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THE FRACTIONAL MIXED FRACTIONAL BROWNIAN MOTION AND FRACTIONAL BROWNIAN SHEET

Charles El-Nouty[1]

Abstract. We introduce the fractional mixed fractional Brownian motion and fractional Brownian sheet, and investigate the small ball behavior of its sup-norm statistic. Then, we state general conditions and characterize the sufficiency part of the lower classes of some statistics of the above process by an integral test. Finally, when we consider the sup-norm statistic, the necessity part is given by a second integral test.

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1. Introduction and main results

Let $\{B_{H_1}(s), s \geq 0\}$ be a fractional Brownian motion (FBM) with index $0 < H_1 < 1$, *i.e.* a centered Gaussian process with stationary increments satisfying $B_{H_1}(0) = 0$, with probability 1, and $\mathbb{E}(B_{H_1}(s))^2 = s^{2H_1}, s \geq 0$. Denote by σ_{H_1} the covariance function of B_{H_1}. Moreover, recall that B_{H_1} can be represented as a random integral, *i.e.*

$$B_{H_1}(s) = \int_{\mathbb{R}} g_{H_1}(s,u)\, \tilde{W}(\mathrm{d}u), \tag{1.1}$$

where $\tilde{W}(u), u \in \mathbb{R}$, is a Wiener process,

$$g_{H_1}(s,u) = k_{2H_1}^{-1}\Big(\max(s-u,0)^{H_1-1/2} - \max(-u,0)^{H_1-1/2}\Big),$$

and k_{2H_1} is a normalizing constant. We refer to Li and Shao [18] for further information on this field.

A natural extension of B_{H_1} in 2-dimensional space is given by

$$B_{H_2,H_3}(s_2,s_3) = \int_{-\infty}^{s_2}\int_{-\infty}^{s_3} g_{H_2}(s_2,u_2)\, g_{H_3}(s_3,u_3)\, W\Big(\mathrm{d}(u_2,u_3)\Big), \tag{1.2}$$

where $W(u_2,u_3), u_2 \in \mathbb{R}, u_3 \in \mathbb{R}$, is a standard Brownian sheet and $0 < H_2, H_3 < 1$. Its covariance function σ_{H_2,H_3} is given by

$$\sigma_{H_2,H_3}\Big((s_2,s_3),(s_2^{'},s_3^{'})\Big) = \sigma_{H_2}(s_2,s_2^{'}) \;\times\; \sigma_{H_3}(s_3,s_3^{'}).$$

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[1] U.F.R. de mathématiques, Université Paris VI, 175 rue du Chevaleret, 75013 Paris, France; elnouty@ccr.jussieu.fr

B_{H_2,H_3} is named the fractional Brownian sheet (FBS). There is a huge literature on this process. We refer to Ayache *et al.* [1], Ayache and Xiao [2], Belinsky and Linde [3], Kühn and Linde [14], Mason and Shi[20] and Xiao and Zhang [25] for further information on the FBS.

The study of the FBM and its extension was motivated by natural time series in economics, fluctuations in solids, hydrology and more recently by new problems in mathematical finance, telecommunication networks and the environment.

In the sequel, we assume that B_{H_1} and B_{H_2,H_3} are independent. Let λ_1 and λ_2 be two real numbers such that $\lambda_1\lambda_2 \neq 0$. In the spirit of Cheridito [5] and El-Nouty [9], we introduce the fractional mixed fractional Brownian motion and fractional Brownian sheet (FMFBMFBS) defined as follows

$$X(w_1, w_2, w_3, s) = \lambda_1\, s^{H_2+H_3} B_{H_1}(w_1) + \lambda_2\, s^{H_1} B_{H_2,H_3}(w_2, w_3),$$

and consider the sup-norm statistic

$$Y(t) = \sup_{0\leq s\leq t}\ \sup_{0\leq w_1,w_2,w_3\leq s} \mid X(w_1, w_2, w_3, s) \mid, \quad t \geq 0.$$

Note first that, by the scaling property, we have for any $\epsilon > 0$

$$I\!P\Big(Y(t) \leq \epsilon\, t^{H_1+H_2+H_3}\Big) = I\!P\Big(\sup_{0\leq s\leq 1}\ \sup_{0\leq w_1,w_2,w_3\leq s} \mid X(w_1, w_2, w_3, s) \mid \leq\ \epsilon\Big)$$

$$= I\!P\Big(Y(1) \leq \epsilon\Big) := \phi(\epsilon),$$

where ϕ is named the small ball function and $\gamma := H_1 + H_2 + H_3$ the scaling factor.

The motivation supporting this paper is threefold:

- The first goal of the FMFBMFBS deals with the potential applications to the above mentioned fields. Since the FMFBMFBS can be analyzed based on the large bodies of knowledge on FBM and FBS, it can be used in the same fields. This may look like a tautology, but this remark applies to fractional mixed fractional Gaussian processes (*i.e.* a suitable combination of some appropriate fractional Gaussian processes). For example, to modelize the discounted stock price, the fractional version of the Samuelson model [23] was studied by Cutland *et al.* [6]. But, since it had also some deficiencies, Cheridito [5] introduced some mixed fractional Gaussian processes. The FMFBMFBS could be used to modelize the diffusion of atmospheric pollutants, either accidental (nuclear, chemical) or not (air pollution). To validate this model, we could compare the theoretical results to those obtained by Gassmann and Bürki [12] and Gassmann *et al.* [13].
- A second application deals with the small ball probability problem of the sum of two joint centered Gaussian random vectors X and Y in a separable Banach space E with norm $\|.\|$. This problem was investigated in Li [16], when X and Y are not necessarily independent and have a standard small ball factor (*cf.* El-Nouty [7, 11]). Here we assume B_{H_1} and B_{H_2,H_3} are independent but B_{H_2,H_3} can have a log-type small ball factor (*cf.* El-Nouty [11]). Thus, the study of the small ball behavior of the FMFBMFBS gives a first answer of the small ball probability problem of the sum of two centered independent Gaussian random vectors, having a log-type small ball factor.
- Last but not least, this paper extends El-Nouty's results [7–11] and consequently answers some new questions. Recall first two definitions of the Lévy classes, stated in Révész [22]. Let $\{Z(t), t \geq 0\}$ be a stochastic process defined on the basic probability space $(\Omega, \mathcal{A})$.

Definition 1.1. The function $f(t), t \geq 0$, belongs to the lower-lower class of the process $Z, (f \in LLC(Z))$, if for almost all $\omega \in \Omega$ there exists $t_0 = t_0(\omega)$ such that $Z(t) \geq f(t)$ for every $t > t_0$.

Definition 1.2. The function $f(t), t \geq 0$, belongs to the lower-upper class of the process $Z, (f \in LUC(Z))$, if for almost all $\omega \in \Omega$ there exists a sequence $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \ldots$ with $t_n \to +\infty$, as $n \to +\infty$, such that $Z(t_n) \leq f(t_n), n \in \mathbb{N}^*$.

In the spirit of Talagrand [24] and El-Nouty [7–11], the main aim of this paper is to characterize the lower classes of Y for any $0 < H_1, H_2, H_3 < 1$. El-Nouty [7] characterized the lower classes of a large class of statistics of the FBM. The role of a log-type small ball factor was studied in El-Nouty [11] by considering the FBS. Here the FMFBMFBS enables us to compare the influence of the FBM and the FBS, and consequently to measure the weight of a standard type small ball factor versus a log-type one.

Set $\alpha = \min(H_1, H_2, H_3)$, which is in $]0,1[$. We introduce the number β taking its values in $\{0, 1 + 1/\alpha\}$. When $H_2 \neq H_3$ or $H_1 < H_2 = H_3$, $\beta = 0$, whereas, when $H_2 = H_3 \leq H_1$, $\beta = 1 + 1/\alpha$. The small ball behavior of the FMFBMFBS is given in the first result.

