PENALIZED ESTIMATORS FOR NON LINEAR INVERSE PROBLEMS

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Abstract. In this article we tackle the problem of inverse non linear ill-posed problems from a statistical point of view. We discuss the problem of estimating an indirectly observed function, without prior knowledge of its regularity, based on noisy observations. For this we consider two approaches: one based on the Tikhonov regularization procedure, and another one based on model selection methods for both ordered and non ordered subsets. In each case we prove consistency of the estimators and show that their rate of convergence is optimal for the given estimation procedure.

1. Introduction

In many problems it is necessary to recover an indirectly observed signal x_0 based on the non linearly transformed and noisy observation model

$$y(t_i) = F(x_0)(t_i) + \varepsilon_i, \ i = 1, \ldots, n. \tag{1.1}$$

We assume $F : X \to Y$ is a known non linear operator between X, Y Hilbert spaces, (t_i) is a fixed observation scheme and ε_i are i.i.d. realizations of a certain random variable ε. Throughout the paper, we shall denote $\mathbf{y} = (y(t_i))_{i=1}^n$. We assume that the observations $y(t_i) \in \mathbb{R}$.

When F is linear, two main estimation procedures have been considered. Namely, regularized estimators such as the Tikhonov estimator, see for instance [4,16], and non linear threshold estimators [6,13]. Rates of convergence for regularized estimators are known and are seen to be optimal for an appropriate selection of the regularization parameter which depends on the regularity of the unknown function. Moreover, based on concentration inequalities it is possible to show that the addition of an appropriate penalization term allows selecting the regularization parameter or the projection subspace based on the data, in such a way as to maintain the best possible rates. Optimal choices for the penalty term can be deduced for example from the general results for Gaussian model selection in [2]. An application of these penalty methods is considered in [15] for general regularization methods. The second approach has its most popular version in the *wavelet-vaguelet* decomposition introduced in [8], further developed in [6,13].

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However, scarce statistical literature exists when F is non linear. Among the few papers available, we point out the works [18] or [19] where some rates are given, and that of Bissantz *et al.* in [3] where they discuss a non linear version of the method of regularization (MOR). Different approaches are also developed in [7] for dynamical inverse problems or in [11] for Maximum Entropy Methods.

Not alike the random setting, the deterministic non linear ill-posed inverse problem literature is extensive. Most commonly, authors propose L^2 regularized Tikhonov type estimators and show that they provide a stable method for approximating the solution. For a general discussion on regularization methods, we refer the reader to [9,21]. As in the linear case, the choice of the regularization parameter is crucial. This issue is often practically solved by numerical methodology which relies on grid methods and iterative algorithms, see for instance [17] or [12]. On the other hand, *a priori* optimal choices of the smoothing sequence are given in the work of [20], leading to estimators of x_0 which attain the best possible rates, whenever its regularity is known beforehand.

Our main goal here is to provide a data driven estimator of the regularization parameter in the Tikhonov procedure and of the "cutoff" point for projection estimators, that behaves as well as the best possible choice in each case. Namely, we consider choosing the regularization parameter or subspace that minimizes the empirical quadratic loss function plus an appropriate penalty, as in the linear case. The fact that F is non linear requires introducing certain technical restrictions which shall be discussed in Section 2.

Given the Tikhonov regularization procedure, which shall be discussed in detail in Section 3, we consider the following estimator of x_0,

$$\hat{x}_{\alpha_{\hat{k}}} = x^\star + (T^*_{m_0} T_{m_0} + \alpha_k I_{m_0})^{-1} T^t_{m_0} y,$$

where $\hat{k}$ is selected as the solution of

$$\hat{k} := \arg\min_{k \in \mathcal{K}} \left\{ \|R_{\alpha_k}(y - F(\hat{x}_{\alpha_k}))\|^2 + \text{pen}(\alpha_k) \right\},$$

for an appropriate penalization term $\text{pen}(\alpha_k)$. Here $T_{m_0} = T\Pi_{Y_{m_0}}$, where Y_{m_0} is a certain subspace of Y and $\Pi_{Y_{m_0}}$ stands for the projection over this subspace, T is a certain linear approximation of F and $x^\star$ is an "initial guess" for x_0. We assume that $x_0 - x^\star$ belongs to the orthogonal complement of the nullspace of T.

Our main result in this section, Theorem 3.1, establishes that the L^2 risk of this data driven estimator $\hat{x}_{\alpha_{\hat{k}}}$ attains the same convergence rates as those obtained for the best possible choice of the regularization parameter α_k, which typically depends on the regularity conditions of the unknown signal x_0.

Our next results deal with projection estimators. Always aiming at constructing data driven estimators of the "cutoff" point, we study two procedures. First, we consider selecting a subspace over a nested collection of linear subspaces Y_m, $m \in M_n$ a collection depending on n. The optimal m is chosen as

$$\hat{m} = \arg\min_{m \in M_n} \|\Pi_{Y_m}(y - F(\hat{x}_m))\|^2_n + \text{pen}(m)$$

where $\text{pen}(m)$ is an appropriate penalization term which shall be discussed in Section 4, Π_{Y_m} stands for the projection over the subspace Y_m, $\| \ \|_n$ is the empirical norm based on the observation scheme (t_i), which shall be discussed in Section 2, and

$$\hat{x}_m := x^\star + \arg\min_{x \in x^\star + T^* Y_m} \|\Pi_{Y_m}(y - F(x))\|^2_n.$$

Given the "initial guess" $x^\star$, we assume $\|x - x^\star\| < \rho$ for some ρ, T is an approximation of $F'(x_0)$ and T^* stands for its adjoint. As above, we assume that $x_0 - x^\star$ belongs to the orthogonal complement of the nullspace of T. We then show, Theorem 4.1, that this data driven selection of m attains the same convergence rates as those obtained for the best possible choice of the "cutoff" point.

Second, we consider the problem of non ordered model selection, which can be thought of as a threshold estimator for the estimated coefficients of x_0 over a certain linear subspace $X_{m_0} := T^*_{m_0} Y$, where T_{m_0} is defined

as in the Tikhonov procedure. More precisely, let m_0 be a certain big collection of possible models. Set

$$\hat{x}_m = x^\star + \arg\min_{x \in X_m} \|(T_{m_0}^t T_{m_0})^{-1} T_{m_0}^t \Pi_{Y_{m_0}}(y - F(x))\|^2.$$

The optimal model is chosen as

$$\hat{m} = \arg\min_{m \subset m_0} \|(T_{m_0}^t T_{m_0})^{-1} T_{m_0}^t \Pi_{Y_{m_0}}(y - F(\hat{x}_m))\|^2 + \text{pen}(m),$$

for a certain penalty function depending on the family of linear operators $\{F'(x), \|x - x^\star\| < \rho\}$, for some ρ and an appropriate "initial guess" $x^\star$, which, as above, is assumed such that $x_0 - x^\star$ belongs to the orthogonal of the nullspace of T. Our main result in this section, Theorem 4.5 states that this data driven choice of m attains the same convergence rates as those obtained by the best possible model choice.

The article is organized as follows. In Section 2 we describe the general framework. In Section 3 we provide an adaptive estimator for the regularization parameter for Tikhonov like estimators, and show the resulting estimator is consistent. Section 4 is devoted to model selection estimators for non linear inverse problems, first with embedded projection spaces and then with non ordered selection. Some conclusions are drawn in Section 5. Section 6 is devoted to technical lemmas which are useful throughout the paper.

2. Basic assumptions and notation

In this section we introduce some general notation and assumptions. These include standard concentration assumptions over the observation noise and some restrictions over the class of operators $F(.)$.

Since the operator F is non linear, our convergence analysis will be a local one. Hence it is necessary to start with an initial guess of the solution. We require that this starting point $x^\star$ allows us to construct a good approximation of the Fréchet derivative of the operator at x_0: $F'(x_0)$. More precisely, we assume F is Fréchet differentiable and that the range of

$$DF(x_1, x_2) = \int_0^1 F'(x_1\theta + x_2(1 - \theta))\mathrm{d}\theta$$

remains unchanged in a neighborhood of the initial solution $x^\star$, *i.e.* in the ball $B_\rho(x^\star) = \{x, \|x - x^\star\| \leq \rho\}$, for a certain $\rho > 0$. The above discussion is summarized by the following condition.

2.1. **AF control over the non linear part of the differential operator**

There exist c_T, a fixed linear operator T (generally $T = F'(x^\star)$) and a linear operator depending on x and x', written $R(x, x')$ such that for $x, x' \in B_\rho(x^\star)$

$$F(x) - F(x') = TR(x, x')(x - x'),$$

with $\|I - R\|^2 \leq c_T$.

