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BRANCHING RANDOM MOTIONS, NONLINEAR HYPERBOLIC SYSTEMS AND TRAVELLING WAVES

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Abstract. A branching random motion on a line, with abrupt changes of direction, is studied. The branching mechanism, being independent of random motion, and intensities of reverses are defined by a particle's current direction. A solution of a certain hyperbolic system of coupled non-linear equations (Kolmogorov type backward equation) has a so-called McKean representation *via* such processes. Commonly this system possesses travelling-wave solutions. The convergence of solutions with Heaviside terminal data to the travelling waves is discussed. The paper realizes the McKean's program for the Kolmogorov-Petrovskii-Piskunov equation in this case. The Feynman-Kac formula plays a key role.

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1. Introduction

Travelling waves for the semilinear heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + f(u) \tag{1}$$

have been extensively studied beginning from the classic papers by Kolmogorov-Petrovskii-Piskunov [22] and Fisher [9] (see detailed review in [37]).

A travelling wave with velocity parameter a is a solution of equation (1) of the form $u = w_a(x - at)$. Here function w_a has the limits $w_a(-\infty) = 0$, $w_a(+\infty) = 1$ and, clearly, solves the ordinary equation

$$\frac{1}{2}w_a'' + aw_a' + f(w_a) = 0.$$

Basically, under certain assumptions on the nonlinearity term $f(u)$ the existence and uniqueness of solution of the initial value problem for (1) are well-known. Moreover, this solution (at least with Heaviside data) converges to the travelling front. More precisely,

$$u(x + m(t),\ t) \to w_{a_*}(x), \qquad t \to \infty \tag{2}$$

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with $a_* = \sqrt{2f'(1)}$ and with some centering term $m = m(t)$.

Since McKean [25, 26] (see also [3–5]) the connection between equation (1) and branching diffusion processes is established and widely applied. This approach is motivated by the following representation. Let $L(t)$ be the position of the left-most particle of a branching Brownian motion and let $g(u)$ be a probability generating function of the branching rule. Then $u = u(x,\ t) = \mathbb{P}(L(t) < x)$ is a solution of equation (1) with Heaviside initial conditions

$$u\mid_{t=0} = \theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0 \end{cases}$$

and with $f(u) = \lambda(g(u) - u)$, where λ is the branching intensity.

Equation (1) arises in physics (especially in combustion theory), chemical kinetics and in a various biological models for gene developments, population dynamics or nerve propagation (see, for instance, [15, 18, 19, 38, 39] and references therein).

Nevertheless this approach has the evident shortages: diffusion particles have infinite velocities and so they lack inertia, directions of their motion in separated time intervals are independent. To remedy these "unphysical" features it is possible to introduce a similar model, which is based on a random motion with finite velocity.

This idea has recently been the object of renewed interest of physicists and mathematicians (see [7, 8, 10, 12–17, 23, 24, 28, 29, 32]). It is applied also to financial market models [6, 36].

To describe these treatments we begin with the so-called telegraph random motion (see [11, 20, 21, 38]). We consider a particle, initially (at time $t = \tau$) situated at point $x \in (-\infty,\ \infty)$, which moves on a line $(-\infty,\ \infty)$ with constant velocity c. At time τ it chooses either initial direction with equal probability. Then it repeatedly takes an opposite direction at random instants $T_1,\ T_2,\ \dots$, which form a Poisson flow. The state of the process at time t is $(X(t),\ \sigma(t))$, where $X(t)$ is the current particle's position and $\sigma(t) = \pm c$ is its current velocity.

Further, we consider the particle, which commences the random motion $(X,\ \sigma)$ for an exponentially distributed holding time S independent of X. At S, the particle splits into a random number of pieces (offsprings). These new particles continue along independent paths of this random motion starting at $X(S)$, and are subject to the same splitting rule as the original particle. After an elapsed time $t - \tau$ we have $n = n(t - \tau)$ particles located at $X_1(t),\ \dots,\ X_n(t)$, where $n(t - \tau)$ is stochastic.

Write $\mathbb{P}_{+,(x,\tau)}$ and $\mathbb{P}_{-,(x,\tau)}$ (with associated expectations $\mathbb{E}_{\pm,(x,\tau)}$) for the laws of this process when it starts at time τ forwards $(+)$ and, respectively, backwards $(-)$, from the initial position $X(\tau) = x$.

Denote

$$u_+(x,\ \tau,\ t) = \mathbb{P}_{+,(x,\tau)}(X_1(t) > 0,\ \dots,\ X_n(t) > 0), \tag{3}$$

$$u_-(x,\ \tau,\ t) = \mathbb{P}_{-,(x,\tau)}(X_1(t) > 0,\ \dots,\ X_n(t) > 0). \tag{4}$$

Using a standard renewal arguments we prove (see Th. 3.1) that the probabilities $u_\pm = u_\pm(x,\ \tau,\ t)$ solve the semilinear hyperbolic system

$$\begin{cases} -\frac{\partial u_+}{\partial \tau} - c\frac{\partial u_+}{\partial x} = \mu_+(u_- - u_+) - \lambda_+ u_+ + \lambda_+ F_+(u_+,\ u_-), \\ -\frac{\partial u_-}{\partial \tau} + c\frac{\partial u_-}{\partial x} = \mu_-(u_+ - u_-) - \lambda_- u_- + \lambda_- F_-(u_+,\ u_-), \end{cases} \quad \tau < t \tag{5}$$

with the terminal conditions

$$u_+\mid_{\tau\uparrow t} = u_-\mid_{\tau\uparrow t} = \theta(x). \tag{6}$$

Here $\mu_+ > 0$ and $\mu_- > 0$ are the intensities of reverses, λ_+ and λ_- are the breeding rates of forward $(+)$ and backward $(-)$ moving particle respectively;

$$F_+(u_+,\ u_-) = \sum_{j+l\ge 2,\ j,l\ge 0} \beta^+_{jl} u_+^j u_-^l, \quad F_-(u_+,\ u_-) = \sum_{j+l\ge 2,\ j,l\ge 0} \beta^-_{jl} u_+^j u_-^l$$

are probability generating functions of breeding rule; β_{jl}^{+} (β_{jl}^{-}) denote the probability of j forward and l backward moving offsprings of a particle, which has forward (backward) direction at a splitting time. We assume that

$$\sum_{j+l\geq 2,\ j,l\geq 0} j\beta_{jl}^{\pm} < \infty, \qquad \sum_{j+l\geq 2,\ j,l\geq 0} l\beta_{jl}^{\pm} < \infty.$$

A weak solution of system (5)-(6) exists and it is unique (see Appendix B). Solution $u_{\pm},\ u_{\pm}(x,\ \tau,\ t) \in [0,\ 1]$ has discontinuities concentrated on characteristics $l_{\pm} = \{x = \pm c(t-\tau)\}$ only. The probabilistic interpretation (3)-(4) motivates following properties of $u_{\pm}$,

$$u_- \equiv 0 \ \text{ for } \ x \leq -c(t-\tau) \quad \text{and} \quad u_- \equiv 1 \ \text{ for } \ x > c(t-\tau); \tag{7}$$

$$u_+ \equiv 0 \ \text{ for } \ x \leq -c(t-\tau) \quad \text{and} \quad u_+ \equiv 1 \ \text{ for } \ x \geq c(t-\tau). \tag{8}$$

Moreover, jump values vanish:

$$u_-(-c(t-\tau)+0,\ \tau,\ t) \equiv 0, \quad u_+(c(t-\tau)-0,\ \tau,\ t) \equiv 1, \tag{9}$$

$$u_-(c(t-\tau)-0,\ \tau,\ t) \to 1, \quad u_+(-c(t-\tau)+0,\ \tau,\ t) \to 0, \tag{10}$$

as $\tau \downarrow -\infty$. The latter is proved in Section 3 (see Cor. 3.3).

System (5) is repeatedly obtained from both a phenomenological viewpoint and irreversible thermodynamics arguments in the papers of Dunbar-Othmer [7], Dunbar [8], Horsthemke [16, 17], Mendez-Camacho [28], Mendez-Compte [29], Othmer-Dunbar-Alt [32]. Many authors have studied the travelling wave-type solutions of (5) emphasizing for stability properties (see review by Fort and Mendez [10] and references therein). Nevertheless, convergence results of the form (2) are still unknown with exception of the very special case of $F_+ = u_+^2,\ F_- = u_-^2$. This nonlinearity corresponds to the following breeding rule. Particles, once born, live forever. At each splitting time S each particle gives birth *only to one* offspring at its own current position $X(S)$ and of *its own current velocity* $\sigma(S)$. The large time behaviour of solutions of the Cauchy problem for (5) (with $F_+ = u_+^2,\ F_- = u_-^2$) researched in details in [23] from both probabilistic and analytic viewpoints (see also [24]).

In this paper we discuss much more general branching rules. The main objective is to study the asymptotic behaviour of probabilistically represented solutions (3)-(4) of (5)-(6) keeping our treatment in the framework of the following three-step McKean's program [25]:

1) proof of existence of limits

$$u_+(x+m_+,\ \tau,\ t) \to w_+(x), \quad u_-(x+m_-,\ \tau,\ t) \to w_-(x), \quad \tau \downarrow -\infty \tag{11}$$

for all $x,\ t \in (-\infty,\ \infty)$, where $m_{\pm}$ are the medians of $u_{\pm}$;

2) stability properties of travelling fronts with respect to the velocity value;

3) identification of the limits in (11) as a travelling-wave solution of (5).

Notice that system (5) has two stationary solutions: $u_+ = u_- \equiv 0$ and $u_+ = u_- \equiv 1$. We assume that

(C1): there are no other stationary solutions of (5), *i.e.* the algebraic system

$$\begin{cases} \mu_+(y-x) + \lambda_+ \left(F_+(x,\ y) - x\right) = 0, \\ \\ \mu_-(x-y) + \lambda_- \left(F_-(x,\ y) - y\right) = 0 \end{cases}$$

has no solutions $x,\ y$, such that $0 \leq x,\ y \leq 1$, excepting $\{0,\ 0\}$ and $\{1,\ 1\}$.

The result of the first step is the following theorem.

