (For a developed presentation and proofs see [6]).

As a pattern we take the symmetric representation of one-dimensional probability distributions that was exploited in [8] in order to reduce solving models with prices of assets given by arbitrary probability distributions over $\mathbb{Z}^1$ to solving such models with two-point distributions. Let $\mathbf{p}$ be a probability distribution over $\mathbb{R}^1$ with zero mean value. Then

$$\mathbf{p} = \mathbf{p}(0)\delta^0 + \int_{x=0^+}^{\infty} \mathbf{p}(\mathrm{d}x) \int_{y=0^+}^{\infty} \frac{x+y}{\int_{t=0}^{\infty} t \cdot \mathbf{p}(\mathrm{d}t)} \cdot \mathbf{p}^0_{x,-y} \cdot \mathbf{p}(-\mathrm{d}y),$$

where $\delta^x$ is the one-point distribution with the support $\{x\}$, and for $x, y > 0$, the distributions $\mathbf{p}^0_{x,-y} = (x \cdot \delta^{-y} + y \cdot \delta^x)/(x+y)$.

Consider the set $\mathbf{P}(\mathbb{R}^2)$ of probability distributions $\mathbf{p}$ over the plane $\mathbb{R}^2 = \{z = (x, y)\}$ with finite first absolute moments. We denote mean values of the distribution $\mathbf{p}$ by $\mathbf{E_p}[x]$ and $\mathbf{E_p}[y]$. We construct symmetric representations of convex sets of distributions with given mean values

$$\Theta(u, v) = \{\mathbf{p} \in \mathbf{P}(\mathbb{R}^2) : \mathbf{E_p}[x] = u, \mathbf{E_p}[y] = v\},$$

as convex hulls of their extreme points, *i.e.* distributions with supports with the same mean values and containing at most three points. For extreme points of convex sets of distributions with given moments see [11]. This is sufficient to give such a decomposition for the set $\Theta(0, 0)$ of centered distributions.

The distribution $\mathbf{p}^0_{z_1,z_2} \in \Theta(0, 0)$ with the two-point support $\{z_1, z_2\}$ such that $(0, 0)$ belongs to the interval $(z_1, z_2)$, *i.e.* $z_1 = ae_\psi$, $z_2 = -be_\psi$ where $e_\psi$ is a unit vector with $\arg e_\psi = \psi$, $a, b \in \mathbb{R}^1_+$, is given by

$$\mathbf{p}^0_{ae_\psi,-be_\psi} = \frac{b \cdot \delta^{ae_\psi} + a \cdot \delta^{-be_\psi}}{a+b}.$$

The distribution $\mathbf{p}^0_{z_1,z_2,z_3} \in \Theta(0, 0)$ with the support $\{z_1, z_2, z_3\}$ such that $(0, 0)$ belongs to the interior of the triangle $\triangle(z_1, z_2, z_3)$ is given by

$$\mathbf{p}^0_{z_1,z_2,z_3} = \frac{\sum_{i=1}^{3} \det[z_{i+1}, z_{i+2}] \cdot \delta^{z_i}}{\sum_{j=1}^{3} \det[z_j, z_{j+1}]},$$

where $\det[z_i, z_{i+1}] = x_i \cdot y_{i+1} - y_i \cdot x_{i+1}$. All arithmetical operations with subscripts are fulfilled in modulo 3.

Consider the set $\Delta^0$ of non-ordered triples $(z_1, z_2, z_3)$ that form triangles containing the point $(0, 0)$:

$$\Delta^0 = \{(z_1, z_2, z_3), z_i \neq (0, 0) : (0, 0) \in \triangle(z_1, z_2, z_3)\}.$$

The set $\Delta^0$ is a manifold with a boundary. Its interior $\mathrm{Int}\Delta^0$ is the set of triples $(z_1, z_2, z_3) \in \Delta^0$ such that $(0, 0)$ belongs to the interior of the $\triangle(z_1, z_2, z_3)$. Its boundary $\partial\Delta^0$ is the set of triples $(z_1, z_2, z_3) \in \Delta^0$ such that $(0, 0)$ belongs to the boundary of the $\triangle(z_1, z_2, z_3)$.

If $(z_1, z_2, z_3) \in \partial\Delta^0$, then there is an index $i$ such that $\det[z_i, z_{i+1}] = 0$. In this case $\arg z_{i+1} = \arg z_i + \pi (\bmod\ 2\pi)$, the point $(0,0) \in [z_i, z_{i+1}]$ and the distribution $\mathbf{p}^0_{z_1,z_2,z_3}$ degenerates into the distribution $\mathbf{p}^0_{z_i,z_{i+1}}$ with the support $\{z_i, z_{i+1}\}$.