This kind of condition is considered in [14] as an alternative to the tangential cone condition. The previous author in [3] shows that this **AF** condition is satisfied, for instance, by the inverse groundwater filtration problem of identifying the transmissivity a in

$$-\nabla(a\nabla u) = f \quad \text{in } \Omega \tag{2.1}$$
$$u = g \quad \text{on } \partial\Omega,$$

on a C^2 domain $\Omega \subset \mathbb{R}^2$, from measurements of the piezometric head u, where F is the parameter-to-solution map $F : a \to u$.

Hence, the non linear operator is replaced by T, a known bounded linear operator that can be viewed as some approximation of $F^{'}(x)$ in a neighborhood $B_\rho(x^\star)$, which we must be able to use in our computations. In contrast, under **AF**, operator R is not required explicitly.

Under such restrictions, the non linearity of the problem does not change the rate of convergence and we still are able to build adaptive estimators. We remark that **AF** is not strictly comparable to the Lipschitz condition required in Theorem 3 in [3], their condition being a global one over the domain of F. However locally **AF** is stronger.

Recall we want to estimate a function $x_0 : \mathrm{I\!R} \to \mathrm{I\!R}$ observed in a inverse model

$$y(t_i) = F(x_0)(t_i) + \varepsilon_i, \ i = 1, \dots, n.$$

It is important to stress that the observations depend on a fixed design $(t_1, \dots, t_n) \in \mathrm{I\!R}^n$. This will require introducing an empirical norm based on this design. Let Q_n be the empirical measure of the co-variables $Q_n = \frac{1}{n} \sum_{i=1}^n \delta_{t_i}$, where δ is the Dirac function. The $L_2(Q_n)$-norm of a function $y \in Y$ is then given by

$$\|y\|_n = \left(\int y^2 \mathrm{d}Q_n \right)^{1/2},$$

and the empirical scalar product by $\langle y, \varepsilon \rangle_n = \dfrac{1}{n} \sum_{i=1}^n \varepsilon_i y(t_i)$. Note that this empirical norm is defined over the observation space Y. Over the solution space X we will consider the norm given by the Hilbert space structure. For the sake of simplicity, we will write $\|.\|_X = \|.\|$ when no confusion is possible. Over a finite dimensional space, the norm $\|.\|$ will always stand for the Euclidean norm and if $v \in \mathbb{R}^d$, v^t will stand for the transpose vector. Likewise for any matrix $A \in \mathbb{R}^{d \times r}$ A^t will stand for the transpose matrix and $A^+ := (A^t A)^{-1} A^t$ for the generalized inverse. Considered as an operator, we will write A^* for the adjoint of the corresponding operator. Given any square matrix B, $\mathrm{Tr}(B)$ and $\rho(B)$ will stand for the trace and the spectral radius respectively.

We also introduce certain standard assumptions on the observation noise.

2.2. **AN moment condition for the errors**

ε is a centered random variable satisfying the moment condition $\mathbb{E}(|\varepsilon|^q/\sigma^q) \leq q!/2$ for all $q \geq 1$, with $\mathbb{E}(\varepsilon^2) = \sigma^2$.

As usual in statistics, assume that X satisfies a certain smoothness condition. In this paper, we assume the following source assumption encountered typically in the inverse problems literature, see for instance [10]. We point out that the source condition is assumed with respect to the linear operator T defined in **AF**.

2.3. **SC source condition**

There exists $0 < \nu \leq 1/2$ such that $x_0 \in \ \mathrm{Range}((T^*T)^\nu) := \mathcal{R}((T^*T)^\nu)$
Moreover consider

$$A_{\nu,\rho} = \{x \in X, x = (T^*T)^\nu \omega, \|\omega\| \leq \rho\}$$

where $0 \leq \nu \leq \nu_0$, $\nu_0 > 0$ and use the further notation

$$A_\nu = \bigcup_{\rho > 0} A_{\nu,\rho} = \mathcal{R}((T^*T)^\nu). \tag{2.2}$$

These sets are usually called source sets, $x \in A_\nu$ is said to have a source representation.

Estimating over all X is in general not possible because we can only observe the functions over the fixed design $(t_1, \ldots, t_n)$. Thus assume that we are equipped with a sequence of linear subspaces Y_m, $m \geq 1$, whose union is dense in Y, . We assume $\dim(Y_m) = d_m$. We are interested in a sub collection of these spaces generated by a set of indices M_n. In this paper, we will use these approximation spaces as projection spaces in order to study the data. So, denote the projection of any space W over any subspace Z by $\Pi_Z W$. Let $\Pi^n_{Y_m}$ stand for the projection in the empirical norm. Set also the corresponding projected operator $T_m = \Pi^n_{Y_m} T$.

Using a sieve of the space Y, we consider the corresponding approximation spaces in the space X, defined as $X_m = T_m^* Y$. By construction

$$\Pi_{X_m} = (\Pi^n_{Y_m} T)^+ \Pi^n_{Y_m} T.$$

We point out that both T_m and its adjoint operator T_m^* depend on the observation sequence t_i. However, we will usually drop this fact from the notation. To illustrate this assertion, consider the following example.

Example 2.1. If Y_m is a collection of nested spaces, generated by some orthonormal basis $\phi = (\phi_1, \ldots, \phi_{d_m})$, with respect to the L^2 norm over Y, and $F = T = \mathrm{Id}$, then

$$\Pi^n_{Y_m} y = \sum_{j=1}^{d_m} y_{j,n} \phi_j,$$

where $y_{j,n} = \langle \Pi^n_{Y_m} y, \phi_j \rangle_n$ are the solution to the projection problem under the empirical measure Q_n. Set $G_m = (\phi_j(t_i))_{i,j}$, $i = 1, \ldots, n$ and $j = 1, \ldots, d_m$. Thus, we may write in matrix notation

$$\Pi^n_{Y_m} \mathbf{y} = (G_m^t G_m)^{-1} G_m^t (y(t_1), \ldots, y(t_n)).$$

Remark that, as above, we can always define T_m in matrix notation and thus $T_m^t \mathbf{y}$ always makes sense. Moreover, if $u \in Y$, we will use indistinctly $T_m^t \mathbf{u} \in \mathbb{R}^{d_m}$ and $T_m^* u \in X_m$, the latter in operator notation.

In our case we will relate the degree of ill-posedness of T to $\Pi^n_{Y_m}$ as follows.

2.4. IP ill-posedness of the operator

Let M_n be an index set. For $m \in M_n$ there exists $p > 0$ such that

$$\gamma^{(m)} := \|(I - \Pi^n_{Y_m})T\| = O(d_m^{-p}).$$

p is the index of ill-posedness of the operator.

To illustrate this condition we include the following example:

Example 2.2. The above assumption can be seen to hold under certain conditions over operator T and matrix G_m defined in example (2.1). Let $(\sigma_j, \phi_j, \varphi_j)_j$ be the singular value decomposition of operator T and assume that there exists $d > 0$ such that $\sigma_j = O(j^{-d})$. Let Y_m be the linear subspace generated by $\{\varphi_j\}_{1 \leq j \leq d_m}$ and assume that the fixed observation design $t_i, i = 1, \ldots, n$ is such that this basis is also orthogonal in the empirical norm. Assume also that $\sup_{j=1,\ldots,d_m} \|\varphi_j\|_\infty < \infty$. Then,

$$I - \Pi^n_{Y_m} = I - \Pi_{Y_m} + \Pi^n_{Y_m}(I - \Pi_{Y_m}),$$

where $\|I - \Pi_{Y_m}\| = O(d_m^{-p})$ and

$$\|\Pi^n_{Y_m}(I - \Pi_{Y_m})u\| = \|\Pi^n_{Y_m}(I - \Pi_{Y_m})u\|_n \leq \|(I - \Pi_{Y_m})u\|_n \leq \sup_{j=1,\ldots,j_m} \|\varphi_j\|_\infty \|(I - \Pi_{Y_m})u\|.$$

Define

$$\nu_m := \|T_m^+ \Pi^n_{Y_m}\|.$$

This quantity controls the amplification of the observation error over the solution space X_m. Consider

$$\gamma_m := \inf_{v \in Y_m, \|v\| = 1} \|T_m^* v\|,$$

which expresses the effect of operator T_m^* over the approximating subspace Y_m. We have as in [17], $\nu_m \geq \gamma_m$.

On the other hand this term is related to the goodness of the approximation scheme. Following the proof in [17], it can be seen that $\gamma_{m+1} \leq \|T^*(I - \Pi_{Y_m}^n)\| = \|(I - \Pi_{Y_m}^n)T\|$.

The next assumption requires that γ_m and $\gamma^{(m)}$ are of the same order, which will be written $\gamma_m \sim \gamma^{(m)}$.

2.5. **AS amplification error**

Assume

$$\gamma_m = O(d_m^{-p}). \tag{2.3}$$

Moreover assume there exists a positive constant U such that

$$\frac{\gamma^{(m)}}{\gamma_m} \leq \sqrt{U}.$$

Remark 2.1. Assumption **AS** thus establishes that the worst amplification of the error over X_m is roughly equivalent to the best approximation over Y_m in the empirical norm.