Theorem 1.1. *There is a constant $K_0, 0 < K_0 \leq 1$, depending on H_1, H_2, H_3, λ_1 and λ_2 only, such that we have for $0 < \epsilon < 1$*

$$\exp\left(- \frac{\left(\log(1/\epsilon)\right)^{\beta}}{K_0 \ \epsilon^{1/\alpha}} \right) \leq \phi(\epsilon) \leq \exp\left(- \frac{K_0 \left(\log(1/\epsilon)\right)^{\beta}}{\epsilon^{1/\alpha}} \right).$$

Note first that the minimum α plays a key role. This is not really surprising. Indeed this phenomenon was already observed in El-Nouty [9]. We can also remark that, when $\beta = 1 + 1/\alpha$, we have a log-type small ball factor. This is a consequence of the small ball behavior of the FBS which was studied by Belinsly and Linde [3] and Mason and Shi [20]. Recall also that the existence of small ball constants for the FBM was showed by Li and Linde [17]. Hence, when $H_1 \leq \min(H_2, H_3)$, Theorem 1.1 (and consequently the sharpness of Ths. 1.2 and 1.3) can be improved by establishing the existence of the small ball constant for the FMFBMFBS.

It appears that the sufficiency part of the lower classes of Y can be stated in a general framework. Roughly speaking, we follow the same lines as those of El-Nouty [7,11].

Let $\{Y_0(t), t \geq 0\}$ be a real-valued statistic of B_{H_1} and B_{H_2,H_3}, such that $Y_0(t)$ is a nondecreasing function of $t \geq 0$.

The following notation is needed. If $\mathbb{K}$ is a Hausdorff compact space, we denote by $C(\mathbb{K})$ the space of all continuous functions from $\mathbb{K}$ to $\mathbb{R}$ equipped with the classical sup-norm. Let $\boldsymbol{X} = C([0,1]) \times C([0,1]^2)$ be the product space equipped with the product topology. Denote by $\mathsf{L}(B_{H_1}, B_{H_2,H_3})$ the Gaussian measure associated to B_{H_1} and B_{H_2,H_3} and defined on $\mathbb{B}$, the Borel σ-field of $\boldsymbol{X}$.

We assume that Y_0 satisfies the three following conditions :
(C1) The scaling condition. There exists $\gamma_0 > 0$ such that

$$\mathbb{P}(Y_0(t) \leq \epsilon t^{\gamma_0}) = \mathbb{P}(Y_0(1) \leq \epsilon) := \phi(\epsilon).$$

(C2) The convexity condition. There exists a convex and $\mathbb{B}$-measurable function $g : (\boldsymbol{X}, \mathsf{L}(B_{H_1}, B_{H_2,H_3})) \to \mathbb{R}$ such that for any $t \geq 0$, $Y_0(t) = g\Big(B_{H_1}(s_1 t), B_{H_2,H_3}(s_2 t, s_3 t); 0 \leq s_1, s_2, s_3 \leq 1\Big)$, and $Y_0(t) < +\infty$, with probability 1.

(C3) The log-type small ball condition. There exist $\alpha_0 \in]0, \gamma_0]$, $\beta_0 \in \mathbb{R}$ and a constant $K, 0 < K \le 1$, depending on γ_0, α_0 and β_0 only, such that we have for $0 < \epsilon < 1$

$$\exp\left(-\frac{\left(\log(1/\epsilon)\right)^{\beta_0}}{K\epsilon^{1/\alpha_0}}\right) \le \phi(\epsilon) \le \exp\left(-\frac{K\left(\log(1/\epsilon)\right)^{\beta_0}}{\epsilon^{1/\alpha_0}}\right).$$

Note that these conditions generalize those of El-Nouty [7,11]. The small ball function still plays a key role. The convexity of the function ψ defined by $\psi(\epsilon) = -\log\phi(\epsilon)$, $0 < \epsilon < 1$, is ensured by the condition $(C2)$ (see Borell [4], p. 243, Ledoux and Talagrand [15] and Lifshits [19], pp. 108–137).

Our second result is given in the following theorem.

Theorem 1.2. *Let $f(t)$ be a positive nondecreasing function of $t \ge 0$. Assume that there exists $m > 0$ such that* $\dfrac{f(t)}{t^{\gamma_0-\alpha_0}}\left(\log\dfrac{t^{\gamma_0}}{f(t)}\right)^{-\beta_0\alpha_0} \ge m.$

If

$$\frac{f(t)}{t^{\gamma_0}} \text{ is bounded and } \int_0^{+\infty} f(t)^{-1/\alpha_0} t^{(\gamma_0/\alpha_0)-1}\left(\log\frac{t^{\gamma_0}}{f(t)}\right)^{\beta_0}\phi\left(\frac{f(t)}{t^{\gamma_0}}\right)\mathrm{d}t < +\infty.$$

then we have

$$f \in LLC(Y_0).$$

The sup-norm statistic Y clearly satisfies the three above conditions with $\gamma_0 = \gamma = H_1 + H_2 + H_3, \alpha_0 = \alpha = \min(H_1, H_2, H_3)$, $\beta_0 = \beta \in \{0, 1 + 1/\alpha\}$ and $K = K_0$. Now, we characterize the necessity part of the lower classes of the FMFBMFBS. Our main result is stated in the following theorem.

Theorem 1.3. *Let $f(t)$ be a positive nondecreasing function of $t \ge 0$ such that $\dfrac{f(t)}{t^{\gamma}}$ is a nonincreasing function of $t > 0$.*

If

$$f \in LLC(Y)$$

then we have

$$\lim_{t\to+\infty}\frac{f(t)}{t^{\gamma}} = 0 \text{ and } \int_0^{+\infty} f(t)^{-1/\gamma}\phi\left(\frac{f(t)}{t^{\gamma}}\right)\mathrm{d}t < +\infty.$$

First, we can notice that Theorem 1.2 depends on γ_0, α_0 and β_0. If $\beta_0 = 0$, Theorem 1.2 looks like Theorem 1 of El-Nouty [7], p. 365, or else like Theorem 1.1 of El-Nouty [11], p. 321. As expected, Theorem 1.3 has the same form as the theorems obtained by Talagrand [24] and El-Nouty [7–11]. The methodology of Talagrand [24] can lead to two integral tests in the study of the lower classes of Y. But Theorems 1.2 and 1.3 are sharp. Indeed, set, if $\beta = 0$,

$$f(t) = \frac{\lambda\, t^{\gamma}}{(\log\log t)^{\alpha}},\ t \ge 3, \lambda > 0.$$

or else (*i.e.* $\beta = 1 + (1/\alpha)$)

$$f(t) = t^{\gamma}\,\frac{\left(\lambda\log\log\log t\right)^{1+\alpha}}{(\log\log t)^{\alpha}},\ t \ge 16, \lambda > 0.$$

If λ is small enough, then Theorem 1.2 yields $f \in LLC(Y)$, and if λ is large enough, then $f \in LUC(Y)$ by applying Theorem 1.3.

In Section 2, we prove Theorem 1.1. The proof of Theorem 1.3 is postponed to Sections 3 and 4 and will be given in details. In Section 4 we establish some key small ball estimates. Note also that these estimates can be of independent interest. The proofs which are modifications of those of El-Nouty [7, 11] will be consequently omitted, in particular the proof of Theorem 1.2.

In the sequel, there is no loss of generality to assume that $H_2 \le H_3$.

2. Proof of Theorem 1.1

The proof will be split into two parts: the lower bound and the upper one.

