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DISCRETE LUNDBERG-TYPE BOUNDS WITH ACTUARIAL APPLICATIONS

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Abstract. Different kinds of renewal equations repeatedly arise in connection with renewal risk models and variations. It is often appropriate to utilize bounds instead of the general solution to the renewal equation due to the inherent complexity. For this reason, as a first approach to construction of bounds we employ a general Lundberg-type methodology. Second, we focus specifically on exponential bounds which have the advantageous feature of being closely connected to the asymptotic behavior (for large values of the argument) of the renewal function. Finally, the last section of this paper includes several applications to risk theory quantities.

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1. Introduction

Discrete approaches in ruin and risk theory are significantly less developed than their continuous counterparts. An example of a semi-discrete approach represents the paper of [2] where a discrete-time risk model, initially proposed by [1], is discussed. Preference is given there to continuous claim amount distributions. Another example is the continuous-time ruin model with discrete claim amounts considered in [11, 12]. Fully discrete models can be traced back to [4], Chapter XIV, where a general gambling problem is considered. Subsequently, a renewal equation involving the probability of ruin arises. Even under such a simple setup it is preferable to avoid using the general solution. One option is to employ bounds instead. Accordingly, an upper and a lower bound are derived there. More recently, another bounding strategy was proposed by [3] who developed bounds depending on the first few moments of the gambler's gains. More elaborate discrete ruin models may be found in [6, 7, 9]. After deriving a discrete renewal equation satisfied by the Gerber-Shiu discounted penalty function in a particular Sparre Andersen model in [6], the same author finds an explicit expression satisfied by the above function in [7]. Also there, discrete renewal equations related to the surplus immediately before ruin, the deficit at ruin, and the claim causing ruin are obtained. The complexity of the analytic formulae encourages employing bounds instead.

In this paper we consider fully discrete models. Motivated by the fact that several quantities of interest in ruin theory are proved to satisfy certain defective renewal equations, we focus our attention on further investigation of such type of equations. Although [7] suggests a way of expressing the solution to a defective renewal equation in terms of compound geometric distributions, the resulting expressions are still difficult to work with. Consequently, we choose to implement a Lundberg-type bounding approach. (For similar topics on continuous bounds see [19].) Further, the resulting general bounds are simplified by considering a special case, namely, the exponential one. The results are then applied to various quantities of interest in risk theory

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and ruin theory. It is noteworthy that the bounds for these quantities are remarkably simple and require only knowledge of all participating parameters and distributions. Judging by the tightens of the respective bounds in several continuous contexts, one may expect that the same will hold in the analogous discrete cases. It is quite unfortunate though, that a numerical evaluation of the tightness is obstructed by minor differences in the discrete-time models studied in the existing literature.

All distributions considered here are counting and are defined either on the nonnegative integers, $\mathbb{N} = \{0, 1, 2, \ldots\}$, or on the positive integers, $\mathbb{N}_+ = \{1, 2, \ldots\}$. Without loss of generality we assume that none of the discussed distributions are degenerate. Aiming to avoid divergent series and consequently to be able to interchange the order of summation whenever needed, we also assume that all utilized distributions have finite means.

We will employ lowercase letters for the probability mass functions or some general functions and uppercase letters for all other functions: cumulative distribution functions, probability generating functions, etc. A bar above the letter designates the tail distribution. We reserve uppercase curly letters for the corresponding generating functions of given discrete functions. An index e of a particular distribution designates that its equilibrium distribution is considered.

In this paper, we give preference to the following definition of a discrete equilibrium distribution.

Definition 1.1. The equilibrium cumulative distribution function $A_e(x)$ of a random variable X with an arbitrary counting distribution $A(x) = 1 - \bar{A}(x)$ is defined by

$$A_e(x) = \begin{cases} 0, & x = 0 \\ \dfrac{\sum_{i=0}^{x-1} \bar{A}(i)}{\sum_{j=0}^{\infty} \bar{A}(j)}, & x \in \mathbb{N}_+. \end{cases}$$

For the alternative of the above, see for example [20].

Let the cumulative distribution function (c.d.f.) $A(x)$, $x \in \mathbb{N}$, have a respective mass function $a(x) = A(x) - A(x-1)$ with $A(-1) = 0$. It is then easily seen that the equilibrium probability generating function (p.g.f.) $\mathcal{A}_e(z) = \sum_{i=1}^{\infty} z^i \bar{A}(i-1)/\mathbb{E}\{X\} = z[1 - \mathcal{A}(z)]/[(1-z)\mathbb{E}\{X\}]$, $|z| \leq R$ for some radius of convergence $R > 0$.

We intend to implement several discrete reliability classes into the bounds. These are the discrete decreasing failure rate (discrete increasing failure rate) or D-DFR (D-IFR), the discrete new worse than used (discrete new better than used) or D-NWU (D-NBU), the discrete new worse in convex ordering (discrete new better in convex ordering) or D-NWUC (D-NBUC), and the discrete increasing mean residual lifetime (discrete decreasing mean residual lifetime) or D-IMRL (D-DMRL). Whenever a number 2 precedes such a class, it designates that the respective equilibrium distribution belongs to the class. Their definitions along with inclusion properties may be found in Appendices A and B. See also [10, 18].

This paper is organized as follows. We start with a general version of a bounding approach in Section 2 and continue with more specific one in the subsequent Section 3. This more specific version is chosen with regard to its practical implementations. Consequently, the last section is dedicated to applications of the Lundberg-type bounds in a variety of actuarial contexts. First, in Section 4.1 the asymptotic behavior and reliability related bounds are considered with regard to the distributions of the surplus immediately before ruin and the deficit at ruin under the compound binomial risk model. The results obtained are complementary to Corollary 3.2 in [9]. Second, in Section 4.2 implications are discussed with respect to quantities of interest arising in relation to the discrete zero-modified compound geometric distribution considered in more detail in Chapter 3 of [8]. Lastly, Section 4.3 is concerned with applications of the Lundberg-type bounds to the stop-loss premium. Two particular bounds are examined and compared in more detail.

It is noteworthy that the discrete setup is particularly sensitive to some technical details in the definitions of the equilibrium distribution and the reliability classes. These need to be chosen to alleviate the differences of our bounding strategy compared with its continuous analogue.

2. General bounds

In this section we present our bounding approach in its full generality. The approach requires the function being bounded to be the solution to a discrete renewal equation. Although the main result seems very similar to its continuous counterpart, it is obvious that a simple discretization would not lead to a correct statement.

Let $r : \mathbb{N} \to [0, \infty)$ be a locally bounded function and $f(x)$ be a zero-truncated probability function (p.f.) with corresponding c.d.f. $F(x) = 1 - \bar{F}(x)$, $x \in \mathbb{N}_+$, and p.g.f. $\mathcal{F}(z) = \sum_{i=1}^{\infty} z^i f(i)$. We remark that the preference of a zero-truncated distribution is only due to the fact that in most actuarial contexts the functions involved happen to be such. Apart from that, the results derived in this and the following sections hold for non-zero-truncated distributions as well.

Proposition 2.1. *The function*

$$m(x) = \phi \sum_{i=1}^{x} m(x-i)f(i) + \phi r(x), \quad x \in \mathbb{N},\ \phi > 0, \tag{2.1}$$

has general solution

$$m(x) = \sum_{n=1}^{\infty} \phi^{n+1} \sum_{i=1}^{x} r(x-i)f^{*n}(i) + \phi r(x), \quad x \in \mathbb{N},\ \phi > 0, \tag{2.2}$$

where f^{*n} denotes the n-fold convolution of f with itself.

The proof of the above proposition follows by [13], Section 3.5.

Note that when the index of a sum runs from a higher value to a lower value, as it happens in equation (2.1) for $x = 0$, we assume that the sum is empty and is consequently equal to zero.

Since the solution (2.2) has a rather complex form, it is preferable to replace it by bounds. Equation (2.1), on the other hand, suggests how those bounds may be applied to quantities of interest, which are known to satisfy discrete renewal equations.

For our intended bounding approach, it is essential to introduce the **Generalized Lundberg Condition** (GLC):

$$\sum_{j=1}^{\infty} \eta(j)f(j) = \frac{1}{\phi}$$

where $\eta : \mathbb{N} \to [0, \infty)$. In particular, in the exponential case considered in Sections 3 and 4 we will assume that a solution ρ to the Lundberg condition exists. Consequently, we will not discuss the complex issue about non-convergent generating functions.

Proposition 2.2. *Let η satisfy the generalized Lundberg condition and*

$$\eta(y)g(x) \geq (\leq) g(x+y) \quad \textit{for all } x, y \in \mathbb{N}, \tag{2.3}$$

where $g : \mathbb{N} \to [0, \infty)$. If $h : \mathbb{N} \to [0, \infty)$ is such that

$$r(s) \leq (\geq) h(x)g(x-s) \sum_{j=s+1}^{\infty} \eta(j)f(j), \quad 0 \leq s \leq x, \tag{2.4}$$

then $m(x) \leq (\geq) g(0)h(x)$.

