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PARTICLE APPROXIMATIONS OF LYAPUNOV EXPONENTS CONNECTED TO SCHRÖDINGER OPERATORS AND FEYNMAN–KAC SEMIGROUPS

Pierre Del Moral[1] and L. Miclo[1]

Abstract. We present an interacting particle system methodology for the numerical solving of the Lyapunov exponent of Feynman–Kac semigroups and for estimating the principal eigenvalue of Schrödinger generators. The continuous or discrete time models studied in this work consists of N interacting particles evolving in an environment with soft obstacles related to a potential function V. These models are related to genetic algorithms and Moran type particle schemes. Their choice is not unique. We will examine a class of models extending the hard obstacle model of K. Burdzy, R. Holyst and P. March and including the Moran type scheme presented by the authors in a previous work. We provide precise uniform estimates with respect to the time parameter and we analyze the fluctuations of continuous time particle models.

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[1] LSP, UMR 5583 du CNRS, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France;
e-mail: `delmoral@cict.fr, miclo@cict.fr`

Introduction

The aim of this work is the design of an interacting particle approach for the numerical estimation of the Lyapunov exponent associated to Feynman–Kac type semigroups on some classical functional Banach spaces. These important spectral quantities characterize decay properties and in some situations they coincide with the principal eigenvalues of Schrödinger generators.

Except in some particular situations such as for the well known harmonic oscillator, explicit descriptions of these quantities are generally not available and we have to resort to some kind of approximations. To motivate this article and show the impact of the particle interpretations presented in this article we give next a brief comparison with different existing "numerical" or "analytic" approximating strategies.

For instance the generator $L^v = L + V$ can be treated as the perturbation of the operator L by a potential V. In some situations the perturbation theory proposes analytic expansions of isolated eigenvalues (see for instance Kato [12]).

Donsker and Varadhan have also presented in a series of papers (see for instance [9]) a theory of large deviations which expresses the well known Raleigh–Ritz representation of the top eigenvalue of L^v in terms of a variational problem in distribution space. In some situations this optimization problem can be solved by using for instance some kind of stochastic global search algorithm or specific Hilbert projection techniques.

Our strategy consists in expressing these spectral exponents in term of the fixed point of a suitably chosen nonlinear dynamical system in distribution space. These key representations bring some new light on connections between spectral theory of Schrödinger operators and nonlinear measure valued processes. There are also the stepping stone of our methodology to produce interacting particle approximating models. To our knowledge the particle Lyapunov approximating exponents described in this article have never been covered in the literature on the subject.

A precise analytic or numerical comparison between particle interpretations with other numerical or analytical methods such as the ones discussed above is an important open problem. The accuracy of these methods depends on the nature of the state space and the form of the generator L^v. Nevertheless we underline that the particle Monte-Carlo strategy does not depend on the dimension of the state space nor on the linearity of the potential function V. Another advantage is that it gives a natural microscopic particle interpretation of these spectral quantities. Furthermore the theoretical uniform estimates presented in this article seem to indicate that these novel particle interpretations will lead in a near future to new practical improvements in the numerical analysis of Feynman–Kac Schrödinger semigroups.

We will present two different types of results:

First we relate the description of the Lyapunov exponent with the asymptotic stability of a nonlinear equation in distribution space. We propose a precise description of the Lyapunov exponent in terms of a fixed point of the corresponding evolution semigroup. We also give rather precise estimates of the decays to equilibrium related to this fixed point representation. We mention that these nonlinear equations arise in various areas and particularly in advanced signal processing and biology. The interesting reader is referred to [4] and references therein.

In the second part of the paper we propose a novel particle strategy to approximate these Lyapunov exponents.

This approach has been influenced by the recent works of Burdzy *et al.* [1, 2], Sznitman [14] and previous works of the authors [4, 6] and earlier joint work of one of the authors with Guionnet [3].

The first two referenced papers propose a Moran/Fleming–Viot approximation of the eigenvalues and eigenfunctions corresponding to the Laplacian with Dirichlet boundary condition on a domain of $\mathbb{R}^d$. As the authors indicate, this study is intimately and essentially related to the properties of the underlying Brownian motion. Furthermore the authors provide no precise rates of convergence. In [14] the author also pointed out the importance of the principal eigenvalue of the Dirichlet Laplacian in the study of the asymptotic behavior of a Brownian particle in a random environment with Poissonnian obstacles. In the last three referenced articles [3, 4, 6] the authors study a class of branching and interacting particle approximations of Feynman–Kac distribution flows with discrete or continuous time index. These articles are essentially concerned around non linear filtering problems and they do not discuss the spectral analysis of Feynman–Kac semigroups and Schrödinger operators. Some comments on applications to physics and biology can be found in [3].

These apparently distinct studies can be related one each other by first expressing the desired eigenvalue in terms of the distribution of a killed Markov particle conditioned by non-extinction and then by applying an interacting particle schemes to solve numerically the evolution of these conditional distributions. Heuristically Dirichlet boundary conditions can also be related to killing at a given rate V by considering the particle among an environment with obstacles. In "soft obstacles" the particle is killed at a rate specified by a nonnegative and "nice" potential V. In regions where the potential function is large the particle will more likely be killed. At the opposite situation when the particle hits an "hard obstacle" it is instantly killed. Intuitively speaking this corresponds to the singular situation where the potential is infinite on the boundary of the obstacle.

These results complement the ones obtained in [3, 4, 6] with providing a novel interpretation of the spectral quantities of Schrödinger operators in terms of the limiting distributions of a non linear Feynman–Kac distribution flow. The class of particle approximating models presented here extend the one discussed in earlier studies. In general this new model contains less randomness than the Moran particle model proposed in [6]. Furthermore, in some situations we will prove that the fluctuation covariance function of these new models remains uniformly bounded with respect to the time parameter. In contrast, this will not be the case for the Moran particle model introduced in [6], for which nontrivial variances converge to infinity as time increases. We also mention that these novel particle interpretations extend the one of Burdzy *et al.* [1, 2] to the soft obstacle situation. We develop a strategy to study the asymptotic behavior of these approximating models in the spirit of [3, 4]. We provide precise uniform estimates yielding what seems to be the first results of this kind in the literature on the subject.

0.1. **Description of the models**

In this study we consider a discrete or continuous time index set $I = \mathbb{N}$ or $I = \mathbb{R}_+$ and in general we will use the subscript $t \in \mathbb{R}_+$ or $n \in \mathbb{N}$ to distinguish the continuous and discrete time index. By $(\Omega, (X_t)_{t\in I}, (F_t)_{t\in I}, (\mathbb{P}_x)_{x\in E})$ we denote a time homogeneous progressively measurable Markov process with time index I taking values in a measurable space $(E, \mathcal{E})$ and whose associated transition semigroup will be designated by $P \stackrel{\text{def.}}{=} (P_t)_{t\in I}$ (for a rigorous definition of this setting, see [7], where a detailed discussion of the required properties is given). The latter operators are considered as acting in the Banach space $\mathcal{B}_b(E)$ of bounded $\mathcal{E}$-measurable functions $f : E \to \mathbb{R}$ endowed with the supremum norm

$$\|f\| = \sup_{x\in E} |f(x)|.$$

We use the notation $\mathbb{E}_x(.)$ for the expectation with respect to $\mathbb{P}_x$ and by $\mathcal{P}(E)$, the set of probability measures on $(E, \mathcal{E})$ with the total variation distance

$$\|\mu - \nu\|_{\text{tv}} = \sup_{A\in\mathcal{E}} |\mu(A) - \nu(A)| = \frac{1}{2} \sup_{\|f\|\leq 1} |\mu(f) - \nu(f)|$$

where the supremum in the right hand side is over the class of $f \in \mathcal{B}_b(E)$ with $\|f\| \leq 1$. As usual, for a finite measure μ, a numerical function f, an integral operator $K(x, \mathrm{d}y)$ on E (and whenever they exist) we denote by $\mu(f)$, $K(f)$, μK the number, the function and the finite measure defined respectively by

$$\mu(f) = \int f(x)\ \mu(\mathrm{d}x)\,, \quad K(f)(x) = \int K(x, \mathrm{d}y)\ f(y)\,, \quad \mu K(\mathrm{d}y) = \int \mu(\mathrm{d}x)\ K(x, \mathrm{d}y).$$

This study is concerned with Schrödinger and Feynman–Kac semigroups on the Banach space $\mathcal{B}_b(E)$ and expressed in terms of the following path integral formulas, for any $f \in \mathcal{B}_b(E)$, $x \in E$ and $t, n \in I$,

$$P_t^v(f)(x) = \mathbb{E}_x\left[f(X_t)\ \exp\left(\int_0^t V(X_s)\mathrm{d}s\right)\right] \qquad \text{when} \quad I = \mathbb{R}_+$$
$$P_n^v(f)(x) = \mathbb{E}_x\left[f(X_n)\ \exp\left(\sum_{p=0}^{n-1} V(X_p)\right)\right] \qquad \text{when} \quad I = \mathbb{N}. \tag{1}$$

Throughout this paper the above appearing potential function V is assumed to belong to $\mathcal{B}_b(E)$ and we write $\mathrm{osc}(V) = \sup\{V(x) - V(y)\ ;\ (x,y) \in E^2\}$ its oscillations. There exist several ways to extend these formulas to more general potential, the interested reader is referred to the book of Sznitman [14] and Section X.11 in Reed and Simon [13].

In discrete time settings P_n^v is clearly the n-time iterate of the integral operator

$$P_1^v(x, \mathrm{d}y) = \mathrm{e}^{V(x)}\ P_1(x, \mathrm{d}y).$$

In the continuous time case, using problems of martingales, it is possible to associate to the above Markov process a natural notion of weak infinitesimal generator L acting on a subspace $\mathcal{D}(L)$ of $\mathcal{B}_b(E)$ (by an approach similar to that of [7], but forgetting there the inhomogeneous time aspect). Then we can give a weak signification (sufficient for our purposes) to the intuition that the generator of the semigroup P^v is the Schrödinger operator defined by

$$\forall\ f \in \mathcal{D}(L), \qquad L^v(f) = L(f) + Vf \quad \in \mathcal{B}_b(E).$$

We mention that the convergence estimates presented in this article are valid in the extended and general set-up described in [7]. To clarify the presentation and underline the main ideas of our constructions we will work here with the following heavier system of assumptions:

- $D(L)$ is a sub-algebra of $\mathcal{B}_b(E)$ generating the underlying σ-algebra $\mathcal{E}$ and $V \in D(L)$;
- for any $t \geq 0$, the operators P_t and P_t^v leave $D(L)$ invariant;
- $L : D(L) \to D(L)$ is an operator such that in the sense of the norm $\|\cdot\|$, we have

$$\forall\ f \in D(L), \qquad \partial_t P_t(f) = L[P_t(f)] \quad = \quad P_t(L[f]);$$

- for any time $T \geq 0$ and any function $f \in D(L)$, the following mapping is bounded

$$[0,T] \times E \ni (t,x) \mapsto L[(P_t^v[f])^2](x).$$

These regularity conditions are not really restrictive. For instance they are met for pure jump processes with bounded rates and $D(L) = \mathcal{B}_b(E)$ as well as for Euclidean diffusions with regular and Lipschitzian coefficients, by considering for $D(L)$ the set of C^∞ functions whose derivatives (and the function itself) are decreasing at infinity faster than any polynomial.

We will rather refer to $(D(L), L)$ as a pregenerator, as in general this operator will not be closed. The advantage of this set of assumptions is that it enables to introduce the "carré du champ" in order to evaluate the quadratic variations of the martingales which will appear. Nevertheless, these computations should only be seen as a heuristic for the general case, as the method developed in [7] permits to dispense with the use of this notion in studies of evolution of empirical measures (by resorting to their tensorizations).

Since the potential is bounded one finds that $P^v = (P_t^v)_{t\in I}$ is a collection of bounded operators on $\mathcal{B}_b(E)$ with norm

$$\|P_t^v\| = \sup_{\|f\|=1} \|P_t^v(f)\| = \|P_t^v(1)\|\ (\leq \exp(t\|V\|))$$

where 1 stands for the unit function. The Lyapunov exponent $\mathrm{Lyap}(P^v) \in [0, +\infty]$ of the semigroup P^v on the Banach space $\mathcal{B}_b(E)$ is the quantity defined by sub-additive arguments as

$$\mathrm{Lyap}(P^v) = \lim_{t\to\infty} \|P_t^v\|^{1/t} \quad = \quad \inf_{t\geq 0} \|P_t^v\|^{1/t}.$$

In the discrete time case the Lyapunov exponent $\mathrm{Lyap}(P^v)$ coincides with the spectral radius of the one step transition P_1^v, that is

$$\mathrm{Spr}(P_1^v) = \lim_{n\to\infty} \|P_n^v\|^{1/n}.$$

It will be convenient to consider also the logarithmic Lyapunov exponent defined by

$$\lambda(P^v) = \ln[\mathrm{Lyap}(P^v)] \qquad \in \bar{\mathbb{R}}.$$

In Section 1 we will discuss some links with the corresponding L_2-spectral quantities, in particular we will see that they coincide under some mixing and symmetry assumption on the semigroup P.

0.2. **Statement of some results**

Our first result concerns the representation of the Lyapunov exponents in terms of the normalized Feynman–Kac distribution flow $\eta = (\eta_t)_{t\in I}$ defined by

$$\eta_t(f) = \gamma_t(f)/\gamma_t(1)\,, \qquad \text{with} \quad \gamma_t(f) = \eta_0 P_t^v(f).$$

The choice of the initial distribution $\eta_0 \in \mathcal{P}(E)$ may vary. When $\eta_0 = \delta_x$ sometimes we write $\gamma_t^{(x)}$ and $\eta_t^{(x)}$ the corresponding measures. To describe these models it is convenient to introduce the nonlinear evolution semigroup $\Phi = (\Phi_t)_{t\in I}$ associated to the flow η, namely for $s, t \in I$,

$$\Phi_t(\eta_s) = \eta_{s+t}.$$

As mentioned in the introduction, the Lyapunov exponent will be expressed in terms of the fixed point of the semigroup Φ. But its existence and the possibility to approximate such a representation will depend on the asymptotic stability properties of Φ. We will use the following assumption:
(Φ) The semigroup Φ is contractive in the sense that

$$\|\Phi_t(\mu) - \Phi_t(\nu)\|_{\mathrm{tv}} \leq \ \alpha_t(\Phi)\ \|\mu - \nu\|_{\mathrm{tv}} \tag{2}$$

for any $t \geq t_0 \geq 0$, $\mu, \nu \in \mathcal{P}(E)$ and some $t_0 \in I$ and $\alpha_t(\Phi) < 1$. In addition, we assume that in discrete time case,

$$\alpha(\Phi) \overset{\text{def.}}{=} \sum_{n\geq t_0} \alpha_n(\Phi) < \infty$$

and in continuous time, that $\mathbb{R}_+ \ni t \mapsto \alpha_t(\Phi)$ is measurable and that

$$\alpha(\Phi) \overset{\text{def.}}{=} \int_{t_0}^{\infty} \alpha_t(\Phi)\ \mathrm{d}t < \infty.$$

Such a regularity property is difficult to check in practice. In [4, 5] several sufficient conditions are proposed. Nevertheless most of these conditions are related to a strong mixing condition, namely:

($\mathcal{P}_1$) The probability measures $P_1(x,.)$, $x \in E$, are mutually absolutely continuous and for some $\varepsilon > 0$ we have for all $x, x' \in E$, $P_1(x', \cdot)$-a.s. in $y \in E$,

$$\forall\ x, x', y \in E, \qquad \frac{\mathrm{d}P_1(x,.)}{\mathrm{d}P_1(x',.)}(y) \geq \epsilon > 0. \tag{3}$$

Under ($\mathcal{P}_1$) condition (Φ) is met and the semigroup Φ is exponentially contractive. In the further development of Section 2.3 we will explicit the dependence of $\alpha_t(\Phi)$ on the parameter ε and on the potential function V. We mention that ($\mathcal{P}_1$) can be relaxed in some ways but we do not detail these extensions, this would be a too great digression here. The interested reader is recommended to consult [4, 5]. The abstract condition (Φ) guarantees the existence of a unique fixed point $\eta_\infty \in \mathcal{P}(E)$ such that for any $t \in I$

$$\Phi_t(\eta_\infty) = \eta_\infty.$$

We are now in position to state our representation of the Lyapunov exponents in terms of the fixed point η_∞ and the mean time average of the flow η_t. The next two theorems give respectively the Feynman–Kac representation of these exponents for discrete and continuous time semigroups.

