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A MARTINGALE CONTROL VARIATE METHOD FOR OPTION PRICING WITH STOCHASTIC VOLATILITY [*, **]

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Abstract. A generic control variate method is proposed to price options under stochastic volatility models by Monte Carlo simulations. This method provides a constructive way to select control variates which are martingales in order to reduce the variance of unbiased option price estimators. We apply a singular and regular perturbation analysis to characterize the variance reduced by martingale control variates. This variance analysis is done in the regime where time scales of associated driving volatility processes are well separated. Numerical results for European, Barrier, and American options are presented to illustrate the effectiveness and robustness of this martingale control variate method in regimes where these time scales are not so well separated.

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Dedicated to Nicole El Karoui in honor of her 60th birthday

Introduction

Monte Carlo pricing for options is a popular approach in particular since efficient algorithms have been developed for optimal stopping problems, see for example [10]. The advantage of Monte Carlo simulations is that it is less sensitive to dimensionality of the pricing problems and suitable for parallel computation; the main disadvantage is that the rate of convergence is limited by the central limit theorem so it is slow.

To increase the efficiency besides parallel computing, quasi Monte Carlo and variance reduction techniques are two possible approaches. We refer to [9] for an extensive review. Quasi Monte Carlo, unlike pseudo-random number generators, forms a class of methods where *low-discrepancy* numbers are generated in deterministic ways. Its efficiency heavily relates to the regularity of the option payoffs, which in most cases are poorly posted. The pros are that such an approach can be always implemented regardless to the pricing problems and it is easy to combine with other sampling techniques such as those involving the Brownian bridge. On the other hand, variance reduction methods seek probabilistic ways to reformulate the pricing problem considered in order to gain significant variance reduction. For example control variate methods take into account correlation properties

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of random variables, and importance sampling methods utilize changes of probability measures. The cons are that the efficiency of these techniques is often restricted to certain pricing problems.

Stochastic volatility models have been an important class of diffusions extending the Black-Scholes model, see [6] for details. Under multifactor stochastic volatility models, this paper aims at generalizing the control variate method proposed by the authors in [4], and studying its variance analysis. Since the proposed control variates are (local) martingales, we shall call this method "martingale control variate method". The pricing problems of European, Barrier and American options are considered in order to demonstrate the effectiveness of our method for a broad range of problems.

The martingale control variate method can be well understood in finance terminology. The constructed control variate corresponds to a continuous (non-self-financing) delta hedge strategy taken by a trader who sells an option. Though perfect replication by *delta* hedging under stochastic volatility models is impossible, the variance of replication error is directly related to the variance induced by the martingale control variate method. This method is also potentially useful to study contracts dealing with volatility or variance risks such as variance swaps.

A variance analysis, presented in the Appendix, deduces an asymptotic result for the variance reduced by martingale control variates. It is based on the singular and regular perturbation method presented in [8]. The paper is organized as follows. In Section 1 we introduce the basic Monte Carlo pricing mechanism and review the martingale control variate method for European options. Section 2 and 3 extends the method to Barrier and American options, respectively. Numerical experiments are included and we conclude this paper in Section 4.

1. Monte Carlo pricing under multiscale stochastic volatility models

Under a risk-neutral pricing probability measure $I\!P^\star$ parametrized by the combined market prices of volatility risk (Λ_1, Λ_2), we consider the following class of multiscale stochastic volatility models:

$$
\begin{aligned}
\mathrm{d}S_t &= rS_t\mathrm{d}t + \sigma_t S_t \mathrm{d}W_t^{(0)*}, \\
\sigma_t &= f(Y_t, Z_t), \\
\mathrm{d}Y_t &= \left[\frac{1}{\varepsilon}c_1(Y_t) + \frac{g_1(Y_t)}{\sqrt{\varepsilon}}\Lambda_1(Y_t, Z_t)\right]\mathrm{d}t + \frac{g_1(Y_t)}{\sqrt{\varepsilon}}\left(\rho_1 \mathrm{d}W_t^{(0)*} + \sqrt{1-\rho_1^2}\,\mathrm{d}W_t^{(1)*}\right), \\
\mathrm{d}Z_t &= \left[\delta c_2(Z_t) + \sqrt{\delta}g_2(Z_t)\Lambda_2(Y_t, Z_t)\right]\mathrm{d}t \\
&\quad + \sqrt{\delta}g_2(Z_t)\left(\rho_2 \mathrm{d}W_t^{(0)*} + \rho_{12}\mathrm{d}W_t^{(1)*} + \sqrt{1-\rho_2^2-\rho_{12}^2}\,\mathrm{d}W_t^{(2)*}\right),
\end{aligned} \tag{1}
$$

where S_t is the underlying asset price process with a constant risk-free interest rate r. Its stochastic volatility σ_t is driven by two stochastic processes Y_t and Z_t varying on the time scales ε and $1/\delta$, respectively (ε is intended to be a short time scale while $1/\delta$ is thought as a longer time scale). The vector $(W_t^{(0)*}, W_t^{(1)*}, W_t^{(2)*})$ consists of three independent standard Brownian motions. The instant correlation coefficients ρ_1, ρ_2, and ρ_{12} satisfy $|\rho_1| < 1$ and $|\rho_2^2 + \rho_{12}^2| < 1$. The volatility function f is assumed to be smooth bounded and bounded away from 0. The coefficient functions of Y_t, namely c_1 and g_1, are assumed to be such that under the physical probability measure, Y_t is ergodic. The Ornstein-Uhlenbeck (OU) process is a typical example by defining $c_1(y) = m_1 - y$ and $g_1(y) = \nu_1\sqrt{2}$ such that $1/\varepsilon$ is the rate of mean reversion, m_1 is the long run mean, and ν_1 is the long run standard deviation. Its invariant distribution is $\mathcal{N}\left(m_1, \nu_1^2\right)$.

The coefficient functions of Z_t, namely c_2 and g_2 are assumed to be smooth enough in order to satisfy existence and uniqueness conditions for diffusions [11]. The combined risk premia Λ_1 and Λ_2 are assumed to be smooth, bounded, and depending on the variables y and z only. Within this setup, the joint process (S_t, Y_t, Z_t) is Markovian. We refer to [8] for a detailed discussion on this class of models.