Part I. The lower bound. We have

$$\phi(\epsilon) \ge I\!P\Bigg(\Big\{ \sup_{0\le s\le 1} \sup_{0\le w_1\le s} \mid \lambda_1\, s^{H_2+H_3}\, B_{H_1}(w_1) \mid \le \frac{\epsilon}{2} \Big\} \bigcap \Big\{ \sup_{0\le s\le 1} \sup_{0\le w_2,w_3\le s} \mid \lambda_2\, s^{H_1}\, B_{H_2,H_3}(w_2,w_3) \mid \le \frac{\epsilon}{2} \Big\} \Bigg).$$

Hence we get by independence and monotonicity

$$\phi(\epsilon) \ge I\!P\Bigg(\sup_{0\le w_1\le 1} \mid B_{H_1}(w_1) \mid \le \frac{\epsilon}{2\mid \lambda_1 \mid} \Bigg) \times I\!P\Bigg(\sup_{0\le w_2,w_3\le 1} \mid B_{H_2,H_3}(w_2,w_3) \mid \le \frac{\epsilon}{2\mid \lambda_2 \mid} \Bigg). \tag{2.1}$$

Combining Belinsky and Linde [3], Mason and Shi [20] Monrad and Rootzen [21] and Talagrand [24] with (2.1), we obtain that:

$$\text{if } H_2 = H_3 \le H_1, \text{ then } \phi(\epsilon) \ge \exp\Bigg(-\frac{\Big(\log(1/\epsilon)\Big)^{1+(1/\alpha)}}{C_1\, \epsilon^{1/\alpha}} \Bigg);$$

$$\text{or else } \phi(\epsilon) \ge \exp\Bigg(-\frac{1}{C_2\, \epsilon^{1/\alpha}} \Bigg),$$

where C_1 and C_2 are strictly positive constants.

The proof of the lower part of Theorem 1.1 is complete.

Part II. The upper bound. By choosing $s = 1$, we get

$$\phi(\epsilon) \le I\!P\Bigg(\sup_{0\le w_1,w_2,w_3\le 1} \mid \lambda_1\, B_{H_1}(w_1) + \lambda_2\, B_{H_2,H_3}(w_2,w_3) \mid \le \epsilon \Bigg).$$

Since $\{(w_1, 0, 0) : 0 \le w_1 \le 1\} \subset [0,1]^3$ and $\{(0, w_2, w_3) : 0 \le w_2, w_3 \le 1\} \subset [0,1]^3$, the following inequality holds

$$\phi(\epsilon) \le \inf\Bigg(I\!P\Bigg(\sup_{0\le w_1\le 1} \mid \lambda_1\, B_{H_1}(w_1) \mid \le \epsilon \Bigg), I\!P\Bigg(\sup_{0\le w_2,w_3\le 1} \mid \lambda_2\, B_{H_2,H_3}(w_2,w_3) \mid \le \epsilon \Bigg) \Bigg). \tag{2.2}$$

By considering the four cases $H_1 \le H_2 < H_3, H_2 < H_3$ and $H_2 \le H_1, H_1 < H_2 = H_3$, and $H_2 = H_3 \le H_1$, we use (2.2) and the results in Belinsky and Linde [3], Mason and Shi [20], Monrad and Rootzen [21] and Talagrand [24]. We emphasize the fact that we obtain a log-type small ball factor if and only if $H_2 = H_3 \le H_1$.

The proof of Theorem 1.1 is now complete. □

3. PROOF OF THEOREM 1.3: PART I

Set $a_t = \dfrac{f(t)}{t^\gamma}$ and $b_t = \phi(a_t)$.

Suppose here that, with probability 1, $f(t) \leq Y(t)$ for all t large enough. We want to prove that $\lim\limits_{t\to+\infty} a_t = 0$ and $\int_0^\infty a_t^{-1/\gamma}\, b_t\, \dfrac{\mathrm{d}t}{t} < +\infty$.

In the sequel, there is no loss of generality in assuming that f is a continuous function of $t \geq 0$. Indeed the set of points at which f is discontinuous is at most countable and therefore has measure zero.

Lemma 3.1. *We have*

$$\lim_{t\to+\infty} a_t = 0. \tag{3.1}$$

Proof. Let $\lim\limits_{t\to\infty} a_t = c \geq 0$. If $c > 0$, then $\lim\limits_{t\to\infty} b_t = \phi(c) > 0$, which is impossible. (3.1) is proved. □

To prove Theorem 1.3, we will show that $f \in LUC(Y)$ when $\int_0^\infty a_t^{-1/\gamma}\, b_t\, \dfrac{\mathrm{d}t}{t} = +\infty$ and $\lim\limits_{t\to+\infty} a_t = 0$.

First recall the following corollary (see Talagrand [24], p. 198).

Corollary A. *Assume that $J \subset \mathbb{N}$ and that, for some numbers K and ϵ, we have for the family of sets $(A_i)_{i\in J}$ in a basic probability space*

$$\forall\, i \in J \qquad \sum_{j>i} \mathbb{P}(A_i \cap A_j) \;\leq\; \mathbb{P}(A_i)\left(K + (1+\epsilon)\sum_{j>i}\mathbb{P}(A_j)\right). \tag{3.2}$$

Then, if

$$\sum_{i\in J} \mathbb{P}(A_i) \;\geq\; \frac{1+2K}{\epsilon}, \tag{3.3}$$

we have

$$\mathbb{P}\Big(\bigcup_{i\in J} A_i\Big) \;\geq\; \frac{1}{1+2\epsilon}. \tag{3.4}$$

Our aim is to construct a suitable set J which satisfies hypothesis (3.2) and (3.3).

Lemma 3.2. *When $\int_0^\infty a_t^{-1/\gamma}\, b_t\, \dfrac{\mathrm{d}t}{t} = +\infty$ and $\lim\limits_{t\to+\infty} a_t = 0$, we can find a sequence $\{t_n, n \geq 1\}$ with the two following properties*

$$t_{n+1} \geq t_n(1 + a_{t_n}^{1/\gamma}), \tag{3.5}$$

and

$$\sum_{n=1}^{\infty} b_{t_n} = +\infty. \tag{3.6}$$

Proof. For the construction of $\{t_n, n \geq 1\}$, we proceed by induction over n. Set $t_1 = 1$. Having constructed t_n, we define

$$s_n = t_n\Big(1 + a_{t_n}^{1/\gamma}\Big).$$

We set $t_{n+1} = s_n(1 + a_{s_n}^{1/\alpha})$.

(3.5) is obviously proved.

To prove (3.6), it is enough to show that, for n sufficiently large, we have

$$I_n = \int_{t_n}^{t_{n+1}} a_t^{-1/\gamma}\, b_t\, \frac{\mathrm{d}t}{t} \le K\, b_{t_n},$$

where K is a constant.

Set $I_{n1} = \int_{t_n}^{s_n} a_t^{-1/\gamma}\, b_t\, \dfrac{\mathrm{d}t}{t}$ and $I_{n2} = \int_{s_n}^{t_{n+1}} a_t^{-1/\gamma}\, b_t\, \dfrac{\mathrm{d}t}{t}$. Consider first I_{n1}. We obtain

$$I_{n1} \le (s_n - t_n) f(t_n)^{-1/\gamma} \phi\Big(a_{t_n}\Big) = b_{t_n}.$$

Consider now I_{n2}. Since $t_{n+1} \le 2s_n$, f is nondecreasing and $a_t^{-1/\gamma} \le a_t^{-1/\alpha}$, we have

$$I_{n2} \le \int_{s_n}^{t_{n+1}} a_t^{-1/\alpha}\, b_t\, \frac{\mathrm{d}t}{t} \le (t_{n+1} - s_n) f(s_n)^{-1/\alpha} t_{n+1}^{(\gamma/\alpha)-1} \phi\Big(a_{s_n}\Big) \le 2^{(\gamma/\alpha)-1}\, b_{t_n}.$$

Hence, we get $I_n = I_{n1} + I_{n2} \le \Big(1 + 2^{(\gamma/\alpha)-1}\Big) b_{t_n}$. □

The sequence $\{t_n, n \ge 1\}$ we have constructed is not yet appropriate. We need a further construction (the reason for which will become apparent only later).

We need the following definition and notation.

Definition 3.1. Consider the interval $A_k = [2^k, 2^{k+1}[,\ k \in I\!N$. If $a_{t_i}^{-1/\gamma} \in A_k,\ i \in I\!N^*$, then we note $u(i) = k$.