Theorem 0.1 (discrete time)**.** *Suppose (Φ) holds for some $t_0 \geq 0$ with $\alpha_n(\Phi) \in (0,1)$ for $n \geq t_0$. We have $\lambda(P^v) = \ln \eta_\infty(\exp V)$ and for any $x \in E$ and $n \geq t_0$,*

$$n\ |\lambda_n^{(x)}(P^v) - \lambda(P^v)| \leq\ t_0\ \mathrm{osc}(V) + 2\mathrm{e}^{\|V\|}\alpha(\Phi) \tag{4}$$

with

$$\lambda_n^{(x)}(P^v) = \frac{1}{n}\sum_{p=0}^{n-1} \ln \eta_p^{(x)}(\exp V).$$

Suppose there exist a P_1-reversible probability measure μ and a positive eigenfunction $h_v \in \mathcal{B}_b(E)$ such that $P_1^v(h_v) = \mathrm{e}^{\lambda(P^v)}\ h_v$, then

$$\eta_\infty(f) = \mu(h_v\ P_1(f))/\mu(h_v).$$

Theorem 0.2 (continuous time)**.** *Suppose (Φ) holds for some $t_0 \geq 0$ with $\alpha_t(\Phi) \in (0,1)$ for $t \geq t_0$. We have $\lambda(P^v) = \eta_\infty(V)$ and for any $x \in E$ and $t \geq t_0$*

$$t\ |\lambda_t^{(x)}(P^v) - \lambda(P^v)| \leq t_0\ \mathrm{osc}(V) + \|V\|\alpha(\Phi) \tag{5}$$

with

$$\lambda_t^{(x)}(P^v) = \frac{1}{t}\int_0^t \eta_s^{(x)}(V)\mathrm{d}s.$$

Suppose there exist a P-reversible probability measure μ and a positive eigenfunction $h_v \in D(L^v)$ such that $L^v(h_v) = \lambda(P^v)\ h_v$, then the fixed point η_∞ is defined for any $f \in D(L^v)$ by

$$\eta_\infty(f) = \mu(h_v\ f)/\mu(h_v).$$

Remark 0.3. When the state space is compact Feng and Kurtz also prove in [10] that this condition ($\mathcal{P}_1$) also guarantees the existence of positive eigenfunctions as needed in Theorem 0.1 and Theorem 0.2.

The interacting particle approximating models are defined in terms of a sequence of Markov processes $\xi = (\xi_t)_{t\in I}$ on a product space E^N and time index I. The particle models are chosen so that the corresponding empirical measures

$$\eta_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_t^i}$$

converge in some sense, as N tends to infinity, to the desired distribution η_t. To define these interacting particle interpretations we need to introduce the nonlinear dynamics of the flows $\eta = (\eta_t)_{t\in I}$.

In the discrete time case they are given by an equation in distribution space the form

$$\eta_{n+1} = \eta_n K_{\eta_n}$$

where $(K_\eta)_{\eta\in\mathcal{P}(E)}$ is a (non unique) collection of Markov transitions on E such that

$$\eta K_\eta(f) = \eta\left(\mathrm{e}^V\ P_1(f)\right)/\eta\left(\mathrm{e}^V\right).$$

The discrete generation particle model associated to a given collection of Markov transitions K_η is the E^N-valued Markov chain $\xi = (\xi_n)_{n\geq 0}$ with elementary transitions

$$\mathbb{P}\left(\xi_{n+1}\in \mathrm{d}y|\xi_n = x\right) = \prod_{i=1}^{N} K_{m(x)}(x^i, \mathrm{d}y^i) \qquad \text{with} \quad m(x) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x^i}$$

where $\mathrm{d}y = \mathrm{d}y^1\times\cdots\times \mathrm{d}y^N$ stands for an infinitesimal neighborhood of the point $y = (y^1,\ldots,y^N)\in E^N$, $x = (x^1,\ldots,x^N)\in E^N$.

In the continuous time case the distribution flow η_t satisfies for any $f\in D(L)$ an equation of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_t(f) = \eta_t(L_{\eta_t}(f))$$

for a (non unique) collection $(L_\eta)_{\eta\in\mathcal{P}(E)}$ of pregenerators on $D(L)$ such that

$$\eta(L_\eta(f)) = \eta(L(f)) + \eta(fV) - \eta(f)\eta(V).$$

In this case the infinitesimal pregenerator of ξ_t is defined for sufficiently regular test function as

$$\mathcal{L}(\varphi)(x^1,\ldots,x^N) = \sum_{i=1}^{N} L_{m(x)}^{(i)}\varphi(x^1,\ldots,x^i,\ldots,x^N)$$

where the superscript (i) indicates that we let $L_{m(x)}$ act on the i^{th} coordinate x_i, for $1\leq i\leq N$.

In both discrete and continuous time situations we suppose the initial system consists of N independent particles with common law η_0. When $\eta_0 = \delta_x$ for some $x\in E$ we write sometimes $\eta_t^{(x,N)}$ the empirical measures associated to the corresponding systems.

As noticed the choice of the transitions K_η and the pregenerators L_η is not unique. The particle models we have chosen to describe in the present work are related to selection/mutation genetic algorithms and Moran particle systems. The precise description of these discrete and continuous time particle models will be given in Section 3. An heuristic comparison between our interacting particle models and the one presented in [2] is given in the end of Section 3.2.1. Although we do not come into the details we already mention that both particle models can be regarded as the motion of N-particles in an environment with obstacles associated to a potential function V. Our particle models corresponds to the "soft obstacles" situation and the one of Burdzy *et al.* is related the boundary "hard obstacles" case.

There are many ways to describe the quality of the N-particle approximating measures η_t^N including central limit theorems, large deviations principles and empirical process convergence theorems (see for instance [4]). Here the most convenient way to guarantee the quality of the particle Lyapunov exponent approximating model is to obtain uniform estimates with respect to the time parameter. This task is extremely difficult and many efforts have been made recently in this direction under the mixing condition $(\mathcal{P}_1)$. The discrete time particle model is known to converge to the desired distribution and uniform L_p rates of natural fluctuation order $1/\sqrt{N}$ have been obtained in [3, 4].

In the continuous time case similar uniform L_p-rates exist with an order N^α with $1 \leq \alpha \leq 1/2$. This estimates have been proved in [4] for a particular Moran particle model. As announced previously we will extend the analysis to a more general class of particle scheme including the soft obstacle version of the model introduced by Burdzy *et al.* in [2]. It is still an open problem to extend the forthcoming estimates to hard obstacle models. Although our approach does not depend on the form of underlying process X it strongly depends on the boundedness of the potential function V. We believe these two models can be treated in a similar fashion but we have not yet succeed to analyze "hard obstacles" with degenerate potentials with our semigroup methodology.

Theorem 0.4 (discrete time). *Suppose the semigroup P satisfies condition $(\mathcal{P}_1)$ for some $\varepsilon > 0$. If we take the particle approximating Lyapunov exponent*

$$\lambda_n^{(x,N)}(P^v) = \frac{1}{n}\sum_{q=0}^{n} \ln \eta_q^{(x,N)}(\exp V)$$

then we have the unbias property

$$\mathbb{E}(\exp(n\lambda_n^{(x,N)}(P^v))) = \exp(n\lambda_n^{(x)}(P^v))$$

and for any $p \geq 1$ we have the uniform estimate

$$\sup_{n\geq 1} \mathbb{E}[|\lambda_n^{(x,N)}(P^v) - \lambda_n^{(x)}(P^v)|^p]^{1/p} \leq \frac{b(p)c_v(\varepsilon)}{\sqrt{N}} \tag{6}$$

for some universal constant $b(p)$ which only depends on p and for some finite constant $c_v(\varepsilon)$ depending on ε and V. In addition, if there exists a P-reversible probability measure μ and a positive eigenfunction $h_v \in \mathcal{B}_b(E)$ then for any $f \in \mathcal{B}_b(E)$ with $\|f\| \leq 1$

$$\sup_{n\geq(\ln N)/(2\epsilon^2)} \sqrt{N}\ \mathbb{E}(|\eta_n^N(f) - \mu(h_v\ P_1(f))/\mu(h_v)|^p)^{1/p} < \infty. \tag{7}$$

If we combine Theorem 0.1 with Theorem 0.4 we clearly obtain the following estimate

$$\sup_{n\geq\sqrt{N}, x\in E} \sqrt{N}\ \mathbb{E}[|\lambda_n^{(x,N)}(P^v) - \lambda(P^v)|^p]^{1/p} < \infty.$$

Theorem 0.5 (continuous time). *Let $\lambda_t^{(x,N)}(P^v)$ be the particle approximating Lyapunov exponent defined by*

$$\lambda_t^{(x,N)}(P^v) = \frac{1}{t}\int_0^t \eta_s^{(x,N)}(V)\ \mathrm{d}s.$$

then we have the unbias property

$$\mathbb{E}(\exp(t\lambda_t^{(x,N)}(P^v))) = \exp(t\lambda_t^{(x)}(P^v)).$$

If the semigroup Φ is exponentially stable then we have the uniform estimate

$$\sup_{t\geq 0}\mathbb{E}(|\lambda_t^{(x,N)}(P^v)-\lambda_t^{(x)}(P^v)|^2)^{1/2}\leq c/N^{\beta/2}$$

for some universal constant $c<\infty$ and some exponent $\beta\in(0,1]$ depending on V. In addition, if there exists a P-reversible probability measure μ and a positive eigenfunction $h_v\in D(L^v)$ then for any $f\in\mathcal{B}_b(E)$ with $\|f\|\leq 1$ we have

$$\sup_{t\geq d\ln N} N^{\beta/2}\ \mathbb{E}(|\eta_t^N(f)-\mu(h_v\ f)/\mu(h_v)|^2)^{1/2}<\infty \tag{8}$$

for some constant d depending on the potential V.

In the continuous time case Theorem 0.2 and Theorem 0.5 lead to the following estimate

$$\sup_{N\geq 1} N\ \sup_{x\in E}\mathbb{E}[|\lambda_N^{(x,\lfloor N^{1/\beta}\rfloor)}(P^v)-\lambda(P^v)|^2]<+\infty$$

where $\lfloor\cdot\rfloor$ stands for the integer part.

1. Spectral interpretations

Our objective in this section is to discuss spectral aspects of the Lyapunov exponent, specially when some properties of reversibility are available. The connection between Lyapunov exponents with the $\mathbb{L}_\infty$ or $\mathbb{L}_2$ spectral radii discussed in this section are probably well known. Nevertheless we have not find a precise reference in the literature which connects precisely these quantities. For these reasons and for the convenience of the reader we have chose to devote a short section on the spectral interpretations of these exponents. The forthcoming analysis also underlines in a precise way various situations in which the particle approximating strategies developed in this article apply.

First we undertake an abstract formulation. Let Q be a bounded operator on the Banach space $(\mathcal{B}_b(E),\|\cdot\|)$ associated to a general underlying measurable space $(E,\mathcal{E})$. We denote by $\|\!|\cdot|\!\|$ its operator norm and by definition its spectral radius $\mathrm{Spr}(Q)$ is the quantity

$$\mathrm{Spr}(Q)\ \stackrel{\text{def.}}{=}\ \lim_{n\to\infty}\|\!|Q^n|\!\|^{1/n}\quad=\quad\inf_{n\geq 1}\|\!|Q^n|\!\|^{1/n}.$$

From now on, we will also assume that Q is non-negativity preserving, in the sense that if $f\in\mathcal{B}_b(E)$ is a nonnegative function, then the same is true for $Q(f)$ (thus Q is almost a generalized Markov operator, only the renormalisation property $Q(1)=1$ is missing). This property permits to give an other characterization of the spectral radius:

Lemma 1.1. *Under the above setting, we have*

$$\mathrm{Spr}(Q)=\lim_{n\to\infty}\|Q^n(1)\|^{1/n}\quad=\quad\inf_{n\geq 1}\|Q^n(1)\|^{1/n}.$$

Proof. It is sufficient to see that in fact, $\|\!|Q|\!\|=\|Q(1)\|$ as the iterates Q^n, $n\in\mathbb{N}^*$, verify the same conditions as Q. The bound $\|\!|Q|\!\|\geq\|Q(1)\|$ is always satisfied, as $\|1\|=1$, and for the reciproque, we take into account the non-negativity preserving property, which shows that for any $f\in\mathcal{B}_b(E)$,

$$Q(f)\leq Q(\|f\|\,1)=\|f\|\,Q(1)$$

and in the same manner that $Q(f)\geq-\|f\|\,Q(1)$, thus trivially implying that $\|\!|Q|\!\|\leq\|Q(1)\|$. $\square$

The above hypothesis admits as another usual consequence that Cauchy–Schwartz inequalities are satisfied:

$$\forall\ f,g\in\mathcal{B}_b(E),\qquad Q(fg)\leq\sqrt{Q(f^2)Q(g^2)}.$$

Next we furthermore make the assumption that there exist a probability μ on $(E,\mathcal{E})$ and a constant $K\geq 0$ such that the following inequalities are verified

$$\forall\ f\in\mathcal{B}_b(E),\qquad \mu(|Q(f)|)\leq K\mu(|f|). \tag{9}$$

In particular, the image $Q(f)$ of a function $f\in\mathcal{B}_b(E)$ negligeable with respect to μ remains negligeable, property which permits to see Q as an operator on $L_\infty(\mu)$. Due to the above bound, the latter can be further uniquely extended as a bounded operator on $L_1(\mu)$. Even on $L_2(\mu)$, as we note that for any $f\in\mathcal{B}_b(E)$,

$$\mu(Q(f)^2)\leq\mu(Q(f^2)Q(1))\leq\|Q(1)\|\,\mu(Q(f^2))\leq K\,\|Q(1)\|\,\mu(Q(f^2)).$$

Thus we are led to consider the corresponding notion of spectral radius,

$$\mathrm{Spr}_{2,\mu}(Q)\ \stackrel{\text{def.}}{=}\ \lim_{n\to\infty}\|Q^n\|_{2,\mu}^{1/n}\quad=\quad\inf_{n\geq 1}\|Q^n\|_{2,\mu}^{1/n}$$

where clearly

$$\|Q^n\|_{2,\mu}^2\ \stackrel{\text{def.}}{=}\ \sup_{f\in L_2(\mu)\setminus\{0\}}\frac{\mu[(Q(f))^2]}{\mu[f^2]}.$$

If E is finite and μ gives positive weight to any of its point, then the equivalence of norms on finite dimensional space (in this case the algebra of $E\times E$ matrices) enables to see that

$$\mathrm{Spr}(Q)=\mathrm{Spr}_{2,\mu}(Q)$$

but this equality is not always satisfied, even when E is finite, as it is easy to device an example for which $\|Q\|>\|Q\|_{2,\mu}$ with a probability μ not charging the whole set E (what is always true in this finite context is that $\|Q\|_{\infty,\mu}=\|Q\|_{2,\mu}$).

Nevertheless, under a symmetry assumption, there is a general bound in that direction:

Lemma 1.2. *Assume that indeed Q is auto-adjoint in $L_2(\mu)$, then we are assured of*

$$\mathrm{Spr}(Q)\geq\mathrm{Spr}_{2,\mu}(Q).$$

Proof. Let a function $f\in L_2(\mu)$ and an integer $n\geq 1$ be given, using the symmetry of Q^n we obtain

$$\mu[(Q^n(f))^2]\leq\mu[Q^n(f^2)Q^n(1)]=\mu[f^2Q^{2n}(1)]\leq\left\|Q^{2n}(1)\right\|\mu[f^2].$$

Taking a supremum over $f\in L_2(\mu)\setminus\{0\}$, this shows that

$$\|Q^n\|_{2,\mu}^{1/n}\leq\left\|Q^{2n}(1)\right\|^{1/(2n)}$$

thus letting n go to infinity we conclude to the previous bound. □

In order to prove a reverse inequality, we assume that Q can be written as a density kernel with respect to μ, namely that there exist a measurable mapping $q\,:\,E\times E\to\mathbb{R}_+$ such that

$$\forall\ f\in\mathcal{B}_b(E),\,\forall\ x\in E,\qquad Q[f](x)=\int q(x,y)f(y)\,\mu(\mathrm{d}y).$$

Lemma 1.3. *Under the hypothesis that*

$$\sup_{x\in E}\int q(x,y)^2\,\mu(\mathrm{d}y) < +\infty$$

we have

$$\mathrm{Spr}(Q) \leq \mathrm{Spr}_{2,\mu}(Q).$$

In particular, if Q is furthermore auto-adjoint (i.e. q is symmetric, $\mu\otimes\mu$-a.s.), then

$$\mathrm{Spr}(Q) = \mathrm{Spr}_{2,\mu}(Q).$$

Proof. We have for any integer number $n \geq 1$ and point $x \in E$, by the Cauchy–Schwartz inequality,

$$Q^n(1)(x) = \int q(x,y)Q^{n-1}(1)(y)\,\mu(\mathrm{d}y) \leq \sqrt{\int q(x,y)^2\,\mu(\mathrm{d}y)}\sqrt{\mu[Q^{n-1}(1)^2]} \leq \sqrt{\int q(x,y)^2\,\mu(\mathrm{d}y)}\,\|\!|Q^{n-1}|\!\|_{2,\mu}.$$

Thus considering the supremum over $x \in E$, taking the n^{th} root and letting n be large, we conclude to the affirmations of the lemma. □

Remarks 1.4.

a) Observe that under both hypotheses of the lemma, the auto-adjoint operator Q is Hilbert–Schmidt in $L_2(\mu)$ and $\mathrm{Spr}_{2,\mu}(Q)$ is indeed its largest eigenvalue.

b) It is sufficient that one of the iterates Q^p, for some $p \geq 1$, satisfies the above hypotheses to derive the same conclusions.