Given the multiscale stochastic volatility model (1), the price of a plain European option with the integrable payoff function H and expiry T is given by

$$P^{\varepsilon,\delta}(t,x,y,z) = I\!E^{\star}{}_{t,x,y,z}\left\{ \mathrm{e}^{-r(T-t)}H(S_T)\right\}, \tag{2}$$

where $I\!E^{\star}{}_{t,x,y,z}$ denotes the expectation with respect to $I\!P^{\star}$ conditioned on the current states $S_t = x$, $Y_t = y$, $Z_t = z$. A basic Monte Carlo simulation estimates the option price $P^{\varepsilon,\delta}(0,S_0,Y_0,Z_0)$ at time 0 by

$$\frac{1}{N}\sum_{i=1}^{N} \mathrm{e}^{-rT} H\left(S_T^{(i)}\right), \tag{3}$$

where N is the total number of independent sample paths and $S_T^{(i)}$ denotes the ith simulated stock price at time T.

Assuming that the European option price $P^{\varepsilon,\delta}(t,S_t,Y_t,Z_t)$ is smooth enough, we apply Ito's lemma to its discounted price $e^{-rt}P^{\varepsilon,\delta}$, and then integrate from time 0 to the maturity T. The following martingale representation is obtained

$$P^{\varepsilon,\delta}(0,S_0,Y_0,Z_0) = \mathrm{e}^{-rT}H(S_T) - \mathcal{M}_0(P^{\varepsilon,\delta}) - \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_1(P^{\varepsilon,\delta}) - \sqrt{\delta}\mathcal{M}_2(P^{\varepsilon,\delta}), \tag{4}$$

where centered martingales are defined by

$$\mathcal{M}_0(P^{\varepsilon,\delta}) = \int_0^T \mathrm{e}^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial x}(s,S_s,Y_s,Z_s) f(Y_s,Z_s) S_s \mathrm{d}W_s^{(0)*}, \tag{5}$$

$$\mathcal{M}_1(P^{\varepsilon,\delta}) = \int_0^T \mathrm{e}^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial y}(s,S_s,Y_s,Z_s) g_1(Y_s) \mathrm{d}\tilde{W}_s^{(1)*}, \tag{6}$$

$$\mathcal{M}_2(P^{\varepsilon,\delta}) = \int_0^T \mathrm{e}^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial z}(s,S_s,Y_s,Z_s) g_2(Z_s) \mathrm{d}\tilde{W}_s^{(2)*}, \tag{7}$$

with the Brownian motions

$$\begin{aligned}
\tilde{W}_s^{(1)*} &= \rho_1 W_s^{(0)*} + \sqrt{1-\rho_1^2}\,W_s^{(1)*},\\
\tilde{W}_s^{(2)*} &= \rho_2 W_s^{(0)*} + \rho_{12}W_s^{(1)*} + \sqrt{1-\rho_1^2-\rho_{12}^2}\,W_s^{(2)*}.
\end{aligned}$$

These martingales play the role of "perfect" control variates for Monte Carlo simulations and their integrands would be the perfect Delta hedges if $P^{\varepsilon,\delta}$ were known and volatility factors traded. Unfortunately neither the option price $P^{\varepsilon,\delta}(s,S_s,Y_s,Z_s)$ nor its gradient at any time $0 \le s \le T$ are in any analytic form even though all the parameters of the model have been calibrated as we suppose here.

One can choose an approximate option price to substitute $P^{\varepsilon,\delta}$ used in the martingales (5, 6, 7) and still retain martingale properties. When time scales ε and $1/\delta$ are well separated, namely $0 < \varepsilon \ll 1 \ll 1/\delta$, an approximation of the Black-Scholes type is derived in [8]:

$$P^{\varepsilon,\delta}(t,x,y,z) \approx P_{BS}(t,x;\bar{\sigma}(z)) \tag{8}$$

with an accuracy of order $\mathcal{O}(\sqrt{\varepsilon},\sqrt{\delta})$ for continuous payoffs. We denote by $P_{BS}(t,x;\bar{\sigma}(z))$ the solution of the Black-Scholes partial differential equation with the terminal condition $P_{BS}(T,x) = H(x)$. The z-dependent *effective volatility* $\bar{\sigma}(z)$ is defined by

$$\bar{\sigma}^2(z) = \int f^2(y,z)\mathrm{d}\Phi(y), \tag{9}$$

where $\Phi(y)$ is the invariant distribution of the fast varying process Y_t. In the OU case, the density Φ is simply the Gaussian density with mean m_1 and variance ν_1^2. Note that the approximate option price $P_{BS}(t,x;\bar{\sigma}(z))$ is independent of the variable y. A martingale control variate estimator is formulated as

$$\frac{1}{N}\sum_{i=1}^{N}\left[\mathrm{e}^{-rT}H\left(S_T^{(i)}\right)-\mathcal{M}_0^{(i)}(P_{BS})-\sqrt{\delta}\mathcal{M}_2^{(i)}(P_{BS})\right]. \tag{10}$$

This is the approach taken by Fouque and Han [4], in which the proposed martingale control variate method is empirically superior to an importance sampling [5] for pricing European options. As control variates $\mathcal{M}_0$ and $\mathcal{M}_2$ are martingales, we shall call them martingale control variates afterwards. Note that there is no $\mathcal{M}_1$ martingale term since the approximation P_{BS} does not depend on y and the y-derivative cancels in (6) with $P^{\varepsilon,\delta}$ replaced by P_{BS}.

Variance analysis of martingale control variates

Since $\mathcal{M}_2(P_{BS})$ is small of order $\sqrt{\delta}$, in a first approximation we can neglect $\mathcal{M}_2(P_{BS})$ in (10). Hence we reduce the number of stochastic integrals or martingale control variates from 2 to 1 and formulate the following unbiased estimator:

$$\frac{1}{N}\sum_{i=1}^{N}\left[\mathrm{e}^{-rT}H\left(S_T^{(i)}\right)-\mathcal{M}_0^{(i)}(P_{BS})\right], \tag{11}$$

where

$$\mathcal{M}_0(P_{BS})=\int_0^T \mathrm{e}^{-rs}\frac{\partial P_{BS}}{\partial x}(s,S_s;\bar{\sigma}(Z_s))f(Y_s,Z_s)S_s\mathrm{d}W_s^{(0)*}.$$