Notation.
1. $I_k = \{i \in I\!N^*, u(i) = k \in I\!N\}$ which is finite by Lemma 3.1;
2. $N_k = \exp\Bigg(K_0\ (\gamma \log 2)^\beta\ k^\beta\ 2^{\gamma(k-1)/\alpha}\Bigg)$, where K_0 has the same value as in Theorem 1.1;
3. $F_{m,k} = \{i \in I\!N^*, i \in I_k, m < i, \mathrm{card}(I_k \cap]m, i]) \le N_k\}, m \in I\!N^*, k \in I\!N$;
4. $k_0 = \inf\left\{n \in I\!N,\ 2^{\gamma n/\alpha} \ge \dfrac{2^{\gamma/\alpha}}{K_0(2^{\gamma/\alpha}-1)(\gamma \log 2)^\beta} + \dfrac{2^{2\gamma/\alpha}}{K_0^2(2^{\gamma/\alpha}-1)}\right\}$, ($k_0$ depends on K_0, γ and α only);
5. $V_m = \bigcup_{k \in I\!N} F_{m,k}$, where m is fixed, $u(m) = k_1$ and $k \ge k_1 + k_0$;
6. $W = \bigcup_{m \ge 1} V_m$.

Now we can define our set J as follows

$$J = I\!N - W.$$

Lemma 3.3. *We have* $\displaystyle\sum_{n \in J} b_{t_n} = +\infty.$ (3.7)

Given $n \in J, m \in J, n < m$, *such that* $\mathrm{card}(I_{u(n)} \cap [n,m]) > \exp(K_0 2^{u(n)-1})$, *we have*

$$\frac{t_m}{t_n} \geq \exp\left(\exp\Big(\frac{K_0}{4}\; 2^{\min(u(n),u(m))}\Big)\right). \tag{3.8}$$

Proof. We show first

$$\sum_{i \in V_m} b_{t_i} \leq \frac{3}{4}\, b_{t_m}. \tag{3.9}$$

We have for any $i \in I_k$

$$\frac{1}{2^{(k+1)\gamma}} \leq a_{t_i} \leq \frac{1}{2^{k\gamma}},$$

and consequently by Theorem 1.1

$$\exp\left(-\frac{(\gamma\log 2)^\beta (k+1)^\beta\, 2^{\gamma(k+1)/\alpha}}{K_0}\right) \leq b_{t_i} \leq \exp\left(-K_0 (\gamma\log 2)^\beta k^\beta\, 2^{\gamma k/\alpha}\right).$$

Hence, we have

$$\sum_{i \in F_{m,k}} b_{t_i} \leq N_k \exp\left(-K_0 (\gamma\log 2)^\beta k^\beta\, 2^{\gamma k/\alpha}\right) \leq \exp\left(-(2^{\gamma/\alpha}-1) K_0 (\gamma\log 2)^\beta k^\beta\, 2^{\gamma(k-1)/\alpha}\right).$$

Then, we obtain

$$\begin{aligned}\sum_{i \in V_m} b_{t_i} &\leq b_{t_m} \sum_{k=k_0+k_1}^{+\infty} \frac{1}{b_{t_m}} \exp\left(-(2^{\gamma/\alpha}-1) K_0 (\gamma\log 2)^\beta k^\beta\, 2^{\gamma(k-1)/\alpha}\right) \\ &\leq b_{t_m} \sum_{k=k_0+k_1}^{+\infty} \exp\left((\gamma\log 2)^\beta (k_1+1)^\beta \Big(\frac{2^{\gamma(k_1+1)/\alpha}}{K_0} - K_0(2^{\gamma/\alpha}-1) 2^{\gamma(k-1)/\alpha}\Big)\right). \end{aligned} \tag{3.10}$$

Setting $l = k - (k_0 + k_1)$ and recalling the definition of k_0, we get

$$\begin{aligned}&(\gamma\log 2)^\beta \left(\frac{2^{\gamma(k_1+1)/\alpha}}{K_0} - K_0(2^{\gamma/\alpha}-1)2^{\gamma(k-1)/\alpha}\right) \\ &\qquad \leq 2^{\gamma k_1/\alpha}\left(\frac{2^{\gamma/\alpha}\,(\gamma\log 2)^\beta}{K_0} - 2^{\gamma l/\alpha} - \frac{2^{\gamma(l+1)/\alpha}\,(\gamma\log 2)^\beta}{K_0}\right) \leq -2^{\gamma(l+k_1)/\alpha} \leq -2^{l+k_1}. \end{aligned} \tag{3.11}$$

Combining (3.10) with (3.11), we get

$$\sum_{i \in V_m} b_{t_i} \;\leq\; b_{t_m} \sum_{l=0}^{+\infty} \exp(-2^l) \;\leq\; b_{t_m} \sum_{l=0}^{+\infty} \frac{3}{4\; 2^{l+1}} \;=\; \frac{3}{4}\, b_{t_m}.$$

Hence (3.9) is proved.

Let $p \in \mathbb{N}, J_p = J \cap \Big(\bigcup_{0 \leq k_1 \leq p} I_{k_1}\Big)$ and $W_p = W \cap \Big(\bigcup_{0 \leq k_1 \leq p} I_{k_1}\Big) = \Big(\bigcup_{0 \leq k_1 \leq p} I_{k_1}\Big) - J_p$.

The definition of W_p and (3.9) yield

$$\sum_{i\in W_p} b_{t_i} \ \le\ \sum_{k_1=0}^{p}\ \sum_{m\in I_{k_1}}\ \sum_{i\in V_m} b_{t_i} \ \le\ \frac{3}{4}\sum_{k_1=0}^{p}\ \sum_{m\in I_{k_1}} b_{t_m} \ =\ \frac{3}{4}\sum_{m\in\left(\bigcup_{0\le k_1\le p}\ I_{k_1}\right)} b_{t_m}.$$

Since $\left(\bigcup_{0\le k_1\le p}\ I_{k_1}\right)$ is finite, we have

$$\sum_{i\in J_p} b_{t_i} \ \ge\ \frac{1}{4}\sum_{m\in\left(\bigcup_{0\le k_1\le p}\ I_{k_1}\right)} b_{t_m}.$$

Let $p\to+\infty$. Then, we have $\frac{1}{4}\sum_{m\in\left(\bigcup_{0\le k_1\le p}\ I_{k_1}\right)} b_{t_m}\to+\infty$ by Lemma 3.2, and $J_p\to J$.

Hence (3.7) is established.

To prove (3.8), set $k=u(n), k_1=u(m)$ and $G=I_k\cap[n,m]=\{i_1,i_2,...,i_z\}$ where $n\le i_1<i_2<...<i_z\le m$. We have

$$\frac{t_m}{t_n}=\frac{t_m}{t_{i_z}}\ \frac{t_{i_z}}{t_{i_{z-1}}}\ ....\ \frac{t_{i_1}}{t_n}. \tag{3.12}$$

Note that, when $i\in I_k$, we have $t_{i+1}\ge t_i(1+a_{t_i}^{1/\gamma})\ge t_i(1+2^{-k-1})$. Moreover, since $\mathrm{card}(G)>\exp(K_0 2^{k-1})$ by hypothesis, (3.12) implies

$$\frac{t_m}{t_n}\ge\exp\biggl(\exp\Bigl(K_0 2^{k-1}\Bigr)\ \ \log\Bigl(1+2^{-k-1}\Bigr)\biggr)\ge\exp\biggl(\exp\Bigl(\frac{K_0}{4}\ 2^k\Bigr)\biggr), \tag{3.13}$$

when n, hence k, are large enough.

Thus, whenever $k_1\le k+k_0$, (3.13) implies (3.8).