Let $S \stackrel{\text{def.}}{=} (S_n)_{n\geq 0}$ be a semigroup of non-negativity preserving bounded operators (*a priori* on $(\mathcal{B}_b(E), \|\!|\cdot|\!\|)$). Since by definition,

$$\mathrm{Lyap}(S) = \mathrm{Spr}(S_1)$$

the above considerations gives conditions for this quantity to be interpreted as a L_2 spectral radius.

Now we are rather interested in a continuous time semigroup $S \stackrel{\text{def.}}{=} (S_t)_{t\geq 0}$ of non-negativity preserving bounded operators and more precisely in its associated Lyapunov constant,

$$\mathrm{Lyap}(S) = \lim_{t\to+\infty} \|\!|S_t|\!\|^{1/t}.$$

As the rhs is a true limit, we also have for any fixed $t_0 > 0$,

$$\mathrm{Lyap}(S) = \lim_{p\to+\infty,\,p\in\mathbb{N}^*} \|\!|S_{pt_0}|\!\|^{1/(pt_0)} = \mathrm{Spr}(S_{t_0})^{1/t_0}$$

(in fact a similar equality also holds in the discrete time case), and under the previous restrictions, this can be used to furnish alternative characterizations of the Lyapunov exponent.

To go a little further, we will work under a stronger hypothesis: assume that S can also be seen as strongly continuous semigroup of auto-adjoint operators on some $L_2(\mu)$ space (as above). Then we can make use of classical spectral calculus (*cf.* for instance [12]). Let $\mathcal{L}_S$ be the generator of S; its domain $\mathcal{D}(\mathcal{L}_S)$ is the set of $f \in L_2(\mu)$ such that $(S_t(f) - f)/t$ converges for small $t > 0$ and the limit is by definition $\mathcal{L}_S(f)$.

This operator is itself auto-adjoint in $L_2(\mu)$ and applying to its opposite the spectral decomposition theorem, we find a spectral family of projections $(E_\lambda)_{\lambda\in\mathbb{R}}$ such that we can write

$$\begin{aligned} \mathcal{D}(\mathcal{L}_S) &= \left\{ f \in L_2(\mu) \,:\, \int \lambda^2 \,\mathrm{d}\mu(E_\lambda(f)f) < +\infty \right\} \\ \forall\, f \in \mathcal{D}(\mathcal{L}_S), \qquad -\mathcal{L}_S(f) &= \int \lambda \,\mathrm{d}E_\lambda(f). \end{aligned}$$

Then for any $t_0 \geq 0$, we are assured of the representation

$$S_{t_0} = \int \exp(-t_0\lambda)\,\mathrm{d}E_\lambda$$

from which we easily deduce that

$$\| S_{t_0} \|_{2,\mu} = \sqrt{\sup\{\exp(-2t_0 r(f)) \,:\, f \in L_2(\mu),\, \mu(f^2) = 1\}}$$

(using the fact that $\mathrm{d}\mu(E_\lambda(f)f)$ is a probability on $\mathbb{R}$ if $\mu(f^2) = 1$), where for any $f \in L_2(\mu)$, we have defined

$$\begin{aligned} r(f) &\overset{\text{def.}}{=} \sup\left\{ l \in \mathbb{R} \,:\, \int_{-\infty}^{l} \exp(-t_0\lambda)\,\mathrm{d}\mu(E_\lambda(f)f) = 0 \right\} \\ &= \sup\left\{ l \in \mathbb{R} \,:\, \int_{-\infty}^{l} \mathrm{d}\mu(E_\lambda(f)f) = 0 \right\} \\ &= \sup\left\{ l \in \mathbb{R} \,:\, \mu(E_l(f)f) = 0 \right\} \cdot \end{aligned}$$

In order to give an alternative formulation, let us introduce the associated Dirichlet form $\mathcal{E}_S$; on the domain

$$\mathcal{D}(\mathcal{E}_S) = \left\{ f \in L_2(\mu) \,:\, \int |\lambda| \,\mathrm{d}\mu(E_\lambda(f)f) < +\infty \right\}$$

it is defined by

$$\forall\, f \in \mathcal{D}(\mathcal{E}_S), \qquad \mathcal{E}_S(f,f) = \int \lambda \,\mathrm{d}\mu(E_\lambda(f)f).$$

Then it appears without much difficulties that we also have

$$\inf\{r(f) \,:\, f \in L_2(\mu) \setminus \{0\}\} = \inf_{f\in L_2(\mu)\setminus\{0\}} \frac{\mathcal{E}_S(f,f)}{\sqrt{\mu(f^2)}}$$

(by convention $\mathcal{E}_S(f,f) = +\infty$ if $f \in L_2(\mu) \setminus \mathcal{D}(\mathcal{E}_S)$), and thus we get that for any $t_0 > 0$,

$$\frac{1}{t_0}\ln(\| S_{t_0} \|_{2,\mu}) = \lambda_{2,\mu}(S) \quad \overset{\text{def.}}{=} \quad - \inf_{f\in L_2(\mu)\setminus\{0\}} \frac{\mathcal{E}_S(f,f)}{\sqrt{\mu(f^2)}}$$

and by consequence, we also obtain $\mathrm{Spr}_{2,\mu}(S_{t_0})^{1/t_0} = \| S_{t_0} \|_{2,\mu}^{1/t_0} = \mathrm{Lyap}_{2,\mu}(S) = \exp(\lambda_{2,\mu}(S))$.

Finally, putting together the previous results, we have shown the next spectral interpretation of the logarithmic Lyapunov exponent.

Proposition 1.5. *Let $S = (S_t)_{t\geq 0}$ be a semigroup of non-negativity preserving bounded operators which can also be seen as a strongly continuous $L_2(\mu)$-semigroup. Assume there exist a time $t_0 > 0$ and a measurable function $q_{t_0} : E \times E \to \mathbb{R}_+$ such that the following representation holds*

$$\forall\ f \in \mathcal{B}_b(E), \forall\ x \in E, \qquad S_{t_0}[f](x) = \int q_{t_0}(x,y) f(y)\,\mu(\mathrm{d}y)$$

with $\sup_{x\in E}\int q_{t_0}(x,y)^2\,\mu(\mathrm{d}y) < +\infty$ *Then we have*

$$\lambda(S) \stackrel{\text{def.}}{=} \ln(\mathrm{Lyap}(S)) = \lambda_{2,\mu}(S).$$

It is time now to investigate the consequences for the models presented in the previous section.

First in the discrete time setting: we associate to our Markovian semigroup $(P_n)_{n\geq 0}$ and our bounded measurable potential V another operator Q^v acting on $\mathcal{B}_b(E)$ by

$$\forall\ f \in \mathcal{B}_b(E), \qquad Q^v(f) \stackrel{\text{def.}}{=} \exp(V/2)P_1[f\exp(V/2)].$$

Using the fact that we can write,

$$\forall\ f \in \mathcal{B}_b(E), \qquad P_1^v(f) \stackrel{\text{def.}}{=} \exp(V/2)Q^v[f\exp(-V/2)]$$

and that the mapping

$$\mathcal{B}_b(E) \ni f \mapsto \exp(V/2)f \in \mathcal{B}_b(E)$$

is an isomorphism of $\mathcal{B}_b(E)$, it is clear that P_1^v and Q^v have the same spectrum, the same eigenvalues and the same spectral radius (notions to be understood in the Banach space $\mathcal{B}_b(E)$, *cf.* for instance Kato [12]).

Meanwhile, the advantage of Q^v is that if P_1 is reversible with respect to a probability μ, namely

$$\forall\ f, g \in \mathcal{B}_b(E), \qquad \mu(fP_1(g)) = \mu(gP_1(f))$$

then the same is true for Q^v, fact which implies that (10) is satisfied with $Q = Q^v$ and $K = 1$, so Q^v can be extended as an operator auto-adjoint in $L_2(\mu)$.

Thus if we assume further that for any $x \in E$, $P_1(x,\cdot) \sim \mu$ and $\sup_{x\in E}\|\mathrm{d}P_1(x,\cdot)/\mathrm{d}\mu\|_{L_2(\mu)} < +\infty$, then these properties will also be verified by Q^v due to the boundedness of V and we end up with

$$\mathrm{Lyap}(P^v) = \mathrm{Spr}(P_1^v) = \mathrm{Spr}(Q^v) = \mathrm{Spr}_{2,\mu}(Q^v).$$

Now we turn to the continuous time case (this is one of the two places in this article where our strong assumptions on the pregenerator $(D(L), L)$ are really needed, the other one will be at the end of the paper, for Prop. 3.7). For that purpose, let us assume furthermore that there exists a probability μ for which the semigroup P is reversible:

$$\forall\ t > 0, \forall\ f, g \in \mathcal{B}_b(E), \qquad \mu(fP_t(g)) = \mu(gP_t(f))$$

(in particular μ is invariant for the semigroup P and (9) is satisfied with $Q = P_t$ and $K = 1$, for all $t \geq 1$). By density of $\mathcal{B}_b(E)$ in $L_2(\mu)$, the above symmetry property and the Cauchy–Schwartz inequality, the operators P_t, for $t \geq 0$, can be extended to $L_2(\mu)$, where they act as non-negativity preserving self-adjoint contractions. Furthermore, since $D(L)$ is an algebra generating the underlying σ-algebra $\mathcal{E}$, it is also dense in $L_2(\mu)$ and we easily deduce from this fact that $(P_t)_{t\geq 0}$ is strongly continuous in $L_2(\mu)$ (and that $D(L)$ is included in the domains $\mathcal{D}(\mathcal{L}_P)$ and $\mathcal{D}(\mathcal{E}_P)$ of the associated generator and Dirichlet form).

Our next objective is to verify directly that the same is true for the semigroup $(P_t^v)_{t\geq 0}$. First we note that for any time $t\geq 0$, any function $f\in\mathcal{B}_b(E)$ and any point $x\in E$,

$$P_t^v(f)(x) = P_t(f)(x) + \int_0^t P_s(\ V\ P_{t-s}^v(f)\ )(x)\mathrm{d}s. \tag{10}$$

Indeed, by our measurability assumption, we have

$$\exp\left(\int_0^t V(X_s)\,\mathrm{d}s\right) - 1 = \int_0^t V(X_s)\ \exp\left(\int_s^t V(X_u)\,\mathrm{d}u\right)\ \mathrm{d}s$$

so integrating with respect to $\mathbb{P}_x$ and using the Markov property at times $0\leq s\leq t$, we obtain the previous affirmation, which can be rewritten in the form

$$P_t^v(f) - f = P_t(f) - f + \int_0^t P_s(VP_{t-s}^v(f))\mathrm{d}s. \tag{11}$$

Taking into account the locally uniform (with respect to $t\geq 0$) boundedness of the operators P_t^v in $L_2(\mu)$ (which is deduced at once from the uniform boundedness of V and P_t, for $t\geq 0$) and the strong continuity of $(P_t)_{t\geq 0}$, the above formula convinces us that for $f\in\mathcal{B}_b(E)$, the mapping $\mathbb{R}_+\ni t\mapsto P_t^v(f)\in L_2(\mu)$ is also continuous. Without difficulty, this result is next seen to hold for any $f\in L_2(\mu)$.

So $(P_t^v)_{t\geq 0}$ already appears as a strongly continuous semigroup of bounded operators in $L_2(\mu)$. Let us show that they are in fact auto-adjoint. For that we remark that by reversibility of μ, for any $t\geq 0$, the couple (X_0, X_t) has the same law that (X_t, X_0) under

$$\mathbb{P}_\mu[\,\cdot\,] \stackrel{\text{def.}}{=} \int_E \mu(\mathrm{d}x)\,\mathbb{P}_x[\,\cdot\,].$$

A standard recursive procedure based on the Markov property proves then that for any $n\in\mathbb{N}$, any $0\leq t_1\leq t_2\leq\cdots\leq t_n\leq t$ and any measurable function $F : E^n\to\mathbb{R}$, the law of $(X_0, F(X_{t_1}, X_{t_2},\ldots,X_{t_n}), X_t)$ is the same as that of $(X_t, F(X_{t_n}, X_{t_{n-1}},\ldots,X_{t_1}), X_0)$. Having resort to a monotonous class argument (and a measurability assumption on the space of possible trajectories, *cf.* [7], which is trivially satisfied if the state space is assumed to be topological and the paths of our Markov process are càdlàg), it appears that the law of $(X_0, \int_0^t V(X_s)\,\mathrm{d}s, X_t)$ is that of $(X_t, \int_0^t V(X_{t-s})\,\mathrm{d}s, X_0) = (X_t, \int_0^t V(X_s)\,\mathrm{d}s, X_0)$ (always under P^μ, of course). The symmetry of P_t^v in $L_2(\mu)$, for $t\geq 0$, follows at once from this equality in law:

$$\begin{aligned}\forall\ f,g\in L_2(\mu),\quad \mu(fP_t^v(g)) &= \mathbb{E}_\mu\left[f(X_0)\exp\left(\int_0^t V(X_s)\,\mathrm{d}s\right)g(X_t)\right] = \mathbb{E}_\mu\left[g(X_0)\exp\left(\int_0^t V(X_s)\,\mathrm{d}s\right)f(X_t)\right]\\ &= \mu(gP_t^v(f)).\end{aligned}$$

We notice that if there exists a time $t_0>0$ such that for all $x\in E$,

$$P_{t_0}(x,\cdot)\sim\mu \quad\text{and}\quad \sup_{y\in E}\|\mathrm{d}P_{t_0}(y,\cdot)/\mathrm{d}\mu\|_{L_2(\mu)} < +\infty$$

then the same will be true for the kernel $P_{t_0}^v$ (due one more time to the boundedness of V), thus we are in position to apply the previous proposition, which shows that in this case,

$$\lambda(P^v) = -\inf_{f\in L_2(\mu)\setminus\{0\}} \frac{\mathcal{E}_{P^v}(f,f)}{\sqrt{\mu(f^2)}}\cdot$$

But let us recall that $\mathcal{D}(\mathcal{E}_{P^v})$ is also the set of functions $f \in L_2(\mu)$ such that $\mu[f(P_t^v(f) - f)]/t$ converge for small $t > 0$ and that the limit is then $\mathcal{E}_{P^v}(f, f)$. Using the formula (11), we see that $\mathcal{D}(\mathcal{E}_{P^v}) = \mathcal{D}(\mathcal{E}_P)$ and that

$$\forall\, f \in D(\mathcal{E}_P), \qquad \mathcal{E}_{P^v}(f, f) = \mathcal{E}_P(f, f) + \mu(V f^2).$$

Thus we end up with the Schrödinger spectral interpretation of the logarithmic Lyapunov exponent:

$$\lambda(P^v) = -\inf_{f \in L_2(\mu)\setminus\{0\}} \frac{\mathcal{E}_P(f, f) + \mu(V f^2)}{\sqrt{\mu(f^2)}}$$

which is also equal to

$$-\inf_{f \in D(L)\setminus\{0\}} \frac{\mu[f(L(f) + V f)]}{\sqrt{\mu(f^2)}}$$

if we assume that $D(L)$ is dense in $\mathcal{D}(\mathcal{E}_P)$ for its natural norm $\sqrt{\mathcal{E}_P(\cdot,\cdot)} + \|\cdot\|_{L_2(\mu)}$.

Remark 1.6. Coming back to our pregenerator $(D(L), L)$ and using formula (10), we can show that in $(\mathcal{B}_b(E), \|\cdot\|)$,

$$\forall\, t \geq 0, \forall\, f \in D(L), \qquad \partial_t P_t^v(f) = P_t^v(L(f) + V f)$$

and furthermore taking into account that the operators P_t^v, for $t > 0$, leave invariant the domain $D(L)$, we also get

$$\forall\, t \geq 0, \forall\, f \in D(L), \qquad \partial_t P_t^v(f) = L[P_t^v(f)] + V P_t^v(f).$$

Nevertheless, let us mention that the corresponding stability property of $D(L)$ by the semigroup P^v is always verified if we work under the setting presented in [7].