For this fixed m_0, let $(\lambda_j, \phi_j, \varphi_j)$, $j = 1, \ldots, d_{m_0}$ be the singular value decomposition of operator T_{m_0}. For any $u \in Y$ we can write $T_{m_0}^* u = \sum_{j=1}^{d_{m_0}} \lambda_j \phi_j \langle u, \varphi_j \rangle_n$, which depends only on $\mathbf{u} = (u(x_1), \ldots, u(x_n))^t$. As previously recall that $G_{m_0} \in M_{d_{m_0}, n}$ is defined by $G = (\varphi_j(x_i))_{j,i}$, $i = 1, \ldots, n$, $j = 1, $, d_{m_0}. Thus, abusing notation we may write $T_{m_0}^t = D(G_{m_0} G_{m_0}^t)^{-1} G_{m_0} : \mathbb{R}^n \to \mathbb{R}^{d_{m_0}}$, where $D = D(\lambda_j)_{j=1,\ldots,d_{m_0}}$ is the diagonal matrix with entries λ_j. Since $T_{m_0}^t \mathbf{u} = T_{m_0}^* u$ the latter in operator notation, both interpretations will be used indistinctly. On the other hand, for $x \in X_{m_0}$, identified with a d_{m_0} dimensional vector, we can think of $(T_{m_0} x(t_1), \ldots, T_{m_0} x(t_n)) = G_{m_0}^t D x$. So that in matrix notation also $T_{m_0}^* T_{m_0} = D^2$. We introduce the following assumptions

SV: There exist positive constants $k_1 < k_2$ such that $k_1 j^{-p} \leq \lambda_j \leq k_2 j^{-p}$.

SF: Let ν_j, $j = 1, \ldots, d_{m_0}$ be the eigenvalues of matrix $G^t G$, then there exist constants $a_1 < a_2$ such that $a_1 n \leq \nu_j \leq a_2 n$.

Remark 2.2. Assumption **SV** is slightly stronger than **AS** as it establishes the exact order of the γ_m. It is seen to hold, for instance, in example (2.2). Assumption **SF** is necessary to assure convergence results further on. It holds also in example (2.2).

Assume also, that the images by the linear operator T of $x^\star$ and x_0 are close, in the sense that:

2.6. **IG identifiability condition**

$x_0 - x^\star \in \text{Ker}(T)^\perp$, where $\text{Ker}(T)^\perp$ is the orthogonal complement of the null space of the operator T.

The following approximation result [14] assures uniqueness of the sought solution if the initial guess is sufficiently close.

Lemma 2.3. *Assume* **AF** *holds with* $c_T < 1/2$ *and assume for* $x, x' \in B_\rho(x^\star)$, $F(x) = F(x')$ *and* $x - x' \in \text{Ker}(T)^\perp$. *Then* $x = x'$.

This lemma guarantees the identifiability of the estimation problem (1.1) since the solution is uniquely chosen. We are now ready to build adaptive estimators for non linear inverse problems.

3. ADAPTIVE TIKHONOV REGULARIZATION FOR NON LINEAR ESTIMATORS

A straightforward generalization of the Tikhonov regularization procedure to non linear operators is to seek $x_m \in x^\star + X_m$, in such a way that $F(x_m)$ is close to the data y, in the sense given by the Tikhonov regularization scheme.

To begin with, project the data onto the finite dimensional space, X_{m_0}, where we assume that m_0 is such that

$$\|(I - \Pi_{X_{m_0}})(x_0 - x^\star)\| \leq \inf_{m \in M_n} \left[\|(I - \Pi_{X_m})(x_0 - x^\star)\| + \sqrt{\frac{d_m}{n}} \frac{1}{\gamma_m} \right].$$

This quantity can be chosen so as to not depend on the unknown regularity of the solution x_0. Under assumption **SC** the above inequality is satisfied if the dimension of the set is such that $d_{m_0}^{2\nu p} \geq n^{\frac{2\nu p}{4\nu p + 2p + 1}}$. Thus it is enough to choose m_0 such that $d_{m_0} \geq n^{1/(2p+1)}$.

Let $\mathcal{K}$ be a set of indices and consider the corresponding smoothing sequences α_k, $k \in \mathcal{K}$. Let T be as defined in assumption **AF** and for a fixed $k \in \mathcal{K}$ consider the non linear Tikhonov regularized estimator as

$$\hat{x}_{\alpha_k} = x^\star + (T_{m_0}^* T_{m_0} + \alpha_k I_{m_0})^{-1} T_{m_0}^t y. \tag{3.1}$$

This estimator belongs to the class of regularized estimators where the regularization operator is given by

$$R_{\alpha_k} = (T_{m_0}^* T_{m_0} + \alpha_k I_{m_0})^{-1} T_{m_0}^t.$$

Let $x_{\alpha_k} = x^\star + R_{\alpha_k} T x_0$ be the regularized true function, which measures the accuracy of the estimation procedure without observation noise. We highlight, that in the non linear case and contrary to the linear case, (3.1) is different from the usual expression for the Tikhonov estimate

$$x^\star + \arg\min_{x \in X_m} \left[\|\Pi_{Y_{m_0}}^n (y - F(x))\|^2 + \alpha_k \|x - x^\star\|^2 \right]. \tag{3.2}$$

In this part, we consider the problem of selecting a data driven smoothing sequence of the regularized estimator (3.1) achieving the best rate of convergence among the set of all sequences. Consider the following penalty

$$\text{pen}(\alpha_k) = r\sigma^2 (1 + L_k) [Tr(R_{\alpha_k}^t R_{\alpha_k}) + \rho^2(R_{\alpha_k})],$$

with $r > 2$. Here, L_k is a sequence of weights chosen to assure that

$$\Sigma(d) := \sum_{k \in \mathcal{K}} 2 \left[\sqrt{\frac{dL_k Tr(R_k^t R_k)}{\rho^2(R_k)}} + 1 \right] \left[\frac{d}{\rho^2(nR_k)} \right]^{-1} e^{-\sqrt{dL_k [Tr(R_k^t R_k) + \rho^2(R_k)]/\rho^2(R_k)}} < \infty \tag{3.3}$$

for d as in Lemma 6.2.

The optimal regularization sequence indexed by $\hat{k}$ is selected as the solution of

$$\hat{k} := \arg\min_{k \in \mathcal{K}} \left\{ \|R_{\alpha_k}(y - F(\hat{x}_{\alpha_k}))\|^2 + \text{pen}(\alpha_k) \right\},$$

which defines the estimator $\hat{x}_{\alpha_{\hat{k}}} = x^\star + R_{\alpha_{\hat{k}}} y$. The following result provides a control over the estimation error of the estimator $\hat{x}_{\alpha_{\hat{k}}}$.

Theorem 3.1. *Assume* **AN, SF, SV, IG** *and* **AF** *are satisfied. For any* $x \in x^\star + X_m$ *and any* k *such that* $d_{m_0} \geq \alpha_k^{-1/(2p)}$, *the following inequality holds true*

$$\mathbb{E}\|\Pi_{X_{m_0}}(\hat{x}_{\alpha_{\hat{k}}} - x_0)\|^2 \leq \frac{1}{1 - c_T} \inf_{k \in \mathcal{K}}[C(1 + c_T)\|\Pi_{X_{m_0}}(x_{\alpha_k} - x_0)\|^2 + 2\mathrm{pen}(\alpha_k)] + \frac{\Sigma(d)}{(1 - c_T)n}, \qquad (3.4)$$

where $\Sigma(d)$ *was defined in (3.3).*

Proof. From condition **AF**, $F(x_1) - F(x_2) = TR(x_1, x_2)(x_1 - x_2)$ with $\|I - R\|^2 \leq c_T$. On the other hand, since R_{α_k} is a sequence of regularization methods (see [9,21]) it follows that there exist constants $C_1 < C_2$, such that $C_1 \leq \|R_{\alpha_k} T\| \leq C_2$. Hence there exists a constant C such that for any k,

$$(1 - c_T)\|\Pi_{X_{m_0}}(x_1 - x_2)\|^2 \leq \|R_{\alpha_k}(F(x_1) - F(x_2))\|^2 \leq C(1 + c_T)\|\Pi_{X_{m_0}}(x_1 - x_2)\|^2.$$

Thus, for any $x \in x^\star + X_{m_0}$,

$$(1 - c_T)\|\Pi_{X_{m_0}}(\hat{x}_{\alpha_{\hat{k}}} - x_0)\|^2 \leq C(1 + c_T)\|\Pi_{X_{m_0}}(x - x_0)\|^2 + 2\mathrm{pen}(\alpha_k) + 2\sup_k[\|R_{\alpha_k}\varepsilon\|^2 - \mathrm{pen}(\alpha_k)].$$

The result follows directly from Lemma 6.2.   □

It is important to stress that computing the estimator (3.1) may be complicated in practice since it involves inverting a matrix and requires a particular attention. But the major drawback in this approach is that the bias term in the oracle inequality $\|\Pi_{X_{m_0}}(\hat{x}_{\alpha_{\hat{k}}} - x_0)\|^2$ may not be sharp, thus leading to other estimation methods.