Proof. Set $m_l(s) = \sum_{n=1}^{l} \phi^{n+1} \sum_{j=1}^{s} r(s-j) f^{*n}(j) + \phi r(s),\ l \in \mathbb{N}$. Then if we multiply this equation when evaluated at $s-i$ by $f(i)$ and after doing so we sum over i from 1 to $s \in \mathbb{N}_+$, it results in

$$\begin{aligned}
\sum_{i=1}^{s} m_l(s-i) f(i) &= \sum_{i=1}^{s} \left[\sum_{n=1}^{l} \phi^{n+1} \sum_{j=1}^{s-i} r(s-i-j) f^{*n}(j) + \phi r(s-i) \right] f(i) \\
&= \sum_{n=1}^{l} \phi^{n+1} \sum_{i=1}^{s} \sum_{j=1}^{s-i} r(s-i-j) f^{*n}(j) f(i) + \phi \sum_{i=1}^{s} r(s-i) f(i) \\
&= \sum_{n=1}^{l} \phi^{n+1} \sum_{i=1}^{s} \sum_{j=i+1}^{s} r(s-j) f^{*\,n}(j-i) f(i) + \phi \sum_{i=1}^{s} r(s-i) f(i) \\
&= \sum_{n=1}^{l} \phi^{n+1} \sum_{j=2}^{s} r(s-j) \sum_{i=1}^{j-1} f^{*\,n}(j-i) f(i) + \phi \sum_{i=1}^{s} r(s-i) f(i) \\
&= \sum_{n=1}^{l} \phi^{n+1} \sum_{j=2}^{s} r(s-j) f^{*(n+1)}(j) \ + \phi \sum_{i=1}^{s} r(s-i) f(i) \\
&= \sum_{n=2}^{l+1} \phi^{n} \sum_{j=1}^{s} r(s-j) f^{*n}(j) + \ \phi \sum_{i=1}^{s} r(s-i) f(i) \\
&= \sum_{n=1}^{l+1} \phi^{n} \sum_{j=1}^{s} r(s-j) f^{*n}(j), \qquad l \in \mathbb{N}, \qquad (2.5)
\end{aligned}$$

which suggests an iterative relationship for $m_l(s)$, namely,

$$m_{l+1}(s) = \phi \sum_{j=1}^{s} m_l(s-j) f(j) + \phi r(s), \qquad s \in \mathbb{N},\ l \in \mathbb{N}.$$

Note that although to derive identity (2.5) we need s to be strictly positive, the last equation is trivially true for $s = 0$.

Since the proof of the lower bound slightly differs from the proof of the upper bound, we will discuss them separately.

Upper bound. We intend to prove by induction that

$$m_l(s) \le h(x) g(x-s), \quad 0 \le s \le x.$$

In the case when $l = 0$, we apply consecutively equation (2.4) and then the GLC after setting s to zero.

$$m_0(s) = \phi r(s) \le \phi\ h(x) g(x-s) \sum_{j=s+1}^{\infty} \eta(j) f(j) \le h(x) g(x-s).$$

Assume now that the desired inequality is true for $l = n$. To prove the result for $l = n+1$, we implement consecutively equation (2.4), the inductive step, equation (2.3), and the GLC. Thus,

$$m_{n+1}(s) \ \le \phi \sum_{j=1}^{s} h(x) g(x-s+j) f(j) + \phi\ h(x) g(x-s) \sum_{j=s+1}^{\infty} \eta(j) f(j)$$

$$
\begin{aligned}
&\leq \phi h(x)g(x-s)\left[\sum_{j=1}^{s}\eta(j)f(j)+\sum_{j=s+1}^{\infty}\eta(j)f(j)\right]\\
&= h(x)g(x-s), \quad 0\leq s\leq x.
\end{aligned}
$$

Taking the limit we achieve $m(s)=\lim_{l\to\infty} m_l(s)\leq h(x)g(x-s)$ and when $s=x$ the desired result follows.

Lower bound. Set $E(s)=1-\bar{E}(s)=\sum_{j=1}^{s} e(j)=\phi\sum_{j=1}^{s}\eta(j)f(j),\ s\in\mathbb{N}_+$, which itself is a c.d.f., and let $E^{*l}(s)=1-\bar{E}^{*l}(s)$ be the c.d.f. of the l-fold convolution of $E(s)$ with itself. Then

$$
\bar{E}^{*(l+1)}(s)=\sum_{j=1}^{s}\bar{E}^{*l}(s-j)e(j)+\bar{E}(s), \quad s\in\mathbb{N}. \tag{2.6}
$$

Again by induction, we will prove that

$$
m_l(s)\geq h(x)g(x-s)\bar{E}^{*(l+1)}(s), \quad 0\leq s\leq x.
$$

Similarly to the upper bound, when $l=0$

$$
m_0(s)=\phi r(s)\geq \phi\ h(x)g(x-s)\sum_{j=s+1}^{\infty}\eta(j)f(j)=h(x)g(x-s)\bar{E}(s).
$$

Assume now that the desired inequality is true for $l=n$. For $l=n+1$, as proved previously,

$$
\begin{aligned}
m_{n+1}(s)&=\phi\sum_{j=1}^{s}m_n(s-j)f(j)+\phi r(s)\\
&\geq \phi\sum_{j=1}^{s}h(x)g(x-s+j)\bar{E}^{*(n+1)}(s-j)f(j)+\ \phi h(x)g(x-s)\sum_{j=s+1}^{\infty}\eta(j)f(j)\\
&\geq\ h(x)g(x-s)\left[\phi\sum_{j=1}^{s}\eta(j)\bar{E}^{*(n+1)}(s-j)f(j)\ +\bar{E}(s)\right]\\
&= h(x)g(x-s)\bar{E}^{*(n+2)}(s)
\end{aligned}
$$

by (2.6) and the definition of $E(s)$.

Since the renewal function $\sum_{l=1}^{\infty}E^{*l}(s)$ is finite[1], $\lim_{l\to\infty}E^{*l}(s)=0$ or equivalently, $\lim_{l\to\infty}\bar{E}^{*l}(s)=1$. Therefore

$$
m(s)=\lim_{l\to\infty}m_l(s)\geq h(x)g(x-s)\lim_{l\to\ \infty}\bar{E}^{*(l+1)}(s)=h(x)g(x-s), \quad 0\leq s\leq x.
$$

Setting $s=x$, we obtain the desired result. □

Observe that contrary to possible expectations, equation (2.4) above is not a discretized version of its continuous counterpart in [19] as summation starts at $s+1$ rather than s.

The following section provides an important approach for several applications.

[1]See Proposition 3.2.2 in [14].

3. Asymptotic behavior and exponential bounds

This section is dedicated to more specific bounding approaches. It is worth noting that both the definition of equilibrium distribution that we chose in Section 1 and the definitions of the D-NWUC and the D-NBUC classes are well suited for our purposes.

When $\eta(y) = \rho^y,\ y \in \mathbb{N}$, where ρ is a real-valued parameter, the general bounding approach produces more specific and meaningful bounds. Moreover, as it will be seen in Section 4, this simplified version is readily applicable in variety of actuarial contexts.

We begin with a discrete version of a well known asymptotic result. (See [13], Sect. 3.11.)

Proposition 3.1. *Suppose there exists $\rho \geq 1$ satisfying $\mathcal{F}(\rho) = 1/\phi$ and such that $\sum_{j=0}^{\infty} \rho^j r(j) < \infty$. Let the greatest common divisor of the integers j for which $f(j) > 0$ be 1. If $m(x)$ is a bounded function defined by the discrete defective renewal equation* (2.1) *then*

$$m(x) \sim \frac{\sum_{i=0}^{\infty} \rho^i r(i)}{\sum_{j=1}^{\infty} j\rho^j f(j)}\, \rho^{-x} \qquad \text{as } x \to \infty.$$

With the bounding approach in mind we define three auxiliary functions. Let $\underline{x} = \inf\{x : F(x) = 1,\ x \in \mathbb{N}_+\}$ and set

$$\alpha(s) = \frac{\rho^s r(s)}{\sum_{j=s+1}^{\infty} \rho^j f(j)}, \quad s \in \mathbb{N},$$
$$\alpha_U(x) = \sup_{\substack{0\leq s\leq x\\ \bar{F}(s)>0}} \alpha(s),\ x \in \mathbb{N}, \qquad \alpha_L(x) = \inf_{\substack{0\leq s\leq x\\ \bar{F}(s)>0}} \alpha(s),\ \ x \in \mathbb{N}.$$

Utilizing these functions, we are able to derive the first pair of an upper and a lower bound.