2. Feynman–Kac representations

This section is essentially concerned with the proof of Theorem 0.1 and Theorem 0.2. In Section 2.1 we present the continuous and discrete time measure valued equations associated to the Feynman–Kac non linear semigroups $\Phi = (\Phi_t)_{t \in I}$. Section 2.2 focuses on the long time behavior and the fixed point of these distribution flows. We also examine the situation where the semigroup P is reversible with respect to some distribution. We propose a representation of the limiting fixed point of Φ in terms of the "eigenvector" associated to the Lyapunov exponent. In Section 2.3 we characterize these exponents in terms of the mean time average of non linear Feynman–Kac distribution flow models. In both Sections 2.2 and 2.3 we design a general and unique strategy to treat in the same fashion the discrete and continuous time case. We mention that the continuous time or discrete generation models described respectively in Section 2.1.1 and Section 2.1.2 extend the ones studied in [3, 4, 6]. They will be also used in the forthcoming development of Section 3 to design a class of particle approximating models. We finally notice that the addition of a constant to the potential V admits as an immediate consequence to add the same constant to the logarithmic Lyapunov exponent. Thus for our purposes there is no real loss of generality in assuming from now on that V is nonnegative.

2.1. Nonlinear measure valued equations

2.1.1. *Continuous time models*

In this short section we briefly present a collection of non linear measure valued evolution models associated to the continuous time semigroup $(\Phi_t)_{t \geq 0}$. We start by noting that in the Radon–Nykodim sense we have the formula

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \gamma_t(1) = \frac{1}{\gamma_t(1)} \gamma_t(V) = \eta_t(V) \quad \text{and} \quad \gamma_t(1) = \exp\left(\int_0^t \eta_s(V)\,\mathrm{d}s\right). \tag{12}$$

One of the consequences of the above exponential formula is the following statement:

Proposition 2.1. *For any $\rho \in [0,1]$ the distribution flow $(\eta_t)_{t\in\mathbb{R}_+}$ satisfies for any $f \in D(L)$ the following time evolutions*

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_t(f) = \eta_t(L^{\rho}_{\eta_t}(f)) \quad \text{with} \quad L^{\rho}_{\eta} = \rho\ L_\eta + (1-\rho)\ L'_\eta$$

and where $(L_\eta)_{\eta\in\mathcal{P}(E)}$ and $(L'_\eta)_{\eta\in\mathcal{P}(E)}$ are two collections of operators on $D(L)$ defined for any $f \in D(L)$ and $x \in E$ by

$$\begin{aligned} L_\eta(f)(x) &= L(f)(x) + \int\ (f(y) - f(x))\ V(y)\ \eta(\mathrm{d}y) \\ L'_\eta(f)(x) &= L(f)(x) + V'(x) \int\ (f(y) - f(x))\ \eta(\mathrm{d}y) \end{aligned}$$

for some bounded nonnegative function V' such that $V'(x) + V(x) = c \in \mathbb{R}$.

Proof. We associate to each $f \in D(L)$ the $\mathbb{P}_{\eta_0}$-martingale

$$M_t(f) = f(X_t) - f(X_0) - \int_0^t\ L(f)(X_s)$$

and we write

$$Z_{\eta,t} = \exp \int_0^t\ [V(X_s) - \eta_s(V)]\mathrm{d}s.$$

The stochastic integration by parts formula gives

$$\mathrm{d}(Z_{\eta,t} f(X_t)) = Z_{\eta,t}\ [L(f)(X_t) + f(X_t)\ [V(X_t) - \eta_t(V)]\mathrm{d}t + Z_{\eta,t}\ \mathrm{d}M_t(f).$$

Hence from the above we have

$$Z_{\eta,t} f(X_t) = f(X_0) + \int_0^t\ Z_{\eta,s}\ [L(f)(X_s) + f(X_s)\ [V(X_s) - \eta_s(V)]\ \mathrm{d}s + \int_0^t Z_{\eta,t}\ \mathrm{d}M_t(f).$$

Integrating by definition of the flow η_t we end up with

$$\eta_t(f) = \eta_0(f) + \int_0^t\ \eta_s(L(f) + f\ [V - \eta_s(V)])\ \mathrm{d}s.$$

We end the proof by noting that for any $\eta \in \mathcal{P}(E)$ and $f \in D(L)$

$$\eta(L(f) + f\ [V - \eta(V)]) = \eta(L_\eta(f)) = \eta(L'_\eta(f)). \qquad \square$$

Remarks 2.2.

- Clearly the potential V' is only determined up to an additive constant. It is also instructive to notice that the normalized Feynman–Kac formulas remain unchanged if we replace V by $-V'$. But from now on, we will further assume that V' is nonnegative (for instance by asking that $c \geq \|V\|$ in the previous proposition), so that $L_{\eta'}$ could be considered as a pregenerator.
- The collection of operators L_η and L'_η are related to nonlinear martingale problems. Their solutions consists of a canonical time inhomogeneous process $\mathcal{X}$ under two probability measures $\mathbb{Q}$ and $\mathbb{Q}'$ with respective infinitesimal pregenerators $(L_{\eta_t})_{t\geq 0}$ and $(L'_{\eta_t})_{t\geq 0}$ and with time marginals

$$\mathbb{Q} \circ \mathcal{X}_t^{-1} = \mathrm{Law}(\mathcal{X}_t) = \eta_t = \mathbb{Q}' \circ \mathcal{X}_t^{-1}$$

for all $t \geq 0$.

The measure $\mathbb{Q}$ is called the McKean measure associated to L_η. We will not come into more details of these problems, this would be a too great digression. We only mention that the choice of the potential V or V' may lead to distinct McKean measure with the same time marginals η_t.
- We also notice that L_η^ρ can be rewritten as follows

$$L_\eta^\rho(f)(x) = L(f)(x) + \int (f(y) - f(x))\ V^\rho(x, y)\ \eta(\mathrm{d}y)$$

with

$$V^\rho(x, y) = \rho\ V(y) + (1 - \rho)\ V'(x).$$

2.1.2. *Discrete time models*

This short section focuses on the dynamical structure of the discrete time non linear semigroup $(\Phi_n)_{n\geq 0}$. In the discrete time case we also have a nonlinear evolution description.

Proposition 2.3. *The Feynman–Kac distribution flow $\eta = (\eta_n)_{n\in\mathbb{N}}$ satisfies a nonlinear dynamical system*

$$\eta_{n+1} = \eta_n K_{\eta_n} \tag{13}$$

where $(K_\eta)_{\eta\in\mathcal{P}(E)}$ is a collection of Markov transitions on E defined by the composition

$$\begin{aligned} K_\eta(x, \mathrm{d}z) &= (S_\eta P_1)(x, \mathrm{d}z) = \int_E \mathcal{S}_\eta(x, \mathrm{d}y)\ P_1(y, \mathrm{d}z) \\ S_\eta(x, \mathrm{d}y) &= \frac{\mathrm{e}^{V^-}(x)}{\eta(\mathrm{e}^V)}\ \delta_x(\mathrm{d}y) + \left(1 - \frac{\mathrm{e}^{V^-(x)}}{\eta(\mathrm{e}^V)}\right)\ \Psi^+(\eta)(\mathrm{d}y) \end{aligned}$$

for any pair of functions (V^+, V^-) with

$$V(x) > V^-(y), \qquad \Psi^+(\eta)(f) = \eta(f\mathrm{e}^{V^+})/\eta(\mathrm{e}^{V^+}) \qquad \text{and} \qquad \mathrm{e}^V = \mathrm{e}^{V^-} + \mathrm{e}^{V^+}.$$

For $V^+ = V$ we take the convention $S_\eta(x, \mathrm{d}y) = \Psi^+(\eta)(\mathrm{d}y)$.

Proof. By the Markov property and by definition of γ_n we have that

$$\eta_n(f) = \frac{\gamma_n(f)}{\gamma_n(1)} = \frac{\gamma_{n-1}(\mathrm{e}^V\ P_1(f))}{\gamma_{n-1}(\mathrm{e}^V)}.$$

Clearly this implies that

$$\eta_n(f) = \frac{\eta_{n-1}(\mathrm{e}^V\ P_1(f))}{\eta_{n-1}(\mathrm{e}^V)} = \Psi(\eta_{n-1})P_1(f) \quad \text{with} \quad \Psi(\eta)(f) = \eta(f\mathrm{e}^V)/\eta(\mathrm{e}^V).$$

On the other hand we check easily that $\Psi(\eta)(f) = \eta(S_\eta(f))$ and the proof is completed. □

Remarks 2.4. a) In the discrete time case the McKean measure associated to K_η and with initial condition η_0 is the probability measure $\mathbb{Q}_{\eta_0}$ on $E^{\mathbb{N}}$ with marginals $\mathbb{Q}_{\eta_0,[0,n]}$ on E^{n+1}

$$\mathbb{Q}_{\eta_0,[0,n]}(\mathrm{d}(x_0,\ldots,x_n)) = \eta_0(\mathrm{d}x_0)\ K_{\eta_0}(x_0,\mathrm{d}x_1)\ \ldots\ K_{\eta_{n-1}}(x_{n-1},\mathrm{d}x_n). \tag{14}$$

Note that for $V^+ = V$ our convention leads to the tensor product McKean measure

$$\mathbb{Q}_{\eta_0} = \otimes_{n\geq 0}\eta_n.$$

b) Since $\mathrm{e}^V = \mathrm{e}^{V^-} + \mathrm{e}^{V^+}$ with $V(x) > V^-(y)$ we have $\mathrm{e}^{V^-(x)} \leq \eta(\mathrm{e}^V)$. It does follow that $S_\eta(x,\mathrm{d}y)$ is a Markov transition.

c) For any $c \geq \mathrm{osc}(V)$ we have

$$V^-(x) = V(x) - c \leq V(x).$$

Therefore we can choose $V^- = V - c$ and $V^+ = V + \ln(1-\mathrm{e}^{-c})$. In this situation we obtain $\Psi^+ = \Psi$ and

$$S_\eta(x,\mathrm{d}y) = \mathrm{e}^{-c}\frac{\mathrm{e}^V(x)}{\eta(\mathrm{e}^V)}\ \delta_x(\mathrm{d}y) + \left(1-\mathrm{e}^{-c}\ \frac{\mathrm{e}^{V(x)}}{\eta(\mathrm{e}^V)}\right)\ \Psi(\eta)(\mathrm{d}y).$$

We end by noting that for any $f \in \mathcal{B}_b(E)$ with $\|f\| \leq 1$ we have

$$\|S_\eta(f) - \Psi(\eta)(f)\| \leq 2\ \mathrm{e}^{-c+\mathrm{osc}(V)} \text{ and } \|S_\eta(x,.) - \Psi(\eta)\|_{\mathrm{tv}} \leq \mathrm{e}^{-c+\mathrm{osc}(V)} \xrightarrow[c\to\infty]{} 0.$$

2.2. Fixed point measures

When condition (Φ) is satisfied for some $t_0 \in I$ and $\alpha_t(\Phi) < 1$ for all $t \geq t_0$, then the Banach fixed point theorem tells us that there exists an unique fixed point $\eta_\infty \in \mathcal{P}(E)$ with $\Phi_t(\eta_\infty) = \eta_\infty$. This implies that for any $s \in I \setminus \{0\}$ the measure η_∞ is also the unique fixed point of Φ_s. Indeed we have that

$$\begin{aligned}\Phi_s(\Phi_t(\eta_\infty)) &= \Phi_s(\eta_\infty) && (\Phi_t(\eta_\infty) = \eta_\infty)\\ &= \Phi_t(\Phi_s(\eta_\infty)) && (\text{semi-group property})\end{aligned}$$

and therefore $\Phi_s(\eta_\infty)$ is a fixed point of Φ_t. By the uniqueness property we end up with $\Phi_s(\eta_\infty) = \eta_\infty$. The proof of the uniqueness is also clear (first for $s \geq t_0$ and next for all $s > 0$, using iterations).

Next we suppose the semigroup $P = (P_t)_{t\in I}$ is reversible with respect to some measure $\mu \in \mathcal{P}(E)$.

In the discrete time case we suppose there exists positive eigenvector $h_v \in \mathcal{B}_b(E)$ such that $P_1^v(h_v) = \mathrm{e}^{\lambda(P^v)}\ h_v$. Let us check that

$$\mu_v(f) = \mu(h_v\ P_1(f))/\mu(h_v)$$

is a fixed point of Φ_1. To prove this claim we simply observe that

$$\begin{aligned}
\Phi_1(\mu_v)(f) &= \mu_v(\mathrm{e}^V P_1(f))/\mu_v(\mathrm{e}^V) \\
&= \mu(h_v\ P_1(\mathrm{e}^V P_1(f)))/\mu(h_v\ P_1(\mathrm{e}^V)) && \text{(definition of } \mu_v) \\
&= \mu(\mathrm{e}^V\ P_1(f)\ P_1(h_v))/\mu(\mathrm{e}^V\ P_1(h_v)) && \text{(reversibility)} \\
&= \mu(P_1(f)\ P_1^v(h_v))/\mu(P_1^v(h_v)) && \text{(definition of } P_1^v) \\
&= \mu(P_1(f)\ h_v)/\mu(h_v) = \mu_v(f) && (h_v \text{ eigenvector}).
\end{aligned}$$

Thus μ_v is a fixed point and by uniqueness we have that $\mu_v = \eta_\infty$.

In the continuous time case we suppose there exists strictly positive eigenvector $h_v \in D(L)$ such that

$$L^v(h_v) = L(h_v) + V h_v = \lambda(P^v)\ h_v. \tag{15}$$

Let us check that the distribution

$$\mu_v(f) = \mu(h_v\ f)/\mu(h_v)$$

is a fixed point of the nonlinear equation (2.1), that is

$$\forall f \in D(L)\,, \qquad \mu_v[L_{\mu_v}(f)] = \mu_v[L(f) + f(V - \mu_v(V))] = 0.$$

First we use the P-reversibility of μ to check that

$$\begin{aligned}
\mu_v[L_{\mu_v}(f)] &= \frac{\mu(h_v\ L(f))}{\mu(h_v)} + \frac{\mu(h_v\ f\ V))}{\mu(h_v)} - \frac{\mu(h_v\ V)}{\mu(h_v)}\ \frac{\mu(h_v\ f)}{\mu(h_v)} \\
&= \frac{1}{\mu(h_v)}\left[\mu([L(h_v) + V h_v]\ f) - \frac{\mu(h_v\ V)}{\mu(h_v)}\ \mu(h_v\ f)\right].
\end{aligned}$$

From (15) we get

$$\begin{aligned}
\mu([L(h_v) + V h_v]\ f) &= \lambda(P^v)\ \mu(h_v\ f) \\
\frac{\mu(h_v\ V)}{\mu(h_v)} &= -\frac{\mu(L(h_v))}{\mu(h_v)} + \lambda(P^v) = \lambda(P^v).
\end{aligned}$$

The last assertion is again due to the P-reversibility of μ (since in this case we recall that $\mu L = 0$). If we combine these two results we end up with $\mu_v[L_{\mu_v}(f)] = 0$ and the proof is completed.

2.3. **Time average models**

The exponential formula (12) leads to the time average description

$$\frac{1}{t}\ln P_t^v(1)(x) = \frac{1}{t}\ln \gamma_t^{(x)}(1) = \frac{1}{t}\int_0^t \eta_s^{(x)}(V)\,\mathrm{d}s$$

whose limit for large $t > 0$ is $\eta_\infty(V)$. Analogously in discrete time settings we have that

$$\gamma_n(1) = \gamma_{n-1}(\mathrm{e}^V) = \eta_{n-1}(\mathrm{e}^V)\ \gamma_{n-1}(1) \qquad \text{and} \quad \gamma_n(1) = \prod_{p=0}^{n-1} \eta_p(\mathrm{e}^V).$$

We also find a time average representation

$$\frac{1}{n}\ln P_n^v(1)(x) = \frac{1}{n}\ln \gamma_n^{(x)}(1) = \frac{1}{n}\sum_{p=0}^{n-1} \ln \eta_p^{(x)}[\exp(V)].$$

As the potential V is nonnegative, we have that $\mathrm{e}^V \geq 1$. To prove (4) we write for $n \geq t_0$,

$$\frac{1}{n}\sum_{p=0}^{n-1}[\ln \eta_p^{(x)}(\mathrm{e}^V) - \ln \eta_\infty(\mathrm{e}^V)] = \frac{1}{n}\sum_{p=0}^{t_0-1}[\ln \eta_p^{(x)}(\mathrm{e}^V) - \ln \eta_\infty(\mathrm{e}^V)] + \frac{1}{n}\sum_{p=t_0}^{n-1}[\ln \eta_p^{(x)}(\mathrm{e}^V) - \ln \eta_\infty(\mathrm{e}^V)].$$

Using the inequality

$$|\ln x - \ln y| \leq |x-y|/(x \wedge y) \tag{16}$$

we arrive at

$$\left|\frac{1}{n}\sum_{p=0}^{n-1}[\ln \eta_p^{(x)}(\mathrm{e}^V) - \ln \eta_\infty(\mathrm{e}^V)]\right| \leq \frac{1}{n}\ (t_0\ \mathrm{osc}(V) + 2\exp(\|V\|)\alpha(\Phi)).$$

To prove (5) we write for $n \geq t_0$

$$\frac{1}{t}\int_0^t [\eta_s^{(x)}(V)\mathrm{d}s - \eta_\infty(V)] = \frac{1}{t}\int_0^{t_0} [\eta_s^{(x)}(V)\mathrm{d}s - \eta_\infty(V)] + \frac{1}{t}\int_{t_0}^t [\eta_s^{(x)}(V)\mathrm{d}s - \eta_\infty(V)]$$

and under (Φ) we end up with

$$\left|\frac{1}{t}\int_0^t \eta_s^{(x)}(V)\mathrm{d}s - \eta_\infty(V)\right| \leq \frac{1}{t}\ (t_0\ \mathrm{osc}(V) + 2\|V\|\alpha(\Phi)).$$

This clearly ends the proof of Theorem 0.2.