4. MODEL SELECTION

4.1. **Ordered selection**

Let M_n be a set of indices and consider $Y_m, m \in M_n$ a sequence of nested subspaces. We consider projection estimators and define ordered selection as the problem of choosing the best m based on the observations. For this we will construct *penalized* estimators that require finding the first m that minimizes

$$\|\Pi^n_{Y_m}(y - F(x_m))\|_n^2 + \mathrm{pen}(m),$$

where $\mathrm{pen}(m)$ is an increasing function. From a deterministic point of view this is essentially equivalent to choosing m based on the *discrepancy principle* (see [17] for an application of the discrepancy principle to non linear problems). More precisely, for a chosen m, define the projection estimator

$$\hat{x}_m := x^\star + \arg\min_{x \in X_m, x \in B_\rho(x^\star)} \|\Pi^n_{Y_m}(y - F(x))\|_n^2.$$

The optimal m is chosen as

$$\hat{m} = \arg\min_{m \in M_n} \|\Pi^n_{Y_m}(y - F(\hat{x}_m))\|_n^2 + \mathrm{pen}(m) \qquad (4.1)$$

where $\mathrm{pen}(m) = r(1 + L)\sigma^2 d_m/n$, $r = 2 + \theta$, for some $\theta > 0$, and $L > 0$.

Numerically, minimization in the above expression is more complicated than it would be in the linear case because we must calculate the projection matrix at each step. However, choosing an efficient sampling scheme will do the job, *i.e.*, such that the projection matrix is diagonal or block diagonal.

We will assume the following condition over the sampling scheme holds true.

CB: There exist constants, $l_1 < l_2$ such that for $u \in Y_{m_0}$

$$l_1\|u\| \leq \|u\|_n \leq l_2\|u\|.$$

The next theorem provides an oracle inequality which proves optimality of the estimator $\hat{x}_{\hat{m}}$.

Theorem 4.1. *Assume* **AN, IP, AS, IG, AF** *and* **CB** *hold true. There exist constants $C(r,\sigma)$, and $k(r,\sigma)$ such that with probability greater than $1 - 2\mathrm{e}^{-ku^{1/(2(p+1))}}$*

$$\|\hat{x}_{\hat{m}} - x_0\|^2 \leq C(r,\sigma) \inf_{m \in M_n} (\|(I - \Pi_{X_m})x_0\|^2 + \frac{\mathrm{pen}(m)}{\gamma_m^2}) + \frac{u}{n}. \tag{4.2}$$

Remark 4.2. For each fixed m, the quantity

$$\Delta_m := \|(I - \Pi_{X_m})x_0\|^2 + \frac{\mathrm{pen}(m)}{\gamma_m^2}$$

is of the same order as the MISE for this problem, that is, $\mathbb{E}\|\hat{x}_m - x_0\|^2$. Hence, the bounds are equivalent to those found in [3] under the appropriate source conditions. We do stress, however, that this knowledge is not required in order to establish our results.

Proof. For each m, assume $x_m = x^\star + z_m$, $z_m \in X_m$, and $x_m \in B_\rho(x^\star)$. Set

$$w(x_m) = \frac{\Pi_{Y_m}^n(F(x_m) - F(x_0))}{\|\Pi_{Y_m}^n(F(x_m) - F(x_0))\|_n}.$$

We divide the proof in a series of steps.

- Control of $d_{\hat{m}}$. Recall the penalization is defined by $\mathrm{pen}(m) = r(1 + L)\sigma^2[d_m + 1]/n$, with $2 < r$ a certain constant.

We have, following standard arguments, that

$$
\begin{aligned}
\|\Pi_{Y_{\hat{m}}}^n(F(\hat{x}_{\hat{m}}) - F(x_0))\|_n^2 + \mathrm{pen}(\hat{m}) &\leq \|\|\Pi_{Y_m}^n(F(x_m) - F(x_0))\|_n^2 + 2\langle\Pi_{Y_{\hat{m}}}^n(F(\hat{x}_{\hat{m}}) - F(x_0)), \varepsilon\rangle_n \\
&\quad -2\langle\Pi_{Y_m}^n(F(x_m) - F(x_0)), \varepsilon\rangle_n - \|\Pi_{Y_{\hat{m}}}^n\varepsilon\|_n^2 + \|\Pi_{Y_m}^n\varepsilon\|_n^2 + \mathrm{pen}(m).
\end{aligned}
$$

Let $0 < \kappa < 1$. Since $2ab \leq \kappa a^2 + \frac{1}{\kappa}b^2$, for any a, b we have for any m and $x_m \in X_m$

$$2\langle\Pi_{Y_m}^n(F(x_m) - F(x_0)), \varepsilon\rangle_n \leq \kappa\|\Pi_{Y_m}^n(F(x_m)) - F(x_0))\|_n^2 + \frac{1}{\kappa}|\langle w(x_m), \varepsilon\rangle_n|^2.$$

Set for $q, x > 0$, $t(m) = c[d_m + 1] + (1 + q)^{-1}x$ and assume κ and g, c are chosen in such a way that $\frac{1}{\kappa}((1 + g) + (1 + 1/g)c) = c_1 < r(1 + L)$. Remark, κ can be chosen very close to one.

Thus,

$$
\begin{aligned}
(1 - \kappa)\|\Pi_{Y_{\hat{m}}}^n(F(\hat{x}_{\hat{m}}) - F(x_0))\|_n^2 + (r(L + 1) - c_1)\frac{\sigma^2[d_{\hat{m}} + 1]}{n} &\leq (1 + \kappa)\|\Pi_{Y_m}^n(F(x_m) - F(x_0))\|_n^2 \\
+ \frac{1}{\kappa}|\langle w(\hat{x}_{\hat{m}}), \varepsilon\rangle_n|^2 - c_1\frac{\sigma^2([d_{\hat{m}} + 1])}{n} - \|\Pi_{Y_{\hat{m}}}^n\varepsilon\|_n^2 + \|\Pi_{Y_m}^n\varepsilon\|_n^2 & \\
+ \frac{1}{\kappa}|\langle w(x_m), \varepsilon\rangle_n|^2 - c_1\frac{\sigma^2([d_m + 1])}{n} + (r(L + 1) + c_1)\frac{\sigma^2[d_m + 1]}{n}. &
\end{aligned}
$$

Now since $\kappa < 1$ and $c_1 = c_1(q) < r(L+1)$, we have,

$$
\begin{aligned}
\frac{\sigma^2(d_{\hat{m}}+1)}{n} \;\leq\; & \frac{2}{r(1+L)-c_1}(\|\Pi^n_{Y_m}(F(x_m)-F(x_0))\|_n^2 + \mathrm{pen}(m)) \\
& + \frac{2}{\kappa(r(1+L)-c_1)} \sup_{m,\|u_m\|_n=1}\left(|\langle u_m,\varepsilon\rangle_n|^2 - \frac{c_1}{2}\kappa\left(\frac{\sigma^2[d_m+1]}{n}\right)\right),
\end{aligned}
$$

for $u_m \in S_m$ a countable dense subset of X_m. Note that

$$
\sup_{y\in Y_m,\ \|y\|_n=1} |\langle \varepsilon, y\rangle_n| = \|\Pi^n_{Y_m}\varepsilon\|_n. \tag{4.3}
$$

Hence,

$$
\sup_{\|u_m\|_n=1}\left(\frac{1}{\kappa}\langle u_m,\varepsilon\rangle_n\right) = c\|\Pi^n_{Y_m}\varepsilon\|_n.
$$

Since $\Pi^n_{Y_m}$ is a projection, $\rho(\Pi^n_{Y_m}{}^t\Pi^n_{Y_m}) = 1$ and $Tr(\Pi^n_{Y_m}{}^t\Pi^n_{Y_m}) = d_m$, and by Lemma 6.2 there are constants d and c_2 such that

$$
\begin{aligned}
P\left[\sup_m \sup_{\|u_m\|=1}\left(|\langle u_m,\varepsilon\rangle_n|^2 - c_1/2\kappa\left(\frac{\sigma^2[d_m+1]}{n}\right)\right) > \frac{x}{n}\right] \leq \\
\sum_m P\left(\sup_{\|u_m\|_n=1}|\langle u_m,\varepsilon\rangle_n|^2 - c_1/2\kappa\left(\frac{\sigma^2[d_m+1]}{n}\right) > \frac{x}{n}\right) \\
\leq \sum_m \exp\left\{-\sqrt{d(x+c_2L[d_m+1])}\right\} \leq C_2\mathrm{e}^{-\sqrt{dx/2}},
\end{aligned}
$$

setting $C_2 = \sum_m \mathrm{e}^{-\sqrt{c_2L[d_m+1]/2}}$.