Next assume $k_1>k+k_0$. Since $m\in J,\ m\notin V_n$, and consequently $n\notin F_{n,k_1}$. Thus $\mathrm{card}(I_{k_1}\cap[n,m])>N_{k_1}$. When n, hence m and k_1, are large enough, we have

$$\mathrm{card}(I_{k_1}\cap[n,m])>N_{k_1}=\exp\biggl(K_0\ \Bigl(\gamma\ \log 2\Bigr)^{\beta}\ k_1^{\beta}\ 2^{(\gamma/\alpha)(k_1-1)}\biggr)\ \ge\ \exp\Bigl(K_0\ 2^{k_1-1}\Bigr).$$

Hence, the arguments leading to (3.13) show that

$$\frac{t_m}{t_n}\ge\exp\biggl(\exp\Bigl(\frac{K_0}{4}\ 2^{k_1}\Bigr)\biggr).$$

The proof of Lemma 3.3 is now complete. □

4. Proof of Theorem 1.3 : Part II

Consider now the events $E_n = \{Y(t_n) < f(t_n)\}$. We have directly $\mathbb{P}(E_n) = b_{t_n}$, and $\sum_{n\in J} b_{t_n} = +\infty$, by Lemma 3.3. Therefore our set J satisfies hypothesis (3.3). To verify (3.2), it suffices to prove the following statement

Given $\epsilon > 0$, there exist a number K and an integer p such that

$$\forall n \in J \quad n \geq p \Rightarrow \sum_{m\in J, m>n} \mathbb{P}\Big(E_n \cap E_m\Big) \leq \mathbb{P}(E_n)\,\Big(K + (1+\epsilon) \sum_{m\in J, m>n} \mathbb{P}(E_m)\Big). \tag{4.1}$$

Given $n \in J,$ J can be rewritten as follows $J = J^{'} \cup \Big(\bigcup_{k\in\mathbb{N}} J_k\Big) \cup J^{"}$, where $J^{'} = \{m \in J, t_n \leq t_m \leq 2t_n\},$ $J_k = \{m \in J \cap I_k, t_m > 2t_n, \mathrm{card}(I_k \cap [n,m]) \leq \exp(K_0 2^{k-1})\}$ and $J^{"} = J - \left(J^{'} \cup \Big(\bigcup_{k\in\mathbb{N}} J_k\Big)\right)$.

Our first key small ball estimate is given in the following lemma.

Lemma 4.1. *Consider* $0 < t < u$, *and* $\theta, \nu > 0$. *Then, we have*

$$\mathbb{P}\Big(\{Y(t) \leq \theta t^{\gamma}\} \cap \{Y(u) \leq \nu\}\Big) \leq \exp(K_5)\; \mathbb{P}\Big(Y(t) \leq \theta t^{\gamma}\Big)\; \exp\left(-\frac{K_5(u-t)}{\nu^{1/\gamma}}\right),$$

where K_5 *depends on* H_1, H_2, H_3, λ_1 *and* λ_2 *only.*

Proof. Set $F_1 = \{Y(t) \leq \theta t^{\gamma}\}$ and $F_2 = \{Y(u) \leq \nu\}$. Denote by $[x]$ the integer part of a real x. Let $\delta > 0$. We consider the sequence $t_k, k \in \{0,..,n\}$, where $t_0 = t, t_{k+1} = t_k + \delta$ and $n = [(u-t)/\delta]$. We have

$$\begin{aligned}
\mathbb{P}\Big(F_1 \cap F_2\Big) &= \mathbb{P}\Big(F_1 \cap \{\sup_{0\leq s\leq u}\ \sup_{0\leq w_1,w_2,w_3\leq s} \mid X(w_1,w_2,w_3,s) \mid \leq \nu\}\Big)\\
&\leq \mathbb{P}\Big(F_1 \cap \{\sup_{t\leq s\leq u}\ \sup_{0\leq w_1,w_2,w_3\leq s} \mid X(w_1,w_2,w_3,s) \mid \leq \nu\}\Big)\\
&\leq \mathbb{P}\Big(F_1 \cap \{\sup_{t\leq s\leq u}\ \sup_{0\leq w_1\leq s} \mid \lambda_1\, s^{H_2+H_3}\, B_{H_1}(w_1) \mid \leq \nu\}\Big).
\end{aligned}$$

Let G_k be the event defined by

$$G_k = F_1 \cap \{\sup_{t\leq s\leq t_k}\ \sup_{0\leq w_1\leq s} \mid \lambda_1\, s^{H_2+H_3}\, B_{H_1}(w_1) \mid \leq \nu\}.$$

We have $F_1 \cap F_2 \subset G_k$.

Moreover, we have

$$G_{k+1} \subset G_k \cap \{\mid \lambda_1\, t_{k+1}^{H_2+H_3}\, B_{H_1}(t_{k+1}) - \lambda_1\, t_{k+1}^{H_2+H_3}\, B_{H_1}(t_k) \mid \leq 2\,\nu\}.$$

$X_k := \lambda_1\, t_{k+1}^{H_2+H_3}\, B_{H_1}(t_{k+1}) - \lambda_1\, t_{k+1}^{H_2+H_3}\, B_{H_1}(t_k)$ can be rewritten by (1.1) as follows $X_k = X_{k,1} + X_{k,2}$, where

$$X_{k,1} = \lambda_1\, t_{k+1}^{H_2+H_3}\, k_{2H_1}^{-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - x)^{H_1 - 1/2}\, \tilde{W}(\mathrm{d}x).$$

Note also that $X_{k,1}$ and $X_{k,2}$ are independent.

Since $I\!P\Big(|\ X_{k,1}+x\ |\leq\ 2\,\nu\Big)$ is maximum at $x=0$ and $X_{k,1}$ and G_k are independent, we have

$$I\!P(G_{k+1})\leq I\!P(G_k)\ I\!P\Big(|\ X_{k,1}\ |\leq\ 2\nu\Big).$$

The integral representation of $X_{k,1}$ implies that $I\!E(X_{k,1})=0$ and

$$\mathrm{Var}X_{k,1}=\lambda_1^2\ t_{k+1}^{2(H_2+H_3)}\ k_{2H_1}^{-2}\ \frac{\delta^{2H_1}}{2H_1}\geq\frac{\lambda_1^2}{2H_1\,k_{2H_1}^2}\ \delta^{2\gamma}:=L^2\ \delta^{2\gamma}.$$

Denote by Φ the distribution function of the absolute value of a standard Gaussian random variable. Then, we obtain

$$I\!P(G_{k+1})\leq I\!P(G_k)\ \Phi\left(\frac{2\,\nu}{L\ \delta^\gamma}\right),$$

and therefore $I\!P(F_1\cap F_2)\leq I\!P(F_1)\ \Phi\left(\dfrac{2\,\nu}{L\ \delta^\gamma}\right)^n.$

Choosing $\delta=\nu^{1/\gamma}$, we get $K_5=-\log\Phi(\dfrac{2}{L})$. Lemma 4.1 is proved. □

Lemma 4.2. $\sum_{m\in J'}\ I\!P(E_n\cap E_m)\leq K'\ b_{t_n}$ *and* $\sum_{m\in(\cup_k\ J_k)}\ I\!P(E_n\cap E_m)\leq K"\ b_{t_n}$, *where* K' *and* $K"$ *are numbers.*

Proof. Setting $u=t_m,t=t_n,\theta=a_{t_n}$ and $\nu=f(t_m)$, Lemma 4.1 implies

$$I\!P(E_n\cap E_m)\leq\exp(K_5)\ b_{t_n}\ \exp\left(-\frac{K_5(t_m-t_n)}{f(t_m)^{1/\gamma}}\right). \tag{4.2}$$

Consider first the case when $m\in J^{'}$.

Lemma 3.2 implies that, for all $i\geq n$, we have $t_{i+1}-t_i\geq t_i\,a_{t_i}^{1/\gamma}=f(t_i)^{1/\gamma}\geq f(t_n)^{1/\gamma}$. Then we can establish

$$t_m-t_n\geq(m-n)f(t_n)^{1/\gamma}\quad\text{and}\quad f(t_m)\leq f(t_n)\left(\frac{t_m}{t_n}\right)^\gamma\leq 2^\gamma f(t_n). \tag{4.3}$$

Combining (4.2) with (4.3), we get

$$I\!P(E_n\cap E_m)\leq\exp(K_5)\ b_{t_n}\ \exp\left(-\frac{K_5(m-n)}{2}\right),$$

which is the first part of Lemma 4.2.

