RAIRO-Oper. Res. 44 (2010) 139–152
DOI: 10.1051/ro/2010011

**RAIRO Operations Research**
www.rairo-ro.org

# EXTENDED VIKOR AS A NEW METHOD FOR SOLVING MULTIPLE OBJECTIVE LARGE-SCALE NONLINEAR PROGRAMMING PROBLEMS

Majeed Heydari[1], Mohammad Kazem Sayadi[1] and Kamran Shahanaghi[1]

**Abstract.** The VIKOR method was introduced as a Multi-Attribute Decision Making (MADM) method to solve discrete decision-making problems with incommensurable and conflicting criteria. This method focuses on ranking and selecting from a set of alternatives based on the particular measure of "closeness" to the "ideal" solution. The multi-criteria measure for compromise ranking is developed from the $l-p$ metric used as an aggregating function in a compromise programming method. In this paper, the VIKOR method is extended to solve Multi-Objective Large-Scale Non-Linear Programming (MOLSNLP) problems with block angular structure. In the proposed approach, the Y-dimensional objective space is reduced into a one-dimensional space by applying the Dantzig-Wolfe decomposition algorithm as well as extending the concepts of VIKOR method for decision-making in continues environment. Finally, a numerical example is given to illustrate and clarify the main results developed in this paper.

**Keywords.** Large-scale systems, multi-criteria decision making, nonlinear programming, compromise programming, ideal solution, VIKOR method.

**Mathematics Subject Classification.** 90C06, 90C30, 90V29.

Received August 11, 2008. Accepted October 9, 2009.

[1] Department of Industrial Engineering, Iran University of Science and Technology, Narmak, Tehran, Iran; m_heidary@ind.iust.ac.ir

Article published by EDP Sciences 

## Introduction

Modeling and optimization of real world problems typically require taking into account considerable and sometimes very large number of variables and parameters which may interrelate in complex and nonlinear manner. In addition, usually simultaneous optimizations of several objectives that may have conflict nature are interested. Increasing the number of variables, objectives and complexity of structures lead to introducing one of the most challenging optimization problems which is called multiple objective large scale nonlinear programming problems (MOLSNLP). In these problems, because of involving large number of variables in nonlinear objectives and constraints besides multiple conflicting objectives, the computational complexity increases sharply and obtaining efficient solutions in a less time and efficient manner becomes harder. However, fortunately when real world problems are modeled as large-scale programming problems, most of them usually have some special structures that can be handled efficiently. Block angular structure is one these special structures. More information about the large scale programming problems and their common structures can be found in [7,11].

In the large scale programming literature, introducing the decomposition algorithm by Dantzig-Wolfe [5,6] had an influentional impact on the subsequent researches on large-scale linear and nonlinear programming problems which have block angular structure. This leads to noticeably increasing the number of researches on the large scale programming problems with block angular structures [8,9,16]. Some of these works focused on extending and applying MCDM models to deal with multi-objective nonlinear programming problems in large-scale context. Abo-sinna *et al.* [1] extended the TOPSIS method for MOLSNLP problems. They used the concept of extended TOPSIS for Multiple Objective Decision Making (MODM) problems introduced by Lai *et al.* [10]. Recently, because of the advantages and high potentials of the VIKOR method [14,15], many researches are conducted to use the VIKOR method for dealing with decision-making problems in different areas. Opricovic developed a fuzzy VIKOR method to solve MADM problem in a fuzzy environment where both criteria and weights could be fuzzy sets [13]. Sayadi *et al.* [17] extended the VIKOR method for solving MADM problem with interval numbers. Buyukozkan *et al.* [3] used the fuzzy VIKOR method for evaluation of suppliers' environmental management performances. Tong *et al.* [18] applied VIKOR method to optimize multi-response processes. Chu *et al.* [4] compared the properties of SAW, TOPSIS and VIKOR methods for knowledge communities' group-decision analysis. They reveal that the VIKOR method produces different rankings than those from TOPSIS and SAW, in addition, it makes easy to choose appropriate strategies.

In this paper, for the first time in continues decision-making literature we extend the VIKOR method to solve MOLSNLP problem. To do this, the Dantzig-Wolfe decomposition algorithm is applied to decompose a $Y$-dimensional objective space with $N$ decision variables to $N$ sub-problems that have $Y$ objective functions with one variable. Afterward, for each sub problem, based on the extended concepts of VIKOR method, objective functions are aggregated as an equation. Finally, these $N$ equations are combined into a single objective optimization problem that can be solved using conventional methods. In the following section, we will give the formulation of MOLSNLP problem with block angular structure for which the Dantzig-Wolfe decomposition algorithm has been successfully applied. The extended VIKOR method is presented in Section 2. For the sake of illustration, a numerical example is given in Section 3. Finally, conclusion is remarked is Section 4.