Hence with probability greater than $1 - C_2\mathrm{e}^{-\sqrt{dx/2}}$ we have

$$
\frac{\sigma^2[d_{\hat{m}}+1]}{n} \leq \inf_m \inf_{x_m\in x^\star+z_m} \frac{2}{r(1+L)-c_1}(\|\Pi^n_{Y_m}(F(x_m)-F(x_0))\|_n^2 + \mathrm{pen}(m)) + \frac{x}{n}. \tag{4.4}
$$

On the other hand let, for any given m, $\tilde{x}_m$ stand for the "projection" of x_0 over $x^\star + X_m$, i.e. $\tilde{x}_m = x^\star + z_m$ is such that

$$
\Pi^n_{Y_m}F(\tilde{x}_m) = \Pi^n_{Y_m}F(x_0). \tag{4.5}
$$

Let $K_2 = \frac{1+c_T}{1-c_T}$ for c_T defined in **AF**. We have the following lemma, the proof of which is exactly as that of Lemma 2 in [14]. $\qquad\square$

Lemma 4.3. *Assume K_2 and $x^\star$ are such that*

$$
K_2\|[I - \Pi_{X_m}](x_0 - x^\star)\| + \|\Pi_{X_m}(x_0 - x^\star)\| \leq \rho.
$$

Then if **IP**, **AF** *and* **IG** *there exists $\tilde{x} \in x^\star + X_m$, such that (4.5) is satisfied.*

Now let $d_{m_{opt}}$ be such that

$$
d_{m_{opt}} = \arg\min_{d_m}\left[(\gamma^{(m)})^2\left(\frac{1+c_T}{1-c_T}\right)^2\|(I - \Pi_{X_m})(x^\star - x_0)\|^2 + \mathrm{pen}(m)\right].
$$

Since we are looking at ordered selection,

$$g(m) = (\gamma^{(m)})^2 \left(\frac{1 + c_T}{1 - c_T} \right)^2 \|(I - \Pi_{X_m})(x^\star - x_0)\|^2$$

is a decreasing sequence, so that the minimizer must be such that $g(m) \sim \mathrm{pen}(m)$. Hence we have

$$P\left[(d_{\hat{m}} + 1 - \frac{4r(1 + L)}{r(1 + L) - c_1}(d_{m_{opt}} + 1))_+ > u \right] \le C_2 \mathrm{e}^{-\sqrt{du/2}},$$

and $d_{\hat{m}} \le \frac{4r(1+L)}{r(1+L) - c_1}(d_{m_{opt}} + 1) - 1 + u$ with probability greater than $1 - C_2 e^{-\sqrt{du/2}}$.

- Error bounds: set

$$\Delta = (\|\Pi_{Y_m}^n(F(\hat{x}_{\hat{m}}) - F(x_0))\|_n^2 - \inf_m \inf_{x_m} \frac{2}{1 - \kappa} \|\Pi_{Y_m}^n(F(x_m) - F(x_0))\|_n^2 + \mathrm{pen}(m))_+.$$

Lemma 6.2 also yields

$$P(\Delta > x/n) \le C_2 \mathrm{e}^{-dx/2}.$$

- We are now able to prove optimal rates for our estimator. For this we need to bound $\|\Pi_{Y_m}^n(F(\hat{x}_{\hat{m}}) - F(x_0))\|_n$ from below. Under **CB**

$$\|\Pi_{Y_m}^n(F(\hat{x}_{\hat{m}}) - F(x_0))\| \le \frac{\|\Pi_{Y_m}^n(F(\hat{x}_{\hat{m}}) - F(x_0))\|_n}{l_1}.$$

Let $\Omega_{\dim}(u)$ be the set such that $d_{\hat{m}} \le \frac{4r(1+L)}{r(1+L) - c_1}[d_{m_{\mathrm{opt}}} + 1] + u - 1$. Let $\Omega_{\mathrm{fit}}(u)$ be the set where $\Delta < u$. In this section we assume we are always in $\Omega_{\dim}(u) \cup \Omega_{\mathrm{fit}}(u)$. We require the following lemma.

Lemma 4.4. *Let $x \in \mathcal{R}(T^*T) + x^\star$. There exists a constant C such that*

$$\|\Pi_{Y_m}^n(F(x) - F(x^\star + \Pi_{X_m}(x - x^\star)))\| \le C(1 + c_T)\gamma^{(m)}\|(I - \Pi_{X_m})(x - x^\star)\|.$$

Proof. Let $x - x^\star = T^*y$, with $y \in \mathcal{R}(T)$. By definition

$$(I - \Pi_{X_m})x = T^*(I - \Pi_{Y_m}^n)y = T^*w,$$

with $w \in Y_m^\perp$.

$$
\begin{aligned}
\|\Pi_{Y_m}^n[F(x) - F(x^\star + \Pi_{X_m}(x - x^\star))]\| &\le \|\Pi_{Y_m}^n TR((x - x^\star) - (\Pi_{X_m}(x - x^\star) - x^\star)))\| \\
&= \|\Pi_{Y_m}^n TR(I - \Pi_{X_m})(x - x^\star)\| \\
&\le \sup_{w \in Y_m^\perp, \|w\|=1} \|\Pi_{Y_m}^n TRT^*w\|\|(I - \Pi_{X_m})(x - x^\star)\|.
\end{aligned}
$$

The first term in the latter is in turn bounded by

$$\|\Pi_{Y_m}^n TR\| \sup_{w \in Y, \|w\|=1} \|T^*(I - \Pi_{Y_m}^n)w\| \le C(1 + c) \sup_{x \in \ker(T)^\perp, \|x\|=1} \|(I - \Pi_{Y_m}^n)Tx\|$$

since T^* is the adjoint operator of T. $\qquad\square$

With this lemma we have, for $x_m = x^\star + z_m$, $z_m \in X_m$

$$\|\Pi_{Y_m}^n (F(x_m) - F(x^\star + \Pi_{X_m}(x_0 - x^\star)))\|^2 \geq \|\Pi_{Y_m}^n TR\Pi_{X_m}(z_m - (x_0 - x^\star))\|^2 - [C(1+c_T)\gamma^{(m)}]^2 \|(I - \Pi_{X_m})(x_0 - x^\star)\|^2.$$

On the other hand, we have

$$
\begin{aligned}
\|\Pi_{Y_m}^n TR\Pi_{X_m}(z_m - (x_0 - x^\star))\|^2 &\geq \|z_m - (x_0 - x^\star)\|^2 \left(\inf_{y \in Y_m} \frac{\|\Pi_{Y_m}^n TRT^*y\|}{\|T^*y\|} \right)^2 \\
&\geq \|z_m - (x_0 - x^\star)\|^2 \left(\inf_{y \in Y_m} \frac{\langle \Pi_{Y_m}^n TRT^*y, y \rangle_n}{\|T^*y\| \|y\|_n} \right)^2 \\
&\geq \|z_m - (x_0 - x^\star)\|^2 \left(\inf_{y \in Y_m} \frac{\langle RT^*y, T^*y \rangle_n}{\|T^*y\| \|y\|_n} \right)^2 \\
&\geq \|z_m - (x_0 - x^\star)\|^2 \left(\inf_{y \in Y_m} \frac{\|T^*y\|^2 - \langle (I - R)T^*y, T^*y \rangle}{\|T^*y\| \|y\|_n} \right)^2 \\
&\geq \|z_m - (x_0 - x^\star)\|^2 (1 - c_T)^2 \left(\inf_{y \in Y_m} \frac{\|T^*y\|}{\|y\|_n} \right)^2 \\
&= \|z_m - (x_0 - x^\star)\|^2 (1 - c_T)^2 \gamma_m^2.
\end{aligned}
$$

Thus, since $(I - \Pi_{X_m})(x_0 - x_m) = (I - \Pi_{X_m})(x_0 - x^\star)$ under assumption **AS**,

$$\|x_m - x_0\|^2 \leq \frac{l_2 \|\Pi_{Y_m}^n (F(x_m) - F(x_0))\|^2}{l_1(1 - c_T)^2 \gamma_m^2} + \left(1 + \frac{C^2(1 + c_T)^2 U}{l_1(1 - c_T)^2} \right) \|(I - \Pi_{X_m})(x_0 - x^\star)\|. \qquad (4.6)$$

Inequality (4.6) is true for whatever $x_m \in x^\star + X_m$. Over $\Omega(u)$, we know

$$d_{\hat{m}} \leq \frac{4r(1 + L)}{r(1 + L) - c_1}[d_{m_{\text{opt}}} + 1] + u - 1.$$