Theorem 1.1. *Let parameters $\lambda_\pm$, $\mu_\pm$ and branching rules $F_\pm$ satisfy conditions C1. Then the limits in (11) exist with the centering terms $m_\pm = m_\pm(\tau)$ which are defined so as to satisfy*

$$u_+(m_+,\ \tau,\ t) = u_-(m_-,\ \tau,\ t) = 1/2. \tag{12}$$

As it follows from assertions (7)-(10), the centering terms $m_\pm$ in (12) are well defined. Theorem 1.1 is proved in Section 4.

We pass the second and the third steps assuming certain restrictions. To describe our assumptions we define the following expected numbers of particles born in each splitting:

$$J_{11} = \sum j\beta^+_{jl}, \quad J_{12} = \sum l\beta^+_{jl}, \quad J_{21} = \sum j\beta^-_{jl}, \quad J_{22} = \sum l\beta^-_{jl}.$$

Note that the matrix

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_+}{\partial u_+} & \frac{\partial F_+}{\partial u_-} \\ \frac{\partial F_-}{\partial u_+} & \frac{\partial F_-}{\partial u_-} \end{pmatrix}_{|u_+=u_-=1}$$

represents the Jacobian of nonlinearity $\{F_+(u_+,\ u_-),\ F_-(u_+,\ u_-)\}$ at $\{1,\ 1\}$.

Let

$$b_{11} = \mu_+ + \lambda_+(1 - J_{11}), \quad b_{22} = \mu_- + \lambda_-(1 - J_{22})$$

$$b_{12} = \mu_+ + \lambda_+ J_{12}, \qquad b_{21} = \mu_- + \lambda_- J_{21}. \tag{13}$$

We assume $b_{ij},\ i,\ j = 1,\ 2$ satisfy the condition

(C2): $\qquad b_{11} + b_{22} < 2\sqrt{b_{12}b_{21}}, \qquad b_{22} > 0.$

A travelling wave solution to (5) is a solution of the form $u_+ = w_+(x - a(t-\tau))$, $u_- = w_-(x - a(t-\tau))$. Define

$$a_* = c\frac{b_{11}^2 - b_{22}^2 + 4\sqrt{b_{12}b_{21}(b_{12}b_{21} - b_{11}b_{22})}}{(b_{11} - b_{22})^2 + 4b_{12}b_{21}}.$$

It can be proved that $0 < a_* < c$ (see Prop. 5.1 below).

The following theorem is proved in Section 5.

Theorem 1.2. *If conditions C1 and C2 hold and $a \in [a_*,\ c)$, then there exists one and, modulo translation, only one wave solution travelling with speed a.*

In Section 6 we try to pass the third step of McKean's program. We prove that the limits in (11) form a properly shifted travelling-wave solution and we determine the value of this shift.

Theorem 1.3. *Let conditions (C1) and (C2) hold and the limit*

$$\lim_{\tau\downarrow-\infty}(m_-(\tau) - m_+(\tau)) = \beta$$

exists. Then

$$\lim_{\tau\downarrow-\infty}(-\dot m_+(\tau)) = \lim_{\tau\downarrow-\infty}(-\dot m_-(\tau)) = a_*.$$

Moreover, if $w_+ = w_+(x)$, $w_- = w_-(x)$ are the limits in (11), then $\{w_+ = w_+(x),\ w_-^* = w_-(x-\beta)\}$ (or $\{w_+^* = w_+(x+\beta),\ w_- = w_-(x)\}$) form a (modulo translation unique) wave solution travelling with the velocity a_*.

Here $\dot m_+$ and $\dot m_-$ denote a derivatives in τ.

Remark 1.4. In this paper we explore the backward equations (5), but it is easy to transfer all results into the results for the forward equation

$$\begin{cases} \frac{\partial u_+}{\partial t} - c\frac{\partial u_+}{\partial x} = \mu_+(u_- - u_+) - \lambda_+ u_+ + \lambda_+ F_+(u_+,\ u_-), \\ \\ \frac{\partial u_-}{\partial t} + c\frac{\partial u_-}{\partial x} = \mu_-(u_+ - u_-) - \lambda_- u_- + \lambda_- F_-(u_+,\ u_-), \end{cases} \quad t > \tau. \tag{14}$$

The unique weak solution of system (14) (with Heaviside initial condition $u_\pm\ |_{t\downarrow\tau} = \theta(x)$) can be interpreted as

$$u_\pm(x,\ \tau,\ t) = \mathbb{P}_{\pm,(0,\tau)}(X_1(t) < x,\ \ldots, X_n(t) < x).$$

Remark 1.5. Recently some results on travelling waves for the branching telegraph-like processes with variable velocities (and for respective hyperbolic systems with variable coefficients $c = c(x)$) have been obtained (see [35]). These results are heavily based on theorems 1.1-1.3.

2. Telegraph processes and Feynman-Kac connection

In this section we remind some properties of the telegraph process (see [21] or [38] for further details).

Let $(\Omega,\ \mathcal{F},\ \mathbb{P})$ be a complete probability space and let $(X(t),\ \sigma(t))$, $t \geq \tau$ be a telegraph process with alternating velocities $\pm c$ and intensities $\mu_\pm$, which is defined in introduction. Denote by $f_\pm$ and $b_\pm$ (generalized) transition probability densities of Markov process $(X(t),\ \sigma(t))$, $t \geq \tau$, *i.e.* for any measurable A

$$\mathbb{P}_{\pm,(x,\tau)}(X(t) \in A,\ \sigma(t) = +c) = \int\limits_A f_\pm(x,\ \tau,\ y,\ t)\mathrm{d}y,$$

$$\mathbb{P}_{\pm,(x,\tau)}(X(t) \in A,\ \sigma(t) = -c) = \int\limits_A b_\pm(x,\ \tau,\ y,\ t)\mathrm{d}y,$$

and denote $f = (f_+ + f_-)/2$, $b = (b_+ + b_-)/2$, $p_+ = f_+ + b_+$, $p_- = f_- + b_-$, $p = f + b = (p_+ + p_-)/2$. Observe that these functions contain Dirac component along characteristics (see *e.g.* [31]).

It is known that $(f_+,\ f_-)$, $(b_+,\ b_-)$ as well as $(p_+,\ p_-)$ are solutions of the system

$$\begin{cases} -\frac{\partial v_+}{\partial \tau} - c\frac{\partial v_+}{\partial x} = \mu_+(v_- - v_+), \\ \\ -\frac{\partial v_-}{\partial \tau} + c\frac{\partial v_-}{\partial x} = \mu_-(v_+ - v_-), \qquad \tau < t. \end{cases} \tag{15}$$

To determine $f_\pm$, $b_\pm$ or $p_\pm$ system (15) should be supplied with the following terminal conditions:

$$f_+\ |_{\tau\uparrow t} = \delta(x - y), \qquad f_-\ |_{\tau\uparrow t} = 0;$$
$$b_+\ |_{\tau\uparrow t} = 0, \qquad b_-\ |_{\tau\uparrow t} = \delta(x - y)$$

and

$$p_+\ |_{\tau\uparrow t} = \delta(x - y), \qquad p_-\ |_{\tau\uparrow t}\ \ \delta(x - y).$$

Moreover, for any bounded left-continuous in y and continuous in t function $g = g(y,\ \sigma,\ t)$, $\sigma = \pm c$, $t \geq \tau$, $y \in (-\infty,\ \infty)$ the expectations

$$v_\pm(x,\ \tau,\ t)\ = \mathbb{E}_{\pm,(x,\tau)} g(X(t),\ \sigma(t),\ t)$$

form the solution of system (15) with the terminal conditions

$$v_+\ |_{\tau\uparrow t} = g_+(x,\ t), \qquad v_-\ |_{\tau\uparrow t} = g_-(x,\ t).$$

Here and everywhere below we repeatedly use g_+ for $g(\,\cdot,\ +c,\ \cdot\,)$ and g_- for $g(\,\cdot,\ -c,\ \cdot\,)$.

In the particular case of $\mu_+ = \mu_- = \mu \equiv$ const, system (15) is equivalent to so-called telegraph equation:

$$\frac{\partial^2 v}{\partial \tau^2} - 2\mu \frac{\partial v}{\partial \tau} = c^2 \frac{\partial^2 v}{\partial x^2}, \tag{16}$$

where $v = (v_+ + v_-)/2$. This transition is known as the Kac trick [21].

Solutions of equations (15) and (16) are well defined with δ-functions in terminal values (see *e.g.* [11], [31] or [33]). For example, the exact expressions for $f_\pm$, $b_\pm$ are well known (see (A.2)-(A.3) and cf. [2]):

$$f_+ = \mathrm{e}^{-\mu_+(t-\tau)}\delta\left(x + c(t-\tau)\right) + E(x,\ \tau,\ t) I_1\left(\mu_*\sqrt{(t-\tau)^2 - x^2/c^2}\right) \frac{\mu_*(t-\tau-x/c)}{\sqrt{(t-\tau)^2 - x^2/c^2}},$$

$$f_- = \mu_- E(x,\ \tau,\ t) I_0\left(\mu_*\sqrt{(t-\tau)^2 - x^2/c^2}\right), \qquad b_+ = \mu_+ E(x,\ \tau,\ t) I_0\left(\mu_*\sqrt{(t-\tau)^2 - x^2/c^2}\right),$$

$$b_- = \mathrm{e}^{-\mu_-(t-\tau)}\delta\left(x - c(t-\tau)\right) + E(x,\ \tau,\ t) I_1\left(\mu_*\sqrt{(t-\tau)^2 - x^2/c^2}\right) \frac{\mu_*(t-\tau+x/c)}{\sqrt{(t-\tau)^2 - x^2/c^2}},$$

where $\mu_* = \sqrt{\mu_+\mu_-}$, $E(x,\ \tau,\ t) = \frac{1}{2c}\mathrm{e}^{-(\mu_++\mu_-)(t-\tau)/2 + (\mu_+-\mu_-)x/(2c)}$.

Under rescaling $c \to \infty$, $\mu \to \infty$ such that $c^2/\mu \to$ const., equation (16) becomes backward Kolmogorov equation for the standard diffusion. More precisely the random motion $X = X(t)$ converges weakly to the Brownian motion (see, for instance, [20] or [34]). This observation motivates us to exploit this random process instead of Brownian motion.