## 1. Problem formulation

Consider a convex Multi-Objective Large-Scale Non Linear Programming problem

$$\max\,(\min) \quad F_y(f_{y1}(x_1), f_{y2}(x_2), \ldots, f_{yN}(x_N)) \qquad y = 1, 2, \ldots, Y,\ Y \geq 2$$
$$S.t. \quad FS = \begin{cases} g_m(x_1) \leq 0 & m = 1,\, \ldots,\, s_1 \\ g_m(x_2) \leq 0 & m = s_1 + 1,\, \ldots,\, s_2 \\ \vdots & \vdots \\ g_m(x_N) \leq 0 & m = s_{r-1} + 1,\, \ldots,\, M \\ H_i(X) = \sum_{j=1}^{N} h_{ij}(x_j) \leq 0 & i = 1,\, \ldots,\, t \end{cases} \tag{1.1}$$

where $X = (x_1,\, \ldots,\, x_N)$is the $N$-dimensional decision vector, $F_y$, $y = 1,\, \ldots,\, Y$ are the objective functions. Note that the set of first $M$ constraints are called common constraints and they are convex real valued functions on $R^N$ . The objective functions and the constraints are also assumed to have an additively separable form. Note that any (or all) of the functions may be nonlinear.

Using the Dantzig-Wolfe decomposition algorithm the MOLSNLP problem (1.1) can be decomposed into $N$ sub-problems as shown in the following lines. The $k$th sub-problem $(P_k)$ for $k = 1,\, \ldots,\, N$ is defined as:

$$\begin{aligned} &\max\,(\min) \quad f_{1k}(x_k) \\ &\max\,(\min) \quad f_{2k}(x_k) \\ &\qquad\qquad \vdots \\ &\max\,(\min) \quad f_{Yk}(x_k) \\ &S.t. \\ &(x_1, x_2, \ldots, x_N) \in FS. \end{aligned} \tag{1.2}$$

## 2. Extension of VIKOR method for MOLSNLP

The VIKOR method was introduced by Opricovic in 1998 [12] as one applicable technique to be implemented within MCDM. It was developed as a multi-attribute decision-making method to solve a discrete decision making problem with incommensurable (different units) and conflicting criteria. This method focuses on ranking and selecting from a set of alternatives, and determines compromise solution for a problem with conflicting criteria, which can help the decision makers to reach a final solution. The compromise solution is a feasible solution, which is the closest to the ideal, and compromise means an agreement established by mutual concessions. The multi-criteria measure for compromise ranking is developed from the $l - p$ metric used as an aggregating function in a compromise programming method [19].

In this section, we extend the VIKOR method to solve MOLSNLP problems formulated as (1.1). To do this, first, the MOLSNLP problem is decomposed into $N$ sub-problems as shown in (1.2). Then the Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS) for $P_k$, $k = 1, \ldots, N$ are computed. Afterward, $S_k$ and $R_k$ for $k = 1, \ldots, N$ are obtained. In the next step $S_k^*$, $S_k^-$, $R_k^*$ and $R_k^-$ are computed. Finally, $Q_k$ for $k = 1, \ldots, N$ are obtained and combined into a single objective optimization problem. Using this approach, we transfer $Y$ incommensurable and conflict objectives into a single objective function that can be solved using the conventional methods. The proposed approach is described as follows:

for $P_k$, $k = 1, \ldots, N$, we indexed the benefit and cost objectives as follows:

$$f_{bk}(x_k) = \text{Benefit objective for maximization} \qquad b \in B,\ B \subseteq Y \tag{2.1}$$

$$f_{ck}(x_k) = \text{Cost objective for maximization} \qquad c \in C,\ C \subseteq Y. \tag{2.2}$$

In order to compute PIS and NIS, following formulas are used:

$$f_{ik}^* = \left\{ \max_{X\in FS} (\min)\, f_{bk}(x_k)\, (f_{ck}(x_k)), \forall b\, (\forall c) \right\} \qquad \text{for } i = 1, \ldots, Y \tag{2.3}$$

$$f_{ik}^- = \left\{ \min_{X\in FS} (\max)\, f_{bk}(x_k)\, (f_{ck}(x_k)), \forall b\, (\forall c) \right\} \qquad \text{for } i = 1, \ldots, Y \tag{2.4}$$

where $b \in B$ and $c \in C$, $B, C \subseteq Y$. $f_k^* = \{f_{1k}^*, \ldots, f_{Yk}^*\}$ and $f_k^- = \{f_{1k}^-, \ldots, f_{Yk}^-\}$ are the sets of individual positive and negative ideal solutions where each of them is a point solution in the Y-dimensional objective functional space.