For $\psi \in [0, 2\pi)$, let $R_\psi$ be the half-line $R_\psi = \{z : \arg z = \psi (\bmod\ 2\pi)\}$. With each value $\psi \in [0, 2\pi)$ we associate the set $\Delta^0(\psi)$ of non-ordered couples

$$\Delta^0(\psi) = \{(z_1, z_2), z_i \neq (0,0) : \forall z \in R_\psi \quad (0,0) \in \triangle(z_1, z_2, z)\}.$$

Let $\mathrm{Int}\Delta^0(\psi)$ and $\partial\Delta^0(\psi)$ be the sets of non-ordered couples $(z_1, z_2)$ such that, for $z \in R_\psi$, the triple $(z_1, z_2, z)$ belongs to $\mathrm{Int}\Delta^0$ and to $\partial\Delta^0$ respectively. We accept that points $(z_1, z_2)$ are indexed counterclockwise. This implies $\det[z_1, z_2] \geq 0$.

Now we introduce the value that plays the role of $\int_{t=0}^{\infty} t \cdot \mathbf{p}(\mathrm{d}t)$, for symmetric representations of distributions over $\mathbb{R}^2$. Set

$$\Phi(\mathbf{p}, \psi) = \int_{\mathrm{Int}\Delta^0(\psi)} \det[z_1, z_2]\mathbf{p}(\mathrm{d}z_1)\mathbf{p}(\mathrm{d}z_2) + 1/2 \int_{\partial\Delta^0(\psi)} \det[z_1, z_2]\mathbf{p}(\mathrm{d}z_1)\mathbf{p}(\mathrm{d}z_2). \tag{6.1}$$

The next fact produces a base for constructing symmetric representations of distributions over $\mathbb{R}^2$.

**Theorem 6.1.** *For any distribution $\mathbf{p} \in \Theta(0,0)$ the quantity $\Phi(\mathbf{p}, \psi)$ does not depend on $\psi$, i.e. this is an invariant $\Phi(\mathbf{p})$ of the distribution $\mathbf{p} \in \Theta(0,0)$.*

**Remark 6.2.** This theorem is a two-dimensional analog of the equality

$$\int_{t=0}^{\infty} t \cdot \mathbf{p}(\mathrm{d}t) = \int_{t=0}^{\infty} t \cdot \mathbf{p}(-\mathrm{d}t)$$

that holds for $\mathbf{p} \in \Theta(0) \subset \mathbf{P}(\mathbb{R}^1)$.

In order that the distribution $\mathbf{p}^0_{z_i,z_{i+1}}$ with the two-point support $(z_i, z_{i+1})$, where $z_i \in R_\psi$ and $z_{i+1} \in R_{\psi+\pi}$, could appear in the decomposition of distribution $\mathbf{p}$ with nonzero probability, it is necessary that the measure $\mathbf{p}(R_\psi)$ and the measure $\mathbf{p}(R_{\psi+\pi})$ be more than zero. This is possible for at most a countable set $\Psi(\mathbf{p})$ of values $\psi$.

Now we formulate the decomposition theorem for bivariate distributions.

**Theorem 6.2.** *Any probability distribution $\mathbf{p} \in \Theta(0,0)$ has the following symmetric representation as a convex combination of distributions with one-, two-, and three-point supports:*

$$\mathbf{p} = \mathbf{p}(0,0) \cdot \delta^0 + \int_{Int\Delta^0} \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3} \mathbf{p}(\mathrm{d}z_1)\mathbf{p}(\mathrm{d}z_2)\mathbf{p}(\mathrm{d}z_3)$$

$$+ \sum_{\Psi(\mathbf{p})} \frac{\partial\Phi(\mathbf{p}, \psi)}{\Phi(\mathbf{p})} \int_{R_\psi} \int_{R_{\psi+\pi}} \frac{r_1 + r_2}{\int_{R_{\psi+\pi}} t\mathbf{p}(\mathrm{d}t)} \mathbf{p}^0_{(r_1,\psi),(r_2,\psi+\pi)} \mathbf{p}(\mathrm{d}r_2)\mathbf{p}(\mathrm{d}r_1), \tag{6.2}$$

*where $\Phi(\mathbf{p})$ is given by (6.1) and $\partial\Phi(\mathbf{p}, \psi) = 1/2 \int_{\partial\Delta^0(\psi)} \det[z_1, z_2]\mathbf{p}(\mathrm{d}z_1)\mathbf{p}(\mathrm{d}z_2)$.*

## 7. Constructing optimal strategies for Player 1

In this section we construct optimal strategies for Player 1 for bidding games for shares of two types with arbitrary distribution with an integer expectation vector $(k, l)$, as a convex combination of his optimal strategies for such games with distributions having at most three-point supports by making use of the decomposition for the initial distribution $\mathbf{p}$ developed above. Further we assume that $\mathbf{E}_{\mathbf{p}}[x] = u$, $\mathbf{E}_{\mathbf{p}}[y] = 0$.