We prove the existence of the limits in (11) by means of Feynman-Kac Lemma. To present this lemma in hyperbolic context let us consider the following linear terminal-value problem:

$$\begin{cases} -\frac{\partial v_+}{\partial \tau} - c\frac{\partial v_+}{\partial x} = \mu_+(x,\ \tau)(v_- - v_+) + k_+(x,\ \tau)v_+, \\[2ex] -\frac{\partial v_-}{\partial \tau} + c\frac{\partial v_-}{\partial x} = \mu_-(x,\ \tau)(v_+ - v_-) + k_-(x,\ \tau)v_-, \quad \tau < t \\[2ex] v_+ \mid_{\tau\uparrow t} = g_+(x,\ t), \qquad v_- \mid_{\tau\uparrow t} = g_-(x,\ t). \end{cases} \tag{17}$$

Here $k_\pm = k_\pm(x,\ \tau)$, $\mu_\pm = \mu_\pm(x,\ \tau)$, $\tau \le t$, $x \in (-\infty,\ \infty)$ are functions with possible discontinuities concentrated on characteristics $x = \pm c(t-\tau)$; $g_\pm = g_\pm(x,\ t)$ are bounded left-continuous in x and continuous in t, $t \ge \tau$ functions. As before we repeatedly unite by $h(\,\cdot,\ \sigma,\ \cdot)$, $\sigma = \pm c$ both h_+ and h_- for all functions h of this type.

A weak solution of (17) exists and it is unique (see *e.g.* [27]).

Let $X = X(t)$, $t \ge \tau$ be the telegraph process with parameters $\mu_\pm$, *i.e.* the transition probability densities $f_\pm$ and $b_\pm$ of X satisfy (15). Let $\{v_+,\ v_-\}$, $v_\pm = v_\pm(x,\ \tau), \tau < t,\ x \in (-\infty,\ \infty)$ be a weak solution to (17).

Theorem 2.1 (Feynman-Kac connection)**.** *Let* $\mathfrak{t}$, $\tau < \mathfrak{t} < t$ *be a stopping time for* X. *Then* v_+, v_- *have the representation*

$$v_+(x,\ \tau) = \mathbb{E}_{+,(x,\tau)} v(X(\mathfrak{t}),\ \sigma(\mathfrak{t}),\ \mathfrak{t}) \exp\left(\int_\tau^{\mathfrak{t}} k(X(s),\ \sigma(s),\ s)\mathrm{d}s\right), \tag{18}$$

$$v_-(x,\ \tau) = \mathbb{E}_{-,(x,\tau)} v(X(\mathfrak{t}),\ \sigma(\mathfrak{t}),\ \mathfrak{t}) \exp\left(\int_\tau^{\mathfrak{t}} k(X(s),\ \sigma(s),\ s)\mathrm{d}s\right). \tag{19}$$

Observe that for $\mathfrak{t} = t =$ const. formulas (18)-(19) connect $v_\pm(\,\cdot\,,\ \tau)$ and $v_\pm(\,\cdot\,,\ t)$ by means of the telegraph process $X = X(t)$. The original Feynman-Kac formula for the parabolic system exploits the Brownian motion for the analogous connection.

Proof. At first, let stopping time $\mathfrak{t}$ be a constant, $\mathfrak{t} = t,\ t > \tau$, and fixed.

Lemma 2.2. *Functions $v_\pm = v_\pm(x,\ \tau)$, which are defined by (18)-(19), satisfy the following system of integral equations:*

$$v_+(x,\ \tau) = v_+^0(x,\ \tau) + \int_\tau^t \mathrm{d}s \int_{-\infty}^{\infty} [f_+(x,\ \tau,\ z,\ s)k_+(z,\ s)v_+(z,\ s) \\ +b_+(x,\ \tau,\ z,\ s)k_-(z,\ s)v_-(z,\ s)]\,\mathrm{d}z, \tag{20}$$

$$v_-(x,\ \tau) = v_-^0(x,\ \tau) + \int_\tau^t \mathrm{d}s \int_{-\infty}^{\infty} [f_-(x,\ \tau,\ z,\ s)k_+(z,\ s)v_+(z,\ s) \\ +b_-(x,\ \tau,\ z,\ s)k_-(z,\ s)v_-(z,\ s)]\,\mathrm{d}z, \tag{21}$$

where

$$v_\pm^0(x,\ \tau) = \mathbb{E}_{\pm,(x,\tau)}g(X(t),\ \sigma(t),\ t).$$

The *proof* of Lemma follows form the evident identity: for any integrable function Φ

$$\exp\left(\int_\tau^t \Phi(s)\mathrm{d}s\right) = 1 + \int_\tau^t \Phi(s)\exp\left(\int_s^t \Phi(r)\mathrm{d}r\right)\mathrm{d}s.$$

To finish the proof of Theorem 3.1 for a constant stopping time it is sufficient to apply $-\frac{\partial}{\partial\tau} - c\frac{\partial}{\partial x}$ to (20) and $-\frac{\partial}{\partial\tau} + c\frac{\partial}{\partial x}$ to (21), exploiting (15) for $v_\pm^0,\ f_\pm$ and $b_\pm$. The passage to the general stopping time $\mathfrak{t}$ is plain (see *e.g.* [30]). □

3. Branching telegraph processes and McKean representation of solutions of nonlinear hyperbolic systems

Let the process $X = X(t)$ to be branching. Assume that the single particle starts at time $t = \tau$ from the point x and performs the telegraph random motion. At exponentially distributed instant $S > \tau$ (with parameter λ_+ for a forward moving particle and with λ_- for a backward moving one) it splits into a several (random number) particles. The descendants start to move from the point $X(S)$ independently one from another. They in turn split and reverse by the same rule.

Suppose that the forward moving particle splits on j forward and l backward moving parts with probability $\beta_{j,\ l}^+,\ j + l \geq 2$. For the backward moving particle the respective probabilities are $\beta_{j,\ l}^-,\ j + l \geq 2$. Denote by $F_+(u_+,\ u_-) = \sum_{j+l\geq 2}\beta_{j,\ l}^+ u_+^j u_-^l$ and $F_-(u_+,\ u_-) = \sum_{j+l\geq 2}\beta_{j,\ l}^- u_+^j u_-^l$ the probability generating functions of splitting rule.

As the result after an elapsed time $t - \tau > 0$ we have n particles situated at $X_1(t), \ldots, X_n(t)$ with the velocities $\sigma_1(t), \ldots, \sigma_n(t)$, $n = n(t - \tau)$.

Consider the expectations

$$u_+(x,\ \tau,\ t) = \mathbb{E}_{+,(x,\tau)}\prod_{i=1}^n g(X_i(t),\ \sigma_i(t),\ t), \tag{22}$$

$$u_-(x,\ \tau,\ t) = \mathbb{E}_{-,(x,\tau)}\prod_{i=1}^n g(X_i(t),\ \sigma_i(t),\ t). \tag{23}$$

As before $g = g(x,\ \sigma,\ t),\ x \in (-\infty,\ \infty),\ \sigma = \pm c,\ t > \tau$ is a bounded left-continuous in x and continuous in t function.

The following theorem is well-known (see [12] and *cf.* [25]).

Theorem 3.1. *Let* $\{u_+,\ u_-\},\ \tau < t,\ x \in (-\infty,\ \infty)$ *be the unique bounded weak solution of system (5) with terminal conditions*

$$u_+ \mid_{\tau\uparrow t} = g_+(x,\ t), \qquad u_- \mid_{\tau\uparrow t} = g_-(x,\ t) \tag{24}$$

(see Appendix B). Then $u_+,\ u_-$ *have representation (22)-(23).*

Corollary 3.2. *If* $L(t) = \min_{1\le i\le n} X_i(t)$ *is the position of the left-most particle and* $u_\pm = u_\pm(x,\ \tau,\ t) = \mathbb{P}_{\pm,(x,\tau)}(L(t) > 0)$*, then* $u_\pm$ *form the solution of (5) with the Heaviside terminal conditions*

$$u_\pm \mid_{\tau\uparrow t} = \theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0. \end{cases} \tag{25}$$

Corollary 3.3. *The solution of problem (5), (25) satisfies the jump conditions (7)–(10).*

Proof. Let us prove the assertion (10) for $u_+(-c(t-\tau)+0,\ \tau,\ t) := \psi_+(\tau)$. From Corollary 3.2 it follows that

$$\psi_+(\tau - \Delta\tau) = (1 - \mu_+\Delta\tau)(1 - \lambda_+\Delta\tau)\psi_+(\tau) + \lambda_+\Delta\tau F_+(\psi_+(\tau),\ 0) + o(\Delta\tau),\ \Delta\tau \to 0,$$

which implies the differential equation

$$\frac{\mathrm{d}\psi_+}{\mathrm{d}\tau} = (\mu_+ + \lambda_+)\psi_+ - \lambda_+ F_+(\psi_+,\ 0),\ \tau < t, \qquad \psi_+ \mid_{\tau\uparrow t} = 1.$$

Hence $\psi_+(\tau) \le \mathrm{e}^{-(\mu_+ + \lambda_+)(t-\tau)} \to 0$, as $\tau \downarrow -\infty$. The complete proof is similar. $\square$

Following Horsthemke [16, 17] we consider three main types of splitting rules.

1. *Isotropic reaction walk*

Let $F_+ = F_- = F(u)$, where $u = (u_+ + u_-)/2$ and $F(u) = \sum_{k\ge 2} \beta_k u^k$. This means that at breeding times a particle splits onto k parts with the probability β_k, which does not depend on the direction of motion. New particles choose either direction with equal probability.

In the particular case of $\mu_+ = \mu_- = \mu(\tau)$ and $\lambda_+ = \lambda_- = \lambda$ (λ is a constant) one can obtain from (5)

$$\begin{cases} \frac{\partial u}{\partial \tau} + c\frac{\partial w}{\partial x} = -\lambda(F(u) - u), \\ \\ \frac{\partial w}{\partial \tau} + c\frac{\partial u}{\partial x} = (2\mu + \lambda)w, \quad \tau < t, \end{cases} \tag{26}$$

$w = (u_+ - u_-)/2$. Eliminating w we have

$$\frac{\partial^2 u}{\partial \tau^2} - 2(\mu + \lambda)\frac{\partial u}{\partial \tau} = c^2\frac{\partial^2 u}{\partial x^2} - \lambda\frac{\partial F(u)}{\partial \tau} + (\lambda^2 + 2\mu\lambda)(F(u) - u). \tag{27}$$

Notice that (26) and (27) are equivalent to the reaction Cattaneo system and to the reaction telegraph equation respectively (see [10, 13, 16, 17]). If $F(u) = u^2$, the hyperbolic version of the classical Kolmogorov-Petrovskii-Piskunov equation arises (see [22, 25]).

