AN ALGORITHM FOR MULTIPARAMETRIC 0-1-INTEGER PROGRAMMING PROBLEMS RELATIVE TO A GENERALIZED MIN MAX OBJECTIVE FUNCTION

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Abstract. The multiparametric 0-1-Integer Programming (0-1-IP) problem relative to the objective function is a family of 0-1-IP problems which are related by having identical constraint matrix and right-hand-side vector. In this paper we present an algorithm to perform a complete multiparametric analysis relative to a generalized min max objective function such that the min sum and min max are particular cases.

Keywords. 0-1-Integer Programming, multiparametric programming, Bottleneck problem.

Mathematics Subject Classification. 90C10, 90C31.

1. Introduction

Uncertainty in the data appears in most real 0-1-Integer Programming (0-1-IP) problems. Researchers have proposed several ideas in the literature in order to solve problems under uncertainty: stochastic programming and robust optimization for example. An alternative approach is to use parametric programming, which address the presence of an uncertain parameter, and multiparametric programming which considers the presence of two or more uncertain parameters. Greenberg [11] published an annotated bibliography for post-solution analysis including parametric IP problems. Greenberg's bibliography is available on the

World Wide Web (WWW) [12]. Another bibliography available on the WWW, that includes references on parametric and multiparametric IP problems, is due to Arsham [4]. A detailed literature review in parametric IP problems can be found in Jenkins [17] and recently in Li and Ierapetritou [19] including multiparametric IP problems.

Jenkins [14–16] has presented a very simple approach to solve parametric IP problems based on Geoffrion and Nauss [10]. His methods work by solving an appropriate sequence of non-parametric problems and joining the solutions to complete the parametrical analysis.

We have used the Jenkins′s approach in order to design algorithms to solve multiparametric 0-1-IP problems relative to the right-hand-side vector (Crema [6]), the min sum objective function (Crema [7]) and the constraint matrix (Crema [8]). The perturbation of the right-hand-side vector, the min sum objective function and the constraint matrix was considered by Crema [9] for 0-1-IP problems and by Li and Ierapetritou [19] for Mixed Integer Programming (MIP) problems.

A theoretical and algorithmic study for parametric 0-1-IP problems relative to the min sum objective function, including complexity results, have been written by Thiongane et al. [28]. The multiparametric problem relative to the min max objective function have been recently considered (Quintero and Crema [23]).

In this paper we present an approach, that can be viewed as a generalization of our previous work (Quintero and Crema [23]), to solve multiparametric 0-1-IP problems relative to a generalized min max objective function that includes the min sum and min max as particular cases.

Let $L \in \mathbb{R}^p$, $U \in \mathbb{R}^p$ with $L \leq U$, let $D \in \mathbb{R}^{q \times p}$, $d \in \mathbb{R}^q$ and let $\Omega = \{\theta \in \mathbb{R}^p : L \leq \theta \leq U, D\theta \leq d\}$, let $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and let $X = \{x : \quad Ax \leq b, \quad x \in \{0,1\}^n\}$.

The multiparametric 0-1-IP problem relative to the objective function is a family of 0-1-IP problems which are related by having identical constraint matrix and right-hand-side vector. A member of the family is defined as

$$(P(\theta)) \qquad \min \ \phi(\theta, x) \quad s.t. \quad x \in X$$

where ϕ is a continuous function on Ω for all $x \in X$, θ is the vector of parameters, $\theta \in \Omega \subseteq \mathbb{R}^p$ and X is the set of feasible solutions which does not depend on the vector θ. Let us suppose that $\Omega \neq \emptyset$ and $X \neq \emptyset$.

The objective function considered in this paper is defined as follows:

$$\phi(\theta, x) = \max\{(F^{(1)}\theta)^t x + d^{(1)^t}\theta + c^{(1)^t}x, \cdots, (F^{(k)}\theta)^t x + d^{(k)^t}\theta + c^{(k)^t}x\}$$

where:

$F^{(j)} \in \mathbb{R}^{n \times p}$, $d^{(j)} \in \mathbb{R}^p$, $c^{(j)} \in \mathbb{R}^n$ $(j = 1, \cdots, k)$ and k is the number of functions used to define ϕ that depends on the particular case considered.