In order to solve discrete decision-making problems using the VIKOR method, the $l - p$ metric with $p = 1$ as $S_k$ and $p = \infty$ as $R_k$ is used. In the same way, for continues decision-making problems we can use the same formulas. In this situation, $S_k$ and $R_k$ are functions not discrete real values. Therefore, the concept of $l - p$ metric distances in continues environment [2,19] are as follows: for $S_k$:

$$S_k = \sum_{b\in B} w_b \left( \frac{f_{bk}^* - f_{bk}(x_k)}{f_{bk}^* - f_{bk}^-} \right) + \sum_{c\in C} w_c \left( \frac{f_{ck}(x_k) - f_{ck}^*}{f_{ck}^- - f_{ck}^*} \right) \qquad k = 1, \ldots, Y \tag{2.5}$$

where, $w_i, i = 1, \dots, Y$ are the weights of objectives that express their relative importance. Note that $S_k$ is interpreted as "group desirability" or "majority" function and can provide the decision makers with information about the measure of "group desirability" in the decision made. The $R_K$ also is the function in terms of $f_{bk}$ or $f_{ck}$ which has maximum distance from the PIS. To obtain $R_k$, the following problem should be solved.

$$\min_{X \in FS} \max \left\{ w_b \left( \frac{f^*_{bk} - f_{bk}(x_k)}{f^*_{bk} - f^-_{bk}} \right), w_c \left( \frac{f_{ck}(x_k) - f^*_{ck}}{f^-_{ck} - f^*_{ck}} \right) \right\} \qquad k = 1, \dots, Y \quad (2.6)$$

which is equivalent to the following $\lambda$-problem:

$$\begin{aligned} &\min \lambda \\ &S.t. \quad w_b \left( \frac{f^*_{bk} - f_{bk}(x_k)}{f^*_{bk} - f^-_{bk}} \right) \leq \lambda \qquad b \in B \\ &\qquad\quad w_c \left( \frac{f_{ck}(x_k) - f^*_{ck}}{f^-_{ck} - f^*_{ck}} \right) \leq \lambda \qquad c \in C \\ &\qquad\quad X = (x_1, x_2, \dots, x_N) \in FS. \end{aligned} \qquad (2.7)$$

If $X^*_k = (x^*_{1k}, \dots, x^*_{Nk})$ is the optimal point of (2.6) and for this point, the inequality constraint $b^+$ ($or\; c^+$) is the active constraint (it is satisfied as equal), then $R_k$ is the left terms of activated constraint as follows:

$$R_k = w_{b^+} \left( \frac{f^*_{b^+k} - f_{b^+k}(x_k)}{f^*_{b^+k} - f^-_{b^+k}} \right) \left\{ or\; w_{c^+} \left( \frac{f_{c^+k}(x_k) - f^*_{c^+k}}{f^-_{c^+k} - f^*_{c^+k}} \right) \right\} \qquad (2.8)$$

where $R_k$ is interpreted as "individual regret" function and can provide the decision makers with information about the measure of "individual regret" in the decision made. Note that if more than one constraint is active we choose the constraint that the values of $R^*_k$ is minimum and if more than one constraint has the same minimum value, we choose the constraint that the values of $R^-_k$ is maximum. Otherwise, we can choose any of them as $R_k$.

For the obtained functions, $S_k$ and $R_k$, the following values are computed:

$$S^*_k = \min_{X \in FS} S_k \qquad S^-_k = \max_{X \in FS} S_k \qquad (2.9)$$

$$R^*_k = \min_{X \in FS} R_k \qquad R^-_k = \max_{X \in FS} R_k. \qquad (2.10)$$

Then $Q_k$ as a function of $x_k$, is obtained as follows:

$$Q_k = \nu \left(\frac{S_k - S_k^*}{S_k^- - S_k^*}\right) + (1-\nu)\left(\frac{R_k - R_k^*}{R_k^- - R_k^*}\right) \tag{2.11}$$

where, $\nu$ is introduced as weight of the strategy of decision-making and can interpreted as "voting by majority rule" (when $\nu > 0.5$), or "by consensus" (when $\nu = 0.5$) or "with veto" (when $\nu < 0.5$). In this situation, the decision maker(s) can impose his/her (their) opinions in the process of decision making by choosing the value of $\nu$.