For the sake of simplicity, we first assume that the instant correlation coefficients, ρ_1, ρ_2 and ρ_{12} in (1), are zero. From (4), the variance of the controlled payoff

$$\mathrm{e}^{-rT}H(S_T)-\mathcal{M}_0(P_{BS}) \tag{12}$$

is simply the sum of quadratic variations of martingales:

$$\begin{aligned}
&\mathrm{Var}\left(\mathrm{e}^{-rT}H(S_T)-\mathcal{M}_0(P_{BS})\right) && (13)\\
&=I\!E^{\star}{}_{0,t,x,y,z}\left\{\int_0^T \mathrm{e}^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x}-\frac{\partial P_{BS}}{\partial x}\right)^2(s,S_s,Y_s,Z_s)f^2(Y_s,Z_s)S_s^2\mathrm{d}s\right.\\
&\quad+\frac{1}{\varepsilon}\int_0^T \mathrm{e}^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial y}\right)^2(s,S_s,Y_s,Z_s)g_1^2(Y_s)\mathrm{d}s\\
&\quad\left.+\delta\int_0^T \mathrm{e}^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial z}\right)^2(s,S_s,Y_s,Z_s)g_2^2(Z_s)\mathrm{d}s\right\}. && (14)
\end{aligned}$$

As in the numerical experiments implemented in [4] and in next sections, we assume that the driving volatility processes Y_t and Z_t are of OU type; namely $c_1(y)=(m_1-y)$, $c_1(z)=(m_2-z)$, $g_1(y)=\nu_1\sqrt{2}$, and $g_2(z)=\nu_2\sqrt{2}$. The volatility premia Λ_1 and Λ_2 are assumed to be smooth and bounded.

Theorem 1.1. *Under the assumptions made above and the payoff function H being continuous piecewise smooth as a call (or a put), for any fixed initial state $(0,x,y,z)$, there exists a constant $C>0$ such that for $\varepsilon\le 1,\delta\le 1$,*

$$Var\left(\mathrm{e}^{-rT}H(S_T)-\mathcal{M}_0(P_{BS})\right)\le C\max\{\varepsilon,\delta\}.$$

The proof of Theorem 1.1 is given in the Appendix.

We comment this theorem:

(1) The assumption of zero instant correlations is not necessary. One can still obtain the same accuracy result with additional cross-variation terms appearing in equation (13).
(2) Adding the next order corrections in $\sqrt{\varepsilon}$ and $\sqrt{\delta}$ to (8), as suggested in [8], and using two martingale control variates as in (10), we would obtain that the variance associated with the estimator is still of the same order as in the theorem. One can obtain next order accurate result for Lemma A.1. However there is no accuracy gain for Lemma A.2 because the next order price approximation is still independent of the fast varying y-variable [8].

Several variance reduction results for pricing European call options can be found in [4], where the martingale control variate method does demonstrate significant variance reduction performance when time scales are well separated.

From the computational viewpoint, since calculating each stochastic integral along a sample path is time consuming, it is useful to reduce the number of stochastic integrals from (10) to (11) and still retain considerable accuracy for the reduced variance. From the finance point of view, the martingale control variate $\mathcal{M}_0(P_{BS})$ represents that a trader, who sells an option, uses the delta hedge strategy continuously. By doing so, the induced error of replicated discounted-payoff $\mathrm{e}^{-rT}H(S_T)-\mathcal{M}_0(P_{BS})$ and its statistical property can be studied through the Monte Carlo simulations (11). Since the martingale control variate method is associated with hedging strategies, it should, in principle, work for all other derivatives pricing problems provided the delta is easy enough to be computed or effectively approximated.

In the next sections, we generalize this method to Barrier and American option pricing problems under stochastic volatility models.

2. Barrier options

The payoff of a barrier option depends on whether the trajectory of the underlying stock hits a pre-specified level or not before the maturity T. For instance a *down and out* call option with the barrier B and the strike K has a payoff

$$\left(S_T - K^+\right)\mathbf{I}_{\{\tau>T\}},$$

where we denote by $\mathbf{I}$ the indicator function and by τ the first hitting time

$$\tau = \inf\{0 \le t \le T, S_t \le B\}.$$

Other popular barrier options such as *down and in*, *up and out* and *up and in* can be defined similarly. Under the risk-neutral probability $I\!P^\star$, a *down and out* barrier call option price at time t conditioning on no *knock-out* before time $t<T$ is given by

$$P^{\varepsilon,\delta}(t,x,y,z) = I\!E^\star{}_{t,x,y,z}\left\{\mathrm{e}^{-r(T-t)}\left(S_T - K^+\right)\mathbf{I}_{\{\tau>T\}}\right\}. \tag{15}$$

The price $P^{\varepsilon,\delta}(t,x,y,z)$ solves a boundary value problem [7]. When parameters ε and δ are small enough, the leading order approximation to $P^{\varepsilon,\delta}$ in (15) is given by

$$P^{\varepsilon,\delta}(t,x,y,z) \approx P^B_{BS}(t,x;\bar\sigma(z)), \tag{16}$$

where $P^B_{BS}(t,x;\bar\sigma(z))$ solves a Black-Scholes partial differential equation for a barrier option problem with the effective volatility $\bar\sigma(z)$, and the boundary conditions $P^B_{BS}(t,B)=0$ for any $0\le t\le T$ and $P^B_{BS}(T,x)=(x-K)^+$

TABLE 1. Parameters used in the two-factor stochastic volatility model (1).

r	m_1	m_2	ν_1	ν_2	ρ_1	ρ_2	ρ_{12}	Λ_1	Λ_2	$f(y,z)$
8%	–1	–1	1	1	–0.2	-0.2	0	0	0	$\exp(y+z)$

for $x > B$. It is known (see for instance [12]) that $P^B_{BS}(t,x;\bar\sigma(z))$ admits the closed form solution

$$P^B_{BS}(t,x;\bar\sigma(z)) = C_{BS}(t,x;\bar\sigma(z)) - \left(\frac{x}{B}\right)^{1-k} C_{BS}(t,B^2/x;\bar\sigma(z)), \tag{17}$$

where $k = 2r/(\bar\sigma^2(z))$ and $C_{BS}(t,x;\bar\sigma(z))$ denotes the Black-Scholes price of a European call option with strike K, maturity T, and volatility $\bar\sigma(z)$.