Theorem 3.2. *Let $\rho > 1$ satisfy $\mathcal{F}(\rho) = 1/\phi$. Then $m(x) \geq \alpha_L(x)\rho^{-x}$, $x \in \mathbb{N}$. Also, if $r(x) = 0$ for all $x \geq \underline{x}$, then $m(x) \leq \alpha_U(x)\rho^{-x},\ x \in \mathbb{N}$.*

Proof. To utilize Proposition 2.2, we set $\eta(x) = \rho^x$ and $g(x) = \rho^x,\ x \in \mathbb{N}$. Restating the definitions of $\alpha,\ \alpha_U$, and α_L we obtain $r(s) \geq \alpha_U(x)\rho^{-s}\sum_{j=s+1}^{\infty}\rho^j f(j)$ and $r(s) \leq \alpha_L(x)\rho^{-s}\sum_{j=s+1}^{\infty}\rho^j f(j)$ even if $x \geq \underline{x}$. Now we make use of Proposition 2.2 with $h(x) = \alpha_L(x)\rho^{-x}$ and $h(x) = \alpha_U(x)\rho^{-x}$ for the two respective bounds, to obtain the desired inequalities. □

If we split further the functions in the definition of α and let

$$\beta_U(x) = \sup_{\substack{0\leq s\leq x\\ \bar{F}(s)>0}} \frac{r(s)}{\bar{F}(s)},\ x \in \mathbb{N}, \qquad \beta_L(x) = \inf_{\substack{0\leq s\leq x\\ \bar{F}(s)>0}} \frac{r(s)}{\bar{F}(s)},\ \ x \in \mathbb{N},$$

$$\sigma_U(x) = \sup_{\substack{0\leq s\leq x\\ \bar{F}(s)>0}} \frac{\rho^s\bar{F}(s)}{\sum_{j=s+1}^{\infty}\rho^j f(j)},\ \ x \in \mathbb{N}, \qquad \sigma_L(x) = \inf_{\substack{0\leq s\leq x\\ \bar{F}(s)>0}} \frac{\rho^s\bar{F}(s)}{\sum_{j=s+1}^{\infty}\rho^j f(j)},\ \ x \in \mathbb{N},$$

other useful bounds may be derived.

Corollary 3.2.1. *Let $\rho > 1$ satisfy $\mathcal{F}(\rho) = 1/\phi$. Then $m(x) \geq \sigma_L(x)\beta_L(x)\rho^{-x}$, $x \in \mathbb{N}$. Also, if $r(x) = 0$ for all $x \geq \underline{x}$, then $m(x) \leq \sigma_U(x)\beta_U(x)\rho^{-x},\ x \in \mathbb{N}$.*

Proof. Simple implementation of the definitions of $\beta_U(x)$ and $\sigma_U(x)$ results in

$$\alpha_U(x) = \sup_{\substack{0\leq s\leq x\\ \bar{F}(s)>0}} \frac{\rho^s r(s)}{\sum_{j=s+1}^{\infty}\rho^j f(j)} \leq \sup_{\substack{0\leq s\leq x\\ \bar{F}(s)>0}} \frac{\rho^s\bar{F}(s)}{\sum_{j=s+1}^{\infty}\rho^j f(j)} \cdot \sup_{\substack{0\leq s\leq x\\ \bar{F}(s)>0}} \frac{r(s)}{\bar{F}(s)} = \sigma_U(x)\beta_U(x).$$

By analogy, $\alpha_L(x) \geq \sigma_L(x)\beta_L(x)$. The desired result follows by Theorem 3.2. □

Further simplifications are achieved when reliability classes are implemented.

Corollary 3.2.2. *Let ρ satisfy $\mathcal{F}(\rho) = 1/\phi$. If $\phi \geq 1$, then $1/\rho \leq \sigma_L(x) \leq \phi$ and $\sigma_U(x) \geq \phi$. Also, if $F(x)$ is D-NWUC and $x \leq \underline{x}$, $\sigma_L(x) = \phi$ and if $F(x)$ is D-NBUC and $x \leq \underline{x}$, $\sigma_U(x) = \phi$. Similarly, if $\phi \leq 1$, then $\phi \leq \sigma_U(x) \leq 1/\rho$ and $\sigma_L(x) \leq \phi$. Also, if $F(x)$ is D-NWUC and $x \leq \underline{x}$, then $\sigma_U(x) = \phi$ and if $F(x)$ is D-NBUC and $x \leq \underline{x}$, then $\sigma_L(x) = \phi$.*

Proof. Let random variable T_s have c.d.f. $F_s(y) = 1 - \frac{\bar{F}(s+y)}{\bar{F}(s)} = 1 - \bar{F}_s(y)$, $s \in \mathbb{N}, y \in \mathbb{N}$, p.f. $f_s(y) = f(s+y)/\bar{F}(s)$, and p.g.f. $\mathcal{F}_s(z) = \sum_{i=1}^{\infty} z^i f_s(i)$. Then,

$$\mathcal{F}_s(\rho) = \sum_{j=1}^{\infty} \rho^j \frac{f(s+j)}{\bar{F}(s)} = \sum_{i=s+1}^{\infty} \rho^{(i-s)} \frac{f(i)}{\bar{F}(s)} = \frac{\sum_{i=s+1}^{\infty} \rho^i f(i)}{\rho^s \bar{F}(s)}.$$

There are two possibilities for ϕ. Consider first the case when $\phi \geq 1$. By the GLC

$$\mathcal{F}(\rho) = \frac{1}{\phi} \leq 1 = \mathcal{F}(1).$$

Since $\mathcal{F}$ is a monotone function, $\rho \leq 1$, the implication being that

$$\mathcal{F}_s(\rho) = \frac{\sum_{j=s+1}^{\infty} \rho^j f(j)}{\rho^s \bar{F}(s)} \leq \frac{\rho^{(s+1)} \bar{F}(s)}{\rho^s \bar{F}(s)} = \rho.$$

Hence, $\sigma_L(x) \geq 1/\rho$ as needed.

Moreover, $\mathcal{F}_0(\rho) = 1/\phi$ by the GLC. Therefore $\sigma_L(x) \leq 1/\mathcal{F}_0(\rho) = \phi$. Now, let $F(x)$ be D-NWUC and $x \leq \underline{x}$. Then equivalent representations lead to

$$\sum_{i=x+y}^{\infty} \bar{F}(i) \geq \bar{F}(x) \sum_{j=y}^{\infty} \bar{F}(j),$$
$$\frac{\sum_{i=y}^{\infty} \bar{F}(x+i)}{\bar{F}(x)} \geq \frac{\sum_{j=y}^{\infty} \bar{F}(j)}{\bar{F}(0)}, \qquad x \leq \underline{x},$$
$$\sum_{i=y}^{\infty} \mathbb{P}\{T_x > i\} \geq \sum_{j=y}^{\infty} \mathbb{P}\{T_0 > j\},$$

i.e., by definition, T_0 is smaller than T_x in convex ordering. This in turn implies that for any decreasing convex function $\xi(x)$ one has $\mathbb{E}\{\xi(T_x)\} \leq \mathbb{E}\{\xi(T_0)\}$[2]. We have already established that $\rho \leq 1$. Therefore $\mathcal{F}_x(\rho) = \mathbb{E}\left\{\rho^{T_x}\right\} \leq \mathbb{E}\left\{\rho^{T_0}\right\} = \mathcal{F}_0(\rho) = 1/\phi$ by the GLC, *i.e.* $1/\mathcal{F}_x(\rho) \geq \phi$. Therefore $\sigma_L(x) \geq \phi,\ x \leq \underline{x}$, but we proved earlier that $\sigma_L(x) \leq \phi$ and hence $\sigma_L(x) = \phi, x \leq \underline{x}$.

If $F(x)$ is D-NBUC and $x \leq \underline{x}$, all inequalities will be reversed and $1/\mathcal{F}_x(\rho) \leq \phi$. Hence $\sigma_U(x) \leq \phi,\ x \leq \underline{x}$, but $\sigma_U(x) \geq \phi$, leading to $\sigma_U(x) = \phi,\ x \leq \underline{x}$.

When $\phi \leq 1$, the proof is similar to the one above. □

We are now ready to proceed with applying the above results.

[2]See Theorem 3.A.1 in [15].

4. Applications of the Lundberg-type bounds

The following subsections consider several applications to risk theory and ruin theory. In particular, the results obtained so far are applied to the zero-modified compound geometric tail and its residual lifetime tail, suggesting complementary bounds to the ones in [8], Section 3.2. Also, in an aggregate claims context, bounds for the surplus before ruin, the deficit at ruin, and the stop-loss premium are derived in Sections 4.1 and 4.3.

4.1. The surplus before ruin and the deficit at ruin

In this subsection we discuss the compound binomial model, considered to be the discrete counterpart of the classical compound Poisson model, and its implications to the surplus immediately before ruin and the deficit at ruin. It is worth noting that the Lundberg condition differs from its continuous analogue and that the same applies to part but not all asymptotic results and bounds. Moreover, consistently with Theorem 4.1 in [9], the Lundberg condition required for deriving the necessary defective renewal equation coincides with the respective requirement in our bounding approach.