To obtain compromise solution of (1.1), we choose the closest solution to the PIS that is equivalent to minimize all of $Q_k$ functions for $k = 1, \ldots, N$. This is based on the assumption that the decision maker would like to choose the decisions that minimize the sum of weighted distances from the optimal group desirability ($S_k^*$) and the optimal individual regret ($R_k^*$). To do this, $N$ objectives ($Q_k, k = 1, \ldots, N$) are transformed into the following single objective problem.

$$\begin{aligned}
&\min\ \alpha\\
&\qquad S.t.\quad Q_1 \le \alpha\\
&\qquad\qquad Q_2 \le \alpha\\
&\qquad\qquad\ \ \vdots\\
&\qquad\qquad Q_N \le \alpha\\
&\qquad \begin{cases}
g_m(x_1) \le 0 & m = 1,\, \ldots,\, s_1\\
g_m(x_2) \le 0 & m = s_1+1,\, \ldots,\, s_2\\
\vdots & \vdots\\
g_m(x_N) \le 0 & m = s_{r-1}+1,\, \ldots,\, M\\
H_i(X) = \sum_{j=1}^N h_{ij}(x_j) \le 0 & i = 1,\, \ldots,\, t.
\end{cases}
\end{aligned} \tag{2.12}$$

After solving this problem, the obtained solution is a compromise solution of the problem (1.1). Then by substituting the compromise solution vector in (1.1), the values of objective functions are computed.

In special cases where, $x_1, x_2, \ldots, x_N$ are independent, we can use the following model instead of model (2.12).

$$\begin{aligned}
&\min\ Q = Q_1 + Q_2 + \ldots + Q_N\\
&\qquad S.t.\\
&\qquad \begin{cases}
g_m(x_1) \le 0 & m = 1,\, \ldots,\, s_1\\
g_m(x_2) \le 0 & m = s_1+1,\, \ldots,\, s_2\\
\vdots & \vdots\\
g_m(x_N) \le 0 & m = s_{r-1}+1,\, \ldots,\, M\\
H_i(X) = \sum_{j=1}^N h_{ij}(x_j) \le 0 & i = 1,\, \ldots,\, t.
\end{cases}
\end{aligned} \tag{2.13}$$

## 3. An illustrative example

In this section, we present a simple example that obviously is not large scale, to illustrate the steps of the proposed approach. Consider the following Vector Optimization Problem (VOP). This example has been adopted from the reference [1].

$$\begin{aligned} &\max\ f_1(X) = x_1^2 + x_2^2 + x_3^2 \\ &\max\ f_2(X) = (x_1-1)^2 + x_2^2 + (x_3-2)^2 \\ &\min\ f_3(X) = 2x_1 + x_2^2 + x_3 \\ &S.t. \quad FS: \{(x_1,x_2,x_3) | x_1 - 3x_2 + 4x_3 \leq 6, 2x_1^2 + 3x_2 + x_3 \leq 10, \\ &\qquad\qquad 0 \leq x_1 \leq 3,\ 0 \leq x_2 \leq 4,\ 0 \leq x_3 \leq 2\}. \end{aligned} \tag{3.1}$$

As mentioned in Section 2, using the Dantzig-Wolfe decomposition algorithm, the VOP is decomposed into the following sub-problems:

$P_1$ :

$$\begin{aligned} &\max\ f_{11}(x_1) = x_1^2 \\ &\max\ f_{21}(x_1) = (x_1-1)^2 \\ &\min\ f_{31}(x_1) = 2x_1 \\ &S.t.\ (x_1,x_2,x_3) \in FS: \end{aligned} \tag{3.2}$$

$P_2$ :

$$\begin{aligned} &\max\ f_{12}(x_2) = x_2^2 \\ &\max\ f_{22}(x_2) = x_2^2 \\ &\min\ f_{32}(x_2) = x_2^2 \\ &S.t.\ (x_1,x_2,x_3) \in FS: \end{aligned} \tag{3.3}$$

$P_3$ :

$$\begin{aligned} &\max\ f_{13}(x_3) = x_3^2 \\ &\max\ f_{23}(x_3) = (x_3-2)^2 \\ &\max\ f_{33}(x_3) = x_3 \\ &S.t.\ (x_1,x_2,x_3) \in FS: \end{aligned} \tag{3.4}$$

Then the following steps are done to solve sub-problems (3.2)–(3.4).