2.1. Martingale control variate estimator for Barrier options

Let $S_0 > B$, one can apply Ito's lemma to the discounted barrier option price, then integrate from time 0 up to the bounded stopping time $\tau \wedge T$ so that

$$P^{\varepsilon,\delta}(0,S_0,Y_0,Z_0) = \mathrm{e}^{-rT}(S_T - B)^+\mathbf{I}_{\{\tau>T\}} - \mathcal{M}_0(P^{\varepsilon,\delta}) - \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_1(P^{\varepsilon,\delta}) - \sqrt{\delta}\mathcal{M}_2(P^{\varepsilon,\delta}) \tag{18}$$

is deduced. The local martingales are defined as in (5, 6, 7) except that the upper bounds are replaced by $\tau \wedge T$.

As in Section 1, we use the barrier price approximation (17) to construct the following local martingale control variate

$$\mathcal{M}_0(P^B_{BS}) = \int_0^{\tau\wedge T} \mathrm{e}^{-rs}\frac{\partial P^B_{BS}}{\partial x}(s,S_s;\bar\sigma(Z_s))f(Y_s,Z_s)S_s\mathrm{d}W_s^{(0)*}.$$

The unbiased martingale control variate estimator by Monte Carlo simulations for the barrier option is

$$\frac{1}{N}\sum_{i=1}^{N}\left[\mathrm{e}^{-rT}(S_T^{(i)} - K)^+\mathbf{I}_{\{\tau^{(i)}>T\}} - \mathcal{M}_0^{(i)}(P^B_{BS})\right].$$

The variance analysis for the estimator $\mathrm{e}^{-rT}(S_T - K)^+\mathbf{I}_{\{\tau>T\}} - \mathcal{M}_0(P^B_{BS})$ can be done similarly as in Theorem 1.1. In fact one can obtain the same accuracy, namely $\mathcal{O}(\varepsilon,\delta)$, because, as shown in [7], the accuracy of the leading order barrier option approximation in (16) is the same as for European options. All other derivations remain the same.

2.2. Numerical results

Several numerical experiments are presented to demonstrate that the martingale control variate method is efficient and robust for barrier option problems even in the regimes where the time scales ε and $1/\delta$ are not so well separated. Relevant parameters and volatility functions for a two-factor stochastic volatility model are chosen as in Table 1. Other values including initial conditions and option parameters are given in Table 2. Option price computations are done with various time scale parameters given in Table 3. The sample size is $N = 10\,000$. Simulated paths are generated based on the Euler discretization scheme [9] with time step size $\Delta t = 10^{-3}$. Figure 1 presents sampled barrier option prices with respect to the number of realizations. The dash line corresponds to basic Monte Carlo simulations, while the solid line corresponds to same Monte Carlo simulations using the martingale control variate $\mathcal{M}_0(P^B_{BS})$.

TABLE 2. Initial conditions and *down and out* barrier call option parameters.

$\$S_0$	Y_0	Z_0	$\$K$	$\$B$	T years
100	–1	–1	110	80	1

TABLE 3. Comparison of standard errors with various ε and δ. The notation Std^{BMC} stands for the standard error estimated from basic Monte Carlo simulations, and Std^{MCV} the standard error from the same Monte Carlo simulations but using the martingale control variate. Numbers within parenthesis in the third and fourth columns are sample means estimated from the two Monte Carlo methods, respectively. The fifth column records the variance reduction ratio, which is calculated by $\left(Std^{BMC}/Std^{MCV}\right)^2$.

$1/\varepsilon$	δ	Std^{BMC}	Std^{MCV}	Variance Reduction Ratio
100	0.01	0.2822 (10.8153)	0.0304 (10.8497)	86
75	0.1	0.2047 (10.7652)	0.0306 (10.7594)	45
50	1	0.2455 (11.2082)	0.0474 (11.0962)	27
25	10	0.2604 (12.6212)	0.0417 (12.4372)	39

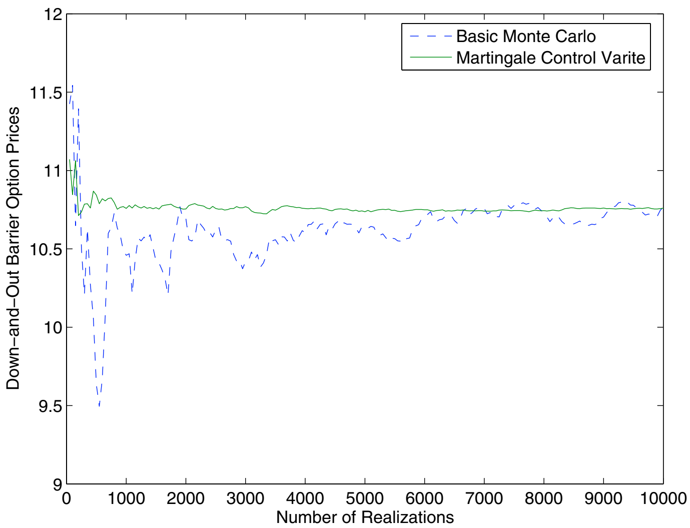

FIGURE 1. Monte Carlo simulations for a *down-and-out* barrier call option price when $1/\varepsilon = 75$ and $\delta = 0.1$. Sampled prices are obtained along the number of realizations.

3. AMERICAN OPTIONS

The most important feature of an American option is that the option holder has the right to exercise the contract early. Under the stochastic volatility models considered, the price of an American option with the payoff function H is given by:

$$P^{\varepsilon,\delta}(t,x,y,z) = (ess) \sup_{t \le \tau \le T} I\!E^{\star}{}_{t,x,y,z} \left\{ e^{-r(\tau-t)} H(S_\tau) \right\}, \tag{19}$$

where τ denotes any stopping time greater than t, bounded by T, adapted to the completion of the natural filtration generated by Brownian motions $(W_t^{(0)*}, W_t^{(1)*}, W_t^{(2)*})$. We consider a typical American put option pricing problem, namely $H(x) = (K - x)^+$, and maturity T. By the connection of optimal stopping problems and variational inequalities [11], $P^{\varepsilon,\delta}(t,x,y,z)$ can be characterized as the solution of the following variational inequalities

$$\begin{cases} \mathcal{L}_{(S,Y,Z)} P^{\varepsilon,\delta}(t,x,y,z) \leq 0 \\ P^{\varepsilon,\delta}(t,x,y,z) \geq (K-x)^+ \\ \mathcal{L}_{(S,Y,Z)} P^{\varepsilon,\delta}(t,x,y,z) \cdot \left(P^{\varepsilon,\delta}(t,x,y,z) - (K-x)^+\right) = 0, \end{cases}$$

where $\mathcal{L}_{(S,Y,Z)}$ denotes the infinitesimal generator of the Markov process (S_t, Y_t, Z_t). The optimal stopping time is characterized by

$$\tau^\star(t) = \inf\left\{t \leq s \leq T, (K - S_s)^+ = P^{\varepsilon,\delta}(s, S_s, Y_s, Z_s)\right\}. \tag{20}$$