Consider the following discrete time ruin model known as the compound binomial model. Suppose that the number of claims arriving at an insurance company is modeled by a discrete renewal process $\{N(t);\, t \in \mathbb{N}\}$ such that the independent and identically distributed (i.i.d.) interclaim times $\{V_1, V_2, \ldots\}$ have a common zero-truncated geometric p.f. $k(j) = (1-\mu)\mu^{j-1},\ j \in \mathbb{N}_+, \mu \in (0,1)$. Suppose further that the individual claim amounts $\{L_1, L_2, \ldots\}$ are i.i.d., independent of $\{V_1, V_2, \ldots\}$, and have a zero-truncated c.d.f. $P(j) = 1-\bar{P}(j),\ j \in \mathbb{N}_+$. Starting with an initial surplus $u \in \mathbb{N}$, the surplus process is defined by $U(t) = u + t - \sum_{i=0}^{N(t)} L_i$ where the premium rate is assumed to be 1 and the insurer charges a relative security loading $\theta > 0$ such that $(1+\theta)\mathbb{E}\{L_1\} = \mathbb{E}\{V_1\}$.

If $T = \inf\{t : U(t) < 0\}$, ruin is considered to occur (in finite time) whenever $T < \infty$. When this is the case, $U(T-)$ is the surplus immediately before ruin and $|U(T)|$ is the amount of the deficit at ruin.

Under the assumptions of the compound binomial model the distributions of both defective cumulative distribution functions of the surplus before ruin, x, and the deficit at ruin, y, satisfy discrete defective renewal equations with respect to the initial surplus u. Those equations may be obtained as limiting cases of the joint defective c.d.f. $F_*(x, y|u) = \mathbb{P}\{U(T-) \le x, |U(T)| \le y, T < \infty | U(0) = u\}$. More specifically, the respective defective renewal equation in [9], p. 275, is

$$
\begin{aligned}
F_*(x, y|u) &= \frac{1}{1+\theta} \sum_{i=0}^{u} F_*(x, y|u-i) p_e(i+1) \\
&\quad + \frac{I\{u < x\}}{1+\theta} \left[\bar{P}_e(u+1) - \bar{P}_e(x+1) - \bar{P}_e(u+y+1) + \bar{P}_e(x+y+1)\right], \qquad x, y, \in \mathbb{N}_+,\ u \in \mathbb{N},
\end{aligned}
$$

where $I\{E\}$ is the indicator function of an event E, equal to 1 when the event occurs and to 0 otherwise. Also, $p_e(i+1) = \bar{P}(i)/\sum_{j=0}^{\infty} \bar{P}(j)$ and $\bar{P}_e(u) = \sum_{i=u+1}^{\infty} p_e(i),\ i \in \mathbb{N}_+$, are the equilibrium p.f. and the equilibrium tail of the claim-size distribution, respectively. Thus,

$$
\begin{aligned}
F_{*,X}(x|u) &= \mathbb{P}\left\{U(T-) \le x, T < \infty \middle| U(0) = u \right\} = \lim_{y \to \infty} F_*(x, y, |u) \\
&= \frac{1}{1+\theta} \sum_{i=0}^{u} F_{*,X}(x|u-i) p_e(i+1) + \frac{I\{u < x\}}{1+\theta} [\bar{P}_e(u+1) - \bar{P}_e(x+1)], \quad x \in \mathbb{N}_+,\ u \in \mathbb{N},
\end{aligned}
$$

and

$$F_{*,Y}(y|u) = \mathbb{P}\left\{|U(T)| \le y, T < \infty \middle| U(0) = u\right\} = \lim_{x\to\infty} F_*(x, y, |u)$$
$$= \frac{1}{1+\theta}\sum_{i=0}^{u} F_{*,Y}(y|u-i)p_e(i+1) + \frac{1}{1+\theta}[\bar{P}_e(u+1) - \bar{P}_e(u+y+1)], \quad y \in \mathbb{N}_+,\ u \in \mathbb{N}.$$

Now we are able to apply the results from Section 3. When the conditions of Proposition 3.1 apply, the two asymptotic formulae are

$$F_{*,X}(x|u) \sim \frac{\sum_{i=0}^{x-1} \rho^i[\bar{P}_e(i+1) - \bar{P}_e(x+1)]}{\sum_{j=1}^{\infty} j\rho^j p_e(j+1)}\,\rho^{-u} \qquad \text{as } u \to \infty$$

and

$$F_{*,Y}(y|u) \sim \frac{\sum_{i=0}^{\infty} \rho^i[\bar{P}_e(i+1) - \bar{P}_e(i+y+1)]}{\sum_{j=1}^{\infty} j\rho^j p_e(j+1)}\,\rho^{-u} \qquad \text{as } u \to \infty$$

for fixed $x, y \in \mathbb{N}_+$. Here ρ is the positive solution to the Lundberg condition $\sum_{i=1}^{\infty} \rho^i p_e(i) = (1+\theta)\rho$.

Consider the ratios $r(s)/\bar{F}(s)$ for $F_{*,X}(x|u)$ and $F_{*,Y}(y|u)$. Note that

$$\frac{r(s)}{\bar{F}(s)} = \frac{I\{s<x\}[\bar{P}_e(s+1) - \bar{P}_e(x+1)]}{\bar{P}_e(s+1)} = I\{s<x\}\left[1 - \frac{\bar{P}_e(x+1)}{\bar{P}_e(s+1)}\right]$$

and

$$\frac{r(s)}{\bar{F}(s)} = \frac{\bar{P}_e(s+1) - \bar{P}_e(s+y+1)}{\bar{P}_e(s+1)} = 1 - \frac{\bar{P}_e(s+y+1)}{\bar{P}_e(s+1)}$$

for the two respective cumulative distribution functions.

If P is 2-D-NBU, *i.e.* its equilibrium distribution is D-NBU, $\bar{P}_e(x+1)/\bar{P}_e(s+1) \le \bar{P}_e(x-s)$ and $\bar{P}_e(s+y+1)/\bar{P}_e(s+1) \le \bar{P}_e(y)$. Therefore $\beta_L(u) \le P_e(x)$ and $\beta_L(u) \le P_e(y)$. Corollary 3.2.1 together with Corollary 3.2.2 result in

$$F_{*,X}(x|u) \ge \frac{P_e(x)}{1+\theta}\,\rho^{-u}, \qquad x \in \mathbb{N}_+,\ u \in \mathbb{N},$$

and

$$F_{*,Y}(y|u) \ge \frac{P_e(y)}{1+\theta}\,\rho^{-u}, \qquad y \in \mathbb{N}_+,\ u \in \mathbb{N},$$

as far as there exists $\rho \ge 1$ satisfying $\sum_{j=1}^{\infty} \rho^j p_e(j) = (1+\theta)\rho$.

It is interesting to note that both defective distributions of the surplus before ruin and the deficit at ruin satisfy the same lower bounds.

Next, we derive upper and lower bounds assuming a stronger condition. On one hand, Corollary 3.2.2 requires P to be 2-D-NBUC in order to have $\sigma_L(u) = 1/(1+\theta)$. On the other hand, if P is assumed to be 2-D-IFR this condition is provided (see Fig. 1 in Appendix B). When the latter applies, $\bar{P}_e(x+1)/\bar{P}_e(s+1)$ and $\bar{P}_e(s+y+1)/\bar{P}_e(s+1)$ are non-increasing and hence $\beta_L(u) = 1 - \bar{P}_e(x+1)/\bar{P}_e(1)$ or $\beta_L(u) = 1 - \bar{P}_e(y+1)/\bar{P}_e(1)$ depending on the c.d.f. considered. Also, $\beta_U(u) = I\{u<x\}[1 - \bar{P}_e(x+1)/\bar{P}_e(u+1)]$ or $\beta_U(u) = 1 - \bar{P}_e(u+y+1)/\bar{P}_e(u+1)$. Then

$$\frac{1}{1+\theta}\left[1 - \frac{\bar{P}_e(x+1)}{\bar{P}_e(1)}\right]\rho^{-u} \le F_{*,X}(x|u) \le I\{u<x\}\left[1 - \frac{\bar{P}_e(x+1)}{\bar{P}_e(u+1)}\right]\rho^{-(u+1)},$$
$$x \in \mathbb{N}_+,\ u \in \{0, 1, ..., \underline{x}\},$$

and

$$\frac{1}{1+\theta}\left[1-\frac{\bar{P}_e(y+1)}{\bar{P}_e(1)}\right]\rho^{-u} \leq F_{*,Y}(y|u) \leq \left[1-\frac{\bar{P}_e(u+y+1)}{\bar{P}_e(u+1)}\right]\rho^{-(u+1)},$$
$$y \in \mathbb{N}_+,\ u \in \{0,1,...,\underline{x}\}.$$

Although similar reasoning may be utilized when P is 2-D-NWU, it is also true that for an arbitrary c.d.f. P, $\beta_U(u) = 1-\bar{P}_e(x+1)/\bar{P}_e(1)$. Now, assuming P to be only 2-D-NWUC

$$F_{*,X}(x|u) \leq \frac{1}{1+\theta}\left[1-\frac{\bar{P}_e(x+1)}{\bar{P}_e(1)}\right]\rho^{-u}, \quad x \in \mathbb{N}_+,\ u \in \mathbb{N}.$$

Therefore this time the additional condition of P being 2-D-NWU benefits only a bound for $F_{*,Y}(y|u)$, where $\beta_U(u) \leq P_e(y)$ and hence

$$F_{*,Y}(y|u) \leq \frac{P_e(y)}{1+\theta}\rho^{-u}, \quad y \in \mathbb{N}_+,\ u \in \mathbb{N},$$

where $\rho \geq 1$ is such that $\sum_{j=1}^{\infty}\rho^j p_e(j) = (1+\theta)\rho$.