**Step 1.** $Q_1$ for sub-problem $P_1$ is obtained as follows:

**Step 1.1.** The PIS and NIS are obtained using (2.3) and (2.4). The results are shown in Tables 1 and 2 respectively.

**Step 1.2.** In order to get numerical solutions, let us assume that the relative importance (weights) of objectives are the same among these objectives ($w_1$ =

TABLE 1. PIS payoff table of $P_1$.

| | $f_1$ | $f_2$ | $f_3$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|---|---|
| max $f_{11}(x_1)$ | $5.0001^*$ | 1.5279 | 4.4722 | 2.2361 | 0 | 0 |
| max $f_{21}(x_1)$ | 5.0001 | $1.5279^*$ | 4.4722 | 2.2361 | 0 | 0 |
| min $f_{31}(x_1)$ | 0 | 1 | $0^*$ | 0 | 0 | 0 |

PIS: $f_1^* = (5.0001, 1.5279, 0)$.

TABLE 2. NIS payoff table of $P_1$.

| | $f_1$ | $f_2$ | $f_3$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|---|---|
| min $f_{11}(x_1)$ | $0^-$ | 1 | 0 | 0 | 0 | 0 |
| min $f_{21}(x_1)$ | 1 | $0^-$ | 2 | 1 | 0 | 0 |
| max $f_{31}(x_1)$ | 5.0001 | 1.5279 | $4.4722^-$ | 2.2361 | 0 | 0 |

NIS: $f_1^- = (0, 0, 4.4722)$.

$w_2 = w_3 = \frac{1}{3}$). In this step $S_1$ and $R_1$ are obtained using the formulas (2.5) and (2.7) respectively. The simplified relation of $S_1$ is obtained as follows:

$$S_1 = -0.2848x_1^2 + 0.5853x_1 + 0.4485$$

in addition, for $R_1$ we have:

$$\begin{aligned} &\min\ \lambda \\ &S.t. \quad \frac{1}{3}\left(\frac{5.0001 - x_1^2}{5.0001 - 0}\right) \leq \lambda \\ &\qquad \frac{1}{3}\left(\frac{1.5279 - (x_1 - 1)^2}{1.5279 - 0}\right) \leq \lambda \\ &\qquad \frac{1}{3}\left(\frac{2x_1 - 0}{4.4722 - 0}\right) \leq \lambda \\ &\qquad X \in FS \end{aligned}$$

where, the optimal point of this problem will be $(1.6388, 0, 0)$ with $\lambda^* = 0.2443$. In the optimal point, the second and third constraints are active and since the values of $R_1^*$ and $R_1^-$ for both constraints are the same, we can choose any of them as $R_1$. Here we choose the second constraint, so simplified $R_1$ is as follows:

$$R_1 = -0.2181x_1^2 + 0.4363x_1 + 0.1152.$$

**Step 1.3.** $S_1^*, S_1^-, R_1^*$ and $R_1^-$ are computed using (2.9) and (2.10). The results are shown in Table 3.

TABLE 3. $S_1^*, S_1^-, R_1^*$ and $R_1^-$ for $P_1$.

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $S_1^* = 0.3333$ | 2.2361 | 0 | 0 |
| $S_1^- = 0.7492$ | 1.0276 | 0 | 0 |
| $R_1^* = 0$ | 2.2361 | 0 | 0 |
| $R_1^- = 0.3333$ | 1 | 0 | 0 |

TABLE 4. PIS payoff table of $P_2$.

| | $f_1$ | $f_2$ | $f_3$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|---|---|
| max $f_{12}(x_2)$ | $11.1109^*$ | 11.1109 | 11.1109 | 0 | 0.3333 | 0 |
| max $f_{22}(x_2)$ | 11.1109 | $11.1109^*$ | 11.1109 | 0 | 0.3333 | 0 |
| min $f_{32}(x_2)$ | 0 | 0 | $0^*$ | 0 | 0 | 0 |

PIS: $f_2^* = (11.1109, 11.1109, 0)$.

TABLE 5. NIS payoff table of $P_2$.

| | $f_1$ | $f_2$ | $f_3$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|---|---|
| min $f_{12}(x_2)$ | $0^-$ | 0 | 0 | 0 | 0 | 0 |
| min $f_{22}(x_2)$ | 0 | $0^-$ | 0 | 0 | 0 | 0 |
| max $f_{32}(x_2)$ | 11.1109 | 11.1109 | $11.1109^-$ | 0 | 0.3333 | 0 |

NIS: $f_2^- = (0, 0, 11.1109)$.

**Step 1.4.** In this step, assuming $\nu = 0.5$, $Q_1$ is obtained using (2.11). The simplified result is as follows:

$$Q_1 = -0.6696x_1^2 + 1.3582x_1 + 0.3112.$$

**Step 2.** Similar to step 1, the following steps are done to obtain $Q_2$ for sub-problem $P_2$.

**Step 2.1.** Using (2.3) and (2.4), PIS and NIS are computed for $p_2$. The results are shown in Tables 4 and 5 respectively.