If the state chosen by chance move is $(0, 0)$, then Player 1 stops the game. In this case he cannot get any profit from his informational advantage.

If the state chosen by chance move is $(x, y) \neq (0, 0)$, then he chooses one or two complementary points by means of a lottery with the conditional probabilities of these complements. Further he plays his optimal strategy for the state $(x, y)$ in the game with a distribution having two- or three-point support that is the state $(x, y)$ and the chosen complement.

Half-lines $R_\psi$ and $R_{\psi+\pi}$ contain points from $\mathbb{Z}^2$ iff $\tan\psi = u/v$ with $(u, v)$ being a relatively prime pair of integers. Here $w_\psi = (u, v)$ and $-w_\psi = w_{\psi+\pi}$ are the nearest to $(0, 0)$ lattice points on these half-lines. Any lattice point of the half-line $z \in \mathbb{Z}^2 \cap R_\psi$ has the form $z = w_\psi \cdot k$ with $k \in \mathbb{N}$. Consequently, for distribution $\mathbf{p}$ over the lattice $\mathbb{Z}^2$

$$\int_{R_\psi} t \cdot \mathbf{p}(\mathrm{d}t) = \sum_{t=1}^{\infty} |w_\psi| t \cdot \mathbf{p}(w_\psi \cdot t).$$

For distributions $\mathbf{p} \in \Theta(0, 0)$ with supports in $\mathbb{Z}^2$ formula (6.2) indicates probabilities $\mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{z_1,z_2,z_3})$ and $\mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{z_1,z_2})$ of appearance of distributions with two-, and three-point supports in their symmetric representations:

$$\mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{z_1,z_2,z_3}) = \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}(z_1)\mathbf{p}(z_2)\mathbf{p}(z_3);$$

$$\begin{aligned}\mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{(r_1,\varphi),(r_2,\varphi+\pi)}) &= \frac{\partial\Phi(\mathbf{p}, \varphi)}{\Phi(\mathbf{p})} \frac{r_1 + r_2}{\sum_{t=1}^{\infty} |w_\varphi| t \cdot \mathbf{p}(-w_\varphi \cdot t)} \mathbf{p}(r_1, \varphi)\mathbf{p}(r_2, \varphi + \pi) \\ &= \frac{\partial\Phi(\mathbf{p}, \varphi + \pi)}{\Phi(\mathbf{p})} \frac{r_1 + r_2}{\sum_{t=1}^{\infty} |w_\varphi| t \cdot \mathbf{p}(w_\varphi \cdot t)} \mathbf{p}(r_1, \varphi)\mathbf{p}(r_2, \varphi + \pi).\end{aligned}$$

It follows that, given one point $z = k \cdot w_\varphi$ in the support of extreme distribution, the conditional probabilities $\mathbf{P}_{\mathbf{p}}(2|z)$ and $\mathbf{P}_{\mathbf{p}}(3|z)$ of two or three points in it are

$$\mathbf{P}_{\mathbf{p}}(2|z) = \frac{\partial\Phi(\mathbf{p}, \varphi)}{\Phi(\mathbf{p})}, \quad \mathbf{P}_{\mathbf{p}}(3|z) = 1 - \frac{\partial\Phi(\mathbf{p}, \varphi)}{\Phi(\mathbf{p})} = \frac{\mathrm{Int}\Phi(\mathbf{p}, \varphi)}{\Phi(\mathbf{p})}. \tag{7.1}$$

The conditional probabilities $\mathbf{P}_{\mathbf{p}}(z_2|z, 2)$ and $\mathbf{P}_{\mathbf{p}}(z_2, z_3|z, 3)$ of complementary points in the support of extreme distribution given the point $z = k \cdot w_\varphi$ in it and the number of points are

$$\mathbf{P}_{\mathbf{p}}(-l \cdot w_\varphi|z, 2) = \frac{l \cdot \mathbf{p}(-l \cdot w_\varphi)}{\sum_{t=1}^{\infty} t \cdot \mathbf{p}(-t \cdot w_\varphi)}; \tag{7.2}$$

$$\mathbf{P}_{\mathbf{p}}(z_2, z_3|z, 3) = \frac{\det[z_2, z_3]\mathbf{p}(z_2)\mathbf{p}(z_3)}{\mathrm{Int}\Phi(\mathbf{p}, \varphi)}. \tag{7.3}$$

Consequently, the optimal strategy of Player 1 is given by the following algorithm:

1. If the state chosen by chance move is $(0, 0)$, then Player 1 stops the game.
2. Let the state chosen by chance move be $z = k \cdot w_\varphi \neq (0, 0)$. Then Player 1 realizes the Bernoulli trial with probabilities (7.1) to choose between two-point and three-point distributions.
3. If two-point distributions are chosen, then Player 1 chooses a point $z_2 = -lw$ by means of the lottery with probabilities (7.2) and plays the optimal strategy $\sigma^*(\cdot|z)$ for the state $z = kw$ in the two-point game $G(\mathbf{p}^0_{kw,-lw})$.
4. If three-point distributions are chosen, then Player 1 chooses a pair of points $z_2, z_3$ by means of the lottery with probabilities (7.3) and plays the optimal strategy $\sigma^*(\cdot|z)$ for the state $z = z_1$ in the three-point game $G(\mathbf{p}^0_{z_1,z_2,z_3})$.

As the optimal strategies $\sigma^*$ ensure Player 1 the gains equal to one half of the sum of component variances $\mathbf{D}_{\mathbf{p}}[u] + \mathbf{D}_{\mathbf{p}}[v]$ in the two and three-point games with $\mathbf{p} \in \Theta(k, l)$, and as the sum of component variances is a linear function over $\Theta(k, l) \cap M^2$, where $M^2$ is the class of distributions with finite second moment, we obtain the following result:

**Theorem 7.1.** *For any distribution $\mathbf{p} \in \Theta(k, l) \cap M^2$ the compound strategy depicted above ensures that Player 1 will gain $1/2 \cdot (\mathbf{D}_{\mathbf{p}}[u] + \mathbf{D}_{\mathbf{p}}[v])$ in the game $G_\infty(\mathbf{p})$.*

## 8. Concluding remarks. Games with $m$ risky assets, $m > 2$.

For games with $m$ risky assets, $m > 2$, it can be shown that if all share prices have finite variances, then values of $n$-stage games do not exceed the piecewise linear function $H$, equal to the one half of the sum of share price variances for distributions with integer mean values of all share prices. This makes it reasonable to consider games of unlimited duration.

The extension of represented approach to games with $m$ risky assets requires resolving two problems:

1) constructing symmetric representations of distributions over $\mathbb{R}^m$ with given mean values as convex combinations of distributions with the same mean values and with supports containing at most $m + 1$ points;
2) constructing strategies of Player 1 that ensure $H$ for games $G(\mathbf{p})$ with distributions $\mathbf{p}$ over $\mathbb{Z}^m$ with supports that contain at most $m + 1$ points.

The solution of the first problem is straightforward for centered distributions over $\mathbb{R}^m$, $m > 2$, that do not include distributions with less than $m$-point supports in their decomposition. These are distributions, such that for any $m - k$-dimensional linear subspace of non-zero measure, $k > 1$, there is a half-subspace of zero measure.

Thus to construct the decomposition of a centered distribution $\mathbf{p}$ over $\mathbb{R}^m$ we represent it as a sum $\mathbf{p} = \sum_{k=1}^{m} \alpha_k \mathbf{p}^k$, where for $k < m$ a distribution $\mathbf{p}^k$ is represented as a convex combination of $k$-point distributions, and $\mathbf{p}^m$ is represented as a convex combination of $m$ and $m+1$-point distributions. In particular, a coefficient $\alpha_1 = \mathbf{p}(0)$ and a distribution $\mathbf{p}^1 = \delta^0$, the characteristic property of a distribution $\mathbf{p}^2$ is that its part disposed on any straight line crossing the origin is a centered substochastic distribution, and so on.

In particular, to construct the decomposition of a centered distribution $\mathbf{p}$ over $\mathbb{R}^3$ we have to exclude the possibility of one- and two-point distributions. So we eliminate an atom at the origin (if there is one). After this elimination there is no more than a countable number of straight lines crossing the origin that have a non-zero measure $\mathbf{p}$. For each of these lines we select a centered substochastic distribution so that on a half-line the whole of distribution is selected. Normalizing the join of selected distributions we get $\mathbf{p}^2$, and normalizing the residual we get $\mathbf{p}^3$.

Constructing optimal strategies of Player 1 for games $G_\infty(\mathbf{p})$ with distributions $\mathbf{p}$ over $\mathbb{Z}^m$ with supports that contain $m+1$ points can be realized in the same way as it was done for games $G_\infty(\mathbf{p})$ with three states. The martingale of posterior expectations generated by the optimal strategy of Player 1 for the game with the $m+1$-point support distribution represents a symmetric random walk over points of integer lattice lying within the simplex spanned across the support points of distribution. The symmetry is broken at the moment that the walk hits the simplex boundary. From this moment, the game turns into one of games with distributions having $m$-point supports.

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