A refinement of the above inequality is obtained when P is 2-D-DFR. In this case $\bar{P}_e(s+y+1)/\bar{P}_e(s+1)$ is non-decreasing in s and hence the respective supremum is $\beta_U(u) = 1-\bar{P}_e(y+1)/\bar{P}_e(1)$. Therefore

$$F_{*,Y}(y|u) \leq \frac{1}{1+\theta}\left[1-\frac{\bar{P}_e(y+1)}{\bar{P}_e(1)}\right]\rho^{-u}, \quad y \in \mathbb{N}_+,\ u \in \mathbb{N}.$$

Note that similar results may be deduced for the claim causing ruin starting with the defective renewal equation provided by Corollary 4.2 in [9]. More generally, with the help of Theorem 4.1 there, analogous arguments are applicable to the Gerber-Shiu function, which is a generalization of several quantities of interest in ruin theory. For more detail, see for instance [5].

4.2. The zero-modified discrete compound geometric model

The tail behavior of the discrete zero-modified compound geometric distribution is examined in connection with quantities of interest in risk theory. A classical illustrative example is provided by the compound binomial model where the probability of ultimate ruin, $\psi(u) = \mathbb{P}\{T < \infty | U(0) = u\}, u \in \mathbb{N}$, is a compound geometric tail with compounding distribution the shifted equilibrium distribution of the claim amounts. The results are complementary to related work in [8]. They are also applicable to the Sparre Andersen model utilized by [6] and the analysis of stop-loss moments associated with discrete modified compound geometric distributions in an aggregate loss context.

We first turn our attention to examining the asymptotic behavior of the zero-modified compound geometric distribution H_α and its residual lifetime c.d.f. $H_{\alpha,x}$. The c.d.f. $H_\alpha(x) = \sum_{i=0}^{x} h_\alpha(i)$ is defined through its tail

$$\bar{H}_\alpha(x) = \sum_{n=1}^{\infty}(1-\alpha)(1-\phi)\phi^{n-1}\bar{F}^{*\ n}(x), \quad x \in \mathbb{N},\ \alpha \in [0,1),\ \phi \in (0,\ 1),$$

which satisfies a defective renewal equation

$$\bar{H}_\alpha(x) = \phi\sum_{i=1}^{x}\bar{H}_\alpha(x-i)f(i) + (1-\alpha)\bar{F}(x), \quad x \in \mathbb{N}. \tag{4.1}$$

The residual lifetime c.d.f. $H_{\alpha,x}(y) = 1 - \bar{H}_{\alpha,x}(y) = 1 - \bar{H}_\alpha(x+y)/\bar{H}_\alpha(x)$, satisfies

$$\bar{H}_{\alpha,x}(y) = \phi \sum_{l=1}^{y} \bar{H}_{\alpha,x}(y-l) f(l) + \phi \bar{F}(y) + (1-\phi)\bar{G}_x(y), \quad x, y \in \mathbb{N},$$

where

$$\bar{G}_x(y) = \frac{\sum_{i=0}^{x} \bar{F}(x+y-i) h_{1-\phi}(i)}{\sum_{j=0}^{x} \bar{F}(x-j) h_{1-\phi}(j)}.$$

(See [17], Prop. 2.2 and Th. 2.1.) Under the compound binomial model, $\bar{G}_{u+1}(y) = \mathbb{P}\left\{|U(T)| > y \middle| U(0) = u, T < \infty\right\} = 1 - F_{*,Y}(y|u)/\psi(u)$ is the proper tail distribution of the deficit at ruin when the initial surplus is u and ruin is known to occur. We first turn our attention to examining the asymptotic behavior of H_α and $H_{\alpha,x}$. To apply Proposition 3.1, we need only the existence of $\rho \geq 1$ satisfying $\mathcal{F}(\rho) = 1/\phi$ and F to be non-arithmetic. The condition $\sum_{j=0}^{\infty} \rho^j r(j) < \infty$ is indirectly implied by the GLC. More specifically, when $\bar{H}_\alpha$ is considered, $r(x) = \frac{1-\alpha}{\phi} \bar{F}(x)$ and by summation by parts

$$\mathcal{F}(\rho) = \sum_{j=1}^{\infty} \rho^j f(j) = (\rho - 1) \sum_{j=0}^{\infty} \rho^j \bar{F}(j) + \bar{F}(0) < \infty.$$

When $\bar{H}_{\alpha,x}(y)$ is examined, $r(y) = \bar{F}(y) + \frac{1-\phi}{\phi} \bar{G}_x(y)$ and

$$\begin{aligned}
\sum_{j=0}^{\infty} \rho^i \bar{G}_x(j) &= \frac{\sum_{j=0}^{\infty} \rho^j \sum_{i=0}^{x} \bar{F}(x+j-i) h_{1-\phi}(i)}{\sum_{j=0}^{x} \bar{F}(x-j) h_{1-\phi}(j)} \\
&= \frac{\sum_{i=0}^{x} h_{1-\phi}(i) \sum_{j=0}^{\infty} \rho^j \bar{F}(x+j-i)}{\sum_{j=0}^{x} \bar{F}(x-j) h_{1-\phi}(j)} \\
&= \frac{\sum_{i=0}^{x} \rho^{i-x} h_{1-\phi}(i) \sum_{j=x-i}^{\infty} \rho^j \bar{F}(j)}{\sum_{j=0}^{x} \bar{F}(x-j) h_{1-\phi}(j)} < \infty.
\end{aligned}$$

Consequently,

$$\bar{H}_\alpha(x) \sim \frac{1-\alpha}{\phi} \cdot \frac{\sum_{i=0}^{\infty} \rho^i \bar{F}(i)}{\sum_{j=1}^{\infty} j \rho^j f(j)} \cdot \rho^{-x} \quad \text{as } x \to \infty,$$

and

$$\bar{H}_{\alpha,x}(y) \sim \frac{\sum_{i=0}^{\infty} \rho^i \bar{F}(i) + \frac{1-\phi}{\phi} \sum_{i=0}^{\infty} \rho^i \bar{G}_x(i)}{\sum_{j=1}^{\infty} j \rho^j f(j)} \cdot \rho^{-y} \quad \text{as } y \to \infty.$$

Further, we obtain bounds by applying Corollaries 3.2.1 and 3.2.2. If $F(x)$ is D-NWUC (D-NBUC) and $x \leq \underline{x}$, then $\sigma_U(x) = \phi$ ($\sigma_L(x) = \phi$). When $\bar{H}_\alpha$ is considered, $\beta_U(x) = \beta_L(x) = (1-\alpha)/\phi$ and hence

$$\bar{H}_\alpha(x) \leq (\geq) \, (1-\alpha)\rho^{-x}, \quad x \in \mathbb{N}. \tag{4.2}$$

In other words, the zero-modified compound geometric tail is bounded by a zero-modified geometric tail.

When $\bar{H}_{\alpha,x}(y)$ is examined, more explicit bounds may be obtained in the case of $F(y)$ being D-DFR (D-IFR) and $y \leq \underline{x}$. Then F is D-NWUC (D-NBUC) as well[3], i.e. $\sigma_U(y) = \phi$ ($\sigma_L(y) = \phi$) is still true. Moreover,

$$\frac{r(s)}{\bar{F}(s)} = 1 + \frac{1-\phi}{\phi} \cdot \frac{\sum_{i=0}^{x} \frac{\bar{F}(s+x-i)}{\bar{F}(s)} h_{1-\phi}(i)}{\sum_{i=0}^{x} \bar{F}(x-i) h_{1-\phi}(i)}.$$

[3]See Figures 1 and 2 in [10] and note that D-IFR implies D-NBU for zero-truncated distributions.