2. *Direction independent reaction walk*

Assume a particle does not die and at the exponentially distributed instant gives a birth to k new particles (with probability β_k). Daughter particles choose either direction with equal probability and move accordingly with the same rule. In this case the nonlinearity of (5) has the form $F_+ = F(u)u_+,\ F_- = F(u)u_-$, where $F(u) = \sum_{k\geq 1}\beta_k u^k$. For the so-called branching-coalescence direction independent kinetic scheme [17] $F(u) = u$.

If $\mu_+ = \mu_- = \mu$, $\lambda_+ = \lambda_- = \lambda$, then system (5) is equivalent to

$$\begin{cases} \frac{\partial u}{\partial \tau} + c\frac{\partial w}{\partial x} = \lambda u(F(u) - 1), \\ \\ \frac{\partial w}{\partial \tau} + c\frac{\partial u}{\partial x} = (2\mu + \lambda)w - \lambda F(u)w, \quad \tau < t. \end{cases}$$

No reaction telegraph equation can be obtained in this case.

3. *Direction dependent reaction walk*

Consider the previous regime, but with some significant modifications. We shall distinguish two main versions.

A. Suppose that each new particle starts strictly in the opposite direction to the direction of the maternal particle. The generating functions are $F_+ = F(u_-)u_+,\ F_- = F(u_+)u_-$. In the particular case $F(u) = u$ (*i.e.* only one new particle arises) we have the branching-coalescence direction dependent kinetic scheme (see [17]).

B. Assume that each new particle starts in the same direction the maternal particle currently moves. The generating functions now are $F_+ = F(u_+)u_+,\ F_- = F(u_-)u_-$. The particular case of $F_+ = F_- = u$ is researched in details by Lyne [23] (see also [24]).

4. Kolmogorov-Petrovskii-Piskunov lemma for hyperbolic systems. Proof of Theorem 1.1

The following proposition plays a key role in the further construction.

Proposition 4.1. *Let $\{u_+,\ u_-\}$ be the McKean solution of (5) with Heaviside terminal data (25) at fixed time horizon t_0:*

$$u_+ \mid_{\tau\uparrow t_0} = u_- \mid_{\tau\uparrow t_0} = \theta(x). \tag{28}$$

Then functions $u_+(x + m_+(\tau),\ \tau,\ t_0)$ and $u_-(x + m_-(\tau),\ \tau,\ t_0)$ increase in τ, if $x > 0$, and decrease in τ, if $x < 0$. Here $m_\pm = m_\pm(\tau)$ are defined by (12).

Proof. (*cf.* Sect. 4 of [25]). First, we need in the following lemma.

Lemma 4.2. *Let $\{v_+,\ v_-\}$ be the solution to the Feynman-Kac system (17) with fixed time horizon t_0. If $v_+(x_0,\ \tau_0) > 0$ or $v_-(x_0,\ \tau_0) > 0$, $\tau_0 < t_0$, then there exists sample path $X_*(x_0,\ t)$, $t \in [\tau_0,\ t_0]$ of the telegraph process, such that*

$$v\left(X_*(x_0, t),\ \sigma_*(x_0, t),\ t\right) > 0 \tag{29}$$

(here $\sigma_*(x_0, t)$ is the velocity of $X_*(x_0, t)$).

Proof (of Lemma). Suppose contrariwise, that the existence of X_* fails. Define the stopping time $\mathfrak{t}$ so as to be the first solution in t, $t > \tau_0$ of

$$v\left(X(x_0, t),\ \sigma(x_0, t),\ t\right) \leq 0.$$

The expectations in the Feynman-Kac formula (18)-(19) (with $x = x_0,\ \tau = \tau_0$) are nonpositive, while $v_+(x_0,\ \tau_0) > 0$ or $v_-(x_0,\ \tau_0) > 0$. This contradiction completes the proof. $\square$

Now we prove Proposition 4.1 for u_+. The proof for u_- is similar.

Fix $\tau_0 < t_0$ and $\alpha > 0$. Denote $x_0 = m_+(\tau_0)$, $x_1 = m_+(\tau_0 - \alpha)$. Set (omitting t_0 from the notations)

$$\begin{aligned} V_+(x,\ \tau) &= u_+(x + x_1,\ \tau - \alpha) - u_+(x + x_0,\ \tau), \\ V_-(x,\ \tau) &= u_-(x + x_1,\ \tau - \alpha) - u_-(x + x_0,\ \tau). \end{aligned} \tag{30}$$

We must prove that

$$V_+(x,\ \tau_0) \le 0 \quad \text{for any} \quad x > 0, \tag{31}$$
$$V_+(x,\ \tau_0) \ge 0 \quad \text{for any} \quad x < 0. \tag{32}$$

First notice that for any $\bar{u}_\pm,\ u_\pm \in [0,\ 1]$

$$F_\pm(\bar{u}_+,\ \bar{u}_-) - F_\pm(u_+,\ u_-) = k_1^\pm \cdot (\bar{u}_+ - u_+) + k_2^\pm \cdot (\bar{u}_- - u_-), \tag{33}$$

where $k_i^\pm,\ i = 1,\ 2$ are some positive analytic functions of $\bar{u}_\pm,\ u_\pm$. By (5) and (33) functions V_+, V_- form the solution of (17) with

$$M_+ = \mu_+ + \lambda_+ k_2^+, \qquad M_- = \mu_- + \lambda_- k_1^-$$

instead of μ_+, μ_-, and

$$K_+ = \lambda_+(k_1^+ + k_2^+ - 1), \qquad K_- = \lambda_-(k_1^- + k_2^- - 1)$$

instead of k_+, k_-.

From terminal conditions (28) it follows

$$\begin{aligned} &V_\pm(x,\ t_0 - 0) \le 0, \quad \text{if} \quad x > -x_0, \\ &V_\pm(x,\ t_0 - 0) \ge 0, \quad \text{if} \quad x < -x_0. \end{aligned} \tag{34}$$

To prove (31) suppose, contrariwise, that $V_+(x_*,\ \tau_0) > 0$ for some $x_* > 0$.

Lemma 4.3. *If $V_+(x,\ \tau_0) > 0$, then there exists a sample path $X_* = X_*(x_*,\ t)$ of the telegraph process which starts at $(x_*,\ \tau_0)$ and passes to $(x,\ t_0)$ with some $x < -x_0$, such that*

$$V(X_*(x_*, t),\ -\sigma_*(x_*, t),\ t) > 0 \tag{35}$$

for all $t,\ \tau_0 \le t \le t_0$.

Proof. Consider the system

$$\begin{cases} -\frac{\partial \bar{V}_-}{\partial \tau} - c\frac{\partial \bar{V}_-}{\partial x} = M_+ \cdot (\bar{V}_+ - \bar{V}_-) + K_+ \cdot \bar{V}_-, \\ -\frac{\partial \bar{V}_+}{\partial \tau} + c\frac{\partial \bar{V}_+}{\partial x} = M_- \cdot (\bar{V}_- - \bar{V}_+) + K_- \cdot \bar{V}_+, \tau < t_0 \end{cases} \tag{36}$$

with the terminal conditions $\bar{V}_+|_{\tau\uparrow t_0} = V_-|_{\tau\uparrow t_0},\ \bar{V}_-|_{\tau\uparrow t_0} = V_+|_{\tau\uparrow t_0}$. Thus $\bar{V}(x,\ \sigma,\ t) \equiv V(x,\ -\sigma,\ t),\ t \in [\tau_0,\ t_0],\ x \in (-\infty,\ \infty)$. Since system (4.3) has the Feynman-Kac form, by Lemma 4.2 there exists the sample path X_* of the telegraph process with the property

$$\bar{V}(X_*,\ \sigma_*,\ t) = V(X_*,\ -\sigma_*,\ t) > 0. \qquad \square$$

To finish the proof of Proposition 4.1 we fix the path X_* and consider the telegraph process $X = X(0,\ t),\ t \in [\tau_0,\ t_0]$ with starting point at 0.

Let $\mathfrak{t} \in [\tau_0,\ t_0]$ be the first moment of intersection of X with $X_* = X_*(x_*,\ t),\ \tau_0 < t < t_0$. It is clear that at the passage time $\mathfrak{t}$ the trajectory X_* continues backwards while X has the forward direction, *i.e.* $\sigma_*(x_*\ \mathfrak{t}) = -c$, $\sigma(0,\ \mathfrak{t}) = +c$. Applying Theorem 3.1 we have

$$V_+(0,\ \tau_0) = \mathbb{E}_{+,(0,\tau_0)} V(X(\mathfrak{t}),\ \sigma(\mathfrak{t}),\ \mathfrak{t}) \exp\left(\int_{\tau_0}^{\mathfrak{t}} K(X(s),\ \sigma(s),\ s)\mathrm{d}s\right). \tag{37}$$

By (35) the expectation in (37) is positive while by the definition (30) we have $V_+(0,\ \tau_0) = 0$. This contradiction completes the proof. □

By Proposition 4.1 the following limits exist:

$$\lim_{\tau\downarrow-\infty} u_+(x + m_+(\tau),\ \tau) = w_+(x), \tag{38}$$

$$\lim_{\tau\downarrow-\infty} u_-(x + m_-(\tau),\ \tau) = w_-(x), \tag{39}$$

which completes the proof of Theorem 1.1.

Note that functions $w_\pm = w_\pm(x)$ increase in x, $w_\pm(0) = 1/2$ and by Lemma 4.1

$$\frac{\partial u_\pm}{\partial \tau}(x + m_\pm(\tau),\ \tau) \le 0 \quad \text{for} \quad x < 0; \tag{40}$$

$$\frac{\partial u_\pm}{\partial \tau}(x + m_\pm(\tau),\ \tau) \le 0 \quad \text{for} \quad x > 0. \tag{41}$$

Now we should establish a connection of w_+ and w_- with travelling-wave solutions of (5). Our plan follows the strategy of McKean [25]. Firstly, we obtain some inequalities for possible velocities of travelling fronts (Sect. 5). Secondly, the upper bound for medians $m_\pm$ is found (Sect. 6). Finally, the direct analytic treatment leads to the main result.