Consider now the case $m\in J_k$.

Combining (4.2) with the definition of J_k, we have

$$I\!P(E_n\cap E_m)\leq\exp(K_5)\ b_{t_n}\ \exp\left(-\frac{K_5}{2(a_{t_m})^{1/\gamma}}\right).$$

Since $u(m) = k$, we get

$$\mathbb{P}(E_n \cap E_m) \le \exp(K_5)\, b_{t_n}\ \exp\Big(-K_5\, 2^{k-1}\Big),$$

and consequently by noting that $\mathrm{card} J_k \le \mathrm{card}(I_k \cap [n,m]) \le \exp(K_0 2^{k-1})$ and by assuming $K_0 < K_5$, we have

$$\sum_{m \in J_k} \mathbb{P}(E_n \cap E_m) \le \exp(K_5)\, b_{t_n}\ \exp\Big((K_0 - K_5)\, 2^{k-1}\Big).$$

Hence, Lemma 4.2 is proved. □

To deal with the set J", we recall the following three lemmas (see El-Nouty [9], p. 117, [11], p. 331, [11], p. 323).

Lemma A is a standard large deviation result for the sup-norm of a real-valued Gaussian process.

Lemma A. *Let $\{Z(t), 0 \le t \le 1\}$ be a separable, centered, real-valued Gaussian process such that $Z(0) = 0$, with probability 1, and with incremental variance satisfying*

$$\Big(\mathbb{E}(Z(t+h) - Z(t))^2\Big)^{1/2} \le f(h) \le c_f h^{\beta_1}, \beta_1 > 0.$$

Then, we have for $c_f^{-1}\delta > 1$

$$\mathbb{P}\Big(\sup_{0 \le t \le 1} \mid Z(t) \mid \ge \delta\Big) \le \frac{1}{C}\ \exp\Big(-C(c_f^{-1}\delta)^2\Big),$$

where C is a positive constant independent of c_f and δ.

Lemma B is the analogue of Lemma A for a two-parameter Gaussian process.

Lemma B. *Let $X = \{X(s_1, s_2), (s_1, s_2) \in [0,1]^2\}$ be a separable real-valued centered Gaussian process such that $X(0,0) = 0$ with Probability 1 and satisfying for any $[s_1, s_1 + h_1] \times [s_2, s_2 + h_2] \subset [0,1]^2$*

$$\left(\mathbb{E}X\Big([s_1, s_1 + h_1] \times [s_2, s_2 + h_2]\Big)^2\right)^{1/2} \le \kappa(h_1, h_2) \le c_\kappa\ h_1^{\alpha_1}\ h_2^{\alpha_2},\ \ \alpha_1 > 0, \alpha_2 > 0,$$

where

$$X\Big([s_1, t_1] \times [s_2, t_2]\Big) = X(t_1, t_2) - X(s_1, t_2) - X(t_1, s_2) + X(s_1, s_2),$$

and we write

$$X\Big([s_1, t_1] \times [s_2, t_2]\Big) = \int_{[s_1,t_1]\times[s_2,t_2]} X(\mathrm{d}(u_1, u_2)).$$

Then, we have for $c_\kappa^{-1}\delta > 1$

$$\mathbb{P}\left(\sup_{(s_1,s_2)\in[0,1]^2} \mid X(s_1, s_2) \mid \ge\ \delta\right) \le \frac{1}{C}\ \exp\Big(-C(c_\kappa^{-1}\delta)^2\Big),$$

where C is a positive constant independent of c_κ and δ.

Lemma C derives from the existence of the right derivative of the convex function $\psi = -\log\phi$ and gives sharp bounds for the increments of ϕ.

Lemma C. *We have for $\epsilon_1 > \epsilon/2$ where ϵ is small enough*

$$\exp\left(-K_3\frac{|\,\epsilon_1-\epsilon\,|\,\Big(\log(1/\epsilon)\Big)^{\beta}}{\epsilon^{1+1/\alpha}}\right) \le \frac{\phi(\epsilon_1)}{\phi(\epsilon)} \le \exp\left(K_3\frac{|\,\epsilon_1-\epsilon\,|\,\Big(\log(1/\epsilon)\Big)^{\beta}}{\epsilon^{1+1/\alpha}}\right), \tag{4.4}$$

where $K_3 > 0$.

Building on Lemmas A, B and C, we can establish our last key small ball estimate in the following result.

Lemma 4.3. *Let λ be a real number such that $1/2 < \lambda < 1$. Set*

$$r = \min\left(\frac{1-\max(H_1,H_2,H_3)}{3}, \frac{(1-\lambda)\alpha}{3}\right).$$

Then, we have for $u \ge 2t$

$$\begin{aligned}
I\!P\Big(Y(t)\le\theta t^{\gamma}, Y(u)\le\nu u^{\gamma}\Big) \;\le\;& \phi(\theta)\phi(\nu)\exp\left(2\left(\frac{t}{u}\right)^{r} K_3\left(\frac{\left(\log\dfrac{1}{\theta}\right)^{\beta}}{\theta^{1+(1/\alpha)}}+\frac{\left(\log\dfrac{1}{\nu}\right)^{\beta}}{\nu^{1+(1/\alpha)}}\right)\right)\\
&+\;3\left(\frac{1}{C_{1,2}}\exp\left(-\frac{C_{1,2}}{4\,\lambda_1^2\,K_{H_1,2}^2}\left(\frac{u}{t}\right)^{r}\right)+\frac{1}{C_{23,2}}\exp\left(-\frac{C_{23,2}}{4\,\lambda_2^2\,K_{H_2,2}^2}\left(\frac{u}{t}\right)^{r}\right)\right)\\
&+\;3\left(\frac{1}{C_{1,1}}\exp\left(-\frac{C_{1,1}}{4\,\lambda_1^2\,K_{H_1,1}^2}\left(\frac{u}{t}\right)^{r}\right)+\frac{1}{C_{23,1}}\exp\left(-\frac{C_{23,1}}{4\,\lambda_2^2\,K_{H_2,1}^2}\left(\frac{u}{t}\right)^{r}\right)\right),
\end{aligned}$$

where $K_{H_1,1}, K_{H_1,2} > 0$ depend on H_1 only, $K_{H_2,1}, K_{H_2,2} > 0$ depend on H_2 ($H_2 \le H_3$) only, $K_3 > 0$ is defined as in Lemma C, $C_{1,1}, C_{1,2} > 0$ are defined as in Lemma A and $C_{23,1}, C_{23,2} > 0$ are defined as in Lemma B.

Proof. Set $Q = I\!P\Big(Y(t)\le\theta t^{\gamma}, Y(u)\le\nu u^{\gamma}\Big)$.

Set $v = \sqrt{ut}$. If $t = o(u)$ then $t = o(v)$ and $v = o(u)$.

Using (1.1) and (1.2), we see that B_{H_1} and B_{H_2,H_3} can be split as follows

$$B_{H_1} = B_{H_1,1} + B_{H_1,2} \;\text{ and }\; B_{H_2,H_3} = B_{H_2,H_3,1} + B_{H_2,H_3,2}, \tag{4.5}$$

where

$$B_{H_1,1}(w_1) = \int_{|x_1|\le v} g_{H_1}(w_1,x_1)\,\tilde{W}(\mathrm{d}x_1),$$

and

$$B_{H_2,H_3,1}(w_2,w_3) = \int_{|x_2|\le v}\int_{-\infty}^{w_3} g_{H_2}(w_2,x_2)\,g_{H_3}(w_3,x_3)\,W\Big(\mathrm{d}(x_2,x_3)\Big).$$

Note that $B_{H_1,1}$ and $B_{H_1,2}$ are independent as $B_{H_2,H_3,1}$ and $B_{H_2,H_3,2}$.