Therefore

$$\beta_U(y) = \left.\frac{r(s)}{\bar{F}(s)}\right|_{s=y} = 1 + \frac{1-\phi}{\phi} \cdot \frac{\sum_{i=0}^{x} \bar{F}_y(x-i)h_{1-\phi}(i)}{\sum_{i=0}^{x} \bar{F}(x-i)h_{1-\phi}(i)}$$

$$\left(\beta_L(y) = \left.\frac{r(s)}{\bar{F}(s)}\right|_{s=y} = 1 + \frac{1-\phi}{\phi} \cdot \frac{\sum_{i=0}^{x} \bar{F}_y(x-i)h_{1-\phi}(i)}{\sum_{i=0}^{x} \bar{F}(x-i)h_{1-\phi}(i)}\right).$$

Finally,

$$\begin{aligned}\bar{H}_{\alpha,x}(y) \quad &\le (\ge) \quad \phi\left[1 + \frac{1-\phi}{\phi} \cdot \frac{\sum_{i=0}^{x} \bar{F}_y(x-i)h_{1-\phi}(i)}{\sum_{i=0}^{x} \bar{F}(x-i)h_{1-\phi}(i)}\right]\rho^{-y} \\ &= \quad \left[\phi + (1-\phi)\frac{\sum_{i=0}^{x} \bar{F}_y(x-i)h_{1-\phi}(i)}{\sum_{i=0}^{x} \bar{F}(x-i)h_{1-\phi}(i)}\right]\rho^{-y}, \quad x, y \in \mathbb{N}.\end{aligned}$$

Luckily, for zero-truncated distributions, which F happens to be, D-DFR (D-IFR) implies D-NWU (D-NBU), and the above bounds for $\bar{H}_\alpha$ and $\bar{H}_{\alpha,x}$ are complementary to the ones given by Corollary 3.17.3 and Corollary 3.17.1 in [8] respectively.

Moreover, these bounds involve mixtures of distributions, which is a convenient representation for several distributions widely employed in actuarial modeling.

4.3. **The stop-loss premium**

As a continuation of the previous subsection, we consider again the zero-modified compound geometric distribution H_α. This time, we pay closer attention to the mean residual lifetime of the “claim-size” c.d.f. F and the stop-loss premium related to H_α. In two different ways we derive upper bounds of the latter. Moreover, the tightness of these two bounds is compared. Related results can be found in [16].

Let X_α be a random variable with c.d.f. H_α. Summing from $x = 0$ to ∞ the discrete renewal equation (4.1), one may find a connection between $\mathbb{E}\{X_\alpha\}$ and the **mean residual lifetime** $r_F(x) = \frac{\sum_{i=x}^{\infty}(i-x)f(i)}{\bar{F}(x)}$ of F :

$$\begin{aligned}\sum_{x=0}^{\infty} \bar{H}_\alpha(x) = &\sum_{x=0}^{\infty} \phi \sum_{i=1}^{x} \bar{H}_\alpha(x-i)f(i) + (1-\alpha)\sum_{x=0}^{\infty} \bar{F}(x) \\ &= \phi \sum_{i=1}^{\infty} f(i) \sum_{x=i}^{\infty} \bar{H}_\alpha(x-i) + (1-\alpha)\sum_{x=0}^{\infty} \bar{F}(x),\end{aligned}$$

or equivalently,

$$\mathbb{E}\{X_\alpha\} = \sum_{x=0}^{\infty} \bar{H}_\alpha(x) = \frac{(1-\alpha)\sum_{x=0}^{\infty} \bar{F}(x)}{1-\phi} = \frac{1-\alpha}{1-\phi} r_F(0). \tag{4.3}$$

Also, $\mathbb{E}\{X_\alpha\} = \frac{1-\alpha}{1-\phi}\mathbb{E}\{Y\}$, where Y is a random variable with c.d.f. F.

Let $R_\alpha(x) = \mathbb{E}\{(X_\alpha - x)_+\} = \sum_{i=x+1}^{\infty}(i-x)h_\alpha(i)$ be the **stop-loss premium**, corresponding to the c.d.f. H_α. Then by summation by parts,

$$R_\alpha(x) = \sum_{i=x+1}^{\infty} (i-x)h_\alpha(i) = \sum_{i=x}^{\infty} \bar{H}_\alpha(i) = \mathbb{E}\{X_\alpha\}\bar{H}_e(x). \tag{4.4}$$

Here we denoted by $\bar{H}_e$ the equilibrium tail of H_α. Since for any $\alpha \in [0, 1)$ the equilibrium distribution is the same, we prefer to simplify the notation and write $\bar{H}_e(x) = 1 - H_e(x)$ instead of $\bar{H}_{e,\alpha}(x) = 1 - H_{e,\alpha}(x)$.

The first upper bound of the stop-loss premium is obtained in the following proposition.

Proposition 4.1. *Let $\rho > 1$ satisfy $\mathcal{F}(\rho) = 1/\phi$ and F be D-NWUC (D-NBUC and $x \le \underline{x}$). Then*

$$R_\alpha(x) \le (\ge) \frac{1-\alpha}{\rho - 1}\rho^{-(x-1)}, \quad x \in \mathbb{N}. \tag{4.5}$$

Proof. First, since F is D-NWUC (D-NBUC), its equilibrium tail $\bar{F}_e$ satisfies $\bar{F}_e(x+j) \ge (\le) \bar{F}_e(x)\bar{F}(j)$, which is equivalent to

$$\frac{\bar{F}_e(x+j)}{\bar{F}_e(x)} \ge (\le)\ \frac{\bar{F}(j)}{\bar{F}(0)}.$$

Therefore

$$\frac{\sum_{j=y}^{\infty} \bar{F}_e(x+j)}{\bar{F}_e(x)} \ge (\le) \frac{\sum_{j=y}^{\infty} \bar{F}(j)}{\bar{F}(0)}.$$

Consequently, the random variable $T_{e,x}$ with c.d.f. $F_{e,x}(j) = 1 - \frac{\bar{F}_e(x+j)}{\bar{F}_e(x)} = 1 - \bar{F}_{e,x}(j)$ and p.f. $f_{e,x}(j) = \frac{f_e(x+j)}{\bar{F}_e(x)}$, $j \in \mathbb{N}_+$, and the random variable T_0 having c.d.f. $F_0(j) = 1 - \frac{\bar{F}(j)}{\bar{F}(0)} = 1 - \bar{F}_0(j)$ are such that

$$\sum_{j=y}^{\infty} \bar{F}_{e,x}(j) \ge (\le) \sum_{j=y}^{\infty} \bar{F}_0(j).$$

Then by definition, $T_{e,x}$ is larger (smaller) in increasing convex ordering than T_0. Hence, by Theorem 3.A.1 in [15], for any positive valued increasing convex function ξ, $\mathbb{E}\{\xi(T_{e,x})\} \ge (\le) \mathbb{E}\{\xi(T_0)\}$, or equivalently,

$$\frac{1}{\mathbb{E}\{\xi(T_{e,x})\}} \le (\ge)\ \frac{1}{\mathbb{E}\{\xi(T_0)\}}.$$

We set $\xi(x) = \rho^x,\ x \in \mathbb{N}_+$. Thus the above leads to

$$\begin{aligned} \frac{1}{\mathbb{E}\{\xi(T_{e,x})\}} \quad &= \quad \frac{1}{\sum_{i=1}^{\infty} \rho^i \frac{f_e(x+i)}{\bar{F}_e(x)}} = \frac{\rho^x \bar{F}_e(x)}{\sum_{j=x+1}^{\infty} \rho^j f_e(j)} \\ &\le (\ge) \quad \frac{1}{\mathbb{E}\{\xi(T_0)\}} = \frac{1}{\sum_{j=1}^{\infty} \rho^j f(j)} = \phi. \end{aligned}$$

Hence

$$\bar{F}_e(x) \le (\ge) \phi \rho^{-x} \sum_{j=x+1}^{\infty} \rho^j f_e(j). \tag{4.6}$$

Second, by summation by parts

$$\sum_{i=x+1}^{\infty} \rho^i f(i) = \rho^x \bar{F}(x) + (\rho - 1) \sum_{i=x}^{\infty} \rho^i \bar{F}(i). \tag{4.7}$$

When $x = 0$, the above equation becomes

$$\begin{aligned} \frac{1}{\phi} &= 1 + \ (\rho - 1)\mathbb{E}\{Y\} \sum_{i=0}^{\infty} \rho^i f_e(i+1) \\ &= 1 + \frac{\rho - 1}{\rho} \mathbb{E}\{Y\} \sum_{j=1}^{\infty} \rho^j f_e(j), \end{aligned}$$

which may be equivalently presented as

$$\mathcal{F}_e(\rho) = \sum_{j=1}^{\infty} \rho^j f_e(j) = \frac{\rho\,(1-\phi)}{\phi(\rho-1)\mathbb{E}\{Y\}} < \infty. \tag{4.8}$$