When ε and δ are small enough, the leading order approximation by a formal expansion is

$$P^{\varepsilon,\delta}(t,x,y,z) \approx P^A_{BS}(t,x;\bar\sigma(z)) \tag{21}$$

while $P^A_{BS}(t,x;\bar\sigma(z))$ solves the homogenized variational inequality

$$\begin{cases} \mathcal{L}_{BS}(\bar\sigma(z)) P^A_{BS}(t,x;\bar\sigma(z)) \leq 0 \\ P^A_{BS}(t,x;\bar\sigma(z)) \geq (K-x)^+ \\ \mathcal{L}_{BS}(\bar\sigma(z)) P^A_{BS}(t,x;\bar\sigma(z)) \cdot \left(P^A_{BS}(t,x;\bar\sigma(z)) - (K-x)^+\right) = 0, \end{cases} \tag{22}$$

where $\mathcal{L}_{BS}(\bar\sigma(z))$ denotes the Black-Scholes operator with the constant volatility $\bar\sigma(z)$. In contrast to typical European and barrier options, there is no closed-form solution for the American put option price under a constant volatility. The derivation of the accuracy of the approximation (21) is still an open problem.

As in the previous sections, we assume that the discounted American option price $\mathrm{e}^{-rt} P^{\varepsilon,\delta}(t, S_t, Y_t, Z_t)$ before exercise is smooth enough to apply Ito's lemma, then we integrate from time 0 to the (bounded) optimal stopping time $\tau^\star$ such that we obtain

$$P^{\varepsilon,\delta}(0, S_0, Y_0, Z_0) = \mathrm{e}^{-rT}(K - S_{\tau^\star})^+ - \mathcal{M}_0(P^{\varepsilon,\delta}) - \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_1(P^{\varepsilon,\delta}) - \sqrt{\delta}\mathcal{M}_2(P^{\varepsilon,\delta}). \tag{23}$$

The local martingales are defined as in (5, 6, 7) except that the upper bounds are replaced by the optimal stopping time $\tau^\star$.

3.1. Martingale control variates for American options

As revealed in (20), the characterization of the optimal stopping time $\tau^\star(t)$ does depend on the American option price, which itself is unknown in advance. This causes an immediate difficulty to implement Monte Carlo simulations because one does not know the time to stop in order to collect the payoff along each realized sample path.

Longstaff and Schwartz [10] took a dynamic programming approach and proposed a least-square regression to estimate the *continuation value* at each in-the-money stock price state.

By comparing the continuation value and the instant exercise payoff, their method exploits a decision rule, denoted by $\underline{\tau}$, for early exercise along each sample path generated. It is shown in [10] that Longstaff-Schwartz' method provides a low-biased American option price estimate for practical Monte Carlo simulations. As the number of least-square basis functions increases to infinity for discrete exercise dates, Clement *et al.* in [3] show that the normalized error in Longstaff-Schwartz' method is asymptotically Gaussian.

Like in previous sections, a local martingale control variate can be in principle constructed as

$$\mathcal{M}_0(P_{BS}^A;\tau^\star) = \int_0^{\tau^\star} \mathrm{e}^{-rs}\frac{\partial P_{BS}^A}{\partial x}(s,S_s;\bar\sigma(Z_s))f(Y_s,Z_s)S_s\mathrm{d}W_s^{(0)*}.$$

Indeed the optimal stopping time $\tau^\star$ is not known. We approximate $\tau^\star$ by the exercise rule $\underline{\tau}$. Note that $\mathcal{M}_0(P_{BS}^A;\underline{\tau})$ may incur a bias but from the sample means in Table 6 this effect seems negligible. In fact, one could build an approximate stopping time from Longstaff-Schwartz' method. This can be done but will be computational expensive.

There is no closed-form solution for the homogenized American option $P_{BS}^A(t,x;\bar\sigma(z))$ either. We introduce an approximation proposed by Barone-Adesi and Whaley [1], denoted by P_{BS}^{BAW}, which is derived from an elliptic-type variational inequalities as an approximation to the parabolic-type variational inequalities (22). The approximation P_{BS}^{BAW} admits the closed-form solution:

$$P_{BS}^{BAW}(t,x;\sigma) = \begin{cases} \lambda x^\alpha + P_{BS}^E(t,x;\sigma), & x > x^* \\ K - x, & x \le x^*, \end{cases}$$

where $P_{BS}^E(t,x;\sigma)$ denotes the corresponding European put option price and the free boundary x^* solves the following nonlinear algebraic equation

$$x^* = |\alpha|\frac{K - P_{BS}^E(t,x^*;\sigma)}{\frac{\partial P_{BS}^E(t,x^*;\sigma)}{\partial x} + 1 + |\alpha|},$$

with

$$\alpha = \frac{1 - \frac{2r}{\sigma^2} - \sqrt{(1-\frac{2r}{\sigma^2})^2 + \frac{8(\kappa r+1)}{\kappa\sigma^2}}}{2}$$

and

$$\lambda = \frac{K - x^* - P_{BS}^E(t,x;\sigma)}{(x^*)^\alpha}.$$

To summarize, we construct the following stopped martingale as a control variate

$$\mathcal{M}_0(P_{BS}^{BAW};\underline{\tau}) = \int_0^{\underline{\tau}} \mathrm{e}^{-rs}\frac{\partial P_{BS}^{BAW}}{\partial x}(s,S_s;\bar\sigma(Z_s))f(Y_s,Z_s)S_s\mathrm{d}W_s^{(0)*}.$$

The Monte Carlo estimator with the martingale control variate is

$$\frac{1}{N}\sum_{i=1}^N\left[\mathrm{e}^{-r\underline{\tau}}(K - S_{\underline{\tau}}^{(i)})^+ - \mathcal{M}_0^{(i)}(P_{BS}^{BAW};\underline{\tau})\right].$$

3.2. Numerical results

We consider American put options under two factor stochastic volatility models, specified in Table 4 and Table 5. Results of variance reduction by the martingale control variate to price American put options are illustrated in Table 6 with various time scale parameters ε and δ. The discrete time step size is $\Delta = 10^{-3}$

TABLE 4. Parameters used in the two-factor stochastic volatility model (1).

r	m_1	m_2	ν_1	ν_2	ρ_1	ρ_2	ρ_{12}	Λ_1	Λ_2	$f(y,z)$
10%	–1	–1	1	1	–0.3	–0.3	0	0	0	$\exp(y+z)$

TABLE 5. Initial conditions and American put option parameters.