To illustrate condition (Φ) we present an easily verifiable sufficient condition and some useful exponential estimates whose proof are essentially given in [4].

Proposition 2.5 [4, 5]**.** *Suppose P_1 satisfies the mixing condition $(\mathcal{P}_1)$ for some $\varepsilon > 0$. Then condition (Φ) is met for any $t \geq t_0 = 2$ with*

$$\alpha_t(\Phi) = a_{\epsilon,v}\ \exp -(b_{\epsilon,v}\ t) \qquad \text{and} \qquad \alpha(\Phi) \leq a_{\epsilon,v}/b_{\epsilon,v}.$$

In addition we have

$$\|\Phi_t(\mu) - \Phi_t(\nu)\|_{\mathrm{tv}} \leq \exp -(b_{\epsilon,v}\ t).$$

In the discrete time case $b_{\epsilon,v} = \epsilon^2$, in the continuous time $b_{\epsilon,v} = \frac{\epsilon^2}{2}\mathrm{e}^{-2\|V\|}$ and in both situations $a_{\epsilon,v} = 2\epsilon^{-2}\mathrm{e}^{4\|V\|}$.

Proof. In the discrete time case the proof is a combination of Theorems 2.3 and 2.7 (pp. 24 and 29) in [4]. In the continuous time case the analog Lipschitz estimate stated in Theorem 2.7 (p. 29) in [4] is proved using the same line of arguments as in the discrete time case. When using Theorem 3.15 (p. 90) in [4] we end up with the same $a_{\epsilon,v}$ but with a constant $b_{\epsilon,v}$ depending on the potential. $\square$

Next we examine two particular situations in which explicit calculations can be done. First of all if the process X does not move, that is $L = 0$ and $X_0 = X_t$ then η_t is the Boltzmann–Gibbs measure with potential V and inverse temperature parameter t, namely

$$\mathrm{d}\eta_t(x) = \frac{1}{\eta_0(\mathrm{e}^{tV})}\ \mathrm{e}^{tV}\ \mathrm{d}\eta_0(x).$$

Now suppose L is a trivial generator in the sense that $L = \mu - Id$ for some $\mu \in \mathcal{P}(E)$ and assume that V a potential function such that $V^2 = V$ (which means that V is the indicator function of a measurable subset). In this case we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\eta_t(V) &= \eta_t(L(V)) + \eta_t(V(V - \eta_t(V)) \\ &= \mu(V) - \eta_t(V) + \eta_t(V) - \eta_t(V)\eta_t(V) = \mu(V) - [\eta_t(V)]^2. \end{aligned} \tag{17}$$

We also notice that the constant function $\sqrt{\mu(V)}$ satisfies (17) since $0 = \mu(V) - [\sqrt{\mu(V)}]^2$. With some easy computations we can check that the solution of (17) is given by

$$\eta_t(V) = \sqrt{\mu(V)}\ \frac{a(V)\mathrm{e}^{2t\sqrt{\mu(V)}} + b(V)}{a(V)\mathrm{e}^{2t\sqrt{\mu(V)}} - b(V)}$$

with

$$a(V) = \eta_0(V) + \sqrt{\mu(V)} \qquad \text{and} \qquad b(V) = \eta_0(V) - \sqrt{\mu(V)}.$$

Thus, for sufficiently large t we conclude that

$$\lim_{t\to\infty} \eta_t(V) = \lambda(P^v) = \sqrt{\mu(V)} \quad \text{and} \quad |\eta_t(V) - \lambda(P^v)| \leq 2\,|b(V)|\ \mathrm{e}^{-2\sqrt{\mu(V)}t}.$$

3. Interacting particle approximating models

In this section we describe and we analyze the asymptotic behavior of a class of discrete and continuous time particle approximating models. The study of the discrete generation models is technically less involved than their continuous time versions. For these reasons we have devoted a separate section to treat each of these situations. Section 3.1 focuses on discrete time genetic type models. We apply the $\mathbb{L}_p$-mean error estimates obtained in earlier studies [4] to obtain uniform convergence estimates for the particle Lyapunov exponents. The final Section 3.2 is concerned with continuous time and Moran type particle models. In a first Section 3.2.1 we connect these novel particle models with the ones presented in [4,6] and with the Fleming–Viot interpretation introduced in [1,2] in the hard obstacle situation. The second Section 3.2.2 focuses on the asymptotic behavior of the particle models as the size of the system tends to infinity. We extend and simplify the semigroup techniques presented in [4] to derive a collection of $\mathbb{L}_2$-estimates for the convergence of these novel particle schemes. We also mention that this semigroup approach gives a natural and transparent proof of the unbias properties stated in Theorem 0.4 and Theorem 0.5. Section 3.2.3 is concerned with a comparison between the covariance function associated to the particle models presented in the former article and the ones associated to the particle models presented in [4] and [6].

3.1. Discrete time and genetic type particle models

The particle approximating model associated to (13) is the Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N)$ on the product space E^N with transitions

$$\mathbb{P}\left(\xi_n \in \mathrm{d}y | \xi_{n-1} = x\right) = \prod_{i=1}^{N}\ K_{m(x)}(x^i, \mathrm{d}y^i) \qquad \text{with} \quad m(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x^i}. \tag{18}$$

Here $dy = dy^1 \times \cdots \times dy^N$ stands for an infinitesimal neighborhood of the point $y = (y^1, \ldots, y^N) \in E^N$, $x = (x^1, \ldots, x^N) \in E^N$ and δ_a denotes the Dirac measure on $a \in E$.

- By convention if we choose $V = V^+$ then we have

$$K_\eta(u, dv) = \Psi(\eta) P_1(dv) \quad \text{and} \quad K_{m(x)}(u, .) = \Phi(m(x)) = \sum_{i=1}^{N} \frac{e^{V(x^i)}}{\sum_{j=1}^{N} e^{V(x^j)}} P_1(x^i, .)$$

  We see that each transition $\xi_n \to \xi_{n+1}$ is decomposed into two separate genetic type mechanisms

$$\xi_n \xrightarrow{\text{Selection}} \widehat{\xi}_n \xrightarrow{\text{Mutation}} \xi_{n+1}.$$

  Given ξ_n the system $\widehat{\xi}_n = (\widehat{\xi}_n^1, \ldots, \widehat{\xi}_n^N)$ consists of N i.i.d. random variables with common law $\Psi(m(\xi_n))$. During the mutation each particle $\widehat{\xi}_n^i$ evolves according the transition P_1. In other words ξ_{n+1} consists of N independent variables ξ_{n+1}^i with law $P_1(\widehat{\xi}_n^i, .)$.
- For more general functions V^+ we again obtain a two step transition. The mutation stage remains the same but we have a slightly distinct selection mechanism.
  In this situation and given ξ_n the system $\widehat{\xi}_n$ consists of N independent random variables $\widehat{\xi}_n^i$ with law

$$S_{m(\xi_n)}(\xi_n^i, .) = e^{V^-(\xi_n^i)}/m(\xi_n)(e^V)\ \delta_{\xi_n^i} + (1 - e^{V^-(\xi_n^i)}/m(\xi_n)(e^V))\ \Psi^+(m(\xi_n))(dy).$$

  In contrast to the latter selection stage each particle $\widehat{\xi}_n^i$ has now a larger probability to remain in the same location.

In some sense, if the initial system consists of N i.i.d. particles with common law η_0 then the empirical measures

$$\eta_n^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_n^i} \qquad \text{and} \qquad \eta_{[0,n]}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\xi_0^i, \ldots, \xi_n^i)}$$

converge as $N \to \infty$ respectively to the Feynman–Kac measures η_n and to the n-marginal of the McKean measure $\mathbb{Q}_{\eta_0,[0,n]}$ defined in (14) (and corresponding to the choice of the transitions K_η). Several asymptotic results including central limit theorems, exponential and large deviations as well as Glivenko or Donsker's type theorems and propagation of chaos for increasing block size and time horizon can be found in [4] and [8]. Here we will use the following uniform convergence estimate:

Lemma 3.1 [4]**.** *For any $n \in \mathbb{N}$ and $p \geq 1$ there exists some finite constants $b(p)$ and $c(n)$ such that for any $f \in \mathcal{B}_b(E)$, $\|f\| \leq 1$,*

$$\mathbb{E}(|\eta_n^N(f) - \eta_n(f)|^p)^{1/p} \leq \frac{1}{\sqrt{N}}\ b(p)\ c(n). \tag{19}$$

Suppose P_1 satisfy the mixing condition $(\mathcal{P}_1)$ and (3) holds for some $\varepsilon > 0$. Then we have the uniform estimates with respect to the time parameter

$$\sup_{n \geq 0} \mathbb{E}(|\eta_n^N(f) - \eta_n(f)|^p)^{1/p} \leq\ b(p)\ \frac{e^{2\operatorname{osc}(V)}}{\varepsilon^4 \sqrt{N}} \cdot \tag{20}$$

Suppose the mixing condition (3) holds for some $\varepsilon > 0$. When the initial particles start at the same point x, Lemma 3.1 tells us that for any $p \geq 1$ there exists some universal and finite constant $b(p)$ such that

$$\sup_{x \in E} \sup_{n \geq 0} \mathbb{E}(|\eta_n^{(N,x)}(\exp V) - \eta_n^{(x)}(\exp V)|^p)^{1/p} \leq\ b(p)\ \frac{e^{2\operatorname{osc}(V) + \|V\|}}{\varepsilon^4 \sqrt{N}} \cdot$$

Using the inequality (16) we conclude that

$$\sup_{n\geq 1} \mathbb{E}[|\lambda_n^{(x,N)}(P^v) - \lambda_n^{(x)}(P^v)|^p]^{1/p} \leq b(p)\, \frac{\mathrm{e}^{2\mathrm{osc}(V)+\|V\|}}{\sqrt{N}\epsilon^4}$$

with

$$\lambda_n^{(x,N)}(P^v) = \frac{1}{n}\sum_{q=0}^{n} \ln \eta_q^N(\exp V) \qquad \text{and} \qquad \lambda_n^{(x)}(P^v) = \frac{1}{n}\sum_{q=0}^{n} \ln \eta_q^{(x)}(\exp V).$$

This yields the estimate (6) with $c_v(\epsilon) = \exp(2\mathrm{osc}(V) + \|V\|)/\epsilon^4$. If we combine this inequality with the estimate stated in theorem 0.1 we end up with (for $n \geq t_0$),

$$\mathbb{E}[|\lambda_n^{(x,N)}(P^v) - \lambda(P^v)|^p]^{1/p} \leq \frac{b(p)}{\sqrt{N}}\frac{\mathrm{e}^{2\mathrm{osc}(V)+\|V\|}}{\epsilon^4} + \frac{2}{n}\left(\mathrm{osc}(V) + \frac{\mathrm{e}^{4\|V\|}}{\varepsilon^4}\right)\cdot$$

To prove (7) we suppose the conditions stated in Theorem 0.4 are satisfied. Using Proposition 2.5 we obtain for any $f \in \mathcal{B}_b(E)$, $\|f\| \leq 1$, and $n \geq 1$ the estimate

$$|\eta_n(f) - \eta_\infty(f)| \leq 2\exp -(\epsilon^2 n).$$

From (20) we conclude that

$$\mathbb{E}(|\eta_n^N(f) - \eta_n(f)|^p)^{1/p} \leq\ b(p)\ \frac{\mathrm{e}^{2\mathrm{osc}(V)}}{\varepsilon^4\sqrt{N}} + 2\exp -(\epsilon^2 n).$$

If we choose $n = n(N) \geq (\ln N)/(2\epsilon^2)$ so that $\mathrm{e}^{-n\epsilon^2} \leq 1/\sqrt{N}$ we end up with

$$\sup_{n\geq(\ln N)/(2\epsilon^2)} \sqrt{N}\mathbb{E}(|\eta_n^N(f) - \mu(h_v\ P_1(f))/\mu(h_v)|^p)^{1/p} \leq 2 + b(p)c_v(\varepsilon).$$

3.2. **Continuous time and Moran type particle systems**

3.2.1. *Description of the models*

As seen in Remark 1.6, under our strong conditions, we have that for any $f \in D(L)$, the mappings $\mathbb{R}_+ \ni t \mapsto P_t(f) \in D(L)$ and $\mathbb{R}_+ \ni t \mapsto P_t^v(f) \in D(L)$ belong to $\mathcal{C}^1([0,\infty), D(L))$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t(f) = L(P_t(f)) = P_t(L(f)) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}P_t^v(f) = L^v(P_t^v(f)) = P_t^v(L^v(f)) \tag{21}$$

(in zero, only the right derivatives are considered). Starting from a collection of operators

$$L_\eta^\rho = \rho\ L_\eta + (1-\rho)\ L'_\eta \qquad \eta \in \mathcal{P}(E)$$

with $\rho \in [0,1]$ and (L_η, L'_η) defined in Proposition 2.1 we consider an interacting particle approximating model $\xi = (\xi_t)_{t\in\mathbb{R}_+}$ which is a Markov process on a product space E^N with infinitesimal pregenerator

$$\mathcal{L}(\varphi)(x^1,\dots,x^N) = \sum_{i=1}^{N} L_{m(x)}^{(\rho,i)}\varphi(x^1,\dots,x^i,\dots,x^N)$$

acting on functions $\varphi \in D(\mathcal{L})(\subset \mathcal{B}_b(E^N))$, the algebra spanned by linear combinations of functions φ of the form

$$\varphi(x^1, \dots, x^N) = \prod_{p=1}^{N} f_i(x^i) \qquad \text{with} \quad f_1, \dots, f_N \in D(L).$$

We recall that we use the notation $L^{(\rho,i)}_{m(x)}$ instead of $L^{\rho}_{m(x)}$ when it acts on the i-th coordinate of $\varphi(x^1, \dots, x^i, \dots, x^N)$ and that for any probability η on $(E, \mathcal{E})$, L^{ρ}_{η} is given by

$$L^{\rho}_{\eta}(f)(u) = L(f)(u) + \int (f(v) - f(u))\ V^{\rho}(u,v)\ \eta(\mathrm{d}v)$$

with $V^{\rho}(u,v) = \rho V(v) + (1-\rho)V'(u)$. It can be rewritten as

$$L^{\rho}_{\eta}(f)(u) = L(f)(u) + a^{\rho}_{\eta}(u) \int (f(v) - f(u))\ K^{\rho}_{\eta}(u, \mathrm{d}v) \tag{22}$$

with $a^{\rho}_{\eta}(u) = \rho\eta(V) + (1-\rho)V'(u)$ and $K^{\rho}_{\eta}(u, \mathrm{d}v) = \dfrac{\rho\ V(v) + (1-\rho)V'(u)}{\rho\eta(V) + (1-\rho)V'(u)}\ \eta(\mathrm{d}v)$.