Third, set $\alpha = 1-\phi$ in (4.2) to obtain

$$\bar{H}_{1-\phi}(x) \leq (\geq) \phi\rho^{-x}, \quad x \in \mathbb{N}.$$

Therefore starting with Remark 7, p. 147, in [8] and consequently applying (4.6) and (4.8), the equilibrium tail $\bar{H}_e(x) = 1 - H_e(x),\ x \in \mathbb{N}$, of H_α satisfies

$$\begin{aligned} \bar{H}_e(x) \quad &= \quad \sum_{i=1}^{x} \bar{H}_{1-\phi}(x-i) f_e(i) + \ \bar{F}_e(x) \\ &\leq (\geq) \quad \phi \sum_{i=1}^{x} \rho^{-(x-i)} f_e(i) + \phi\ \rho^{-x} \sum_{j=x+1}^{\infty} \rho^j f_e(j) \\ &= \quad \phi\rho^{-x} \sum_{j=1}^{\infty} \rho^j f_e(j) = \frac{1-\phi}{(\rho-1)\mathbb{E}\{Y\}}\, \rho^{-(x-1)}. \end{aligned} \tag{4.9}$$

Finally,

$$R_\alpha(x) = \mathbb{E}\{X_\alpha\}\bar{H}_e(x) = \ \frac{1-\alpha}{1-\phi}\mathbb{E}\{Y\}\bar{H}_e(x) \leq (\geq)\ \frac{1-\alpha}{\rho-1}\,\rho^{-(x-1)}$$

as required. □

In order to obtain a second upper bound of the stop-loss premium, we need first to derive a defective renewal equation satisfied by it.

Proposition 4.2. *$R_\alpha(x)$ is the solution to the* **discrete defective renewal equation**

$$R_\alpha(x) = \phi \sum_{i=1}^{x} R_\alpha(x-i) f(i) + \mathbb{E}\{X_\alpha\}[\phi\bar{F}(x) + (1-\phi)\bar{F}_e(x)], \qquad x \in \mathbb{N}.$$

Proof. Since $R_\alpha(x) = \mathbb{E}\{X_\alpha\}\bar{H}_e(x)$,

$$\sum_{x=0}^{\infty} z^x R_\alpha(x) = \mathbb{E}\{X_\alpha\} \sum_{x=0}^{\infty} z^x \bar{H}_e(x) = \mathbb{E}\{X_\alpha\} \frac{1-\mathcal{H}_e(z)}{1-z}$$

by [4], p. 265, where $\mathcal{H}_e(z)$ is the equilibrium p.g.f. of X_α.

Now, we will show that employing the defective renewal equation, we obtain the same quantity.

$$\begin{aligned} \sum_{x=0}^{\infty} z^x R_\alpha(x) = \ &\phi \sum_{x=0}^{\infty} z^x \sum_{i=1}^{x} R_\alpha(x-i) f(i) + \ \mathbb{E}\{X_\alpha\} \left[\phi \sum_{x=0}^{\infty} z^x\ \bar{F}(x) + (1-\phi) \sum_{x=0}^{\infty} z^x \bar{F}_e(x)\right] \\ &= \phi \sum_{i=1}^{\infty} f(i) \sum_{x=i}^{\infty} z^x\ R_\alpha(x-i) + \ \frac{\mathbb{E}\{X_\alpha\}}{1-z} \{\phi[1-\mathcal{F}(z)] + (1-\phi)[1-\mathcal{F}_e(z)]\} \\ &= \phi\mathcal{F}(z) \sum_{x=0}^{\infty} z^x R_\alpha(x) + \frac{\mathbb{E}\{X_\alpha\}}{1-z} \left\{\phi\,[1-\mathcal{F}(z)] + (1-\phi)\,[1-\mathcal{F}_e(z)]\right\}, \end{aligned}$$

again by [4], p. 265, where $\mathcal{F}_e$ is the equilibrium p.g.f. of Y. Equivalently,

$$\begin{aligned}\sum_{x=0}^{\infty} z^x R_\alpha(x) &= \frac{\mathbb{E}\{X_\alpha\}}{1-z} \cdot \frac{[1-\phi\mathcal{F}(z)] - (1-\phi)\mathcal{F}_e(z)}{1-\phi\mathcal{F}(z)} \\ &= \frac{\mathbb{E}\{X_\alpha\}}{1-z} \cdot [1-\mathcal{H}_{1-\phi}(z)\mathcal{F}_e(z)] = \frac{\mathbb{E}\{X_\alpha\}}{1-z} [1-\mathcal{H}_e(z)]\end{aligned}$$

where the last two equalities follow since $H_{1-\phi}(x)$ is a discrete compound geometric distribution and by (4.9). □

We now have the necessary preliminary result which was needed for obtaining the other upper bound. The bound itself is provided by the following proposition.

Proposition 4.3. *Let $\rho > 1$ satisfy $\mathcal{F}(\rho) = 1/\phi$. If F is D-IMRL (D-DMRL and $x \leq \underline{x}$),*

$$R_\alpha(x) \leq (\geq) [\phi\mathbb{E}\{X_\alpha\} + (1-\alpha) r_F(x)]\rho^{-x}, \quad x \in \mathbb{N}. \tag{4.10}$$

Equality is achieved when either $x = 0$, or Y is a zero-truncated geometric random variable with geometric parameter $\vartheta \in (0,1)$.

Proof. We prove the result for only the upper bound since its lower bound version is analogous. Taking Proposition 4.2 into account, we have by Corollary 3.2.1 that $R_\alpha(x) \leq \sigma_U(x)\beta_U(x)\rho^{-x}$ where

$$\beta_U(x) = \sup_{\substack{0\leq s\leq x \\ \bar{F}(s)>0}} \frac{r(s)}{\bar{F}(s)} = \sup_{\substack{0\leq s\leq x \\ \bar{F}(s)>0}} \frac{\mathbb{E}\{X_\alpha\}[\phi\bar{F}(s) + (1-\phi)\bar{F}_e(s)]}{\phi\bar{F}(s)}.$$

It is also true that

$$\begin{aligned}\frac{r(s)}{\bar{F}(s)} &= \mathbb{E}\{X_\alpha\}\left[1 + \frac{1-\phi}{\phi} \cdot \frac{\bar{F}_e(s)}{\bar{F}(s)}\right] = \mathbb{E}\{X_\alpha\}\left[1 + \frac{1-\phi}{\phi} \cdot \frac{\sum_{j=s}^{\infty} \bar{F}(j)}{\bar{F}(s)\mathbb{E}\{Y\}}\right] \\ &= \mathbb{E}\{X_\alpha\}\left[1 + \frac{1-\phi}{\phi} \cdot \frac{\sum_{j=s}^{\infty} \bar{F}(j)}{\bar{F}(s)\frac{1-\phi}{1-\alpha}\mathbb{E}\{X_\alpha\}}\right],\end{aligned}$$

which is non-decreasing since F is D-IMRL. Hence,

$$\beta_U(x) = \mathbb{E}\{X_\alpha\} + \frac{1-\alpha}{\phi} \cdot \frac{\sum_{j=x}^{\infty} \bar{F}(j)}{\bar{F}(x)}$$

but through summation by parts

$$r_F(x) = \frac{\sum_{i=x}^{\infty} (i-x) f(i)}{\bar{F}(x)} = \frac{\sum_{i=x}^{\infty} \bar{F}(i)}{\bar{F}(x)}.$$

Thus, $\beta_U(x) = \mathbb{E}\{X_\alpha\} + \frac{1-\alpha}{\phi} r_F(x)$.