$\$S_0$	Y_0	Z_0	$\$K$	T years
90	–1	–1	100	1

TABLE 6. Comparison of standard errors with various ε and δ. The notation Std^{BMC} stands for the standard error estimated from basic Monte Carlo simulations and Std^{MCV} the standard error from same Monte Carlo simulations but using the martingale control variate. Numbers within the parenthesis in the third and fourth columns are sample means estimated from the two Monte Carlo methods, respectively. The fifth column records the variance reduction ratio, which is calculated by $(Std^{BMC}/Std^{MCV})^2$.

$1/\varepsilon$	δ	Std^{BMC}	Std^{MCV}	Variance Reduction Ratio
100	0.01	0.2354 (21.4340)	0.0240 (21.5942)	96
75	0.1	0.2564 (21.4791)	0.0286 (21.8001)	81
50	1	0.2571 (21.5217)	0.0350 (21.6319)	54
25	10	0.2606 (21.9621)	0.0453 (21.3243)	32

and the total sample size is $N = 5000$. Figure 2 presents sampled American put option prices with respect to the number of realizations. The dash line corresponds to basic Monte Carlo simulations, while the solid line corresponds to same Monte Carlo simulations using martingale control variate $\mathcal{M}_0(P_{BS}^{BAW})$.

4. Conclusion

In the context of multifactor stochastic volatility models, we propose a martingale control variate method to price options by Monte Carlo simulations. A theoretical variance analysis is provided to characterize the size of the variance reduced by martingale control variate in the case that driving volatility time scales are well separated. Comparing to plain Monte Carlo simulations, significant variance reduction ratios for European, Barrier and American options are obtained even in regimes where volatility time scales are not so well separated. These results confirm the robustness of our method based on martingale control variates constructed as in delta hedging strategies. The effectiveness of our method depends on option price approximations to the pricing problem considered.

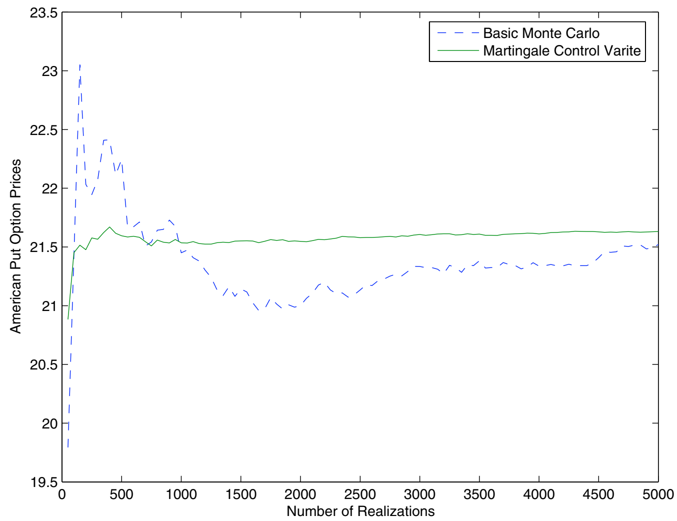

FIGURE 2. Monte Carlo simulations for an American put option price when $1/\varepsilon = 50$ and $\delta = 1$. Sampled prices are obtained along the number of realizations.

APPENDIX A. DERIVATION OF THE ACCURACY OF THE VARIANCE ANALYSIS

In order to prove Theorem 1.1, we need the following three lemmas.

Lemma A.1. *Under the assumptions of Theorem 1.1, for any fixed initial state $(0, x, y, z)$, there exists a positive constant $C_1 > 0$ such that for $\varepsilon \leq 1$ and $\delta \leq 1$, one has*

$$\mathbb{E}^\star{}_{0,t,x,y,z}\left\{\int_0^T \mathrm{e}^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}}{\partial x}\right)^2 (s, S_s, Y_s, Z_s) f^2(Y_s, Z_s) S_s^2 \mathrm{d}s\right\} \leq C_1 \max\{\varepsilon, \delta\}.$$

Proof. By Cauchy-Schwartz inequality we have

$$\begin{aligned}\mathbb{E}^\star{}_{0,t,x,y,z}&\left\{\int_0^T \mathrm{e}^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}}{\partial x}\right)^2 (s, S_s, Y_s, Z_s) f^2(Y_s, Z_s) S_s^2 \mathrm{d}s\right\} \\ &\leq \sqrt{\mathbb{E}^\star \int_0^T \left(\frac{\partial P^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}}{\partial x}\right)^4 (s, S_s, Y_s, Z_s) \mathrm{d}s} \\ &\quad \times \sqrt{\int_0^T \mathbb{E}^\star\left\{f^4(Y_s, Z_s)\left(\mathrm{e}^{-rs} S_s\right)^4\right\} \mathrm{d}s},\end{aligned} \tag{24}$$

where we omitted the sub-scripts under the expectation $\mathbb{E}^\star$. The second factor on the right hand side of this inequality is bounded by

$$\sqrt{\int_0^T \mathbb{E}^\star\left\{f^4(Y_s, Z_s)\left(\mathrm{e}^{-rs} S_s\right)^4\right\} \mathrm{d}s} \leq C_f \sqrt{\int_0^T \mathbb{E}^\star\left\{\left(\mathrm{e}^{-rs} S_s\right)^4\right\} \mathrm{d}s} \tag{25}$$

for some constant C_f, as the volatility function f is bounded. Using the notation $\sigma_t = f(Y_t, Z_t)$ as in (1), and $W^{(0)*} = W$ for simplicity, one has

$$\mathrm{e}^{-rs} S_s = x \mathrm{e}^{\int_0^s \sigma_u \mathrm{d}W_u - \frac{1}{2}\int_0^s \sigma_u^2 \mathrm{d}u},$$