The choice of a_η and K_η is not unique, we can alternatively write

$$L^{\rho}_{\eta}(f)(u) = L(f)(u) + b^{\rho} \int (f(v) - f(u))\ G^{\rho}_{\eta}(u, \mathrm{d}v)$$

with for any $\|V\| \leq b$, $\|V'\| \leq b'$

$$b^{\rho} = \rho b + (1-\rho)b' \quad \text{and} \quad G^{\rho}_{\eta}(u, \mathrm{d}v) = (a^{\rho}_{\eta}(u)/b^{\rho})\ K^{\rho}_{\eta}(u, \mathrm{d}v) + (1 - a^{\rho}_{\eta}(u)/b^{\rho})\ \delta_u(\mathrm{d}v).$$

We notice that

$$\eta(L^{\rho}_{\eta}(f)) = \eta(L(f)) + \eta(fV) - \eta(f)\eta(V)$$

and the "carré du champ" $\Gamma_{L^{\rho}_{\eta}}$ associated to L^{ρ}_{η} is given by

$$\begin{aligned} \Gamma_{L^{\rho}_{\eta}}(f,f)(u) &= \Gamma_L(f,f)(x) + a^{\rho}_{\eta}(u) \int (f(v) - f(u))^2\ K^{\rho}_{\eta}(u, \mathrm{d}v) \\ &= \rho\ \Gamma_{L_\eta}(f,f)(u) + (1-\rho)\ \Gamma_{L'_\eta}(f,f)(u). \end{aligned} \tag{23}$$

For $\rho = 1$ and $\rho = 0$ we also have that

$$\eta(\Gamma_{L_\eta}(f,f)) = \eta(\Gamma_L(f,f)) + \eta(Vf^2) + \eta(V)\eta(f^2) - 2\eta(Vf)\eta(f) \tag{24}$$

$$\begin{aligned} \eta(\Gamma_{L'_\eta}(f,f)) &= \eta(\Gamma_L(f,f)) + \eta(V'f^2) + \eta(V')\eta(f^2) - 2\eta(V'f)\eta(f) \\ &= \eta(\Gamma_L(f,f)) + 2c\ \eta((f - \eta(f))^2) - [\eta(Vf^2) + \eta(V)\eta(f^2) - 2\eta(Vf)\eta(f)]. \end{aligned} \tag{25}$$

To have some "local" comparison let us mention that

$$c \geq 2\|V\| \Longrightarrow V' \geq V \Longrightarrow \eta(\Gamma_{L'_\eta}(f,f)) \geq \eta(\Gamma_{L_\eta}(f,f)).$$

If the potential V is constant then for the choice $c \equiv V(x)$ we have

$$\eta(\Gamma_{L'_\eta}(f,f)) = \eta(\Gamma_L(f,f)) \quad \text{but} \quad \eta(\Gamma_{L_\eta}(f,f)) = \eta(\Gamma_L(f,f)) + 2c\ \eta((f - \eta(f))^2).$$

When the potential V is constant $V(x) = a$ the pregenerators L_η still contain a jump part

$$L_\eta(f)(u) = L(f)(u) + a \int (f(v) - f(u))\ \eta(\mathrm{d}u)$$

but if we choose $c = a$ we have $V'(x) = 0$ and $L'_\eta = L$. (Nevertheless, the case of constant V can also be reduced directly to $V \equiv 0$, *via* the observation at the beginning of Sect. 2.)

Since the potentials V and V' are bounded positive functions, it is possible to rewrite their pregenerators in the following form

$$\mathcal{L}^\rho = \tilde{\mathcal{L}} + \widehat{\mathcal{L}}^\rho \quad \text{with} \quad \tilde{\mathcal{L}}(\varphi)(x) = \sum_{i=1}^{N} L^{(i)}(\varphi)(x)$$

and the interacting jump part

$$\widehat{\mathcal{L}}^\rho(\varphi)(x) = \mathrm{d}^\rho(x) \int_{E^N} (\varphi(y) - \varphi(x))\ \mathcal{K}^\rho(x, \mathrm{d}y)$$

with

$$\mathrm{d}^\rho(x) = N\ m(x)^{\otimes 2}(V^\rho)\ (= N\ [\rho\ m(x)(V) + (1-\rho)\ m(x)(V')])$$
$$\mathcal{K}^\rho(x, \mathrm{d}y) = \sum_{i,j=1}^{N,N} \frac{V^\rho(x^i, x^j)}{\sum_{k,l=1}^{N,N} V^\rho(x^k, x^l)}\ \delta_{x_{(i,j)}}(\mathrm{d}y)$$

(we make the convention that $\mathcal{K}^\rho(x, \mathrm{d}y) = \delta_x(\mathrm{d}y)$ if the denominator ($= \mathrm{d}^\rho(x)$) in the rhs is null), and where for $1 \le i, j \le N$ and $x = (x^1, \cdots, x^N) \in E^N$, $x_{(i,j)}$ is defined by replacing in x the i-th coordinate x^i by the j-th coordinate x^j, that is

$$\forall k \neq i \qquad x^k_{(i,j)} = x^k \qquad \text{and} \qquad x^i_{(i,j)} = x^j.$$

In particular when $V^\rho \equiv 0$ (for instance if $\rho = 0$ and $V' \equiv 0$ or $\rho = 1$ and $V \equiv 0$), we have $\mathcal{L}^\rho = \tilde{\mathcal{L}}$.

Remarks 3.2. Alternatively we can also use the next two equivalent formulations

$$1) \qquad \widehat{\mathcal{L}}^\rho(\varphi)(x) = \sum_{i=1}^{N} a^\rho_{m(x)}(x^i) \int_{E^N} (\varphi(y) - \varphi(x))\ K^{(\rho,i)}_{m(x)}(x^i, \mathrm{d}y)$$

with

$$K^{(\rho,i)}_{m(x)}(x^i, \mathrm{d}y) = \sum_{j=1}^{N} \frac{\rho\ V(x^j) + (1-\rho)V'(u)}{\sum_{k=1}^{N} \rho V(x^k) + (1-\rho) N V'(x^i)}\ \delta_{x_{(i,j)}}(\mathrm{d}v)$$

$$2) \qquad \widehat{\mathcal{L}}^\rho(\varphi)(x) = (Nb^\rho) \int_{E^N} (\varphi(y) - \varphi(x))\ \mathcal{G}^\rho(x, \mathrm{d}y) \tag{26}$$

with $\|V\| \le b$, $\|V'\| \le b'$, $b^\rho = \rho b + (1-\rho) b'$ and

$$\mathcal{G}^\rho(x, \mathrm{d}y) = m(x)^{\otimes 2}\,(V^\rho / b^\rho)\ \mathcal{K}^\rho(x, \mathrm{d}y) + \left(1 - m(x)^{\otimes 2}\,(V^\rho / b^\rho)\right)\ \delta_x(\mathrm{d}y).$$

In view of previous observations there exists different equivalent descriptions of the motion of the particles. We have chosen to present the one associated to the representation (26) and to the particular choice $\rho = 1$.

As in the discrete time situation the motion is decomposed into two separate mechanisms. Let τ_n , $n \geq 1$, be a collection of independent and exponential random times with common intensity (Nb).

- Between theses dates each particle evolves independently of each other (and independently of the previously defined random times) according to an L-motion.

- At each random times τ_n with a probability $m(\xi_{\tau_n -})\,(V/b)$ we sample a new configuration ζ_n with probability $\mathcal{K}(\xi_{\tau_n -}, \,.\,)$ and a we set $\xi_{\tau_n} = \zeta_n$ and with a probability $1 - m(\xi_{\tau_n -})\,(V/b)$ the configuration does not change and we set $\xi_{\tau_n} = \xi_{\tau_n -}$ (as our state space is not assumed to be topological, $\xi_{\tau_n -}$ designates the position of the system if we would have not considered a resampling at time τ_n, *cf.* [7]).

In our framework the particle model presented in the hard obstacle case in [2] corresponds to the interacting particle systems with $\rho = 0$. More precisely the particle model with pregenerator $\mathcal{L}^0$ is the analog of the one of [2] in the "soft obstacle" case. To clarify the connections between these two models we briefly indicate how the N particles evolve in such an environment. When a particle ξ^i_τ has just been born at a given time τ we sample a reference exponential random variable e^i_τ with parameter 1. Then it evolves randomly with an L-motion up to its death time

$$\zeta^i = \inf\left\{ t \geq \tau \; ; \; \int_\tau^t \; V(\xi^i_s)\mathrm{d}s \geq \mathrm{e}^i_\tau \right\}\cdot$$

When it is killed a different particle instantly splits into two offsprings.

In the first algorithm ($\rho = 1$) the interacting jump pregenerator $\widehat{L}^1$ is intended to improve the quality of the system by giving to individuals with higher V-fitness more chance to be copied in the next generation. In the second model ($\rho = 0$) the interacting jump pregenerator $\widehat{L}^0$ prevents particles from visiting death living regions with higher V'-obstacles. Recalling that $V(x) + V'(x) = c$ these two interaction mechanisms lead to the same competitive selection by allocating reproductive opportunities to individuals living in regions with higher V-fitness.

When V is a constant function the flow η_t represents the state distributions of Markov process X_t at time t. In this degenerate situation we can choose a null potential $V' = 0$ and the N-particle approximating model with $\rho = 0$ consists of N independent copies of the Markov process X. In contrast to this situation for $\rho = 1$ the motion of the particles still contain an interacting jump part. In this sense we can say that the latter contains an extra degree of randomness.

3.2.2. *L_2-estimates*

To each $\varphi \in \mathcal{C}^1(\mathbb{R}_+, D(\mathcal{L}))$ we associate an F-martingale $\mathcal{M}(\varphi)$

$$\mathcal{M}_t(\varphi) = \varphi(t, \xi_t) - \varphi(0, \xi_0) - \int_0^t \; \left[\frac{\partial \varphi}{\partial s}(s, \,.\,) + \mathcal{L}^\rho(\varphi(s, \,.\,)) \right](\xi_s) \; \mathrm{d}s \tag{27}$$

with increasing process $\langle \mathcal{M}(\varphi) \rangle$

$$\langle \mathcal{M}(\varphi) \rangle_t = \int_0^t \; \Gamma_{\mathcal{L}^\rho}(\varphi(s, \,.\,), \varphi(s, \,.\,))(\xi_s) \; \mathrm{d}s$$

where $\Gamma_{\mathcal{L}^\rho}$ is the "carré du champ" associated to $\mathcal{L}^\rho$ and defined for any $\psi \in D(\mathcal{L})$ by

$$\forall \; x \in E^N, \qquad \Gamma_{\mathcal{L}^\rho}(\psi, \psi)(x) = \mathcal{L}^\rho(\psi^2)(x) - 2\psi(x)\mathcal{L}^\rho(\psi)(x) = \mathcal{L}^\rho\left([\psi - \psi(x)]^2 \right)(x).$$

For test functions of the form $\psi(x) = m(x)(f)$, $f \in D(L)$, we find that

$$\begin{aligned}\mathcal{L}^\rho(\psi)(x) &= m(x)[L^\rho_{m(x)}(f)] \\ \Gamma_{\mathcal{L}^\rho}(\psi,\psi)(x) &= \mathcal{L}^\rho\left([\psi-\psi(x)]^2\right)(x) = \frac{1}{N^2}\sum_{i=1}^N L^\rho_{m(x)}\left([f-f(x^i)]^2\right)(x^i) \\ &= \frac{1}{N}\, m(x)(\Gamma_{L^\rho_{m(x)}}(f,f)).\end{aligned}$$

If we introduce the notation $\eta^N_t = m(\xi_t)$ then for $\varphi(t,x) = m(x)(f(t,.))$ and $f \in \mathcal{C}^1(\mathbb{R}_+, D(L))$, we have

$$\mathcal{M}_t(\varphi) = \eta^N_t(f(t,.)) - \eta^N_0(f(0,.)) - \int_0^t \eta^N_s\left[\frac{\partial f}{\partial s}(s,.) + L^\rho_{\eta^N_s}(f(s,.))\right]\mathrm{d}s \tag{28}$$

and

$$\langle\mathcal{M}(\varphi)\rangle_t = \frac{1}{N}\int_0^t \eta^N_s \Gamma_{L^\rho_{\eta^N_s}}(f(s,.),f(s,.))\,\mathrm{d}s.$$

Let $\Phi = (\Phi_{s,t})_{0\le s\le t}$ be the nonlinear semigroup on distribution space and associated to the flow η_t, that is

$$\Phi_{s,t}(\eta_s)(f) = \eta_t(f) = \frac{\mathbb{E}_{s,\eta_s}(f(X_t)\exp\int_s^t V(X_u)\mathrm{d}u)}{\mathbb{E}_{s,\eta_s}(\exp\int_s^t V(X_u)\mathrm{d}u)}$$

where $\mathbb{E}_{s,\mu}(.)$, $s \in \mathbb{R}_+$, $\mu \in \mathcal{P}(E)$, denotes the expectation with respect to the distribution $\mathbb{P}_{s,\mu} = \int \mu(\mathrm{d}x)\,\mathbb{P}_{s,x}$ and $\mathbb{P}_{s,x} = \mathbb{P}_{0,x}\circ\theta_s^{-1}$ with the usual shifting operators θ_s, $s \in \mathbb{R}_+$.

To clarify notations we also write $P^v_{s,t}$ instead of P^v_{t-s}, that is

$$P^v_{s,t}(f)(x) = \mathbb{E}_{s,x}\left(f(X_t)\exp\int_s^t V(X_u)\mathrm{d}u\right)\cdot$$

With this notations we have for any distribution $\mu \in \mathcal{P}(E)$ and $f \in \mathcal{B}_b(E)$

$$\Phi_{s,t}(\mu)(f) = \frac{\mu P^v_{s,t}(f)}{\mu P^v_{s,t}(1)}\cdot$$

Arguing as in Section 2.1 we notice that for any $f \in \mathcal{B}_b(E)$

$$\gamma_t(f) = \eta_t(f)\ \exp\int_0^t \eta_s(V)\mathrm{d}s.$$

It is therefore natural to define the N-approximating measures and the F-martingale $\mathcal{N}(\varphi)$

$$\gamma^N_t(f) = \eta^N_t(f)\ \exp\int_0^t \eta^N_s(V)\mathrm{d}s \qquad \text{and} \qquad \mathcal{N}_t(\varphi) = \int_0^t \gamma^N_s(1)\ \mathrm{d}\mathcal{M}_s(\varphi).$$

Until the end of this section and unless otherwise stated we fix a final time horizon $T \ge 0$, a test function $f \in D(L)$ such that $\|f\| \le 1$. For any $x \in E^N$ and $0 \le t \le T$ we write

$$\varphi^{(0)}(t,x) = m(x)(P^v_{t,T}(1))\,, \quad \varphi^{(1)}(t,x) = m(x)(P^v_{t,T}(f))\,, \quad \varphi^{(2)}(t,x) = m(x)([P^v_{t,T}(f)]^2).$$

When there is no possible confusion, we slightly abuse notations and we write $\mathbb{E}(\bullet)$ the expectation with respect to the law of the process ξ. We also fix a parameter $\rho \in [0,1]$ and we set $v(\rho) = (1/2+\rho)\,\|V\| + (1-\rho)\,\|V'\|$.