(4.5) implies that the FMFBMFBS X can be rewritten as follows $X = X_1 + X_2$ where

$$X_1(w_1, w_2, w_3, s) = \lambda_1 \, s^{H_2+H_3} B_{H_1,1}(w_1) + \lambda_2 \, s^{H_1} B_{H_2,H_3,1}(w_2, w_3).$$

Set

$$Y_i(t) = \sup_{0\le s\le t} \; \sup_{0\le w_1,w_2,w_3\le s} \mid X_i(w_1, w_2, w_3, s) \mid, \quad t \ge 0, \quad i \in \{1, 2\}.$$

Then, given $\delta > 0$, we have (Talagrand [24], pp. 210–211)

$$Q \le \phi(\theta + 2\delta) \; \phi(\nu + 2\delta) + \; 3 \; I\!P\Big(Y_2(t) > \delta t^{\gamma}\Big) + \; 3 \; I\!P\Big(Y_1(u) > \delta u^{\gamma}\Big).$$

(4.4) implies

$$\phi(\theta + 2\delta) \le \phi(\theta) \; \exp\left(2 \; \delta \; K_3 \; \left(\frac{\left(\log \dfrac{1}{\theta}\right)^{\beta}}{\theta^{1+(1/\alpha)}}\right)\right),$$

and consequently

$$\phi(\theta + 2\delta)\phi(\nu + 2\delta) \le \phi(\theta)\phi(\nu) \; \exp\left(2 \; \delta \; K_3 \; \left(\frac{\left(\log \dfrac{1}{\theta}\right)^{\beta}}{\theta^{1+(1/\alpha)}} + \frac{\left(\log \dfrac{1}{\nu}\right)^{\beta}}{\nu^{1+(1/\alpha)}}\right)\right).$$

If we choose $\delta = \left(\dfrac{t}{u}\right)^{r}$, then we get the first term of the RHS of Lemma 4.3.

Next, we want to obtain an upper bound of

$$I\!P\Big(Y_2(t) > \delta t^{\gamma}\Big) = I\!P\Big(\sup_{0\le s\le 1} \; \sup_{0\le w_1,w_2,w_3\le s} \mid \lambda_1 \, s^{H_2+H_3} L_{H_1,2}(w_1) + \lambda_2 \, s^{H_1} L_{H_2,H_3,2}(w_2, w_3) \mid > \delta\Big),$$

where

$$L_{H_1,2}(w_1) = \int_{|x_1|\ge v/t} g_{H_1}(w_1, x_1) \; \tilde{W}(\mathrm{d}x_1),$$

and

$$L_{H_2,H_3,2}(w_2, w_3) = \int_{|x_2|\ge v/t} \int_{-\infty}^{w_3} g_{H_2}(w_2, x_2) \; g_{H_3}(w_3, x_3) \; W\Big(\mathrm{d}(x_2, x_3)\Big).$$

We can show, by standard computations, that

$$I\!P\Big(Y_2(t) > \delta t^{\gamma}\Big) \le I\!P\left(\sup_{0\le w_1\le 1} \mid L_{H_1,2}(w_1) \mid > \frac{\delta}{2 \mid \lambda_1 \mid}\right) + \; I\!P\left(\sup_{0\le w_2,w_3\le 1} \mid L_{H_2,H_3,2}(w_2, w_3) \mid > \frac{\delta}{2 \mid \lambda_2 \mid}\right). \tag{4.6}$$

There exists $K_{H_1,2} > 0$ such that

$$\mathrm{Var}\Big(L_{H_1,2}(w_1 + h_1) - L_{H_1,2}(w_1)\Big) \le K^2_{H_1,2} \; \left(\frac{v}{t}\right)^{2H_1-2} \; h_1^2.$$

Hence, we may apply Lemma A and we get

$$\mathbb{P}\left(\sup_{0\le w_1\le 1} \mid L_{H_1,2}(w_1)\mid > \frac{\delta}{2\mid \lambda_1\mid}\right) \le \frac{1}{C_{1,2}}\ \exp\left(-\frac{C_{1,2}}{K^2_{H_1,2}(v/t)^{2H_1-2}}\ \frac{\delta^2}{4\lambda_1^2}\right). \tag{4.7}$$

Set $\sigma_{H_2,2}$ the covariance function of the process $\{L_{H_2,2}(s), 0\le s\le 1\}$. Since

$$\mathbb{E}\left(L_{H_2,H_3,2}(w_2,w_3)\ L_{H_2,H_3,2}(w_2^{'},w_3^{'})\right) = \sigma_{H_2,2}(w_2,w_2^{'})\ \times\ \sigma_{H_3}(w_3,w_3^{'}),$$

we have for any $[s_2,s_2+h_2]\times[s_3,s_3+h_3]\subset[0,1]^2$

$$\begin{aligned}
&\mathbb{E}\left(L_{H_2,H_3,2}\Big([s_2,s_2+h_2]\times[s_3,s_3+h_3]\Big)^2\right)\\
&= \mathbb{E}\left(\int_{[s_2,s_2+h_2]\times[s_3,s_3+h_3]} L_{H_2,H_3,2}(\mathrm{d}(w_2,w_3)) \times \int_{[s_2,s_2+h_2]\times[s_3,s_3+h_3]} L_{H_2,H_3,2}(\mathrm{d}(w_2^{'},w_3^{'}))\right)\\
&\le \int_{s_2}^{s_2+h_2}\int_{s_2}^{s_2+h_2} \mid \sigma_{H_2,2}(w_2,w_2^{'})\mid\ \mathrm{d}w_2\,\mathrm{d}w_2^{'}\ \times \int_{s_3}^{s_3+h_3}\int_{s_3}^{s_3+h_3}\mid \sigma_{H_3}(w_3,w_3^{'})\mid\ \mathrm{d}w_3\,\mathrm{d}w_3^{'}\\
&:= I\times II.
\end{aligned} \tag{4.8}$$

Consider II first. We get by the inequality of Cauchy-Schwarz

$$II \le \int_{s_3}^{s_3+h_3}\int_{s_3}^{s_3+h_3} w_3^{H_3}\ w_3^{'H_3}\ \mathrm{d}w_3\ \ \mathrm{d}w_3^{'} \le h_3^2. \tag{4.9}$$

Let us turn to I.

A straight computation implies that there exists $K_{H_2,2}>0$ depending on H_2 such that

$$\mathbb{E}\Big(L_{H_2,2}(w_2)\Big)^2 \le K^2_{H_2,2}\ w_2^2\ (v/t)^{2H_2-2},$$

and consequently , by the inequality of Cauchy-Schwarz,

$$\mid \sigma_{H_2,2}(w_2,w_2^{'})\mid \le\ K^2_{H_2,2}\ w_2\ w_2^{'}\ (v/t)^{2H_2-2}.$$

So we get

$$I\ \le\ K^2_{H_2,2}\ (v/t)^{2H_2-2}\ h_2^2. \tag{4.10}$$

Hence, combining (4.8) with (4.9) and (4.10), we have

$$\mathbb{E}\left(L_{H_2,H_3,2}\Big([s_2,s_2+h_2]\times[s_3,s_3+h_3]\Big)^2\right) \le K^2_{H_2,2}\ (v/t)^{2H_2-2}\ h_2^2\ h_3^2.$$

An application of Lemma B with $\alpha_1 = \alpha_2 = 1$, $c_\kappa = K_{H_2,2}(\frac{v}{t})^{H_2-1}$ and $c_\kappa^{-1}\delta > 1$, yields

$$\mathbb{P}\Big(\sup_{0\le w_2,w_3\le 1} \mid L_{H_2,H_3,2}(w_2,w_3)\mid > \frac{\delta}{2\mid\lambda_2\mid}\Big) \le \frac{1}{C_{23,2}}\ \exp\Big(-\frac{C_{23,2}}{K^2_{H_2,2}\Big(\frac{v^2}{t^2}\Big)^{H_2-1}}\,\frac{\delta^2}{4\lambda_2^2}\Big). \tag{4.11}$$

Set $\delta = (\frac{t}{u})^r$. Recall that $v^2 = ut$ and $r \le \dfrac{1-\max(H1,H_2,H_3)}{3}$. Combining (4.6) with (4.7) and (4.11), we get

$$\mathbb{P}\Big(Y_2(t) > \delta t^\gamma\Big) \le \frac{1}{C_{1,2}}\ \exp\Big(-\frac{C_{1,2}}{4\,\lambda_1^2\,K^2_{H_1,2}}\,\Big(\frac{u}{t}\Big)^r\Big) + \frac{1}{C_{23,2}}\ \exp\Big(-\frac{C_{23,2}}{4\,\lambda_2^2\,K^2_{H_2,2}}\,\Big(\frac{u}{t}\Big)^r\Big),$$

which is the second term of the RHS of Lemma 4.3.