and therefore

$$I\!E^\star \left\{ \left(\mathrm{e}^{-rs} S_s\right)^4 \right\} = x^4 I\!E^\star \left\{ \mathrm{e}^{6\int_0^s \sigma_u^2 \mathrm{d}u} \mathrm{e}^{\int_0^s 4\sigma_u \mathrm{d}W_u - \frac{1}{2}\int_0^s 16\sigma_u^2 \mathrm{d}u} \right\} \leq C_f' x^4 I\!E^\star \left\{ \mathrm{e}^{\int_0^s 4\sigma_u dW_u - \frac{1}{2}\int_0^s 16\sigma_u^2 \mathrm{d}u} \right\} = C_f' x^4,$$

where we have used again the boundedness of f, and the martingale property. Combined with (25) we obtain

$$\sqrt{\int_0^T I\!E^\star \left\{ f^4(Y_s, Z_s) \left(\mathrm{e}^{-rs} S_s\right)^4 \right\} \mathrm{d}s} \leq C_2, \tag{26}$$

for some positive constant C_2.

In order to study the first factor on the right hand side of the inequality (24), we have to control the "delta" approximation, $\frac{\partial P^{\varepsilon,\delta}}{\partial x} \to \frac{\partial P_{BS}}{\partial x}$, as opposed to the option price approximation, $P^{\varepsilon,\delta} \to P_{BS}$, studied in [8] for European options, or in [7] for digital-type options.

By pathwise differentiation (see [9] for instance), the chain rule can be applied and we obtain

$$\frac{\partial P^{\varepsilon,\delta}}{\partial S_t}(t, S_t, Y_t, Z_t) = I\!E^\star \left\{ \mathrm{e}^{-r(T-t)} \mathbf{I}_{\{S_T > K\}} \frac{\partial S_T}{\partial S_t} \mid S_t, Y_t, Z_t \right\}.$$

At time $t = 0$,

$$\mathrm{e}^{-rT} \frac{\partial S_T}{\partial S_0} = \mathrm{e}^{\int_0^T \sigma_t \mathrm{d}W_t^{(0)*} - \frac{1}{2}\int_0^T \sigma_t^2 \mathrm{d}t} \tag{27}$$

gives an exponential martingale, and therefore one can construct a $I\!P^\star$-equivalent probability measure $\tilde{P}$ by Girsanov theorem. As a result, the delta $\frac{\partial P^{\varepsilon,\delta}}{\partial S_t}(t, S_t, Y_t, Z_t)$ has a probabilistic representation under the new measure $\tilde{P}$ corresponding to the digital-type option

$$\frac{\partial P^{\varepsilon,\delta}}{\partial S_t}(t, S_t, Y_t, Z_t) = \tilde{E} \left\{ \mathbf{I}_{\{S_T > K\}} \mid S_t, Y_t, Z_t \right\},$$

where the dynamics of S_t becomes

$$\mathrm{d}S_t \quad = \quad \left(r + f^2\left(Y_t, Z_t\right)\right) S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}\tilde{W}_t,$$

with $\tilde{W}$ being a standard Brownian motion under $\tilde{P}$. The dynamics of Y_t and Z_t remain the same as in (1) because we have assumed here zero correlations between Brownian motions. Then one can apply the accuracy result in [7] for digital options to claim that

$$\left| \tilde{E} \left\{ \mathbf{I}_{\{S_T > K\}} \mid S_t, Y_t, Z_t \right\} - \bar{E} \left\{ \mathbf{I}_{\{\bar{S}_T > K\}} \mid \bar{S}_t = S_t, Z_t \right\} \right| \leq C_3(Y_t) \max\{\sqrt{\varepsilon}, \sqrt{\delta}\},$$

where the constant C_3 may depend on Y_t, and the "homogenized" stock price $\bar{S}_t$ satisfies

$$\mathrm{d}\bar{S}_t \quad = \quad \left(r + \bar{\sigma}^2(Z_t)\right) \bar{S}_t \mathrm{d}t + \bar{\sigma}(Z_t) \bar{S}_t \mathrm{d}\bar{W}_t$$

with $\bar{W}_t$ being a standard Brownian motion [6]. In fact, the homogenized approximation $\bar{E}\left\{\mathbf{I}_{\{\bar{S}_T>K\}} \mid \bar{S}_t, Z_t\right\}$ is a probabilistic representation of the homogenized "delta", $\frac{\partial P_{BS}}{\partial x}$. Consequently, we obtain the accuracy result for delta approximation:

$$\left|\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x}-\frac{\partial P_{BS}}{\partial x}\right)(t,S_t,Y_t,Z_t)\right| \le C_3(Y_t)\max\{\sqrt{\varepsilon},\sqrt{\delta}\}.$$

The existence of moments of Y_t ensures the existence of the fourth moment of $C_3(Y_t)$, and therefore the first factor on the right hand side of (24) is bounded by

$$\sqrt{I\!\!E^\star\left\{\int_0^T\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x}-\frac{\partial P_{BS}}{\partial x}\right)^4(s,S_s,Y_s,Z_s)\mathrm{d}s\right\}} \le C_4\max\{\varepsilon,\delta\} \tag{28}$$

for some positive constant C_4. From (24), (28) and (26), we conclude that

$$I\!\!E^\star\left\{\int_0^T \mathrm{e}^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x}-\frac{\partial P_{BS}}{\partial x}\right)^2(s,S_s,Y_s,Z_s)f^2(Y_s,Z_s)S_s^2\mathrm{d}s\right\} \le C_1\max\{\varepsilon,\delta\}$$

for some constant C_1. □

Lemma A.2. *Under the assumptions of Theorem 1.1, for any fixed initial state $(0,x,y,z)$, there exists ε a positive constant C such that for $\varepsilon \le 1$ and $\delta \le 1$, one has*

$$\int_0^T \mathrm{e}^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial y}\right)^2(s,S_s,Y_s,Z_s)g_1^2(Y_s)\mathrm{d}s \le C\,\varepsilon^2. \tag{29}$$