We distinguish then two cases according to whether $\hat{m} < m_{opt}$ or not.
  – In the first case it is clear that $x_{\hat{m}} \in x^\star + X_{m_{\text{opt}}}$ and m in (4.6) can be replaced by m_{opt}.
  – In the second case we have:

$$\|(I - \Pi_{X_{\hat{m}}})(x_0 - x^\star)\| \leq \|(I - \Pi_{X_{m_{\text{opt}}}})(x_0 - x^\star)\|.$$

In any case we have, over $\Omega(u) \cap \Omega_{\text{fit}}(u)$

$$
\begin{aligned}
\|\hat{x}_{\hat{m}} - x_0\|^2 &\leq \|(I - \Pi_{X_{m_{\text{opt}}}})(x_0 - x^\star)\|^2 \left(1 + \frac{C^2(1 + c_T)^2 U}{l_1(1 - c_T)^2} \right) + \frac{2(\text{pen}(m_{\text{opt}}) + u/n)}{(1 - \kappa)(1 - c_T)^2 \gamma_{\hat{m}}^2} \\
&\leq \|(I - \Pi_{X_{m_{\text{opt}}}})(x_0 - x^\star)\|^2 \left(1 + \frac{C^2(1 + c_T)^2}{l_1(1 - \kappa)(1 - c_T)^2} \right) \\
&\quad + \frac{2(\text{pen}(m_{\text{opt}}) + u/n)}{(1 - \kappa)(1 - c_T)^2 \gamma_{m_{\text{opt}}}^2} \frac{1}{d_{m_{\text{opt}}}^p} \left(\frac{4r(1 + L)}{r(1 + L) - c_1}[d_{m_{\text{opt}}} + 1] + u - 1 \right)^p \\
&\leq C(r) \left(\|(I - \Pi_{X_{m_{\text{opt}}}})(x_0 - x^\star)\|^2 + \frac{\text{pen}(m_{\text{opt}})}{\gamma_{m_{\text{opt}}}^2} \right) + K(r) \frac{u^{p+1}}{n},
\end{aligned}
$$

for some appropriate constants $C(r, \sigma)$ and $K(r, \sigma)$. Thus,

$$P\left(\left(\|\hat{x}_{\hat{m}} - x_0\|^2 - C(r, \sigma)\left\{\|(I - \Pi_{X_{m_{\text{opt}}}})(x_0 - x^\star)\|^2 + \frac{\text{pen}(m_{\text{opt}})}{\gamma_{m_{\text{opt}}}^2}\right\}\right)_+ > \frac{K(r, \sigma)u^{p+1}}{n}\right) \leq 2\text{e}^{-\sqrt{d/2u}},$$

which ends the proof.

4.2. Non ordered selection

Ordered selection has the advantage of working directly on the observation space. It has the disadvantage that the expansion of the solution x_0 over the resulting subspace X_m might not be efficient. This introduces the need for non ordered selection, or equivalently, for threshold methods. The combination of both ill-posedness and non linearity makes this a difficult problem. Indeed, the former yields that it is no longer possible to work on the observation space as this would require simultaneous control of γ_m and d_m. Working on the solution space requires considering the inverse of a certain matrix. The goodness of fit of the estimator is then defined by the trace and spectral radius of this inverse matrix restricted to the sequence of subspaces, which in turn depends on the degree of non linearity of the problem.

More precisely as in Section 3, let m_0 be such that

$$\|(I - \Pi_{X_{m_0}})(x_0 - x^\star)\| \leq \inf_m \left[\|(I - \Pi_{X_m})(x_0 - x^\star)\| + \sqrt{\frac{d_m}{n}}\frac{1}{\gamma_m}\right].$$

For this fixed m_0, let M_n be a set of indices such that $\{Y_m\}_{m \in M_n} \subset Y_{m_0}$ be a collection of not necessarily nested subspaces. We will use the notation $m \subset m_0$ to express the embedding of such subsets. Our goal is to find the best subspace along this collection using penalized estimation.

Set $A_{m_0} = T_{m_0}^+ \Pi_{Y_{m_0}}^n$. For fixed x, $D_m(x) := \Pi_{X_{m_0}} R(x)\Pi_{X_m}$ is a linear operator, $D_m(x) : X_m \to X_{m_0}$. Let S_m be the matrix whose entries are defined by

$$S_m(i, j) = \sup_{x \in B_\rho(x^\star)} |A_{m_0}^t D_m(x)(i, j)|.$$

Set $\rho_m = \rho(S_m^t S_m)$ and $t_m = \text{Tr}(S_m^t S_m)$. Let $R_m = t_m/\rho_m$. From the definition of A_{m_0} under **SF** and **AF** both $n\rho_m$ and nt_m do not depend on n. Introduce L_m a certain weight factor and in the notation of Lemma 6.2, assume

$$\Sigma_1 := \sum_{m \subset m_0} \text{e}^{-\sqrt{d/2rL_m(R_m+1)}} < \infty \tag{4.7}$$

and for $q \geq 1$

$$\begin{aligned}\Sigma_2(q) &:= \sum_{m \subset m_0} (n\rho_m\sigma^2/d)^q[(d/2rL_m(R_m+1))^{q-1/2} + (d/2rL_m(R_m+1))^{q-1}]\text{e}^{-\sqrt{d/2rL_m(R_m+1)}} \\ &< \infty.\end{aligned} \tag{4.8}$$

As before we will consider penalized estimation. The penalty term in this case will be set to

$$\text{pen}(m) = r\sigma^2(1 + L_m)[t_m + \rho_m],$$

with $r > 2$. The projection estimator is defined for a fixed m as

$$\hat{x} = x^\star + \arg\min_{x \in X_m} \|A_{m_0}(y - F(x))\|^2.$$

The optimal model is chosen as

$$\hat{m} = \arg\min_{m \subset m_0} \|A_{m_0}(y - F(\hat{x}_m))\|^2 + \text{pen}(m). \tag{4.9}$$

Then, the following theorem proves optimality of the corresponding model selection estimator $\hat{x}_{\hat{m}}$.

Theorem 4.5. *Assume* **AN, IG, AF** *and* **SF** *are satisfied. Assume (4.7) and (4.8) hold true. Then, if $0 < \kappa < 1$, with probability greater than $1 - \Sigma_1 e^{-\sqrt{d/(2n\rho_{m_0})u}}$*

$$\|\hat{x}_{\hat{m}} - x_0\|^2 \le \|(I - \Pi_{X_{m_0}})(x_0 - x^\star)\|^2 + \frac{2}{(1 - c_T - \kappa)} \inf_{m \subset m_0} \{\arg\min_{x_m \in X_m} (1 + c_T)\|x_m - x_0\|^2 + \text{pen}(m)\} + \sigma^2 u/n$$

and for $q \ge 1$ there exists a constant C_q depending on q such that

$$\mathbb{E}[\|\hat{x}_{\hat{m}} - x_0\|^2]^q \le [\|(I - \Pi_{X_{m_0}})(x_0 - x^\star)\|^2 + \frac{2}{1 - c_T - \kappa} \inf_{m \subset m_0} \{\arg\min_{x_m \in X_m} (1 + c_T)\|x_0 - x_m\|^2 + \text{pen}(m)\}]^q + \frac{C_q \Sigma_2(q)}{n^q}.$$

The estimator $\hat{x}_{\hat{m}}$ looks complicated, but in the linear case, it has a nice expression. Indeed, in this case $T = F$ and $x^\star = 0$, so the minimization problem in (4.9) is similar to a hard thresholding scheme, since it is equivalent to minimizing

$$
\begin{aligned}
\hat{x}_{\hat{m}} &= \arg\min_{m \subset m_0} \arg\min_{x_m \in X_m} \{-2\langle A_{m_0}y, A_{m_0}Tx_m\rangle + \|A_{m_0}Tx_m\|^2\} + \text{pen}(m) \\
&= \arg\min_{m \subset m_0} \arg\min_{x_m \in X_m} \{-2\langle \Pi_{X_m}A_{m_0}y, x_m\rangle + \|x_m\|^2\} + \text{pen}(m).
\end{aligned}
$$

Let $\{e_j\}_{j \in m}$ be the canonical base over X_m. Define for each m,

$$x_{m,j} = \langle A_{m_0}y, e_j\rangle = \langle \mathbf{y}, A_{m_0}^t e_j\rangle, \; j = 1, \ldots, m.$$

Thus, m is selected by minimizing

$$-\sum_{j \in m} x_{m,j}^2 + r\sigma^2(1 + L_m) \left[\sum_{j \in m} \lambda_{m,j} + \sup_{j \in m} \lambda_{m,j}\right].$$

In the non linear case the problem is equivalent to minimizing

$$\arg\min_{m \subset m_0} \arg\min_{x_m \in X_m} \{-2 < A_{m_0}(y - F(x^\star)), \Pi_{X_{m_0}} R(x_m, x^\star)(x_m - x^\star) > + \|x_m - x^\star\|^2\} + \text{pen}(m).$$

Set $\mathbf{F}(x^\star) = (F(x^\star)(t_i))_{i=1}^n$. Hence,

$$
\begin{aligned}
(x_m - x^\star)_j &= \langle A_{m_0}(y - F(x^\star)), \Pi_{X_{m_0}} R(x_m, x^\star)e_j\rangle \\
&= \langle \mathbf{y} - \mathbf{F}(x^\star), A_{m_0}^t \Pi_{X_{m_0}} R(x_m, x^\star)e_j\rangle, \; j = 1, \ldots, m.
\end{aligned}
$$

Then m is chosen as above. We point out that, in this case, the problem must be solved numerically which is troublesome as A_{m_0} is a badly conditioned matrix. However practical implementation of the estimator is a difficult task which lies beyond the scope of this paper and will be tackled in a future work.