The min sum and min max are particular cases as follows. Let $k = 1, p = n, d^{(1)} = 0, c^{(1)} = 0, F^{(1)} = I$, then $\phi(\theta, x) = \theta^t x$ and we have the min sum case. Let $k = n, p = n, (F^{(j)}\theta)^t = (0, \cdots, \theta_j, \cdots, 0), d^{(j)} = 0, c^{(j)} = 0, (j = 1, \cdots, k)$, then $\phi(\theta, x) = \max\{\theta_1 x_1, \cdots, \theta_n x_n\}$ and we have the min max case. We consider other interesting cases in our computational experience.

Note that since X is a finite set then there exists an optimal solution for $P(\theta)$ for all $\theta \in \Omega$. All the problems considered in this paper have optimal solutions. We use the following standard notation: if T is an optimization problem $v(T)$ denotes its optimal value.

We say that $x^{(1)}, \cdots, x^{(r)}$ is a multiparametrical solution if:

(i) $x^{(i)} \in X$ for all $i = 1, \cdots, r$;

(ii) $v(P(\theta)) = \min\{\phi(\theta, x^{(1)}), \cdots, \phi(\theta, x^{(r)})\}$ for all $\theta \in \Omega$.

Note that the index r denotes the number of solutions used to define the multiparametrical solution.

In Section 2 we present the theory that allows us to design the algorithm presented in the same section. In Section 3 we present a general approach to write as a 0-1-MIP problem a non-linear 0-1-MIP problem that appears if we use our algorithm. Computational experience is presented in Section 4. A summary and further extensions is given in Section 5.

2. THEORETICAL RESULTS AND THE ALGORITHM

This section is a summary of our previous results in solving multiparametric 0-1-IP problems relative to the objective function (Quintero and Crema [23]) that are independent of the particular case of ϕ to be considered.

Suppose that $x^{(i)} \in X$ for all $i = 1, \cdots, r$ and let $g^{(r)}(\theta)$ be an upper bound function of $v(P(\theta))$ defined by $g^{(r)}(\theta) = \min\{\phi(\theta, x^{(1)}), \cdots, \phi(\theta, x^{(r)})\}$. Note that for all $\theta \in \Omega$ we have $g^{(r)}(\theta) \geq v(P(\theta))$ and if $x^{(r+1)} \in X$ then $g^{(r+1)}(\theta) \leq g^{(r)}(\theta)$.

Let $Q^{(r)}$ be a problem in (θ, x, y, z) defined as:

$$(Q^{(r)}) \quad \max \ y - z \ \ s.t. \ \ \theta \in \Omega, \ \ x \in X$$

$$y = g^{(r)}(\theta)$$
$$z = \phi(\theta, x)$$
$$y \in \mathbb{R}, z \in \mathbb{R}.$$

Observe that θ is a vector of decision variables in $Q^{(r)}$.

Note that with $Q^{(r)}$ we are looking for the maximal difference between $v(P(\theta))$ and $g^{(r)}(\theta)$. If the maximal difference is zero then we have found $v(P(\theta))$ for all $\theta \in \Omega$ and the analysis was completed, otherwise our algorithm finds $x^{(r+1)}$ and θ^* such that:

(i) $g^{(r+1)}(\theta) \leq g^{(r)}(\theta)$ for all $\theta \in \Omega$

(ii) $g^{(r+1)}(\theta^*) < g^{(r)}(\theta^*)$.

Note that this approach has been presented in Crema [7] and Quintero and Crema [23] in a different context.

Lemma 2.1. *(i) There exists an optimal solution for $Q^{(r)}$. (ii) $v(Q^{(r)}) \geq 0$. (iii) If $v(Q^{(r)}) = 0$ then $v(P(\theta)) = g^{(r)}(\theta)$ for all $\theta \in \Omega$. (iv) If $(\theta^*, x^{(r+1)})$ is an optimal solution for $Q^{(r)}$ then $x^{(r+1)}$ is an optimal solution for $P(\theta^*)$. (v) Let $(\theta^*, x^{(r+1)})$ be an optimal solution for $Q^{(r)}$. If $v(Q^{(r)}) > 0$ then $x^{(i)}$ is not an optimal solution for $P(\theta^*)$ for all $i = 1, \cdots, r$ and $g^{(r+1)}(\theta^*) < g^{(r)}(\theta^*)$.*

The proof of Lemma 2.1 may be seen in Quintero and Crema [23]. Since X is a finite set, the lemma proves that the next algorithm (the multiparametric algorithm) provide us a complete multiparametrical analysis:

Step 0. Find $\theta^{(1)} \in \Omega$. Solve $P(\theta^{(1)})$. Let $x^{(1)}$ be an optimal solution.