Proof. Conditioning on the path of volatility process and by iterative expectations, the price of a European option can be expressed as

$$\begin{aligned} P^{\varepsilon,\delta}(t,x,y,z) &= I\!\!E^\star{}_{t,x,y,z}\left\{I\!\!E^\star\left\{\mathrm{e}^{-r(T-t)}\left(S_T-K\right)^+ \mid \sigma_s, t\le s\le T\right\}\right\} \\ &= I\!\!E^\star{}_{t,x,y,z}\left\{P_{BS}\left(t,x;K,T;\sqrt{\overline{\sigma^2}}\right)\right\}, \end{aligned} \tag{30}$$

where the realized variance is denoted by $\overline{\sigma^2}$:

$$\overline{\sigma^2} = \frac{1}{T-t}\int_t^T f(Y_s,Z_s)^2\mathrm{d}s. \tag{31}$$

Taking a pathwise derivative for $P^{\varepsilon,\delta}$ [9] with respect to the fast varying variable y, we deduce by the chain rule

$$\frac{\partial P^{\varepsilon,\delta}}{\partial y}(t,x,y,z) = I\!\!E^\star{}_{t,x,y,z}\left\{\frac{\partial P_{BS}}{\partial\sigma}\left(t,x;K,T;\sqrt{\overline{\sigma^2(y,z)}}\right)\frac{\partial\sqrt{\overline{\sigma^2}}}{\partial y}\right\}. \tag{32}$$

Inside of the expectation the first derivative, known as Vega,

$$\frac{\partial P_{BS}}{\partial\sigma} = \frac{x\mathrm{e}^{-\mathrm{d}_1^2/2}\sqrt{T-t}}{\sqrt{2\pi}},$$

with $d_1 = \frac{\log(x/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, is uniformly bounded in σ. Using the chain rule one obtains

$$\frac{\partial\sqrt{\overline{\sigma^2}}}{\partial y} = \frac{1}{(T-t)\sqrt{\overline{\sigma^2}}}\int_t^T \left[\frac{\partial f}{\partial y}(Y_s,Z_s)\frac{\partial Y_s}{\partial y} + \frac{\partial f}{\partial z}(Y_s,Z_s)\frac{\partial Z_s}{\partial y}\right] f(Y_s,Z_s)\mathrm{d}s. \tag{33}$$

In order to control the growth rate of $\frac{\partial Y_s}{\partial y}$ and $\frac{\partial Z_s}{\partial y}$ we consider their dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\frac{\partial Y_s}{\partial y}\\ \frac{\partial Z_s}{\partial y}\end{pmatrix} = \begin{pmatrix} -\frac{1}{\varepsilon}+\frac{\nu_1\sqrt{2}}{\sqrt{\varepsilon}}\frac{\partial\Lambda_1}{\partial y}(Y_s,Z_s) & \frac{\nu_1\sqrt{2}}{\sqrt{\varepsilon}}\frac{\partial\Lambda_1}{\partial z}(Y_s,Z_s)\\ \sqrt{\delta}\nu_2\frac{\partial\Lambda_2}{\partial y}(Y_s,Z_s) & -\delta+\sqrt{\delta}\nu_2\sqrt{2}\frac{\partial\Lambda_2}{\partial z}(Y_s,Z_s)\end{pmatrix}\begin{pmatrix}\frac{\partial Y_s}{\partial y}\\ \frac{\partial Z_s}{\partial y}\end{pmatrix} \tag{34}$$

with the initial condition $(\frac{\partial Y_t}{\partial y},\frac{\partial Z_t}{\partial y}) = (1,0)$.

Rescaling the system (34) by defining $\tilde{Y}_s^\varepsilon = Y_{s\varepsilon}$ and $\tilde{Z}_s^\varepsilon = Z_{s\varepsilon}$, we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\frac{\partial\tilde{Y}_s^\varepsilon}{\partial y}\\ \frac{\partial\tilde{Z}_s^\varepsilon}{\partial y}\end{pmatrix} &= \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{\partial\tilde{Y}_s^\varepsilon}{\partial y}\\ \frac{\partial\tilde{Z}_s^\varepsilon}{\partial y}\end{pmatrix}\\ &+ \sqrt{\varepsilon}\begin{pmatrix}\nu_1\sqrt{2}\frac{\partial\tilde{\Lambda}_1}{\partial y}(\tilde{Y}_s^\varepsilon,\tilde{Z}_s^\varepsilon) & \frac{\partial\tilde{\Lambda}_1}{\partial z}(\tilde{Y}_s^\varepsilon,\tilde{Z}_s^\varepsilon)\\ \sqrt{\delta\varepsilon}\nu_2\frac{\partial\tilde{\Lambda}_2}{\partial y}(\tilde{Y}_s^\varepsilon,\tilde{Z}_s^\varepsilon) & -\delta\sqrt{\varepsilon}+\sqrt{\delta\varepsilon}\nu_2\sqrt{2}\frac{\partial\tilde{\Lambda}_2}{\partial z}(\tilde{Y}_s^\varepsilon,\tilde{Z}_s^\varepsilon)\end{pmatrix}\begin{pmatrix}\frac{\partial\tilde{Y}_s^\varepsilon}{\partial y}\\ \frac{\partial\tilde{Z}_s^\varepsilon}{\partial y}\end{pmatrix}.\end{aligned}$$

The functions $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are defined according to the rescaling and they are smooth and bounded as Λ's. By a classical stability result [2], we obtain $|\frac{\partial Y_s}{\partial y}| < C_5\mathrm{e}^{-(s-t)/\varepsilon}$ and $|\frac{\partial Z_s}{\partial y}| < C_6\delta\varepsilon$ for some constants C_5 and C_6. Applying these estimates to (33) and by the smooth boundness of f, we obtain

$$\frac{\partial\sqrt{\overline{\sigma^2}}}{\partial y} \le C\varepsilon$$

for some C, and consequently a similar bound for $\frac{\partial P^{\varepsilon,\delta}}{\partial y}(t,x,y,z)$ in (32). Finally, as $g_1 = \nu_1\sqrt{2}$, Lemma A.2 follows. □

Lemma A.3. *Under the assumptions of Theorem 1.1, for any fixed initial state* $(0,x,y,z)$*, there exists a positive constant* C *such that for* $\varepsilon \le 1$ *and* $\delta \le 1$*, one has*

$$\int_0^T \mathrm{e}^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial z}\right)^2(s,S_s,Y_s,Z_s)g_2^2(Z_s)\mathrm{d}s \le C.$$

Proof. The proof is similar to Lemma A.2. □

From the bounds in Lemma A.1, A.2 and A.3, we deduce Theorem 1.1.

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