Remark 4.6. For each fixed m, the quantity

$$\Delta_m' := \|x_m - x_0\|^2 + \text{pen}(m)$$

is of the same order as the MISE for this problem, and the same comment as that at the end of Theorem 4.1 applies.

Proof. From the definition for any m and x_m,

$$\|A_{m_0}(F(x_0) - F(\hat{x}_{\hat{m}}))\|^2 \leq \|A_{m_0}(F(x_0) - F(x_m))\|^2 + \text{pen}(m)$$
$$+2\langle \varepsilon, A_{m_0}^t A_{m_0}(F(x_0) - F(\hat{x}_{\hat{m}}))\rangle_n + 2\langle \varepsilon, A_{m_0}^t A_{m_0}(F(x_0) - F(x_m))\rangle_n - \text{pen}(\hat{m}).$$

We have

$$A_{m_0}(F(x_1) - F(x_2)) = \Pi_{m_0} R(x_1, x_2)(x_1 - x_2).$$

Hence, the left hand side is bounded from below by $(1 - c_T)\|\Pi_{X_{m_0}}(\hat{x}_{\hat{m}} - x_0)\|^2$ and

$$\|A_{m_0}(F(x_0) - F(x_m))\|^2 \leq (1 + c_T)\|\Pi_{X_{m_0}}(x_m - x_0)\|^2.$$

Thus,

$$(1 - c_T)\|\Pi_{X_{m_0}}(\hat{x}_{\hat{m}} - x_0)\|^2 \leq (1 + c_T)\|\Pi_{X_{m_0}}(x_m - x_0)\|^2 + \text{pen}(m)$$
$$+2\langle \varepsilon, A_{m_0}^t R(\hat{x}_{\hat{m}}, x_m)(\hat{x}_{\hat{m}} - x_m)\rangle_n - \text{pen}(\hat{m}).$$

For any m, m' set $\Pi_{m \setminus m'} = \Pi_{X_m \setminus X_{m'}}$ and $\Pi_{m \cap m'} = \Pi_{X_m \cap X_{m'}}$. With this notation

$$\|x_{m'} - x_m\|^2 = \|\Pi_{m \cap m'}(x_{m'} - x_m)\|^2 + \|\Pi_{m \setminus m'}(x_{m'} - x_m)\|^2 + \|\Pi_{m' \setminus m}(x_{m'} - x_m)\|^2,$$

and

$$|\langle \varepsilon, A_{m_0}^t R(\hat{x}_{\hat{m}}, x_m)(\hat{x}_{\hat{m}} - x_m)\rangle| = |\langle \varepsilon, A_{m_0}^t R(\hat{x}_{\hat{m}}, x_m)\Pi_{m \cap \hat{m}}(\hat{x}_{\hat{m}} - x_m)\rangle_n$$
$$+\langle \varepsilon, A_{m_0}^t R(\hat{x}_{\hat{m}}, x_m)\Pi_{m \setminus \hat{m}}(\hat{x}_{\hat{m}} - x_m)\rangle_n$$
$$+\langle \varepsilon, A_{m_0}^t R(\hat{x}_{\hat{m}}, x_m)\Pi_{\hat{m} \setminus m}(\hat{x}_{\hat{m}} - x_m)\rangle_n|$$
$$\leq \kappa\|\hat{x}_{\hat{m}} - x_m\|^2 + 2/\kappa\|\varepsilon S_{\hat{m}}\|^2 + 1/\kappa\|\varepsilon S_m\|^2.$$

The first term in the latter is bounded by

$$\kappa[\|\hat{x}_{\hat{m}} - (\Pi_{X_{m_0}} x_0 + x^\star)\|^2 + \|x_m - (\Pi_{X_{m_0}} x_0 + x^\star)\|^2].$$

The proof then follows directly from Lemma 6.2. □

5. Concluding remarks

Penalized estimation enables to construct estimators which are optimal in the sense that without prior knowledge of the regularity of x_0 they achieve the best estimation error for all the possible choice of tuning sequences. Rates of convergence depend on the choice of the projection spaces and their approximation properties. Under **SC** we have if $\nu \leq 1/2$,

$$\|(I - \Pi_{X_m})x_0\| \leq \|(I - \Pi_{Y_m}^n)T\|^{2\nu} = O(d_m^{-2\nu p}).$$

This leads to the rate

$$\|\hat{x}_{\hat{m}} - x_0\| = O_{\mathbf{P}}(n^{-\frac{2\nu p}{4\nu p + 2p + 1}}). \tag{5.1}$$

Interpreting this rate in the statistical literature reads $s = 2\nu p$: the regularity depends on the ill-posedness of the problem. Similar rates can be encountered in the statistical literature under similar kind of assumptions, see for instance Bissantz *et al.* [4]. Hence, the rates of convergence under source conditions **SC** are the same as in the case of linear inverse problems.

So, what is the price to pay here for non linearity? Since we consider the operator T, a linear expansion of the operator F in a neighborhood of x_0, the introduction of non linearity requires controlling the linear part of the Fréchet differential operator F' in balls around the true solution x_0. As opposed to linear problems, this

fact entails the need of finding a "good" initial guess which is denoted by $x^\star$. Moreover, the ill-posedness of the problem requires relating the non linearity to the smoothing properties of $F'(x^\star)$, see condition **AF**. We remark that this kind of condition is at the heart of probabilistic control of noise amplification.

So, the whole procedure, heavily relies on the assumption that there exists this good starting point, namely $x^\star$. Asking for a good initial point is a common requirement in the literature for ill-posed inverse problems in a deterministic setting, see for instance [14]. From a practical point of view, this means that some prior knowledge about the solution is required. Another method to find this initial starting approximation point is to use, either a preliminary estimate, or to iterate the estimation process, starting from a randomly chosen point.

6. Appendix

In this section we give some technical lemmas.

The following result is a deviation inequality based on a functional exponential inequality (Thm. 7.4) due to [5] (2003).

Lemma 6.1. *Set* $\eta(A) = \sup_{\|u\|=1} \sum_{i=1}^{n} \varepsilon_i (A^t u)_i$ *for* $A : \mathbb{R}^n \to \mathbb{R}^k$. *Let*

$$v = \mathbb{E} \sum_{i=1}^{n} \sup_{\|u\|=1} \frac{(A^t u)_i^2}{\rho(A^t A)} \left(\frac{\varepsilon_i}{\sigma}\right)^2 + 2\mathbb{E}\eta(A)/(\sigma\rho^{1/2}(A^t A)).$$

Then,

$$P\left(\frac{\eta(A)}{\sigma\rho^{1/2}(A^t A)} > \mathbb{E}\frac{\eta(A)}{\sigma\rho^{1/2}(A^t A)} + \sqrt{2vx} + x\right) \le \mathrm{e}^{-x}.$$

Proof. Since the application $u \to A^t u$ is continuous, we have $\eta(A) = \sup_{u \in S} \sum_{i=1}^{n} \varepsilon_i (A^t u)_i$ for S some countable subset of the unit ball. On the other hand,

$$\sup_{\|u\|=1} [A^t u]_i \le \sup_{\|u\|=1} \|A^t u\| \le \rho(A).$$

Thus $\sup_{\|u\|\le 1} |(A^t u)_i/\rho^{1/2}(A^t A)| \le 1$. Also, following the proof of Corollary 5.1 in [1]

$$\begin{aligned}
\sup_{\|u\|=1} \frac{(A^t u)_i^2}{\rho(A^t A)} &\le \sup_{\|u\|=1} \frac{(\sum_{j=1}^{m} u_j (A^t e_j)_i^2)}{\rho(A^t A)} \\
&\le \sup_{\|u\|=1} \frac{(\sum_{j=1}^{m} (A^t e_j)_i^2) \sum_{j=1}^{m} u_j^2}{\rho(A^t A)} \\
&:= z_i.
\end{aligned}$$

Set $Z = Z(\varepsilon_1, \ldots, \varepsilon_n) = \eta(A)/(\sigma\rho^{1/2}(A^t A))$. Let $\mathbb{E}_j$ stand for the conditional expectation given ε_i for $i \ne j$. Hence, in the proof of Theorem 7.4 in [5] we may bound

$$|Z - \mathbb{E}_j Z|^q \le \frac{|\varepsilon_j|^q}{\sigma^q} \sup_{\|u\|=1} \frac{(A^t u)_j^2}{\rho(A^t A)} \sup_{\|u\|=1} \max_i \left(\frac{(A^t u)_i^2}{\rho(A^t A)}\right)^{q-2} \le (|\varepsilon_j|/\sigma)^q z_j.$$

Thus, $\mathbb{E}|Z - \mathbb{E}_j Z|^q \le z_j q!/2$. Finally, note that

$$\sum_{j=1}^{n} z_j = \frac{Tr(A^t A)}{\rho(A^t A)}.$$

Thus, the proof follows from Theorem 7.4 in [5].   □

As a corollary, we have the following lemma.