Step 1. $r = 1$

Step 2. Solve $Q^{(r)}$ and let $(\theta^*, x^{(r+1)})$ be an optimal solution.

Step 3. If $v(Q^{(r)}) = 0$ STOP (with $v(P(\theta)) = g^{(r)}(\theta)$ for all $\theta \in \Omega$), otherwise let $r = r + 1$ and return to step 2.

In the min sum case $P(\theta^{(1)})$ is a 0-1-IP problem. With ϕ the objective function considered in this paper $P(\theta^{(1)})$ may be rewritten as a 0-1-MIP problem. In order to use the algorithm based on Lemma 2.1 we need an algorithm to solve $Q^{(r)}$ that was presented as a non-linear 0-1-MIP problem.

3. An approach to solve $Q^{(r)}$

The main contribution of this paper is an approach to solve $Q^{(r)}$ in the general case presented.

Problem $Q^{(r)}$ to be solved was defined as follows:

$$(Q^{(r)}) \quad \max \ y - z \ \ s.t. \ \ \theta \in \Omega, \ \ x \in X$$
$$y = g^{(r)}(\theta)$$
$$z = \phi(\theta, x)$$
$$y \in \mathbb{R}, z \in \mathbb{R}.$$

We present a sequence of problems that are equivalent to $Q^{(r)}$ until we reach an appropriate model to be solved as a 0-1-MIP problem.

We use the definition of the upper bound function, $g^{(r)}(\theta)$, and the definition of $\phi(\theta, x)$ in order to rewrite $Q^{(r)}$ as follows:

$$(Q1^{(r)}) \quad \max \ y - z \ \ s.t. \ \ \theta \in \Omega, \ \ x \in X$$

$$y = \min\{\max\{(F^{(j)}\theta)^t x^{(i)} + d^{(j)t}\theta + c^{(j)t}x^{(i)} : j = 1, \cdots, k\} : i = 1, \cdots, r\}$$

$$z = \max\{(F^{(j)}\theta)^t x + d^{(j)t}\theta + c^{(j)t}x : j = 1, \cdots, k\}$$

$$y \in \mathbb{R}, z \in \mathbb{R}.$$

Let $(Q2^{(r)})$ be a problem in (θ, x, y, z) defined as follows:

$$(Q2^{(r)}) \quad \max \ y - z \ \ s.t. \ \ \theta \in \Omega, \ \ x \in X$$

$$y \leq (F^{(j)}\theta)^t x^{(i)} + d^{(j)t}\theta + c^{(j)t}x^{(i)} \text{ for some } j \in \{1, \cdots, k\} \quad (i = 1, \cdots, r)$$

$$z \geq (F^{(j)}\theta)^t x + d^{(j)t}\theta + c^{(j)t}x \quad (j = 1, \cdots, k)$$

$$y \in \mathbb{R}, z \in \mathbb{R}.$$

Since maximization is the optimization criterion then problems $Q1^{(r)}$ and $Q2^{(r)}$ are equivalent: (θ, x, y, z) is an optimal solution to $Q1^{(r)}$ if and only if (θ, x, y, z) is an optimal solution to $Q2^{(r)}$ and $v(Q1^{(r)}) = v(Q2^{(r)})$.

Finally we use 0-1 auxiliary variables in order to rewrite the constraints relative to y. Let $Q3^{(r)}$ be a problem in (θ, x, y, z, w) defined as follows:

$$(Q3^{(r)}) \quad \max \ y - z \ \ s.t. \ \ \theta \in \Omega, \ \ x \in X$$

$$y \leq \sum_{j=1}^{k}((F^{(j)}\theta)^t x^{(i)} + d^{(j)t}\theta + c^{(j)t}x^{(i)})w_j^{(i)} \quad (i = 1, \cdots, r)$$

$$\sum_{j=1}^{k} w_j^{(i)} = 1 \ \ (i = 1, \cdots, r)$$

$$w_j^{(i)} \in \{0, 1\} \ \ (i = 1, \cdots, r) \ \ (j = 1, \cdots, k)$$

$$z \geq (F^{(j)}\theta)^t x + d^{(j)t}\theta + c^{(j)t}x \quad (j = 1, \cdots, k)$$

$$y \in \mathbb{R}, z \in \mathbb{R}.$$

If (θ, x, y, z, w) is an optimal solution to $Q3^{(r)}$ then (θ, x, y, z) is an optimal solution to $Q2^{(r)}$. If (θ, x, y, z) is an optimal solution to $Q2^{(r)}$ then (θ, x, y, z, w) is an optimal solution to $Q3^{(r)}$ with w defined as follows:

(i) let s be an index such that:
$\max\{(F^{(j)}\theta)^t x^{(i)} + d^{(j)t}\theta + c^{(j)t}x^{(i)} : j = 1, \cdots, k\} = (F^{(s)}\theta)^t x^{(i)}$
$+ d^{(s)t}\theta + c^{(s)t}x^{(i)})$, with the ties broken arbitrarily,

(ii) $w_s^{(i)} = 1$ and $w_j^{(i)} = 0$ if $j \neq s$ $(i = 1, \cdots, r$ and $j = 1, \cdots, k)$.

Hence, $Q2^{(r)}$ and $Q3^{(r)}$ are equivalent problems and $v(Q2^{(r)}) = v(Q3^{(r)})$. Therefore $Q^{(r)}$ and $Q3^{(r)}$ are equivalent problems and $v(Q^{(r)}) = v(Q3^{(r)})$

Note that $Q3^{(r)}$ includes non-linear terms. However, all of them may be rewritten with an standard method (Adams, Forrester and Glover [2]) in order to solve $Q3^{(r)}$ as a 0-1-MIP problem:

For all $i = 1, \cdots, r$ and $j = 1, \cdots, k$ use $\alpha_j^{(i)} w_j^{(i)}$ instead of $\left(F^{(j)}\theta\right)^t x^{(i)} w_j^{(i)}$ where $\alpha_j^{(i)}$ is a new continuous variable and add to $Q3^{(r)}$ the constraints:

$$V_j^{(i)}(1 - w_j^{(i)}) \leq \left(F^{(j)}\theta\right)^t x^{(i)} - \alpha_j^{(i)} \leq S_j^{(i)}(1 - w_j^{(i)})$$
$$V_j^{(i)} w_j^{(i)} \leq \alpha_j^{(i)} \leq S_j^{(i)} w_j^{(i)}$$
$$\alpha_j^{(i)} \in \mathbb{R} \ (i = 1, \cdots, r) \ (j = 1, \cdots, k)$$

where $V_j^{(i)}$ and $S_j^{(i)}$ are bounds such that:

$$S_j^{(i)} \geq \max\{\left(F^{(j)}\theta\right)^t x^{(i)} : \ \theta \in \Omega\}$$

$$V_j^{(i)} \leq \min\{\left(F^{(j)}\theta\right)^t x^{(i)} : \ \theta \in \Omega\}.$$

Terms $d^{(j)t}\theta w_j^{(i)}$ and $\left(F^{(j)}\theta\right)^t{}_l x_l$ may be replaced by using the same method for all $i = 1, \cdots, r$, $j = 1, \cdots, k$ and $l = 1, \cdots, n$.

We present the standard method because it is appropriate to the general objective function considered, however, others methods, that depend on the particular case of objective function, may be used to replace the non-linear terms (Adams *et al.* [2], Adams and Forrester [1], Oral and Kettani [22]).

To the best of our knowledge there are no other algorithms to be applied in the general case considered. Our algorithm may be implemented by using any software capable of solving MIP problems.

4. Computational experience

Previous computational experiences in the min sum case and min max cases were presented in Crema [7] and Quintero and Crema [23]. The problems considered were the multiconstrained knapsack problem, with the min sum objective, and the bottleneck generalized assignment problem, with the min max objective function (Martello and Toth [20]). In both cases the linearization process of $Q^{(r)}$ did not use the standard method presented in this paper because of the particular objective function considered.

Our algorithm has been implemented in C++ by using the OSL package of IBM [13,21] that uses a Branch and bound algorithm based on linear relaxations to solve MIP problems. The new experiments with the general objective function were performed on a PC Pentium IV with 2 Ghz and 256 MB of RAM.