**Step 2.2.** $S_2$ and $R_2$ are obtained using (2.5) and (2.8) respectively as follows:

$$S_2 = -0.0300x_2^2 + 0.6667$$

also similar to the Step 1.2, $R_2$ is obtained as:

$$R_2 = -0.0300x_2^2.$$

**Step 2.3.** $S_2^*, S_2^-, R_2^*$ and $R_2^-$ are computed using (2.9) and (2.10). The results are shown in Table 6.

TABLE 6. $S_2^*, S_2^-, R_2^*$ and $R_2^-$ for $P_2$

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $S_2^* = 0.3333$ | 0 | 3.3333 | 0 |
| $S_2^- = 0.6667$ | 0 | 0 | 0 |
| $R_2^* = 0$ | 0 | 0 | 0 |
| $R_2^- = 0.3333$ | 0 | 3.3333 | 0 |

TABLE 7. PIS payoff table of $P_3$.

| | $f_1$ | $f_2$ | $f_3$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|---|---|
| max $f_{13}(x_3)$ | $4^*$ | 0 | 2 | 0 | 2 | 2 |
| max $f_{23}(x_3)$ | 0 | $4^*$ | 0 | 0 | 0 | 0 |
| min $f_{33}(x_3)$ | 0 | 4 | $0^*$ | 0 | 0 | 0 |

PIS: $f_3^* = (4, 4, 0)$.

TABLE 8. NIS payoff table of $P_3$.

| | $f_1$ | $f_2$ | $f_3$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|---|---|
| min $f_{12}(x_2)$ | $0^-$ | 4 | 0 | 0 | 0 | 0 |
| min $f_{22}(x_2)$ | 4 | $0^-$ | 2 | 0 | 2 | 2 |
| max $f_{32}(x_2)$ | 4 | 0 | $2^-$ | 0 | 2 | 2 |

NIS: $f_3^- = (0, 0, 2)$.

**Step 2.4.** Then $Q_2$ is obtained using (2.11) as follows:

$$Q_2 = 0.00001x_2^2 + 0.5.$$

**Step 3.** Similar to the above steps, we obtain $Q_3$ for $P_3$.
**Step 3.1.** PIS and NIS are computed using (2.3) and (2.4)for $p_3$. The results are shown in Tables 7 and 8 respectively.
**Step 3.2.** Then $S_3$ and $R_3$ are obtained using (2.5) and (2.8 respectively as follows:

$$S_3 = -0.1667x_3^2 + 0.5x_3 + 0.3333$$

and $R_3$ is obtained as:

$$R_3 = -0.0833x_3^2 + 0.3333.$$

**Step 3.3.** $S_3^*, S_3^-, R_3^*$ and $R_3^-$ are computed using (2.9) and (2.10). The results are shown in Table 9.
**Step 3.4.** Then $Q_3$ is obtained using (2.11) as follows:

$$Q_3 = -0.3473x_3^2 + 0.6668x_3 + 0.5.$$

TABLE 9. $S_3^*, S_3^-, R_3^*$ and $R_3^-$ for $P_3$.

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $S_2^* = 0.3333$ | 0 | 0 | 0 |
| $S_2^- = 0.7082$ | 0 | 0 | 1.4997 |
| $R_2^* = 0.0001$ | 0 | 2 | 2 |
| $R_2^- = 0.3333$ | 0 | 0 | 0 |

As mentioned before, in order to obtain the compromise solution of (3.1), we need to minimize $Q_1$, $Q_2$ and $Q_3$. To do this, we use model (2.12).

Now, we use $Q_1$, $Q_2$ and $Q_3$ in the following model:

$$\begin{aligned}
&\min\ \alpha\\
&\quad S.t. -0.6696x_1^2 + 1.3582x_1 + 0.3112 \le \alpha\\
&\qquad 0.00001x_2^2 + 0.5 \le \alpha\\
&\qquad -0.3473x_3^2 + 0.6668x_3 + 0.5 \le \alpha\\
&\qquad x_1 - 3x_2 + 4x_3 \le 6\\
&\qquad 2x_1^2 + 3x_2 + x_3 \le 10\\
&\qquad 0 \le x_1 \le 3\\
&\qquad 0 \le x_2 \le 4\\
&\qquad 0 \le x_3 \le 2.
\end{aligned} \tag{3.5}$$