Implementing equation (4.7) twice,

$$\begin{aligned}\sum_{j=s+1}^{\infty}\rho^j f(j) &= \rho^s\bar{F}(s) + (\rho-1)\sum_{j=s}^{\infty}\rho^j\bar{F}(j) = \rho^s\bar{F}(s) + (\rho-1)\mathbb{E}\{Y\}\sum_{j=s}^{\infty}\rho^j f_e(j+1)\\ &= \rho^s\bar{F}(s) + \frac{\rho-1}{\rho}\mathbb{E}\{Y\}\sum_{j=s+1}^{\infty}\rho^j f_e(j)\\ &= \rho^s\bar{F}(s) + \frac{\rho-1}{\rho}\mathbb{E}\{Y\}\left[\rho^s\bar{F}_e(s) + (\rho-1)\sum_{j=s}^{\infty}\rho^j\bar{F}_e(j)\right].\end{aligned}$$

Dividing by $\rho^s\bar{F}(s)$ we obtain

$$\frac{\sum_{j=s+1}^{\infty}\rho^j f(j)}{\rho^s\bar{F}(s)} = 1 + \frac{\rho-1}{\rho}\mathbb{E}\{Y\}\left[\frac{\bar{F}_e(s)}{\bar{F}(s)} + (\rho-1)\sum_{j=0}^{\infty}\rho^j\frac{\bar{F}_e(s+j)}{\bar{F}(s)}\right].$$

But

$$\frac{\bar{F}_e(s+j)}{\bar{F}(s)} = \frac{\bar{F}_e(s+j)}{\bar{F}_e(s)}\cdot\frac{\bar{F}_e(s)}{\bar{F}(s)} = \frac{1}{\mathbb{E}\{Y\}}\cdot\frac{\bar{F}_e(s+j)}{\bar{F}_e(s)}\cdot\frac{\sum_{i=s}^{\infty}\bar{F}(i)}{\bar{F}(s)}, \qquad j\in\mathbb{N},$$

is a non-decreasing function of s since F is both D-IMRL and 2-D-DFR. (For the equivalence of the D-IMRL and 2-D-DFR classes see Lem. 2.1 in [10].) Thus $\frac{\sum_{j=s+1}^{\infty}\rho^j f(j)}{\rho^s\bar{F}(s)}$ is non-decreasing, which is the same as $\frac{\rho^s\bar{F}(s)}{\sum_{j=s+1}^{\infty}\rho^j f(j)}$ being non-increasing. The latter means that

$$\sigma_U(x) = \sup_{\substack{0\le s\le x\\ \bar{F}(s)>0}}\frac{\rho^s\bar{F}(s)}{\sum_{j=s+1}^{\infty}\rho^j f(j)} = \left.\frac{\rho^s\bar{F}(s)}{\sum_{j=s+1}^{\infty}\rho^j f(j)}\right|_{s=0} = \phi,$$

i.e.,

$$R_\alpha(x) \le [\phi\mathbb{E}\{X_\alpha\} + (1-\alpha)r_F(x)]\rho^{-x}, \qquad x\in\mathbb{N}.$$

To prove that we have equality when $x=0$, we employ equation (4.4) with $x=0$ and we obtain $R_\alpha(0)=\mathbb{E}\{X_\alpha\}$. The right hand side of (4.10) becomes with the help of (4.3)

$$\phi\mathbb{E}\{X_\alpha\} + (1-\alpha)r_F(0) = \phi\mathbb{E}\{X_\alpha\} + (1-\alpha)\cdot\frac{1-\phi}{1-\alpha}\mathbb{E}\{X_\alpha\} = \mathbb{E}\{X_\alpha\},$$

which establishes the equality.

When Y is a zero-truncated geometric random variable, $\bar{F}_e(x)=\bar{F}(x)=\vartheta^x$ and (4.9) is almost identical with (4.1) when $\alpha=1-\phi$. Therefore $\bar{H}_e(x)=\frac{1}{\phi}\bar{H}_{1-\phi}(x)$. Now, noting that the p.g.f. of $X_{1-\phi}$ is $\mathcal{H}_{1-\phi}(z)=[1-\phi]/[1-\phi\mathcal{F}(z)]$ and that by [4], p. 265,

$$\sum_{x=0}^{\infty}z^x\bar{H}_{1-\phi}(x) = \frac{1-\mathcal{H}_{1-\phi}(z)}{1-z} = \frac{\phi[1-\mathcal{F}(z)]}{(1-z)[1-\phi\mathcal{F}(z)]},$$

where $\mathcal{F}(z)=[(1-\vartheta)z]/[1-\vartheta z]$ is the p.g.f. of Y, we obtain

$$\sum_{x=0}^{\infty}z^x\bar{H}_{1-\phi}(x) = \frac{\phi[1-\vartheta\, z-(1-\vartheta)z]}{(1-z)[1-\vartheta\, z-\phi(1-\vartheta)z]} = \frac{\phi}{1-[\vartheta+\phi(1-\vartheta)]z} = \phi\sum_{x=0}^{\infty}z^x[\theta+\phi(1-\theta)]^x.$$

Hence,

$$\bar{H}_e(x) = [\vartheta + \phi(1-\vartheta)]^x, \quad x \in \mathbb{N}.$$

Thus (4.4) implies that $R_\alpha(x) = \mathbb{E}\{X_\alpha\}[\vartheta + \phi(1-\vartheta)]^x$. Knowing that

$$r_F(x) = \frac{\sum_{i=x}^{\infty} \bar{F}(i)}{\bar{F}(x)} = \frac{1}{1-\vartheta} = \mathbb{E}\{Y\} = r_F(0),$$

we see that the righthand side of (4.10) becomes $\mathbb{E}\{X_\alpha\}\rho^{-x}$ by (4.3). Since

$$\frac{1}{\phi} = \mathcal{F}(\rho) = \frac{(1-\vartheta)\rho}{1-\vartheta\rho},$$

which is the same as $\rho^{-1} = \vartheta + \phi(1-\vartheta)$, the equality is established. □

The last two results in this section provide a comparison of the two pairs of bounds of the stop-loss premium. If we let $x_0 = \inf\left\{x \in \mathbb{N} \mid r_F(x) \le \frac{\rho^2}{\rho-1} - \mathbb{E}\{X_{1-\phi}\}\right\}$ with $x_0 = \infty$ if $r_F(x) > \frac{\rho^2}{\rho-1} - \mathbb{E}\{X_{1-\phi}\}$ for all $x \in \mathbb{N}$, it turns out that x_0 is the cutting point for estimating the tightness of the two bounds.

Proposition 4.4. *Let F be D-IMRL. If $x \le x_0$ $(x > x_0)$, then the upper bound provided by* (4.10) *is tighter (weaker) than the upper bound of* (4.5).

Proof. Since the D-IMRL property implies the D-NWUC one (see Fig. 1 in [10]), (4.5) is true if F is D-IMRL. The existence of x_0 is easily seen by equating the right hand sides of the two alternative bounds (4.5) and (4.10).

To complete the proof, we will mention that by the definition of the D-IMRL class $r_F(x)$ is non-decreasing, which implies that the cutting point x_0 is unique. □

A respective statement for the D-DMRL case is given by the following result.

Proposition 4.5. *Let F be D-DMRL and $x \le \underline{x}$. If $x \le x_0$ $(x > x_0)$, then the lower bound in* (4.5) *is tighter (weaker) than the lower bound in* (4.10).

The proof of this last proposition differs from the one of Proposition 4.4 in just one aspect namely, that the arguments follow only for a zero-truncated distribution, which F is.

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Appendix A: Definitions of discrete reliability classes

Let $A(x) = 1 - \bar{A}(x)$, $x \in \mathbb{N}$, be a discrete c.d.f. with corresponding probability mass function a.

Definition A.1. The distribution A is called **Discrete Decreasing Failure Rate (Discrete Increasing Failure Rate)** or D-DFR (D-IFR) if its failure rate $h_A(x) = a(x)/[a(x) + \bar{A}(x)]$ is non-decreasing (non-increasing) for $x \in \mathbb{N}_+$.

Definition A.2. The distribution A is called **Discrete New Worse than Used (Discrete New Better than Used)** or D-NWU (D-NBU) if $\bar{A}(x+y) \ge (\le)\, \bar{A}(x)\bar{A}(y)$, $x, y \in \mathbb{N}$.

Definition A.3. The distribution A is called **Discrete Increasing Mean Residual Lifetime (Discrete Decreasing Mean Residual Lifetime)** or D-IMRL (D-DMRL) if its mean residual lifetime $r_A(x) = \sum_{i=x}^{\infty} \bar{A}(i)/\bar{A}(x)$ is non-decreasing (non-increasing) in $x \in \mathbb{N}$.

Definition A.4. A discrete distribution A is called **Discrete New Worse than Used in Convex ordering** or D-NWUC (**Discrete New Better than Used in Convex ordering** or D-NBUC) if

$$\sum_{i=x+y}^{\infty} \bar{A}(i) \geq (\leq) \ \bar{A}(x) \sum_{j=y}^{\infty} \bar{A}(j) \ \text{ for all } x, y \in \mathbb{N},$$

or equivalently

$$\bar{A}_e(x+y) \geq (\leq) \, \bar{A}(x)\bar{A}_e(y) \ \text{ for all } x, y \in \mathbb{N}.$$

Appendix B: Class inclusion properties

Figure 1 shows how the reliability classes are related to each other. An arrow denotes that the left-hand class is a subclass of the right-hand one and a non-solid arrow indicates that the inclusion property requires the additional condition that the distributions concerned are zero-truncated.

$$\begin{tikzcd}
 & D-IMRL \arrow[dr] & \\
D-DFR \arrow[ur] \arrow[dr] & & D-NWUC \\
 & D-NWU \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & D-DMRL \arrow[dr, dashed] & \\
D-IFR \arrow[ur] & & D-NBUC \\
 & D-NBU \arrow[ur] &
\end{tikzcd}$$

Figure 1. Inclusion relationships of the reliability classes.

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