The problem considered in this paper is the m-Bottleneck assignment (m-BA) problem as follows:

Given n *tasks* and m *agents*, the *penality*, c_{ij}, corresponding to the assignment of the task j to agent i and given $a_{ij} \in \{0, 1\}$ such that the task j may be assigned to agent i if and only if $a_{ij} = 1$ $(i = 1, \cdots, m$ and $j = 1, \cdots, n)$, the m-BA problem

is to assign each task to one agent so that the maximum penality of the agents is minimized where the penality of the agent i is computed as $\sum_{j \text{ assigned to } i} \theta_{ij}$.

By introducing binary variables x_{ij} with $x_{ij} = 1$ if and only if task j is assigned to agent i, the problem can be formulated as follows:

$$\min \ \max\left\{\sum_{j=1}^{n} c_{ij}x_{ij} : \ i = 1, \cdots, m\right\} \ s.t.$$

$$\sum_{i=1}^{m} a_{ij}x_{ij} = 1 \ (j = 1, \cdots, n)$$

$$x_{ij} \in \{0, 1\}, \ (i = 1, \cdots m)(j = 1, \cdots, n).$$

Let us suppose that some of the penalities are uncertain parameters and then we rewrite the problem by using θ_{ij} instead of c_{ij}. Let K be the index set of elements of the objective function that are uncertain parameters.

The experiments were designed with $\Omega = \{\theta_{ij} : (1 - \beta)c_{ij} \leq \theta_{ij} \leq (1 + \beta)c_{ij} \ \ for \ all \ (i, j) \in K, \ \ \theta_{ij} = c_{ij} \ \ for \ all \ (i, j) \notin K\}$ where $0 < \beta < 1$ is a parameter used to define the level of perturbation.

The indexes that belong to K were randomly selected from the set $\{(i, j) : a_{ij} = 1, \ i = 1, \cdots, n \text{ and } j = 1, \cdots, m\}$.

In our experimets the elements c_{ij} and a_{ij} were generated as follows:

(i) Uncorrelated case: the elements c_{ij} were drawn from a uniform distribution on $[1, c\max]$ with $c\max > 1$. If $c_{j_1 j} \leq c_{j_2 j} \leq \cdots \leq c_{j_m j}$ then we use $a_{j_1 j} = a_{j_2 j} = 1$ and $a_{ij} = 0$ if $i \notin \{j_1, j_2\}$.

(ii) Correlated case: given n *towns* (the tasks) and m *service centers* (the agents), the distance, d_{ij}, from town j until service center i and the population, P_j, of town j,then the *penality*, c_{ij}, corresponding to the assignment of the town j to service center i is defined as $c_{ij} = d_{ij}P_j$. The location of town j and service center i were drawn from a uniform distribution on the square $[0, 1] \times [0, 1]$ and d_{ij} is the euclidean distance from the location of i until the location of j ($i = 1, \cdots, m$ and $j = 1, \cdots, n$). The elements P_j were drawn from a uniform distribution on $[1, P\max]$ with $P\max > 1$. If $d_{j_1 j} \leq d_{j_2 j} \leq \cdots \leq d_{j_m j}$ then we use $a_{j_1 j} = a_{j_2 j} = 1$ and $a_{ij} = 0$ if $i \notin \{j_1, j_2\}$.

The results are reported in Table 1 (uncorrelated case) and Table 2 (correlated case). The notation used in the tables is as follows: p is an index to identify the problem, r is the number of $Q^{(r)}$ problems solved in order to complete the multiparametrical analysis, $Si1$ is the number of simplex iterations computed to

TABLE 1. Computational experience to the uncorrelated case.

p	m	n	n_K	cmax	β	r	Si1	N1	Si	N	t
1	2	10	10	1000	0.20	2	18	7	1507	99	13.130
2		20				3	32	10	4317	332	356.149
3		30				4	58	19	926836	17577	9145.388
4	2	10	10	1000	0.05	2	23	9	456	22	4.032
5					0.10	2	22	10	617	25	12.167
6					0.15	4	25	11	832	31	13.672
7					0.25	4	27	15	706	96	15.523
8					0.30	5	21	9	3618	362	315.517
9					0.35	5	20	6	1622	92	25.512
10					0.40	8	18	5	3526	331	427.735
11	2	10	5	1000	0.20	2	19	5	435	20	9.541
12			15			3	24	9	945	45	24.478
13			20			4	24	10	1566	91	36.376
14	2	10	10	10	0.20	3	20	6	736	36	8.151
15				100		3	19	3	625	28	6.292
16				10000		4	19	2	1240	46	15.261
17	3	20	10	1000	0.20	7	35	12	934622	26333	9532.336
18	5					10	69	25	1822633	61226	17457.672