As seen, the above problems is a quadratic problem and if we use the Lingo software to solve this problem, the compromise solution is obtained as $X^* = (1.7331, 0.7802, 1.6518)$ with $\lambda^* = 0.6538$, that is the compromise solution of VOP (3.1). Note that for $X^*$ the first and second constraints of $FS$ are active. For this point, the values of objectives are computed as follows:

$$F_{\text{Extended VIKOR}} = (6.3408, 1.2647, 5.7267).$$

As mentioned in the introduction, Abo Sinna *et al.* [1] proposed the extended TOPSIS method to solve MOLSNLP problems. To make comprehensive comparisons between extended VIKOR and extended TOPSIS we used above example to illustrate the solution procedure of the extended TOPSIS in solving the MOLSNLP problems and clarify the advantages of the proposed method. The solution procedure of the extended TOPSIS summarily is as follows: in the extended TOPSIS, as similar to extended VIKOR, for each sub-problem $P_k$ using the (2.3) and (2.4) the PIS and NIS are obtained and then using the $l - p$ metric with $p = 2$ two following distances as function distances from the PIS and NIS are computed.

$$d_2^{PIS} = \sum_{b \in B} w_b^2 \left( \frac{f_{bk}^* - f_{bk}(x_k)}{f_{bk}^* - f_{bk}^-} \right)^2 + \sum_{c \in C} w_c^2 \left( \frac{f_{ck}(x_k) - f_{ck}^*}{f_{ck}^- - f_{ck}^*} \right)^2 \tag{3.6}$$

$$d_2^{NIS} = \sum_{b \in B} w_b^2 \left( \frac{f_{bk}(x_k) - f_{bk}^-}{f_{bk}^* - f_{bk}^-} \right)^2 + \sum_{c \in C} w_c^2 \left( \frac{f_{ck}^- - f_{ck}(x_k)}{f_{ck}^- - f_{ck}^*} \right)^2 . \tag{3.7}$$

The compromise solution of the extended TOPSIS is a solution that minimizes $d_2^{PIS}$ and simultaneously maximizes $d_2^{NIS}$. In order to obtain a compromise solution, the following bi-objective problem with two commensurable (but often conflicting) objectives must be solved:

$$\begin{aligned} \min \quad & d_2^{PIS} \\ \max \quad & d_2^{NIS} \\ & S.t.\ X \in FS. \end{aligned} \tag{3.8}$$

Finally, the concept of membership function of fuzzy set theory is used to represent the satisfaction level for both criteria. Then, by applying the max – min decision model which is proposed by Bellman and Zadeh and extended by Zimmermann [20] the compromise solution of extended TOPSIS is obtained.

Following the above steps for the example at hand and solving (3.8) for each sub-problem $P_1$,$P_2$ and $P_3$, The compromise solution of the extended TOPSIS for VOP (3.1) is as follows:

$$X^*_{\text{Extended TOPSIS}} = (0, 0, 1.1722)$$

where, for this point, the values of objectives are obtained as follows:

$$F^*_{\text{Extended TOPSIS}} = (1.3741, 1.6853, 1.1722).$$

As we can see, the compromise solution of the extended TOPSIS is completely different from the compromise solution of the proposed approach. This difference arises from the different philosophies of conventional TOPSIS and conventional VIKOR methods that elaborately discussed in [14]. Note that, the obtained compromise solutions are non-dominated and each of them can be chosen as a pareto optimal solution of VOP (3.1). However, the proposed approach has advantages that convince the decision maker or analyzer to choose the proposed method. The main advantages of this method are as follows:

The extended VIKOR method uses the linear $l - p$ metric ($P = 1$ and $P = \infty$) and helps that the complexity (the degree of nonlinearity) of the aggregated objective function ($Q$) remains unchanged. Whereas, because of the application of $l - p$ metric with $P = 2$, the complexity of aggregated functions in the extended TOPSIS in comparison with the objective functions of main problem quadratically increase that is a major concern in the handling of nonlinear problems especially large scale ones. In addition, the proposed method reduce a multi-dimensional objective space to a one-dimensional space whereas, the reduced objective space in the extended TOPSIS has two dimensions and it is still needs to other reduction to reduce bi-objective space to a single objective space.

Moreover, in the process of decision making of the proposed approach, two type weights are considered, one is that of the objective functions and the other is the weight of the strategy of decision making ($\nu$). The weight of strategy enables the decision maker to impose his/her thought about the relative importance and the role of "majority rule" and "individual regret" in the decision-making process. Clearly, in this situation, the decision maker can choose the value of $0 \leq \nu \leq 1$ that is satisfies his/her willingness in a higher level.