Lemma 3.3. *For any time $0 \le t \le T$ we have*

$$\gamma_t^N(P^v_{t,T}(f)) = \eta_0^N(P^v_{0,T}(f)) + \mathcal{N}_t(\varphi^{(1)}) \quad \text{and} \quad N \sup_{t\le T} \mathbb{E}(\langle \mathcal{N}(\varphi^{(1)})\rangle_t) \le \exp{(4Tv(\rho))}.$$

Proof. With some obvious abusive notations we first use (21) and we observe that for $\mu \in \mathcal{P}(E)$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + L_\mu\right)(P^v_{t,T}(f)) &= -(L+V)(P^v_{t,T}(f)) + L(P^v_{t,T}(f)) + \mu(VP^v_{t,T}(f)) - \mu(V)P^v_{t,T}(f)\\ &= \mu(VP^v_{t,T}(f)) - V(P^v_{t,T}(f)) - \mu(V)P^v_{t,T}(f).\end{aligned} \tag{29}$$

In much the same way, recalling that $V' = c - V$, we get

$$\begin{aligned}\left(\frac{\partial}{\partial t} + L'_\mu\right)(P^v_{t,T}(f)) &= -(L+V)(P^v_{t,T}(f)) + L(P^v_{t,T}(f)) + V'\ [\mu(P^v_{t,T}(f)) - P^v_{t,T}(f)]\\ &= -V\mu(P^v_{t,T}(f)) + c\ [\mu(P^v_{t,T}(f)) - P^v_{t,T}(f)].\end{aligned} \tag{30}$$

For any $\rho \in [0,1]$, this yields that

$$\mu\left[\left(\frac{\partial}{\partial t} + L^\rho_\mu\right)(P^v_{t,T}(f))\right] = -\mu(V)\mu(P^v_{t,T}(f)).$$

Hence by (28) we conclude that

$$\begin{aligned}\mathrm{d}\eta_t^N(P^v_{t,T}f) &= \eta_t^N\left[\left(\frac{\partial}{\partial t} + L^\rho_{\eta_t^N}\right)(P^v_{t,T}(f))\right]\mathrm{d}t + \mathrm{d}\mathcal{M}_t(\varphi^{(1)})\\ &= -\eta_t^N(V)\eta_t^N(P^v_{t,T}(f))\ \mathrm{d}t + \mathrm{d}\mathcal{M}_t(\varphi^{(1)})\end{aligned}$$

and therefore

$$\mathrm{d}\gamma_t^N(P^v_{t,T}f) = \gamma_t^N(1)\ \mathrm{d}\mathcal{M}_t(\varphi^{(1)}) = \mathrm{d}\mathcal{N}_t(\varphi^{(1)}).$$

This completes the proof of the first part of the lemma. By (29) we also notice that

$$\begin{aligned}\Gamma_{L_\mu}(P^v_{t,T}f, P^v_{t,T}f) &= \left(\frac{\partial}{\partial t} + L_\mu\right)(P^v_{t,T}(f)^2) - 2P^v_{t,T}(f)\left(\frac{\partial}{\partial t} + L_\mu\right)(P^v_{t,T}(f))\\ &= \left(\frac{\partial}{\partial t} + L_\mu\right)(P^v_{t,T}(f)^2) - 2P^v_{t,T}(f)[\mu(VP^v_{t,T}(f)) - V(P^v_{t,T}(f)) - \mu(V)P^v_{t,T}(f)]\end{aligned}$$

and hence

$$\begin{aligned}\mu[\Gamma_{L_\mu}(P^v_{t,T}f, P^v_{t,T}f)] = \mu\left(\frac{\partial}{\partial t} + L_\mu\right)((P^v_{t,T}f)^2) + 2[\mu(V(P^v_{t,T}(f))^2)\\ + \mu(V)\mu((P^v_{t,T}(f))^2) - \mu(VP^v_{t,T}(f))\mu(P^v_{t,T}(f))].\end{aligned} \tag{31}$$

In an equivalent way of reasoning we get by (30)

$$\begin{aligned}\Gamma_{L'_\mu}(P^v_{t,T}f,P^v_{t,T}f) &= \left(\frac{\partial}{\partial t}+L'_\mu\right)(P^v_{t,T}(f)^2)-2P^v_{t,T}(f)\left(\frac{\partial}{\partial t}+L'_\mu\right)(P^v_{t,T}(f))\\ &= \left(\frac{\partial}{\partial t}+L'_\mu\right)(P^v_{t,T}(f)^2)-2P^v_{t,T}(f)[-V\mu(P^v_{t,T}(f))+c\ [\mu(P^v_{t,T}(f))-P^v_{t,T}(f)]]\end{aligned}$$

and thus

$$\mu[\Gamma_{L'_\mu}(P^v_{t,T}f,P^v_{t,T}f)]=\mu\left(\frac{\partial}{\partial t}+L'_\mu\right)((P^v_{t,T}f)^2)+2[\mu(VP^v_{t,T}f)\mu(P^v_{t,T}f)+c\ \mu([P^v_{t,T}f-\mu(P^v_{t,T}f)]^2)]. \tag{32}$$

By (23) we combine (31) and (32) to conclude that

$$\begin{aligned}\mu[\Gamma_{L^\rho_\mu}(P^v_{t,T}f,P^v_{t,T}f)] &= \mu\left(\frac{\partial}{\partial t}+L^\rho_\mu\right)((P^v_{t,T}f)^2)+2\rho[\mu(V(P^v_{t,T}(f))^2)+\mu(V)\mu((P^v_{t,T}(f))^2)\\ &\quad-\mu(VP^v_{t,T}(f))\mu(P^v_{t,T}(f))]+2(1-\rho)[\mu(VP^v_{t,T}f)\mu(P^v_{t,T}f)+c\ \mu([P^v_{t,T}f-\mu(P^v_{t,T}f)]^2)].\end{aligned} \tag{33}$$

On the other hand from (33) we notice that the increasing process $\langle\mathcal{N}(\varphi^{(1)})\rangle$ may be rewritten as

$$\begin{aligned}N\mathrm{d}\langle\mathcal{N}(\varphi^{(1)})\rangle_t &= \gamma^N_t(1)^2\ N\mathrm{d}\langle\mathcal{M}(\varphi^{(1)})\rangle_t=\gamma^N_t(1)^2\ \eta^N_t(\Gamma_{L^\rho_{\eta^N_t}}(P^v_{t,T}f,P^v_{t,T}f)\mathrm{d}t\\ &= \gamma^N_t(1)^2\ \mathrm{d}\eta^N_t((P^v_{t,T}f)^2)-\gamma^N_t(1)^2\ \mathrm{d}\mathcal{M}_t(\varphi^{(2)})+2\gamma^N_t(1)^2\ \rho\ [\eta^N_t(V(P^v_{t,T}(f))^2)\\ &\quad+\eta^N_t(V)\ \eta^N_t((P^v_{t,T}(f))^2)-\eta^N_t(VP^v_{t,T}(f))\ \eta^N_t(P^v_{t,T}(f))]\\ &\quad+2\gamma^N_t(1)^2\ (1-\rho)\ [\eta^N_t(VP^v_{t,T}f)\eta^N_t(P^v_{t,T}f)+c\ \eta^N_t([P^v_{t,T}f-\eta^N_t(P^v_{t,T}f)]^2)].\end{aligned}$$

Since

$$\mathrm{d}\gamma^N_t(1)^2=2\eta^N_t(V)\ \gamma^N_t(1)^2\mathrm{d}t$$

a simple integration by part yields

$$\gamma^N_t(1)^2\ \mathrm{d}\eta^N_t((P^v_{t,T}f)^2)=-2\eta^N_t(V)\gamma^N_t(1)^2\eta^N_t((P^v_{t,T}f)^2)\mathrm{d}t+\mathrm{d}(\gamma^N_t(1)^2\ \eta^N_t((P^v_{t,T}f)^2))$$

and

$$\begin{aligned}N\mathrm{d}\langle\mathcal{N}(\varphi^{(1)})\rangle_t &= \mathrm{d}[\gamma^N_t(1)^2\ \eta^N_t((P^v_{t,T}f)^2)]-\gamma^N_t(1)^2\ \mathrm{d}\mathcal{M}_t(\varphi^{(2)})\\ &\quad+2\rho\gamma^N_t(1)^2\ [\eta^N_t(V(P^v_{t,T}(f))^2)-\eta^N_t(VP^v_{t,T}(f))\ \eta^N_t(P^v_{t,T}(f))]\mathrm{d}t\\ &\quad+2(1-\rho)\gamma^N_t(1)^2\ [\eta^N_t(VP^v_{t,T}(f))\eta^N_t(P^v_{t,T}(f))-\eta^N_t(V)\ \eta^N_t((P^v_{t,T}(f))^2)]\mathrm{d}t\\ &\quad+2(1-\rho)\gamma^N_t(1)^2\ c\eta^N_t([P^v_{t,T}(f)-\eta^N_t(P^v_{t,T}f)]^2)\mathrm{d}t.\end{aligned}$$

By definition of V' we arrive at

$$\begin{aligned}N\mathrm{d}\langle\mathcal{N}(\varphi^{(1)})\rangle_t &= \mathrm{d}[\gamma^N_t(1)^2\ \eta^N_t((P^v_{t,T}f)^2)]-\gamma^N_t(1)^2\ \mathrm{d}\mathcal{M}_t(\varphi^{(2)})\\ &\quad+2\rho\gamma^N_t(1)^2\ [\eta^N_t(V(P^v_{t,T}(f))^2)-\eta^N_t(VP^v_{t,T}(f))\ \eta^N_t(P^v_{t,T}(f))]\mathrm{d}t\\ &\quad+2(1-\rho)\gamma^N_t(1)^2\ [\eta^N_t(V')\ \eta^N_t((P^v_{t,T}(f))^2)-\eta^N_t(V'P^v_{t,T}(f))\eta^N_t(P^v_{t,T}(f))]\mathrm{d}t.\end{aligned}$$

This implies (after noting that $\mathcal{M}(\varphi^{(2)})$ is a bounded martingale according to our last hypothesis on the pregenerator L) that

$$\begin{aligned}N\mathbb{E}(\langle\mathcal{N}(\varphi^{(1)})\rangle_t) &= \mathbb{E}(\gamma_t^N(1)^2\ \eta_t^N((P_{t,T}^v f)^2)) - \mathbb{E}(\eta_0^N((P_{0,T}^v f)^2))\\ &+2\rho\mathbb{E}\left(\int_0^t \gamma_s^N(1)^2\ [\eta_s^N(V(P_{s,T}^v(f))^2) - \eta_s^N(VP_{s,T}^v(f))\ \eta_s^N(P_{s,T}^v(f))]\mathrm{d}s\right)\\ &+2(1-\rho)\mathbb{E}\left(\int_0^t \gamma_s^N(1)^2\ [\eta_s^N(V')\ \eta_s^N((P_{s,T}^v f)^2) - \eta_s^N(V'P_{s,T}^v f)\eta_s^N(P_{s,T}^v f)]\mathrm{d}s\right) \quad (34)\end{aligned}$$

and clearly

$$\begin{aligned}N\mathbb{E}(\langle\mathcal{N}(\varphi^{(1)})\rangle_t) &\le \mathrm{e}^{2t\|V\|+2(T-t)\|V\|}\\ &+ 4\rho\,\|V\|\int_0^t \mathrm{e}^{2\|V\|s+2(T-s)\|V\|}\mathrm{d}s + 4(1-\rho)\,\|V'\|\int_0^t \mathrm{e}^{2\|V\|s+2(T-s)\|V\|}\mathrm{d}s\\ &= \mathrm{e}^{2T\|V\|}(1+4(\rho\,\|V\|+(1-\rho)\,\|V'\|)T) \le \exp(4Tv(\rho)).\end{aligned}$$

□

Proposition 3.4. *The N-approximating measures γ_t^N have no bias, namely $\mathbb{E}(\gamma_t^N(f)) = \gamma_t(f)$, and we have the mean square estimates*

$$\sqrt{N}\ \mathbb{E}\left(\sup_{t\le T}|\gamma_t^N(P_{t,T}^v f) - \gamma_t(P_{t,T}^v f)|^2\right)^{1/2} \le 3\exp(2v(\rho)T).$$

Proof. The first assertion is a direct consequence of Lemma 3.3. To prove the mean square estimates we use the decomposition

$$\begin{aligned}\gamma_t^N(P_{t,T}^v f) - \gamma_t(P_{t,T}^v f) &= \gamma_t^N(P_{t,T}^v f) - \eta_0^N(P_{0,T}^v f) + \eta_0^N(P_{0,T}^v f) - \eta_0(P_{0,T}^v f)\\ &= \eta_0^N(P_{0,T}^v f) - \eta_0(P_{0,T}^v f) + \mathcal{N}_t(\varphi^{(1)}).\end{aligned}$$

Using Burkholder–Davis–Gundy inequality and Lemma 3.3 we get

$$\mathbb{E}\left(\sup_{t\le T}\mathcal{N}_t^2(\varphi^{(1)})\right) \le 4\ \mathbb{E}(\langle\mathcal{N}(\varphi^{(1)})\rangle_T) \le \frac{4}{N}\ \exp(4Tv(\rho)).$$

Since the initial configuration consists of N-independent random variables with law η_0 by the weak law of large numbers we also have that

$$\mathbb{E}([\eta_0^N(P_{0,T}^v f) - \eta_0(P_{0,T}^v f)]^2) = \frac{1}{N}\ \eta_0([P_{0,T}^v f - \eta_0(P_{0,T}^v f)]^2) \le \frac{\mathrm{e}^{2T\|V\|}}{N}\cdot$$

If we combine these two upper bounds we end up with the desired estimate. More precisely we have that

$$\begin{aligned}\mathbb{E}\left(\sup_{t\le T}|\gamma_t^N(P_{t,T}^v f) - \gamma_t(P_{t,T}^v f)|^2\right)^{1/2} &\le \mathbb{E}([\eta_0^N(P_{0,T}^v f) - \eta_0(P_{0,T}^v f)]^2)^{1/2} + \mathbb{E}\left[\sup_{t\le T}\mathcal{N}_t^2(\varphi^{(1)})\right]^{1/2}\\ &\le \frac{1}{\sqrt{N}}(\mathrm{e}^{T\|V\|} + 2\mathrm{e}^{2Tv(\rho)})\end{aligned}$$

and the end of the proof of the proposition is now clear. □

Proposition 3.5. *For any fixed time parameters $s \leq T$, we have*

$$\sqrt{N}\ \mathbb{E}\left(\sup_{t\leq T} |\Phi_{t,T}(\eta_t^N)(f) - \Phi_{t,T}(\eta_t)(f)|^2\right)^{1/2} \leq 6\exp\left([\|V\| + 2v(\rho)]T\right) \tag{35}$$

$$\sqrt{N}\ \mathbb{E}\left(\sup_{s\leq t\leq T} |\Phi_{t,T}(\eta_t^N)(f) - \Phi_{s,T}(\eta_s^N)(f)|^2 |F_s\right)^{1/2} \leq 4\exp\left(2v(\rho)(T-s)\right). \tag{36}$$

Proof. To prove the first estimate we use the decomposition

$$\begin{aligned}\Phi_{t,T}(\eta_t^N)(f) - \Phi_{t,T}(\eta_t)(f) &= \frac{\gamma_t^N(P_{t,T}^v f)}{\gamma_t^N(P_{t,T}^v 1)} - \frac{\gamma_T(f)}{\gamma_T(1)}\\ &= \frac{\gamma_T(1)}{\gamma_t^N(P_{t,T}^v 1)} \times \gamma_t^N P_{t,T}^v\left(\frac{1}{\gamma_T(1)}(f - \eta_T(f))\right).\end{aligned}$$

It is clear that

$$0 \leq \gamma_T(1)/\gamma_t^N(P_{t,T}^v 1) \leq \mathrm{e}^{T\,\mathrm{osc}(V)} \qquad \text{and} \qquad \|(f - \eta_T(f))/\gamma_T(1)\| \leq 2.$$

By Proposition 3.4 we conclude that

$$\sqrt{N}\ \mathbb{E}\left(\sup_{t\leq T} |\Phi_{t,T}(\eta_t^N)(f) - \Phi_{t,T}(\eta_t)(f)|^2\right)^{1/2} \leq 2\mathrm{e}^{T\,\mathrm{osc}(V)}\ 3\mathrm{e}^{2v(\rho)T} = 6\mathrm{e}^{[\|V\|+2v(\rho)]T}.$$

To prove (36) we first recall that

$$\gamma_t^N(P_{t,T}^v f) - \gamma_s^N(P_{s,T}^v f) = \int_s^t \gamma_r^N(1)\ \mathrm{d}\mathcal{M}_r(\varphi^{(1)})$$

and dividing by $\gamma_s^N(1)$ this implies that

$$\frac{\gamma_t^N(1)}{\gamma_s^N(1)}\ \eta_t^N(P_{t,T}^v f) - \eta_s^N(P_{s,T}^v f) = \frac{1}{\gamma_s^N(1)}[\mathcal{N}_t(\varphi^{(1)}) - \mathcal{N}_s(\varphi^{(1)})] = \int_s^t \frac{\gamma_r^N(1)}{\gamma_s^N(1)}\ \mathrm{d}\mathcal{M}_r(\varphi^{(1)}).$$

If we define

$$\gamma_{s,t}^N(1) = \gamma_t^N(1)/\gamma_s^N(1) \qquad \text{and} \qquad \overline{\mathcal{N}}_{s,t} = \frac{1}{\gamma_s^N(1)}[\mathcal{N}_t - \mathcal{N}_s]$$

we obtain the following decomposition

$$\begin{aligned}\Phi_{t,T}(\eta_t^N)(f) - \Phi_{s,T}(\eta_s^N)(f) &= \frac{\gamma_{s,t}^N(1)\eta_t^N(P_{t,T}^v f)}{\gamma_{s,t}^N(1)\eta_t^N(P_{t,T}^v 1)} - \frac{\eta_s^N(P_{s,T}^v f)}{\eta_s^N(P_{s,T}^v 1)}\\ &= \frac{1}{\eta_s^N(P_{s,T}^v 1)}\ [(\gamma_{s,t}^N(1)\eta_t^N(P_{t,T}^v f) - \eta_s^N(P_{s,T}^v f)) + \Phi_{t,T}(\eta_t^N)(f)(\eta_s^N(P_{s,T}^v 1) - \gamma_{s,t}^N(1)\eta_t^N(P_{t,T}^v 1))]\end{aligned}$$

thus,

$$\Phi_{t,T}(\eta_t^N)(f) - \Phi_{s,T}(\eta_s^N)(f) = \frac{1}{\eta_s^N(P_{s,T}^v 1)}\ [\overline{\mathcal{N}}_{s,t}(\varphi^{(1)}) - \Phi_{t,T}(\eta_t^N)(f)\ \overline{\mathcal{N}}_{s,t}(\varphi^{(0)})].$$