solve $P(\theta^1)$, N1 is the number of nodes generated by the branch and bound algorithm to solve $P(\theta^1)$, Si is the number of simplex iterations computed to solve $P(\theta^1), Q^{(1)}, \cdots Q^{(r)}$. The number of nodes generated by the branch and bound algorithm to solve $P(\theta^1), Q^{(1)}, \cdots Q^{(r)}$ is denoted with N and t is the CPU time in seconds to solve $P(\theta^1), Q^{(1)}, \cdots Q^{(r)}$. Both, $Si1$ and Si, include the number of simplex iterations computed to solve the relaxations of subproblems in the branch and bound algorithm.

Non-linear terms were replaced following a generalization of the procedure used to the min max case (Quintero and Crema [23]) based on Oral and Kettanni [22], instead of the standard method, in order to generate a model with fewer constraints.

The values of $\theta_{ij}^{(1)}$ were defined as follows:

$\theta_{ij}^{(1)} = c_{ij}$ if $(i,j) \notin K$ and $\theta_{ij}^{(1)} = (1 - \beta)c_{ij}$ if $(i,j) \in K$ $(i = 1, \cdots, n)$, $(j = 1, \cdots, m)$.

Let n_K be the cardinality of K. The experiments were designed in order to evaluate the perfomance of the algorithm as m increases (problems 17–18 and 37–39), n increases (problems 1–3 and 19–23), n_K increases (problems 11–13 and 31–33), c max increases (problems 14–16), P max increases (problems 34–36), or β increases (problems 4–10 and 24–30). As we can expect the effort increases as the dimensions (m and n), the number of parameters (n_K) and the level of perturbation (β) increase. As we hope the uncorrelated problems seem to be more

TABLE 2. Computational experience to the correlated case.

p	m	n	n_K	Pmax	β	r	Si1	N1	Si	N	t
19	3	10	10	1000	0.20	5	51	21	6564	336	3.241
20		20				8	98	39	7031	521	6.127
21		30				10	265	76	13259	822	17.230
22		40				15	411	95	21346	1336	234.216
23		50				17	623	121	45165	3845	2561.377
24	3	10	10	1000	0.05	5	67	22	7126	429	5.040
25					0.10	7	78	29	8655	784	9.954
26					0.15	7	92	26	11278	832	51.265
27					0.25	6	54	17	9188	812	34.167
28					0.30	11	99	27	19424	1239	191.637
29					0.35	10	156	34	17823	1038	125.527
30					0.40	14	178	36	21296	1495	436.163
31	3	10	5	1000	0.20	5	55	19	7527	362	7.362
32			15			8	94	35	13349	923	74.396
33			20			15	114	47	25562	2156	924.169
34	3	10	10	10	0.20	7	45	19	13636	938	7.321
35				100		8	56	17	16723	1326	54.252
36				10000		10	72	25	20362	1934	152.241
37	5	30	10	1000	0.20	16	158	37	56578	4020	932.241
38	7					21	466	64	96377	7625	9448.478
39	9					26	593	91	674744	59272	23363.644

difficult than the correlated problems. However, more problems should be solved to make definitive conclusions.

5. SUMMARY AND FURTHER EXTENSIONS

We designed and implemented an algorithm to solve the multiparametric 0-1-IP problem relative to a very general objective function (min sum and min max are particular cases). Computational experience was presented for m-BA problems with uncorrelated and correlated data. Our algorithm works by choosing an appropriate finite sequence of non-parametric MIP problems in order to obtain a complete multiparametrical analysis and this explains that the computer storage was not a problem for our algorithm, that is: if we can solve the non-parametric $P(\theta^{(1)})$ problem then we can expect no problems to perform a complete multiparametrical analysis. The algorithm may be implemented by using any software capable of solving MIP problems. To the best of our knowledge there are no other implementations of algorithms to solve the multiparametric min max 0-1-IP problem relative to the very general objective function considered and for this reason we did not compare the performance of our algorithm with any other.