On the other hand, beside the relative advantages of extended VIKOR method, Extended TOPSIS method is based upon the principle that the compromise solution should have the shortest distance from the PIS and the farthest from the NIS. While, in the proposed approach the distance from the ideal solution is a major concern that can be the rationale of human choice. Because, being far away from negative ideal solution could be a goal only in a particular situations. Therefore, in general, it is logical that the decision maker wants to choose the closest compromise solution to the ideal solution.

## 4. Conclusion

In the present paper, the VIKOR method has been extended to solve Multi-Objective Large-Scale Nonlinear Programming (MOLSNLP) problems with block angular structure. In the proposed method, first, the Dantzig-Wolfe decomposition algorithm was applied to decompose MOLSNLP into sub problems. Then the extended concepts of VIKOR method was used to obtain an equation for each sub problem. Afterward, these equations were combined into a single objective problem that could be solved by conventional methods. The analysis of the proposed method reveal that, the extended VIKOR method has good advantages in comparison with the same methods and it is a good alternative to handle MOLSNLP problems.

## References

[1] M.A. Abo Sinna and A.H. Amer, Extensions of TOPSIS for multi-objective large-scale nonlinear programming problems, *Appl. Math. Comput.* **162** (2005) 243–256.

[2] V.J. Bowman, On the relationship of the Tchebycheff norm and the efficient frontier of multiple criteria objectives, *Lect. Notes Econ. Math.* **135** (1976) 76–85.

[3] G. Buyukozkan and O. Feyzioglu, Evaluation of suppliers' environmental management performances by a fuzzy compromise ranking technique. *J. Multiple-Valued Logic and Soft Computing* **14** (2008) 309–323.

[4] M.T. Chu, J. Shyu, G.H. Tzeng and R. Khosla, Comparison among three analytical methods for knowledge communities group-decision analysis. *Expert Syst. Appl.* **33** (2007) 1011–1024.

[5] G. Dantzig, *Linear Programming and Extensions.* Princeton University Press, Princeton (1963).

[6] G. Dantzig and P. Wolfe, The decomposition algorithm for linear programming. *Econometrical* **29** (1961) 767–778.
[7] M. Geoffrion, Elements of large scale mathematical programming: Part II: Synthesis of algorithms and bibliography. *Manage. Sci.* **16** (1970) 676–691.
[8] J.K. Ho and R.P. Sundarraj, An advanced implementation of the Dantzig-Wolf decomposition algorithm for linear programming. *Math. Program.* **20** (1981) 303–326.
[9] J.K. Ho and R.P. Sundarraj, Computational experience with advanced implementation of decomposition algorithm for linear programming. *Math. Program.* **27** (1983) 283–290.
[10] Y.J. Lai, T.Y. Liu and C.L. Hwang, TOPSIS for MODM. *Eur. J. Oper. Res.* **76** (1994) 486–500.
[11] L.S. Lasdon, *Optimization theory for large systems.* Macmillan, New York, USA (1970).
[12] S. Opricovic, *Multi-criteria optimization of civil engineering systems*, Faculty of Civil engineering, Belgrade (1998).
[13] S. Opricovic, A fuzzy compromise solution for multi-criteria problems. *Int. J. Unc. Fuzz. Knowl. Based Syst.* **15** (2007) 363–380.
[14] S. Opricovic and G.H. Tzeng, Compromise solution by MCDM methods; a comparative analysis of VIKOR and TOPSIS. *Eur. J. Oper. Res.* **156** (2004) 445–455.
[15] S. Opricovic and G.H. Tzeng, Extended VIKOR method in comparison with outranking methods. *Eur. J. Oper. Res.* **178** (2007) 514–529.
[16] M. Sakawa, *Large Scale Interactive Multi-objective Programming Decomposition Approaches.* Physica-Verlag, New York (2000).
[17] M.K. Sayadi, M. Heydari and K. Shahanaghi, Extension of VIKOR method for decision making problem with interval numbers. *Appl. Math. Model.* **33** (2009) 2257–2262.
[18] L.I. Tong, C.C. Chen and C.H. Wang, Optimization of multi-response processes using the VIKOR method. *Adv. Manuf. Tech.* **31** (2007) 1049–1057.
[19] M. Zeleny, Compromise programming, in Multiple Criteria Decision Making edited by J.L. Cochrane, M. Zeleny. University of South Carolina, SC (1973) pp. 262–300.
[20] H.J. Zimmermann, *Fuzzy sets,decision making and expert systems.* Kluwer Academic Publishers, Boston, USA (1987).