5. Wave solutions. Proof of Theorem 1.2

In this section we study stability properties of travelling-wave solutions. We suppose here $\mu_+,\ \mu_-$ to be constant. The travelling-wave solution of system (5) is a solution of the form $u_\pm(x,\ \tau,\ t) = w_\pm(x - a(t-\tau))$. Functions w_+ and w_- describe a travelling wave, if

$$\begin{cases} -(c+a)w_+' = \mu_+(w_- - w_+) - \lambda_+ w_+ + \lambda_+ F_+(w_+,\ w_-), \\ \\ (c-a)w_-' = \mu_-(w_+ - w_-) - \lambda_- w_- + \lambda_- F_-(w_+,\ w_-). \end{cases} \tag{42}$$

We are interested in probabilistic solutions of (42), *i.e.* $0 \le w_\pm \le 1$, $\lim\limits_{z\to-\infty} w_\pm(z) = 0$, $\lim\limits_{z\to+\infty} w_\pm(z) = 1$.

The states $\{0,\ 0\}$ and $\{1,\ 1\}$ are clearly equilibriums of system (42). According with assumption C1 there are no other equilibrium points. We show in this section that condition C2 guarantees the point $\{0,\ 0\}$ to be unstable and the point $\{1,\ 1\}$ to be stable.

More precisely, we should prove, that there exists a monotone wave solution travelling with the speed a, $a_* \le a < c$ from $\{0,\ 0\}$ to $\{1,\ 1\}$, where a_* is some positive bound which depends on parameters $b_{ij},\ i,\ j = 1,\ 2$ (see (13)). This solution $\{w_+,\ w_-\}$ of (42) is modulo translation unique.

The proof splits onto the following parts.

5.1. **Phase portrait at** $\{0,\ 0\}$

A linearization of (42) at point $\{0,\ 0\}$ has the form

$$\begin{cases} w_+' = \frac{\lambda_+ + \mu_+}{c+a} w_+ - \frac{\mu_+}{c+a} w_-, \\ \\ w_-' = \frac{\mu_-}{c-a} w_+ - \frac{\lambda_- + \mu_-}{c-a} w_-. \end{cases} \tag{43}$$

Eigenvalues of (43) are the roots of the equation

$$\zeta^2 + \left[\frac{\lambda_- + \mu_-}{c-a} - \frac{\lambda_+ + \mu_+}{c+a}\right]\zeta - \frac{\lambda_+\lambda_- + \mu_+\lambda_- + \mu_-\lambda_+}{c^2-a^2} = 0.$$

Clearly, if $a^2 < c^2$, the eigenvalues have opposite signs. After some easy algebra one can find that for a positive ζ eigenvector $\mathbf{e} = (e_1,\ e_2)$ satisfies $e_1 > e_2 > 0$. Thus $\mathbf{e}$ is directed into $\{(w_+,\ w_-) :\ 0 < w_- < w_+\}$. Hence $\{0,\ 0\}$ is a saddle point with positive outgoing orbit.

5.2. **Phase portrait at** $\{1,\ 1\}$

The linear part of (42) at $\{1,\ 1\}$ has the matrix

$$A = \begin{pmatrix} \frac{b_{11}}{c+a} & -\frac{b_{12}}{c+a} \\ & \\ \frac{b_{21}}{c-a} & -\frac{b_{22}}{c-a} \end{pmatrix},$$

where $b_{ij},\ i,\ j = 1,\ 2$ are defined in (13). Our aim is to show that if $a_* < a < c$ with a suitable $a_* > 0$, then assumption C2 imply the state $\{1,\ 1\}$ to be a stable node. Moreover eigenvectors of A have positive entries.

To check this note that matrix A has two negative eigenvalues if and only if

$$\begin{cases} \mathrm{tr}A < 0, \\ \det A > 0, \\ (\mathrm{tr}A)^2 - 4\det A > 0. \end{cases} \tag{44}$$

Here $\mathrm{tr}A$ is the trace and $\det A$ is the determinant of matrix A.

Inequalities (44) read in details as follows:

$$\alpha B > b, \tag{45}$$

$$4\mu_*^2 > B^2 - b^2, \tag{46}$$

$$f(\alpha) \equiv \alpha^2(b^2 + 4\mu_*^2) - 2\alpha bB + B^2 - 4\mu_*^2 > 0. \tag{47}$$

Here we use the following notations: $B = b_{11} + b_{22}$, $b = b_{11} - b_{22}$, $\mu_*^2 = b_{12}b_{21}$, $\alpha = a/c$.

Proposition 5.1. *Let condition C2 to be hold. Then (46) fulfilled and (47) is equivalent to* $\alpha_* < \alpha < 1$ *with*

$$\alpha_* = \frac{bB + 4\mu_*\sqrt{\mu_*^2 - b_{11}b_{22}}}{b^2 + 4\mu_*^2}. \tag{48}$$

Furthermore, $0 < \alpha_* < 1$ *and inequality (45) holds for any* $\alpha,\ \alpha_* < \alpha < 1$.

Proof. First note that C2 leads to (46): if $B \geq 0$, then $4\mu_*^2 > B^2$ by the first part of C2 and thus we have (46); if $B < 0$, then (by $b_{22} > 0$) $b^2 > B^2$ and (46) is obviously fulfilled.

Now we study the intersection of parabola (47) and the horizontal axis. Notice that $f(1) = 4b_{22}^2 > 0$ and $f(-1) = 4b_{11}^2 \geq 0$ and, if $bB \neq 0$, then

$$f(b/B) = (B^2 - b^2)(B^2 - b^2 - 4\mu_*^2)/B^2, \tag{49}$$

$$f(B/b) = 4\mu_*^2\left(B^2/b^2 - 1\right). \tag{50}$$

Furthermore, if $b_{11} \geq 0$, then $B^2 \geq b^2$, and by (46) and (49) we have $f(b/B) \leq 0$; if $b_{11} \leq 0$, then $B^2 \leq b^2$, and by (50) it leads to $f(B/b) \leq 0$. Finally, in the case $bB = 0$ by (46) we have $f(0) < 0$.

Summarizing, we conclude that inequality (47) is equivalent to

$$\alpha_* < \alpha < 1, \tag{51}$$

where α_* is the greater root of the equation $f(\alpha) = 0$, *i.e.* α_* is defined by (48). Moreover, if $b_{11} \geq 0$, then $\alpha_* > b/B$, and if $b_{11} \leq 0$, then $\alpha_* > B/b$.

From C2 it follows that $\alpha_* > 0$. Indeed, if $B > 0$, then $f(0) = B^2 - 4\mu_*^2 < 0$ and thus α_* is positive; if $B \leq 0$, then $b < 0$ and thus by (50) $f(B/b) < 0$, where $0 < B/b < 1$. Hence $\alpha_* > B/b > 0$.

If condition C2 fails, then α_* can be negative. This case is not considered in this paper and will be elsewhere reported later.

To check (45) note that in the case $b_{11} \geq 0$ we have $B > 0$, thus inequality (45) follows from $\alpha_* > b/B$. If $b_{11} < 0$, then $b < 0$ and for $B \leq 0$ we have $1 > \alpha > \alpha_* > B/b$. Then $B > \alpha b$ and thus $\alpha B > \alpha^2 b > b$, which is required. The case $b < 0,\ B > 0$ is evident. □

Remark 5.2. It follows from our above explanations that as α_* is the greater root of $f(\alpha) = 0$,

$$B - \alpha_* b = 2\mu_* \sqrt{1 - \alpha_*^2}. \tag{52}$$

It is important to note that an eigenvectors of A has the right entries.

Proposition 5.3. *Let ζ be an eigenvalue of matrix A and $\mathbf{e} = \{e_1,\ e_2\}$ be an eigenvector with the eigenvalue ζ. If C2 holds, then $e_1 e_2 > 0$.*

Proof. The entries $e_1,\ e_2$ of $\mathbf{e}$ satisfy the equation

$$\frac{b_{11}e_1 - b_{12}e_2}{c + a} = \zeta e_1,$$

where

$$\zeta = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2} = \frac{1}{2(1-\alpha^2)c}\left[b - \alpha B \pm \sqrt{f(\alpha)}\right].$$

Therefore

$$\left[B - \alpha b \pm \sqrt{f(\alpha)}\right] e_1 = 2(1-\alpha) b_{12} e_2.$$

We can note that

$$B - \alpha b > 0. \tag{53}$$

Indeed, if $b \geq 0$, then this inequality follows from (45). For $b < 0$ and $B \leq 0$ we obtained (53) above (see the last paragraph of the proof of Proposition 4.1). In the case $b < 0$ and $B > 0$ inequality (53) is evident. Hence $B - \alpha b \pm \sqrt{f(\alpha)} = B - \alpha b \pm \sqrt{(B - \alpha b)^2 - 4\mu_*^2(1-\alpha^2)} > 0$ and the proposition follows from $b_{12} > 0$. □

If $\alpha = \alpha_*$, the negative double eigenvalue $\zeta = \frac{b - \alpha B}{2(1-\alpha^2)c} < 0$ arises. Clearly, as α decreases through α_* the two negative eigenvalues $\zeta_1,\ \zeta_2 < 0$ of A coalesce and, at least for α sufficiently close to α_*, become a complex conjugate pair with negative real part. This corresponds to an eigenvector $\mathbf{e} = (e_1,\ e_2)$ with correct signs of entries:

$$e_2/e_1 = \frac{B - \alpha b}{2(1-\alpha)b_{12}} > 0.$$

Notice that by the first equation of (42) one can see, that $w'_+ \mid_{\{0 \leq w_+ \leq 1, w_- = 0\}} > 0,\ \ w'_+ \mid_{\{w_+ = 1, 0 \leq w_- \leq 1\}} > 0$, $w'_+ \mid_{\{0 \leq w_+ = w_- \leq 1\}} > 0$. Hence the travelling wave solution can not leave the square $[0,\ 1] \times [0,\ 1]$.

This completes the proof of Theorem 1.2. □

It is interesting to interpret these results for the main examples which were introduced in Section 3.

5.3. **Examples**

We suppose here that $\mu_+ = \mu_- = \mu > 0$ and $\lambda_+ = \lambda_- = \lambda > 0$.