We prefer to think that the multiparametric algorihtm is a general metodology and problems $Q^{(r)}$ would be solved with specialized algorithms associated to the

structure of the non-parametric problems $P(\theta)$ in order to solve hard problems with higher dimensions and more parameters if necessary.

In a near future we will intend to develop a procedure to generate the non-dominated set of a multiobjective 0-1-IP problem by using our multiparametric algorithm. In order to present some basic theory of multiobjective programming we use the presentation of Alves and Climaco [3] as follows:

Consider the following multiobjective 0-1-IP problem:

$$(P) \quad \max \ c^{(1)^t}x, \cdots, c^{(q)^t}x \ \ s.t. \ x \in X.$$

A solution $\bar{x}$ is efficient for problem P if and only if there is no $x \in X$ such that $c^{(i)^t}x \geq c^{(i)^t}\bar{x}$ for all $i = 1, \cdots, q$ and $c^{(i)^t}x > c^{(i)^t}\bar{x}$ for at least one i.

Let X_E be the set of efficient solutions ($X_E = \{x \in X : x \text{ is efficient}\}$).

A solution $\bar{x}$ is weakly efficient for problem P if and only if there is no $x \in X$ such that $c^{(i)^t}x > c^{(i)^t}\bar{x}$ for all $i = 1, \cdots, q$.

Let X_W be the set of weakly efficient solutions ($X_W = \{x \in X : x \text{ is weakly efficient}\}$).

Let $Z \subset \mathbb{R}^q$ be the image of the feasible set X in the objective functions space (the criteria space). A point z corresponding to a efficient (weakly efficient) solution x is called non-dominated (weakly non-dominated).

Let Z_E be the non-dominated set defined as follows:

$$Z_E = \{z \in \Re^q : \text{ there exists } x \in X_E \text{ such that } z_i = c^{(i)^t}x \ (i = 1, \cdots, q)\}.$$

Let Z_W be the weakly non-dominated set defined as follows:

$$Z_W = \{z \in \Re^q : \text{ there exists } x \in X_W \text{ such that } z_i = c^{(i)^t}x \ (i = 1, \cdots, q)\}.$$

Let $\bar{C}$ denotes a reference point of the criteria space: $\bar{C}_i > c^{(i)^t}x$ for all $x \in X$ and for all $i = 1, \cdots, q$.

The θ-weighted Tchebycheff norm computed in $x \in X$ is defined as: $\max\{\theta_i(\bar{C}_i - c^{(i)^t}x) : \ i = 1, \cdots, q\}$ where $\theta \in \Omega = \{\theta : \ \theta \in \mathbb{R}^q, \ \theta \geq 0, \ \sum_{i=1}^q \theta_i = 1\}$.

Bowman [5] proved that the multiparametric analysis of:

$$(P(\theta)) \quad \min \ \max\{\theta_i(\bar{C}_i - c^{(i)^t}x) : \ i = 1, \cdots, q\}$$

generates the non-dominated set as follows: for all $\bar{z} \in Z_E$ there exists $\bar{\theta} \in \Omega$ such that $\bar{z}_i = c^{(i)^t}\bar{x} \ (i = 1, \cdots, q)$ where $\bar{x} \in X_E$ is an optinal solution of $P(\bar{\theta})$.

However, some weakly non-dominated solutions may appear if there exists $\bar{\theta} \in \Omega$ such that the optinal solution of $P(\bar{\theta})$ is not an efficient solution.

The generation of weakly non-dominated solutions is avoided by considering the augmented θ-weighted Tchebycheff norm (Steuer and Choo [24]) and the corresponding multiparametric analysis:

$$(P(\theta)) \quad \min \quad \max\{\theta_i(\bar{C}_i - c^{(i)^t}x) - \alpha \sum_{i=1}^{q} c^{(i)^t}x : \quad i = 1, \cdots, q\}$$

where α is small enough positive value.

As you can see our algorithm may be used to generate the non-dominated set. However some computational experience suggests that the linearization method must be carefully designed. Also, an approach to generate a representative subset of the non-dominated set may be designed like in other generating methods (Sylva and Crema [25], Sylva and Crema [26], Sylva and Crema [27], Laumanns *et al.* [18]).

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