Finally, we want to establish a similar result for $\mathbb{P}\Big(Y_1(u) > \delta u^\gamma\Big)$, which is smaller than

$$\mathbb{P}\Big(\sup_{0\le w_1\le 1} \mid L_{H_1,1}(w_1)\mid > \frac{\delta}{2\mid\lambda_1\mid}\Big) + \mathbb{P}\Big(\sup_{0\le w_2,w_3\le 1} \mid L_{H_2,H_3,1}(w_2,w_3)\mid > \frac{\delta}{2\mid\lambda_2\mid}\Big), \tag{4.12}$$

where

$$L_{H_1,1}(w_1) = \int_{|x_1|\le v/u} g_{H_1}(w_1,x_1)\ \tilde{W}(\mathrm{d}x_1),$$

and

$$L_{H_2,H_3,1}(w_2,w_3) = \int_{|x_2|\le v/u}\int_{-\infty}^{w_3} g_{H_2}(w_2,x_2)\ g_{H_3}(w_3,x_3)\ W\Big(\mathrm{d}(x_2,x_3)\Big).$$

Since $1/2 < \lambda < 1$, there exists $K_{H_1,1} > 0$ such that

$$\mathrm{Var}\Big(L_{H_1,1}(w_1+h_1) - L_{H_1,1}(w_1)\Big) \le K^2_{H_1,1}\ \Big(\frac{v}{u}\Big)^{2H_1-2\lambda H_1}\ h_1^{2\lambda H_1}.$$

Hence, we may apply Lemma A and we get

$$\mathbb{P}\Big(\sup_{0\le w_1\le 1} \mid L_{H_1,1}(w_1)\mid > \frac{\delta}{2\mid\lambda_1\mid}\Big) \le \frac{1}{C_{1,1}}\ \exp\Big(-\frac{C_{1,1}}{K^2_{H_1,1}(v/u)^{2H_1-2\lambda H_1}}\,\frac{\delta^2}{4\lambda_1^2}\Big). \tag{4.13}$$

Set $\sigma_{H_2,1}$ the covariance function of the process $\{L_{H_2,1}(s), 0\le s\le 1\}$. Since

$$\mathbb{E}\Big(L_{H_2,H_3,1}(w_2,w_3)\ L_{H_2,H_3,1}(w_2',w_3')\Big) = \sigma_{H_2,1}(w_2,w_2')\ \times\ \sigma_{H_3}(w_3,w_3'),$$

we have for any $[s_2,s_2+h_2]\times[s_3,s_3+h_3]\subset[0,1]^2$

$$\mathbb{E}\Big(L_{H_2,H_3,1}\Big([s_2,s_2+h_2]\times[s_3,s_3+h_3]\Big)^2\Big)$$

$$\le \int_{s_2}^{s_2+h_2}\int_{s_2}^{s_2+h_2} \mid\sigma_{H_2,1}(w_2,w_2')\mid\ \mathrm{d}w_2\,\mathrm{d}w_2'\ \times \int_{s_3}^{s_3+h_3}\int_{s_3}^{s_3+h_3} \mid\sigma_{H_3}(w_3,w_3')\mid\ \mathrm{d}w_3\,\mathrm{d}w_3'.$$

Then, there exists $K_{H_2,1} > 0$ such that

$$\mathbb{E}\bigg(L_{H_2,H_3,1}\Big([s_2, s_2+h_2] \times [s_3, s_3+h_3]\Big)^2\bigg) \le K^2_{H_2,1}\ (v/u)^{2H_2-2\lambda H_2}\ h_2^2\ h_3^2.$$

An application of Lemma B with $\alpha_1 = \alpha_2 = 1$, $c_\kappa = K_{H_2,1}\Big(\dfrac{v}{u}\Big)^{H_2-\lambda H_2}$ and $c_\kappa^{-1}\delta > 1$, yields

$$\mathbb{P}\bigg(\sup_{0\le w_2,w_3\le 1} \mid L_{H_2,H_3,1}(w_2,w_3)\mid > \frac{\delta}{2\mid\lambda_2\mid}\bigg) \le \frac{1}{C_{23,1}}\ \exp\Bigg(-\frac{C_{23,1}}{K^2_{H_2,1}\Big(\dfrac{v}{u}\Big)^{2H_2-2\lambda H_2}}\ \frac{\delta^2}{4\lambda_2^2}\Bigg). \tag{4.14}$$

Set $\delta = \Big(\dfrac{t}{u}\Big)^r$. Recall that $v^2 = ut$ and $r \le \dfrac{(1-\lambda)\,\alpha}{3}$. Combining (4.12) with (4.13) and (4.14), we get

$$\mathbb{P}\Big(Y_1(u) > \delta u^\gamma\Big) \le \frac{1}{C_{1,1}}\ \exp\bigg(-\frac{C_{1,1}}{4\,\lambda_1^2\,K^2_{H_1,1}}\ \Big(\frac{u}{t}\Big)^r\bigg) + \frac{1}{C_{23,1}}\ \exp\bigg(-\frac{C_{23,1}}{4\,\lambda_2^2\,K^1_{H_2,1}}\ \Big(\frac{u}{t}\Big)^r\bigg),$$

which concludes the proof of Lemma 4.3. □

Finally, we state the last technical lemma.

Lemma 4.4. *There exists an integer p such that, if $n > \sup_{s\le p}\Big(\sup I_s\Big)$, then, for $m \in J"$, $m > n$, given $\epsilon > 0$, we have $\mathbb{P}(E_n \cap E_m) \le (1+\epsilon)\ b_{t_n} b_{t_m}$.*

Proof. Let $u(n) = k'$ and $u(m) = k_1$. We have by Lemma 3.3

$$\frac{t_m}{t_n} \ge \exp\bigg(\exp\Big(\frac{K_0}{4}\ 2^{\min(k',k_1)}\Big)\bigg).$$

Let $p \in \mathbb{N}$. Then $k' > p$ and $k_1 > p$. Thus, we have $\min(k', k_1) > p$.

Set $t = t_n, u = t_m, \theta = a_{t_n}$ and $\nu = a_{t_m}$. Note that $\log\frac{1}{\theta} \le \frac{1}{\theta} \le 2^{(k'+1)\alpha}$, $\log\frac{1}{\nu} \le \frac{1}{\nu} \le 2^{(k_1+1)\alpha}$ and $\frac{1}{b_{t_n} b_{t_m}} = \exp(\psi(\theta) + \psi(\nu))$.
By using Lemma 4.3 and letting $p \to +\infty$, we complete the proof of Lemma 4.4. □

Combining Lemma 4.2 and Lemma 4.4, we get (4.1). Since our set J satisfies hypothesis (3.2) and (3.3), (3.4) implies that, given $\epsilon > 0$,

$$\frac{1}{1+2\epsilon} \le \mathbb{P}(\bigcup_{n\in J} E_n) = \mathbb{P}(\bigcup_{n\in J}\{Y(t_n) \le f(t_n)\}),$$

and consequently $f \in LUC(Y)$. The proof of Theorem 1.3 is now complete. □

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