1. *Isotropic reaction walk*

Assume that $F_+ = F_- = F(u)$, $u = \frac{u_+ + u_-}{2}$. Let $F'(1) = \sum k\beta_k = q > 1$ be the expected number of descendants in a single birth. We have $J_{11} = J_{12} = J_{21} = J_{22} = q/2$ and $b_{11} = b_{22} = \mu + \lambda(1 - q/2)$, $b_{12} = b_{21} = \mu + \lambda q/2$.

In this example condition C2 reads

$$2\mu > \lambda(q-2)$$

(it disappears, if $q \le 2$) and the critical velocity value is

$$\alpha_* = \frac{\sqrt{\lambda(2\mu+\lambda)(q-1)}}{\mu + \lambda q/2}. \tag{54}$$

2. *Direction independent reaction walk*

The reaction terms are $F_+ = F(u)u_+$ and $F_- = F(u)u_-$. In this case $J_{11} = J_{22} = 1 + q/2,\ J_{12} = J_{21} = q/2$. Here and below in the third example $q = F'(1) > 0$ is the mean number of descendants (maternal particle is not taking into account). So $b_{11} = b_{22} = \mu - \lambda q/2$, $b_{12} = b_{21} = \mu + \lambda q/2$ and thus C2 means $2\mu > \lambda q$. The critical velocity value is

$$\alpha_* = \frac{\sqrt{2\lambda\mu q}}{\mu + \lambda q/2}. \tag{55}$$

3. *Direction dependent reaction walk*

For the version A we supposed $F_+ = F(u_-)u_+$ and $F_- = F(u_+)u_-$. Hence $J_{11} = J_{22} = 1,\ J_{12} = J_{21} = q$. Thus $b_{11} = b_{22} = \mu$, $b_{12} = b_{21} = \mu + \lambda q$. In this case we have

$$\alpha_* = \frac{\sqrt{\lambda q(2\mu + \lambda q)}}{\mu + \lambda q}. \tag{56}$$

The critical values of velocities of travelling waves (54)–(56) coincide with respective estimations for similar models due to Mendez *et al.* [28, 29] and Horsthemke [17].

For the version B (where $F_+ = F(u_+)u_+,\ F_- = F(u_-)u_-$, thus $J_{11} = J_{22}1 + q,\ J_{12} = J_{21} = 0$ and $b_{11} = b_{22} = \mu - \lambda q,\ b_{12} = b_{21} = \mu$) in the same manner as before one can obtain

$$\mu > \lambda q, \qquad \alpha_* = \frac{\sqrt{\lambda q(2\mu - \lambda q)}}{\mu} \tag{57}$$

(*cf.* Lyne [23]).

Remark 5.4. Observe that under the standard scaling

$$c,\ \mu \to \infty, \qquad c^2/\mu \to 1$$

system (5) is equivalent to nonlinear heat equation (1). The critical wave speed in the case of (1) is $\alpha_* = \sqrt{2f'(1)}$ [22]. In the McKean's interpretation $f'(1) = \lambda(Q-1)$, where λ is the intensity of the birth process, Q is the expected number of descendants in a single birth.

In the hyperbolic model all four formulas (54)–(57) lead to the same result:

$$\alpha_* c \to \sqrt{2\lambda(Q-1)}.$$

6. Upper bound for medians and convergence to travelling waves

Fix time horizon t. To obtain an upper bound of $m_\pm(\tau)$ we use comparison arguments and the results of Appendix A.

Theorem 6.1. *Let condition C2 to be hold. Then functions $m_\pm(\tau)$ satisfy the following inequalities*

$$m_\pm(\tau) \le \alpha_* c(t-\tau) - \gamma \ln(t-\tau), \tag{58}$$

where α_* is defined by (48) and γ is some positive constant.

Proof. Functions $u_+ = \mathbb{P}_{+,(x,\tau)}(L(t) > 0)$ and $u_- = \mathbb{P}_{-,(x,\tau)}(L(t) > 0)$, where $L(t)$ is the position of the left-most particle, satisfy system (5) with Heaviside terminal conditions $u_\pm \mid_{\tau\uparrow t} = \theta(x)$. Hence $\bar{u}_\pm = 1 - u_\pm = \mathbb{P}_{\pm,(x,\tau)}(L(t) \le 0)$ solve the system

$$\begin{cases} -\frac{\partial \bar{u}_+}{\partial \tau} - c\frac{\partial \bar{u}_+}{\partial x} = -b_{11}\bar{u}_+ + b_{12}\bar{u}_- - \lambda_+ R_+(\bar{u}_+,\ \bar{u}_-), \\ -\frac{\partial \bar{u}_-}{\partial \tau} + c\frac{\partial \bar{u}_-}{\partial x} = b_{21}\bar{u}_+ - b_{22}\bar{u}_- - \lambda_- R_-(\bar{u}_+,\ \bar{u}_-), \\ \tau < t, \qquad x \in (-\infty,\ +\infty) \end{cases} \tag{59}$$

with the terminal data

$$\bar{u}_\pm \mid_{\tau\uparrow t} = \theta(-x), \qquad x \in (-\infty,\ \infty).$$

Here $b_{ij},\ i,\ j = 1,\ 2$ are defined in (13) and

$$R_+(x,\ y) \equiv F_+(1-x,\ 1-y) + J_{11}x + J_{12}y - 1,$$

$$R_-(x,\ y) \equiv F_-(1-x,\ 1-y) + J_{21}x + J_{22}y - 1.$$

Notice that, due to convexity of $F_\pm$ (see *e.g.* [1]), functions $R_\pm(x,\ y) \ge 0$ for $(x,\ y) \in [0,\ 1] \times [0,\ 1]$.

Let $\bar{v}_\pm$ solve the respective linear system

$$\begin{cases} -\frac{\partial \bar{v}_+}{\partial \tau} - c\frac{\partial \bar{v}_+}{\partial x} = -b_{11}\bar{v}_+ + b_{12}\bar{v}_-, \\ -\frac{\partial \bar{v}_-}{\partial \tau} + c\frac{\partial \bar{v}_-}{\partial x} = b_{21}\bar{v}_+ - b_{22}\bar{v}_-, \qquad \tau < t, \quad x \in (-\infty,\ +\infty) \end{cases}$$

with the same terminal conditions.

Thus $\bar{u}_+ \le \bar{v}_+$ and $\bar{u}_- \le \bar{v}_-$. Now to finish the proof it is sufficient to note that from Proposition A.2 it follows

$$\bar{v}_\pm(\alpha_* c(t-\tau) - \gamma \ln(t-\tau),\ \tau,\ t) \to 0$$

as $\tau \downarrow -\infty$. □

Notice that (58) implies

$$\limsup_{\tau\downarrow-\infty} [-\dot{m}_\pm] \le \alpha_* c. \tag{60}$$

Recall that here and everywhere below $\dot{m} = \mathrm{d}m/\mathrm{d}\tau$.

Fix $t \in (-\infty,\ \infty)$ and consider functions $U_\pm$ and $U_\pm^*$ of the following form:

$$U_+(x,\ \tau) = u_+(x + m_+,\ \tau,\ t), \quad U_-(x,\ \tau) = u_-(x + m_-,\ \tau,\ t),$$

$$U_+^*(x,\ \tau) = u_+(x + m_-,\ \tau,\ t), \quad U_-^*(x,\ \tau) = u_-(x + m_+,\ \tau,\ t).$$

Clearly, $U_+(0,\ \tau) = U_-(0,\ \tau) = 1/2$.

In these notations system (5) leads to

$$\begin{cases} -\frac{\partial U_+}{\partial \tau} - (c-\dot m_+)\,\frac{\partial U_+}{\partial x} = \mu_+(U_-^* - U_+) + \lambda_+(F_+(U_+,\ U_-^*) - U_+), \\ -\frac{\partial U_-}{\partial \tau} + (c+\dot m_-)\,\frac{\partial U_-}{\partial x} = \mu_-(U_+^* - U_-) + \lambda_-(F_-(U_+^*,\ U_-) - U_-), \\ -\frac{\partial U_+^*}{\partial \tau} - (c-\dot m_-)\,\frac{\partial U_+^*}{\partial x} = \mu_+(U_- - U_+^*) + \lambda_+(F_+(U_+^*,\ U_-) - U_+^*), \\ -\frac{\partial U_-^*}{\partial \tau} + (c+\dot m_+)\,\frac{\partial U_-^*}{\partial x} = \mu_-(U_+ - U_-^*) + \lambda_-(F_-(U_+,\ U_-^*) - U_-^*). \end{cases} \tag{61}$$

The following theorem gives a simple sufficient condition for a convergence of $U_\pm$ and $U_\pm^*$ to travelling waves. Denote $\psi(\tau) = m_-(\tau) - m_+(\tau)$.

Theorem 6.2. *If the limit*

$$\lim_{\tau\downarrow-\infty} \psi(\tau) = \beta \tag{62}$$

exists, then there exist the limits

$$\lim_{\tau\downarrow-\infty}(-\dot m_+(\tau)) = a_+, \qquad \lim_{\tau\downarrow-\infty}(-\dot m_-(\tau)) = a_-,$$

$a_+ = a_-$ *and*

$$\lim_{\tau\downarrow-\infty} U_+^*(x,\ \tau) = w_+(x+\beta), \ \lim_{\tau\downarrow-\infty} U_-^*(x,\ \tau) = w_-(x-\beta).$$

Moreover, pair $\{w_+(x),\ w_-(x-\beta)\}$ *(and* $\{w_+(x+\beta),\ w_-(x)\}$*) form a travelling-wave solution.*

Proof. First note that by (38)-(39) and (62) the following limits exist

$$\lim_{\tau\downarrow-\infty} U_+(x,\ \tau) = w_+(x), \qquad \lim_{\tau\downarrow-\infty} U_-(x,\ \tau) = w_-(x),$$

$$\lim_{\tau\downarrow-\infty} U_+^*(x,\ \tau) = w_+(x+\beta), \qquad \lim_{\tau\downarrow-\infty} U_-^*(x,\ \tau) = w_-(x-\beta).$$

Integrating the first two equations of (61) in τ from $\tau-1$ to τ and in x from 0 to x and passing to the limit as $\tau\downarrow-\infty$ we obtain

$$\begin{cases} -(c+a_+)(w_+(x)-1/2) = \int\limits_0^x [\mu_+(w_-(x'-\beta) - w_+(x')) \\ +\lambda_+(F_+(w_+(x'),\ w_-(x'-\beta)) - w_+(x'))]\,\mathrm{d}x', \\ (c-a_-)(w_-(x)-1/2) = \int\limits_0^x [\mu_-(w_+(x'+\beta) - w_-(x')) \\ +\lambda_-(F_-(w_+(x'+\beta),\ w_-(x')) - w_-(x'))]\,\mathrm{d}x'. \end{cases}$$

Similarly, from the last two equations of (61) it follows

$$\begin{cases} -(c+a_-)(w_+(x+\beta) - w_+(x_0+\beta)) = \int\limits_{x_0}^x [\mu_+(w_-(x') - w_+(x'+\beta)) \\ +\lambda_+(F_+(w_+(x'+\beta),\ w_-(x')) - w_+(x'+\beta))]\,\mathrm{d}x', \\ (c-a_+)(w_-(x-\beta) - w_-(x_0-\beta)) = \int\limits_{x_0}^x [\mu_-(w_+(x') - w_-(x'-\beta)) \\ +\lambda_-(F_-(w_+(x'),\ w_-(x'-\beta)) - w_-(x'-\beta))]\,\mathrm{d}x'. \end{cases}$$

Differentiating these two pairs of coupled equations we conclude that the pair $\{w_+(x),\ w_-(x-\beta)\}$ forms a travelling-wave solution with velocity a_+, and the pair $\{w_+(x+\beta),\ w_-(x)\}$ is a travelling wave with velocity a_-.