Lemma 6.2.

- *There exists a positive constant d that depends on r such that the following inequality holds*

$$
\begin{aligned}
P(\eta^2(A) &\geq \sigma^2[Tr(A^tA) + \rho(A^tA)]r/2(1+L) + \sigma^2 u) \\
&\leq \exp\{-\sqrt{d(1/\rho(A^tA)u + r/2L[Tr(A^tA)/\rho(A^tA) + 1])}\}.
\end{aligned}
\tag{6.1}
$$

- *Set $k_1 = d/(\rho(A^tA)\sigma^2)$ and $k_2 = dr/2L[Tr(A^tA)/\rho(A^tA) + 1]$. Then, there exists a constant C_q, which depends only on q, such that,*

$$
\mathbb{E}[\eta^2(A) - \sigma^2[Tr(A^tA) + \rho(A^tA)]r/2(1+L)]_+^q \leq C_q k_1^{-q}[k_2^{q-1/2} + k_2^{q-1}]e^{-\sqrt{k_2}}
\tag{6.2}
$$

holds.

Proof. As a first step we will bound v. Since $\mathbb{E}Z \leq \mathbb{E}^{1/2}Z^2$, we have

$$
v \leq \mathbb{E}\sum_{i=1}^n z_i \left(\frac{\varepsilon_i}{\sigma}\right)^2 + 2\sqrt{\mathbb{E}\sum_{i=1}^n z_i \left(\frac{\varepsilon_i}{\sigma}\right)^2} \leq (1+\nu)\mathbb{E}\sum_{i=1}^n z_i \left(\frac{\varepsilon_i}{\sigma}\right)^2 + Tr(A^tA)/\rho(A^tA).
$$

Moreover, following, [1] p. 480, for all $q \geq 2$, the following version of Rosenthal's inequality holds:

$$
\mathbb{E}^{q/2}\sum_{i=1}^n z_i \left(\frac{\varepsilon_i}{\sigma}\right)^2 \leq 2^{q/2}Tr(A^tA)/\rho(A^tA)\mathbb{E}\frac{|\varepsilon_1|^q}{\sigma^q}.
$$

Hence, we have

$$
v \leq (1+\nu)Tr(A^tA)/\rho(A^tA) + \frac{1}{\nu}
$$

and

$$
v^2 \leq 2\left[2^2(1+\nu)^2 Tr(A^tA)/\rho(A^tA)\mathbb{E}\frac{|\varepsilon^4|}{\sigma^4} + \left(\frac{1}{\nu}\right)^2\right].
$$

Set $0 < \alpha < 1$. Choose δ and β such that if

$$
2^2 4!\delta^2(1+1/\alpha)(1-\nu)^2 < c_1,
$$

$$
2\delta^2(1+1/\alpha)\left(\frac{1}{\nu}\right)^2 < c_2
$$

and $c = \max((1+\beta)\max(c_1, c_2), (1+\beta)(1+\alpha))$, then $r/2 > c$. Let $u > 0$ and without loosing generality, assume $\sigma = 1$. Thus,

$$
\begin{aligned}
P(\eta^2(A) &\geq (Tr(A^tA) + \rho(A^tA))r/2(1+L) + u) \\
&\leq P(\eta^2(A) \geq (Tr(A^tA)(1+\alpha) + (1+1/\alpha)\delta^2\nu^2\rho(A^tA))(1+\beta) \\
&\quad + [r/2 - c](Tr(A^tA) + \delta^2 v^2\rho(A^tA)) + r/2L(Tr(A^tA) + \rho(A^tA)) + u) \\
&\leq P(\eta^2(A) \geq (Tr(A^tA)(1+\alpha) + (1+1/\alpha)\delta^2\nu^2\rho(A^tA))(1+\beta) \\
&\quad + r/2L(Tr(A^tA) + \rho(A^tA)) + u).
\end{aligned}
$$

Set

$$
x' = (1 + \frac{1}{\beta})^{-1}\left[\frac{r}{2L}\left(\frac{Tr(A^tA)}{\rho(A^tA)} + 1\right) + \frac{u}{\rho(A^tA)}\right].
$$

The last term is equal to

$$P\left(\frac{\eta^2(A)}{\rho(A^tA)} \geq \left(\frac{\text{Tr}(A^tA)}{\rho(A^tA)}(1+\alpha)\right.\right.$$
$$\left.\left. + (1+1/\alpha)v^2\delta^2\right)(1+\beta) + r/2L\frac{\text{Tr}(A^tA)}{\rho(A^tA)} + 1) + u\right)$$
$$= P\left(\frac{\eta^2(A)}{\rho(A^tA)} \geq \left(\frac{\text{Tr}(A^tA)}{\rho(A^tA)}(1+\alpha) + (1+1/\alpha)v^2\delta^2\right)(1+\beta) + (1+1/\beta)x'\right).$$

Finally, we may bound

$$\leq P\left(\frac{\eta^2(A)}{\rho(A^tA)} \geq \left(\mathbb{E}\frac{\eta(A)}{\rho^{1/2}(A^tA)} + \delta v\right)^2 (1+\beta) + (1+1/\beta)x'\right)$$
$$\leq P\left(\frac{\eta^2(A)}{\rho(A^tA)} \geq \left(\mathbb{E}\frac{\eta(A)}{\rho^{1/2}(A^tA)} + \delta v + \sqrt{x'}\right)^2\right)$$
$$= P\left(\frac{\eta(A)}{\rho^{1/2}(A^tA)} \geq \mathbb{E}\frac{\eta(A)}{\rho^{1/2}(A^tA)} + \delta v + (1+2/\delta)x''\right)$$
$$\leq P\left(\frac{\eta(A)}{\rho^{1/2}(A^tA)} \geq \mathbb{E}\frac{\eta(A)}{\rho^{1/2}(A^tA)} + \sqrt{2vx''} + x''\right) \leq e^{-x''}$$
$$= e^{-\sqrt{g(A)}},$$

where we have used repeatedly that for any constant $c > 0, ca^2 + 1/cb^2 \geq 2ab$ and set

$$g(A) = ((1+1/\beta)^{-1}(1+2/\delta)^2)(r/2L[\text{Tr}(A^tA)/\rho(A^tA) + 1] + u/\rho(A^tA)).$$

Set also $d = [(1+1/\beta)^{-1}(1+2/\delta)^2]^{-1}$ and $b(A) = \text{Tr}(A^tA)/\rho(A^tA)$. Thus we have shown the first part of the lemma.

Moreover, using the above inequality,

$$\mathbf{E}[\eta^2(A) - \sigma^2(\text{Tr}(A^tA) + \rho(A^tA))r/2(1+L)]_+^q \leq \int_0^\infty \sigma^{2q}qu^{q-1}e^{-\sqrt{dr/2L[b(A)+1]+du/(\rho(A^tA))}}\text{d}u.$$

Consider the change of variable $w = \text{d}u/(\rho(A^tA)) + \text{d}r/2L[b(A)+1]$, so that

$$\mathbb{E}[\eta^2(A) - \sigma^2(\text{Tr}(A^tA) + \rho(A^tA))r/2(1+L)]_+^q \leq$$
$$\left(\frac{\sigma^2\rho(A^tA)}{d}\right)^q \int_{\text{d}r/2L[b(A)+1]}^\infty (w - \text{d}r/2L[b(A)+1])^{q-1}e^{-\sqrt{w}}\text{d}w.$$

The last expression is in turn bounded by

$$\left(\frac{\sigma^2\rho(A^tA)}{d}\right)^q \int_{\text{d}r/2L[b(A)+1]}^\infty e^{-\sqrt{w}}[w^{q-1} + (\text{d}r/2L[b(A)+1])^{q-1}]\text{d}w \leq C_q k_1^{-q}[k_2^{q-1/2} + k_2^{q-1}]e^{-\sqrt{k_2}},$$

ending the proof.   □

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