From results of Section 5 (Prop. 5.1) it follows

$$a_\pm \geq \alpha_* c.$$

From (60) we have

$$a_\pm \leq \alpha_* c.$$

Therefore $a_+ = a_- = \alpha_* c$ and the theorem is proved. □

Remark 6.3. In general, the question whether the limit (62) really exists is still open. Nevertheless it is easy to check (62) at least for isotropic reaction walk.

Proposition 6.4. *Let $\mu_+ = \mu_- = \mu$, $\lambda_+ = \lambda_- = \lambda$ and $J_{11} = J_{22} = J_{12} = J_{21}$. Then*

$$\lim_{\tau \downarrow -\infty} \psi(\tau) = \frac{2c}{2\mu + \lambda}\cdot \tag{63}$$

Proof. Let S be the first breeding time. Notice that particles forget its original direction at the breeding time. Hence variables $m_+(\tau - S) + X_+(S)$ and $m_-(\tau - S) + X_-(S)$ are identically distributed (for sufficiently large $-\tau$). Here X_+ and X_- are the telegraph processes initially moving forwards and backwards respectively. It is easy to see that $\mathbb{E}X_+(S) = -\mathbb{E}X_-(S) = \frac{c}{\lambda + 2\mu}$. Hence

$$\lim_{\tau \downarrow -\infty} \psi(\tau) = \frac{2c}{2\mu + \lambda}\cdot$$

□

Appendix A. Solutions of linear hyperbolic systems

The objective of this part is to propose the exact formulas for solutions to linear hyperbolic systems and to obtain some inequalities desired in Section 6. The following proposition is well-known (*cf.* [13]).

Proposition A.1. *Solution $v_\pm$ of the system*

$$\begin{cases} -\frac{\partial v_+}{\partial \tau} - c\frac{\partial v_+}{\partial x} = -b_{11}v_+ + b_{12}v_-, \\ -\frac{\partial v_-}{\partial \tau} + c\frac{\partial v_-}{\partial x} = b_{21}v_+ - b_{22}v_-, \\ \tau < t, \qquad x \in (-\infty,\ +\infty) \end{cases} \tag{A.1}$$

with terminal conditions $v_\pm \mid_{\tau \uparrow t} = g_\pm(x),\ x \in (-\infty,\ \infty)$ has the form

$$v_+ = \mathrm{e}^{-b_{11}(t-\tau)} g_+(x + c(t-\tau)) + \frac{1}{2} e^{-B(t-\tau)/2} \int\limits_{-(t-\tau)}^{t-\tau} \mathrm{e}^{-bs/2} \left[b_{12} g_-(x + cs) I_0\left(\mu_* \sqrt{(t-\tau)^2 - s^2}\right) \right. \tag{A.2}$$

$$\left. + \mu_* g_+(x + cs) I_1\left(\mu_* \sqrt{(t-\tau)^2 - s^2}\right) \frac{t - \tau + s}{\sqrt{(t-\tau)^2 - s^2}} \right] \mathrm{d}s$$

and

$$v_- = \mathrm{e}^{-b_{22}(t-\tau)}g_-(x-c(t-\tau)) + \frac{1}{2}\mathrm{e}^{-B(t-\tau)/2}\int\limits_{-(t-\tau)}^{t-\tau}\mathrm{e}^{-bs/2}\left[b_{21}g_+(x+cs)I_0\left(\mu_*\sqrt{(t-\tau)^2-s^2}\right)\right. \tag{A.3}$$

$$\left.+\mu_* g_-(x+cs)I_1\left(\mu_*\sqrt{(t-\tau)^2-s^2}\right)\frac{t-\tau-s}{\sqrt{(t-\tau)^2-s^2}}\right]\mathrm{d}s.$$

Here $B = b_{11} + b_{22}$, $b = b_{11} - b_{22}$ *and* $\mu_* = \sqrt{b_{12}b_{21}}$; $I_0(z) = \sum\limits_{n=0}^{\infty}\frac{z^{2n}}{2^{2n}(n!)^2}$ *is the zero-order Bessel function of imaginary argument,* $I_1(z) = I_0'(z)$.

Corollary. *For the Heaviside terminal conditions* $g_\pm = \theta(-x)$ *formulas* (A.2)-(A.3) *take the form*

$$v_+ = \mathrm{e}^{-b_{11}(t-\tau)}\theta(-x-c(t-\tau)) + \frac{1}{2}\mathrm{e}^{-B(t-\tau)/2}\int\limits_{-(t-\tau)}^{\min(-x/c,t-\tau)}\mathrm{e}^{-bs/2}\left[b_{12}I_0\left(\mu_*\sqrt{(t-\tau)^2-s^2}\right)\right.$$

$$\left.+\mu_* I_1\left(\mu_*\sqrt{(t-\tau)^2-s^2}\right)\frac{t-\tau+s}{\sqrt{(t-\tau)^2-s^2}}\right]\mathrm{d}s\theta(c(t-\tau)-x) \tag{A.4}$$

$$v_- = \mathrm{e}^{-b_{22}(t-\tau)}\theta(-x+c(t-\tau)) + \frac{1}{2}\mathrm{e}^{-B(t-\tau)/2}\int\limits_{-(t-\tau)}^{\min(-x/c,t-\tau)}\mathrm{e}^{-bs/2}\left[b_{21}I_0\left(\mu_*\sqrt{(t-\tau)^2-s^2}\right)\right.$$

$$\left.+\mu_* I_1\left(\mu_*\sqrt{(t-\tau)^2-s^2}\right)\frac{t-\tau-s}{\sqrt{(t-\tau)^2-s^2}}\right]\mathrm{d}s\theta(c(t-\tau)-x). \tag{A.5}$$

Proposition A2. *Let* $\{v_+,\ v_-\}$ *be a solution of* (A.1) *with Heaviside terminal data* $v_\pm\mid_{\tau\uparrow t}= \theta(-x)$. *If condition C2 holds, then as* $\tau\downarrow -\infty$

$$v_\pm(\alpha_* c(t-\tau) - \gamma c\ln(t-\tau),\ \tau,\ t) \to 0, \tag{A.6}$$

where α_* *is defined by* (48).

Proof. Keeping in mind formulas (A.4) and (A.5) it is sufficient to prove that for $x(T) = \alpha_* T - \gamma\ln T$

$$V_1(T) = \int\limits_{-T}^{-x(T)}\mathrm{e}^{-bs/2}I_0(\mu_*\sqrt{T^2-s^2})\mathrm{d}s = o(\mathrm{e}^{BT/2}), \tag{A.7}$$

$$V_2(T) = \int\limits_{-T}^{-x(T)}\mathrm{e}^{-bs/2}\frac{\partial}{\partial T}I_0(\mu_*\sqrt{T^2-s^2})\mathrm{d}s = o(\mathrm{e}^{BT/2}) \tag{A.8}$$

as $T\to\infty$.

We prove here (A.7) (for (A.8) a similar idea can be applied). We split integral $V_1(T)$ into two parts: $V_1(T) = V_{11}(T) + V_{12}(T)$, where

$$V_{11}(T) = \int\limits_{-T}^{-\alpha T}\mathrm{e}^{-bs/2}I_0(\mu_*\sqrt{T^2-s^2})\mathrm{d}s, \quad V_{12}(T) = \int\limits_{-\alpha T}^{-x(T)}\mathrm{e}^{-bs/2}I_0(\mu_*\sqrt{T^2-s^2})\mathrm{d}s.$$

Here $\alpha \in (\alpha_*,\ 1)$. First note that

$$I_0(z) \le \mathrm{e}^z, \qquad z \in (-\infty,\ \infty).$$

Thus

$$\begin{aligned}
V_{11} &\le \int_{-T}^{-\alpha T} \exp\left(-\frac{bs}{2} + \mu_* \sqrt{T^2 - s^2}\right) \mathrm{d}s \\
&= \int_{-T}^{-\alpha T} \exp\left(\mu_* T - \frac{bs}{2} - \frac{\mu_* s^2}{T + \sqrt{T^2 - s^2}}\right) \mathrm{d}s \\
&\le \int_{-T}^{-\alpha T} \exp\left(\mu_* T - \frac{bs}{2} - \frac{\mu_* s^2}{T(1 + \sqrt{1 - \alpha^2})}\right) \mathrm{d}s
\end{aligned}$$

$$= \exp\left(\mu_* T + \frac{b^2 T}{16\mu_*}\left(1 + \sqrt{1-\alpha^2}\right)\right) \int_{\beta_-}^{\beta_+} \exp\left(-\frac{\mu_* s^2}{T\left(1 + \sqrt{1-\alpha^2}\right)}\right) \mathrm{d}s, \tag{A.9}$$

where

$$\beta_+ = -\alpha T + \frac{bT}{4\mu_*}\left(1 + \sqrt{1-\alpha^2}\right), \quad \beta_- = -T + \frac{bT}{4\mu_*}\left(1 + \sqrt{1-\alpha^2}\right).$$

Note that, if $\alpha_* < \alpha < 1$, then $\beta_+ < 0$. Indeed, if $b \le 0$, then it is evident. In the case $b > 0$ the inequality $\alpha_* > b/B$ is hold (see the proof of Prop. 5.1), and thus by condition C2 ($2\mu_* > B$) we have

$$4\mu_*\alpha > 4\mu_*\alpha_* > 4\mu_* \frac{b}{B} > 2b > b\left(1 + \sqrt{1-\alpha^2}\right),$$

which is desired.

By the inequality

$$\int_{-\infty}^{-A} \mathrm{e}^{-x^2/2\sigma^2} dx \le \frac{\sigma^2}{A^2}\mathrm{e}^{-A^2/2\sigma^2}, \quad A > 0$$

the right hand side of (A.9) can be estimated by

$$\mathrm{const} \cdot \exp\left(\frac{b\alpha T}{2} + \mu_* T\sqrt{1-\alpha^2}\right) = o(\mathrm{e}^{BT/2}) \tag{A.10}$$

(see (47)).

To estimate V_{12} we apply the inequality

$$I_0(z) \le \frac{\mathrm{e}^z}{\sqrt{2\pi z}}, \qquad z \to \infty.$$

Thus

$$V_{12} \le \frac{1}{(2\pi\mu_* T)^{1/2}(1-\alpha^2)^{1/4}} \int_{-\alpha T}^{-\alpha_* T + \gamma \ln T} \exp\left(-\frac{bs}{2} + \mu_* \sqrt{T^2 - s^2}\right) \mathrm{d}s.$$

In the same way as before one can obtain

$$V_{12} \le \mathrm{const} \cdot T^{-1/2} \exp\left(\frac{b\alpha_* T}{2} - \frac{b\gamma \ln T}{2} + \mu_* \sqrt{T^2 - (-\alpha_* T + \gamma \ln T)^2}\right)$$

$$\leq \text{const} \cdot T^{-1/2} \exp\left(\frac{b\alpha_* T}{2} + \mu_* T\sqrt{1-\alpha_*^2} - \frac{b\gamma \ln T}{2} + \frac{\mu_*\alpha_*\gamma \ln T}{\sqrt{1-\alpha_*^2}}\right)$$

$$= \text{const} \cdot T^{-1/2+\gamma\left(-b/2+\mu_*\alpha_*/\sqrt{1-\alpha_*^2}\right)} \exp\left(\frac{b\alpha_* T}{2} + \mu_* T\sqrt{1-\alpha_*^2}\right)$$

$$= \delta_T e^{BT/2},$$

where $\delta_T = \text{const} \cdot T^{-1/2+\gamma\left(-b/2+\mu_*\alpha_*/\sqrt{1-\alpha_*^2}\right)} \to 0$ for suitably chosen positive γ. In the last line above we use (52). Consequently, property (A.7) follows from this and from (A.10). The proposition is proved. □

Appendix B. Existence and uniqueness results

We shall prove here the existence and uniqueness of the solution of system (5) with measurable terminal data $g_\pm$.

First notice that the linear system

$$\begin{cases} -\frac{\partial v_+}{\partial \tau} - c\frac{\partial v_+}{\partial x} = \mu_+(v_- - v_+) - \lambda_+ v_+, \\ -\frac{\partial v_-}{\partial \tau} + c\frac{\partial v_-}{\partial x} = \mu_-(v_+ - v_-) - \lambda_- v_- \end{cases} \quad \tau < t \tag{B.1}$$

(with the terminal condition $v_\pm \mid_{\tau\uparrow t} = g_\pm,\ g_\pm \in \mathcal{C}^1$) has the unique solution in $\mathcal{C}^1$. Moreover this solution can be represented in the Fenmann-Kac form (see Th. 2.1)

$$v_\pm(x,\ \tau,\ t) = \mathbb{E}_{\pm,(x,\tau)}\left[g(X(t),\ \sigma(t),\ t)\exp\left(-\int_\tau^t \lambda_{\sigma(s)}\mathrm{d}s\right)\right] := U_{\tau,t}(g), \tag{B.2}$$

where $(X,\ \sigma)$ is the telegraph process with intensities $\mu_\pm$. Fix $t,\ t \in (-\infty,\ \infty)$. Solution $v = U_{\tau,t}(g),\ \tau \leq t$ forms a semigroup acting on $\mathcal{C}^1 \times \mathcal{C}^1$.

From (B.2) it follows the monotonicity of semigroup U: if $f_+ \leq g_+$ and $f_- \leq g_-$, then $U_{\tau,t}(f)(x,\ \sigma) \leq U_{\tau,t}(g)(x,\ \sigma),\ -\infty < x < \infty,\ \sigma = \pm$. Moreover, if $g(x,\ \sigma) \leq C$, then $U_{\tau,t}(g)(x,\ \sigma) \leq C\mathbb{E}_\sigma \exp\left(-\int_\tau^t \lambda_{\sigma(s)}\mathrm{d}s\right)$.

By the Riezs representation theorem semigroup $U_{\tau,t},\ \tau \leq t$ has a canonical extension on $\mathcal{B} \times \mathcal{B}$, the space of bounded measurable functions. In particular, if function g is peace-wise continues, then the solution $v = v(x,\ \sigma,\ \tau,\ t) = U_{\tau,t}(g)(x,\sigma)$ has discontinuities propagating along characteristics.

We regard solution u of non-linear system (5) as a weak solution. It means that for $\tau < t$ and a test function $\varphi = \varphi(x,\ \sigma,\ \tau),\ \varphi_\pm \in \mathcal{C}_0^1$,

$$\begin{cases} \int_\tau^t \int u_+(x,\ s,\ t)\left[\frac{\partial\varphi_+}{\partial s} + c\frac{\partial\varphi_+}{\partial x} + (\mu_+ + \lambda_+)\varphi_+\right](x,\ s)\mathrm{d}s\mathrm{d}x - \mu_+\int_\tau^t \int u_-(x,\ s,\ t)\varphi_+(x,\ s)\mathrm{d}s\mathrm{d}x \\ -\lambda_+\int_\tau^t \int F_+(u_+,\ u_-)\varphi_+(x,\ s)\mathrm{d}s\mathrm{d}x = \int u_+(x,\ \tau)\varphi_+(x,\ \tau)\mathrm{d}x - \int g_+(x,\ t)\varphi_+(x,\ t)\mathrm{d}x, \\ \\ \int_\tau^t \int u_-(x,\ s,\ t)\left[\frac{\partial\varphi_-}{\partial s} - c\frac{\partial\varphi_-}{\partial x} + (\mu_- + \lambda_-)\varphi_-\right](x,\ s)\mathrm{d}s\mathrm{d}x - \mu_-\int_\tau^t \int u_+(x,\ s,\ t)\varphi_-(x,\ s)\mathrm{d}s\mathrm{d}x \\ -\lambda_-\int_\tau^t \int F_-(u_+,\ u_-)\varphi_-(x,\ s)\mathrm{d}s\mathrm{d}x = \int u_-(x,\ \tau)\varphi_-(x,\ \tau)\mathrm{d}x - \int g_-(x,\ t)\varphi_-(x,\ t)\mathrm{d}x. \end{cases} \tag{B.3}$$

Theorem B.1. *If u is the solution of system* (B.3)*, then u satisfies*

$$u = U_{\tau,t}(g) - \int\limits_{\tau}^{t} U_{t+\tau-s,t}(\lambda F(u)\mid_s)\mathrm{d}s \tag{B.4}$$

for for each $\tau < t$ and for almost every x.

Proof. Consider the system

$$\begin{cases} -\frac{\partial\psi_+}{\partial s} + c\frac{\partial\psi_+}{\partial x} = -(\mu_+ + \lambda_+)\psi_+ + \mu_-\psi_-, \\ \\ -\frac{\partial\psi_-}{\partial s} - c\frac{\partial\psi_-}{\partial x} = -(\mu_- + \lambda_-)\psi_- + \mu_+\psi_+ \end{cases} \quad s < t \tag{B.5}$$

with the terminal condition $\psi_\pm\mid_{s\uparrow t} = \psi_\pm^0,\ \psi_\pm^0 \in \mathcal{C}_0^\infty$.

Solution $\psi_\pm,\ \psi(x,\ s,\ t) = U_{s,t}^*(\psi^0)$ of this system is defined by semigroup $U_{s,t}^*$ dual to $U_{s,t}$. By the statement that $U_{\tau,t}$ and $U_{\tau,t}^*$ are dual we mean that for $\tau < t$ and bounded measurable functions $g,\ \psi$

$$\int g(x,\ t) \cdot U_{\tau,t}^*(\psi)(x)\mathrm{d}x = \int U_{\tau,t}(g)(x) \cdot \psi(x,\ t)\mathrm{d}x,$$

where $f \cdot g = f_+g_+ + f_-g_-$. To finish the of Theorem it is sufficient to substitute $\varphi(x,\ s) = \psi(x,\ t+\tau-s)$ in system (B.3). $\square$

Notice that the solution of system (B.5) can be expressed by means of the reversed underlying telegraph process $X^*(s) = X(t-s)$. More precisely,

$$\varphi_\pm(x,\ s) = \mathbb{E}\left[\psi^0(X^*(s),\ \sigma^*(t),\ t)\exp\left(-\int\limits_s^t \lambda_{\sigma^*(s')}\mathrm{d}s'\right)\ \middle|\ \sigma^*(s) = \pm,\ X^*(s) = x\right].$$

Theorem B.2. *Let $0 \le g_\pm \le 1,\ g \in \mathcal{B}\times\mathcal{B}$. Then the solution $u \in \mathcal{B}\times\mathcal{B},\ 0 \le u_\pm \le 1$ of equation* (B.4) *exists and it is unique.*

Proof. Define

$$u^{(0)} = U_{\tau,t}(g)$$

$$u^{(n)} = U_{\tau,t}(g) - \int\limits_\tau^t U_{t+\tau-s,t}(\lambda F(u^{(n-1)})\mid_s)\mathrm{d}s, \qquad n \ge 1.$$

Then, applying the usual contraction technique, one can prove the convergence $u^{(n)} \to u$ uniformly on $(\tau,\ t]$, which gives the exact solution of equation (B.4). $\square$

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