Since

$$\eta_s^N(P_{s,T}^v 1) \geq 1$$

it does follows that

$$|\Phi_{t,T}(\eta_t^N)(f) - \Phi_{s,T}(\eta_s^N)(f)| \leq |\overline{\mathcal{N}}_{s,t}(\varphi^{(1)})| + |\overline{\mathcal{N}}_{s,t}(\varphi^{(0)})|. \tag{37}$$

On the other hand using Burkholder–Davis–Gundy inequality we have

$$\begin{aligned}\mathbb{E}\left(\sup_{s\leq t\leq T} \overline{\mathcal{N}}^2_{s,t}(\varphi^{(1)})|F_s\right) &= \frac{1}{\gamma_s^N(1)^2}\ \mathbb{E}(\sup_{s\leq t\leq T}(\mathcal{N}_t(\varphi^{(1)}) - \mathcal{N}_s(\varphi^{(1)}))^2|F_s)\\ &\leq \frac{4}{\gamma_s^N(1)^2}\ \mathbb{E}(\langle\mathcal{N}(\varphi^{(1)})\rangle_T - \langle\mathcal{N}(\varphi^{(1)})\rangle_s|F_s).\end{aligned}$$

Using (34) we obtain

$$\begin{aligned}N\mathbb{E}(\sup_{s\leq t\leq T} \overline{\mathcal{N}}^2_{s,t}(\varphi^{(1)})|F_s) &\leq 4\mathbb{E}(\mathrm{e}^{2\int_s^T \eta_r^N(V)\mathrm{d}r}\ \eta_T^N(f^2)|F_s) - 4\eta_s^N((P_{s,T}^v f)^2)\\ &+ 8\rho\mathbb{E}\left(\int_s^T e^{2\int_s^r \eta_u^N(V)\mathrm{d}u}[\eta_r^N(V(P_{r,T}^v(f))^2) - \eta_r^N(VP_{r,T}^v(f))\ \eta_r^N(P_{r,T}^v(f))]\mathrm{d}r|F_s\right)\\ &+ 8(1-\rho)\mathbb{E}\left(\int_s^T \mathrm{e}^{2\int_s^r \eta_u^N(V)du}[\eta_r^N(V')\eta_r^N((P_{r,T}^v(f))^2) - \eta_r^N(V'P_{r,T}^v(f))\ \eta_r^N(P_{r,T}^v(f))]\mathrm{d}r|F_s\right).\end{aligned}$$

From the above we get the upper bound

$$\begin{aligned}N\ \mathbb{E}(\sup_{s\leq t\leq T} \overline{\mathcal{N}}^2_{s,t}(\varphi^{(1)})|F_s) &\leq 4\left[\mathrm{e}^{2(T-s)\|V\|} + 4\,\|V\|\,\rho\int_s^T \mathrm{e}^{2(r-s)\|V\|+2(T-r)\|V\|}\mathrm{d}r\right.\\ &\qquad\left.+4\,\|V'\|\,(1-\rho)\int_s^T \mathrm{e}^{2(r-s)\|V\|+2(T-r)\|V\|}\mathrm{d}r\right]\\ &= 4\mathrm{e}^{2(T-s)\|V\|}(1+4(T-s)(\rho\,\|V\| + (1-\rho)\,\|V'\|)) \leq 4\mathrm{e}^{4(T-s)v(\rho)}.\end{aligned}$$

By (37) and the triangle inequality we conclude that

$$\sqrt{N}\ \mathbb{E}\left(\sup_{s\leq t\leq T}|\Phi_{t,T}(\eta_t^N)(f) - \Phi_{s,T}(\eta_s^N)(f)|^2|F_s\right)^{1/2} \leq 4\mathrm{e}^{2(T-s)v(\rho)}$$

and the proof is completed. □

3.2.3. *Uniform estimates and related fluctuations*

Theorem 3.6. *Suppose the semigroup Φ is exponentially stable in the sense that for any $\mu,\nu \in \mathcal{P}(E)$ and $t \geq t_0$*

$$\frac{1}{t}\ln\|\Phi_t(\mu) - \Phi_t(\nu)\|_{\mathrm{tv}} \leq -\sigma$$

for some $t_0 > 0$ *and some parameter* $\sigma > 0$. *Then for any* $\ln N \geq 2t_0(\sigma + 2v(\rho) + \|V\|)$ *and* $f \in D(L)$ *with* $\|f\| \leq 1$ *and* $\rho \in [0,1]$ *we have the uniform estimate*

$$\sup_{t\geq 0} \mathbb{E}(|\eta_t^N(f) - \eta_t(f)|^2)^{1/2} \leq 8/N^{\beta(\rho)/2} \qquad \text{with} \quad \beta(\rho) = \sigma/(\sigma + 2v(\rho) + \|V\|).$$

Proof. Using Proposition 3.5 we have for any $t \leq h \geq t_0$

$$\mathbb{E}(|\eta_t^N(f) - \eta_t(f)|^2)^{1/2} \leq \frac{6}{\sqrt{N}} \mathrm{e}^{[2v(\rho)+\|V\|]h}.$$

Next we introduce the decomposition

$$\eta_{t+h}^N - \eta_{t+h} = \eta_{t+h}^N - \phi_h(\eta_t^N) + \phi_h(\eta_t^N) - \phi_h(\eta_t)$$

for any $t \geq 0$ and $h \geq t_0$. Again from Proposition 3.5 we obtain

$$\mathbb{E}(|\eta_{t+h}^N(f) - \Phi_h(\eta_t^N)(f)|^2)^{1/2} \leq \frac{4}{\sqrt{N}} \mathrm{e}^{2hv(\rho)} \leq \frac{6}{\sqrt{N}} \mathrm{e}^{[2v(\rho)+\|V\|]h}$$

and

$$|\Phi_h(\eta_t^N)(f) - \Phi_h(\eta_t)(f)| \leq 2\mathrm{e}^{-\sigma h}.$$

This clearly implies that

$$\mathbb{E}(|\eta_{t+h}^N(f) - \eta_{t+h}(f)|^2)^{1/2} \leq \frac{6}{\sqrt{N}}\ \mathrm{e}^{[2v(\rho)+\|V\|]h} + 2\mathrm{e}^{-\sigma h}.$$

Therefore we conclude that for any $h \geq t_0$

$$\sup_{t\geq 0} \mathbb{E}(|\eta_t^N(f) - \eta_t(f)|^2)^{1/2} \leq \frac{6}{\sqrt{N}}\ \mathrm{e}^{[2v(\rho)+\|V\|]h} + 2\mathrm{e}^{-\sigma h}.$$

If we choose $h = h(N)$ and $N \geq 1$ such that

$$h(N) = \frac{\ln N}{2(\rho + 4\,\|V\|)} \geq t_0$$

we have $\frac{1}{\sqrt{N}}\ \exp\left(h(N)[2v(\rho) + \|V\|]\right) = \exp^{-(\sigma h(N))}$ and

$$\mathrm{e}^{\rho h(N)} = N^{\beta(\rho)/2} \qquad \text{with} \qquad \beta(\rho) = \sigma/(\sigma + 2v(\rho) + \|V\|).$$

The end of the proof is now straightforward. □

Our next objective is to study the fluctuations of the random fields $W_{t,T}^{\gamma,N}$ and $W_{t,T}^{\eta,N}$ defined by

$$W_{t,T}^{\gamma,N}(f) = \sqrt{N}\ (\gamma_t^N(P_{t,T}^v f) - \gamma_t(P_{t,T}^v f)), \quad W_{t,T}^{\prime\gamma,N}(f) = \sqrt{N}\ (\gamma_t^{\prime N}(P_{t,T}^v f) - \gamma_t(P_{t,T}^v f))$$

which are parameterized by functions $f \in D(L)$. We remark that by our last hypothesis on the pregenerator L, the mapping

$$[0,T] \times E \ni (t,x) \mapsto \Gamma[P_{t,T}^v(f), P_{t,T}^v(f)](x)$$

is easily seen to be bounded, fact which enables to show the same property for

$$[0,T] \times E \ni (t,x) \mapsto \Gamma_{L^\rho_\eta}[P^v_{t,T}(f), P^v_{t,T}(f)](x)$$

uniformly in $0 \leq \rho \leq 1$ and in the probability η.

The computations of Section 3.2.2 then imply that for any $f \in D(L)$ and $T \geq 0$, there exists a constant $K_T(f) \geq 0$ such that

$$\mathbb{E}[\langle \mathcal{M}(\varphi^{(2)})\rangle_T] \leq \frac{K_T(f)}{N}$$

and thus using Doob's inequality, we see that

$$\sup_{0\leq t\leq T}\left|\int_0^t (\gamma_s^N(1))^2\,\mathrm{d}\mathcal{M}_s(\varphi^{(2)})\right|$$

is converging to zero for large N, in L_2 or a.s. This is the main ingredient for the proof of the next proposition.

Proposition 3.7. *In the sense of convergence of finite distributions, the random field* $(W^{\gamma,N}_{t,T}(f))_{f\in D(L)}$*, respectively* $(W^{\eta,N}_{t,T}(f))_{f\in D(L)}$*, converges as* N *tends to infinity to a centered Gaussian field* $(W^{\gamma}_{t,T}(f))_{f\in D(L)}$*, respectively* $(W^{\eta}_{t,T}(f))_{f\in D(L)}$*, with variance functions given for all* $f \in D(L)$ *by*

$$\mathbb{E}(W^\gamma_{t,T}(f)^2) = \gamma_t(1)\gamma_t((P^v_{t,T}f)^2) - \gamma_T(f)^2 + 2\rho\int_0^t [\gamma_s(1)\ \gamma_s(V(P^v_{s,T}f)^2) - \gamma_s(V(P^v_{s,T}f))\gamma_T(f)]\mathrm{d}s$$
$$2(1-\rho)\int_0^t [\gamma_s(V')\ \gamma_s((P^v_{s,T}f)^2) - \gamma_s(V'(P^v_{s,T}f))\gamma_T(f)]\mathrm{d}s$$

and

$$\mathbb{E}(W^\eta_{t,T}(f)^2) = \eta_t([\overline{P}^v_{t,T}(f-\eta_T(f))]^2) + 2\rho\int_0^t \eta_s(V[\overline{P}^v_{s,T}(f-\eta_T(f))]^2)\ \mathrm{d}s$$
$$+ 2(1-\rho)\int_0^t \eta_s(V')\ \eta_s([\overline{P}^v_{s,T}(f-\eta_T(f))]^2)\ \mathrm{d}s$$

with the "normalized" semigroup $(\overline{P}^v_{t,T})_{t\leq T}$ *defined by*

$$\overline{P}^v_{t,T}(f)(x) = \frac{P^v_{t,T}(f)(x)}{\eta_t(P^v_{t,T}1)} = \mathbb{E}_{t,x}\left[f(X_T)\ \exp\left(\int_t^T [V(X_s) - \eta_s(V)]\mathrm{d}s\right)\right].$$

Proof. Arguing as in the proof of Lemma 3.33 (p. 107) in [4] (based on Th. 3.11, p. 432 in [11]), to get a central limit theorem for the martingales

$$t \in [0,T] \mapsto W^{\gamma,N}_{t,T}(f) = \sqrt{N}[\eta_0^N(P^v_{0,T}(f)) - \eta_0(P^v_{0,T}(f))] + \sqrt{N}\mathcal{N}_t(\varphi^{(1)})$$

it is sufficient to show such a result for the initial value $W^{\gamma,N}_{0,T}(f) = \sqrt{N}[\eta_0^N(P^v_{0,T}(f)) - \eta_0(P^v_{0,T}(f))]$ and to verify that the increasing process of $(\sqrt{N}\mathcal{N}_t(\varphi^{(1)}))_{0\leq t\leq T}$ converges in probability to a deterministic and continuous limit. The first point is clear since it is an application of the usual central limit theorem for independent and identically distributed L_2 (even bounded here) variables, and it appears that the limit variance for these terms is

$$\eta_0((P^v_{0,T}(f))^2) - (\eta_0(P^v_{0,T}(f)))^2.$$

Meanwhile, in the proof of Lemma 3.3, we computed that

$$\begin{aligned}
N\langle\mathcal{N}(\varphi^{(1)})\rangle_t &= \gamma_t^N(1)^2\ \eta_t^N((P^v_{t,T}f)^2) - \gamma_0^N(1)^2\ \eta_0^N((P^v_{0,T}f)^2) - \int_0^t \gamma_s^N(1)^2\ \mathrm{d}\mathcal{M}_s(\varphi^{(2)})\\
&+2\rho\int_0^t \gamma_s^N(1)^2\ [\eta_s^N(V(P^v_{s,T}(f))^2) - \eta_s^N(VP^v_{s,T}(f))\ \eta_s^N(P^v_{s,T}(f))]\,\mathrm{d}s\\
&+2(1-\rho)\int_0^t \gamma_s^N(1)^2\ [\eta_s^N(V')\ \eta_s^N((P^v_{s,T}(f))^2) - \eta_s^N(V'P^v_{s,T}(f))\eta_s^N(P^v_{s,T}(f))]\,\mathrm{d}s
\end{aligned}$$

which converges in probability (using for $0\le s\le t$ the convergences of γ_s^N and η_s^N proved previously and the observation above the proposition) to

$$\begin{aligned}
\gamma_t(1)^2\ \eta_t((P^v_{t,T}f)^2) - \gamma_0(1)^2\ \eta_0((P^v_{0,T}f)^2) + 2\rho\int_0^t \gamma_s(1)^2\ [\eta_s(V(P^v_{s,T}(f))^2) - \eta_s(VP^v_{s,T}(f))\ \eta_s(P^v_{s,T}(f))]\,\mathrm{d}s\\
+\,2(1-\rho)\int_0^t \gamma_s(1)^2\ [\eta_s(V')\ \eta_s((P^v_{s,T}(f))^2) - \eta_s(V'P^v_{s,T}(f))\eta_s(P^v_{s,T}(f))]\,\mathrm{d}s.
\end{aligned}$$

Thus we can conclude to the first convergence announced in the proposition and to the validity of the expression for the variance of the corresponding limit (using that for any $0\le s\le T$, $\eta_s(P^v_{s,T}(f)) = \gamma_T(f)/\gamma_s(1)$).

To study the convergence of the random field $W^{\eta,N}_{t,T}$ we introduce the decomposition

$$\begin{aligned}
\Phi_{t,T}(\eta_t^N)(f) - \Phi_{t,T}(\eta_t)(f) &= \frac{\gamma_t^N(P^v_{t,T}f)}{\gamma_t^N(P^v_{t,T}1)} - \frac{\gamma_t(P^v_{t,T}f)}{\gamma_t(P^v_{t,T}1)}\\
&= \frac{\gamma_t(P^v_{t,T}1)}{\gamma_t^N(P^v_{t,T}1)}\times\gamma_t^N\left(\frac{P^v_{t,T}}{\gamma_t(P^v_{t,T}1)}(f-\Phi_{t,T}(\eta_t)(f))\right).
\end{aligned}$$

Noticing that

$$\gamma_t\left(\frac{P^v_{t,T}}{\gamma_t(P^v_{t,T}1)}(f-\Phi_{t,T}(\eta_t)(f))\right) = 0$$

we see that $W^{\eta,N}_{t,T}$ and $W^{\gamma,N}_{t,T}$ are connected by the formula

$$W^{\eta,N}_{t,T}(f) = \frac{\gamma_t(P^v_{t,T}1)}{\gamma_t^N(P^v_{t,T}1)}\times W^{\gamma,N}_{t,T}\left((f-\eta_T(f))/\gamma_T(1).\right).$$

If follows from Proposition 3.4 that

$$\lim_{N\to\infty}\gamma_t(P^v_{t,T}1)/\gamma_t^N(P^v_{t,T}1) = 1$$

in probability sense. This implies that $W^{\eta,N}_{t,T}$ converges as $N\to\infty$ (and in the sense of convergence of finite distributions) to the centered Gaussian field

$$W^{\eta}_{t,T}(f) = W^{\gamma}_{t,T}\left((f-\eta_T(f))/\gamma_T(1)\right).$$

Then we are assured of

$$\begin{aligned}
\mathbb{E}(W^{\eta}_{t,T}(f)^2) &= \mathbb{E}([W^{\gamma}_{t,T}((f-\eta_T(f))/\gamma_T(1))]^2) \\
&= \gamma_t(1)^2\ \eta_t(\ (P^v_{t,T}(f-\eta_T(f))/\gamma_T(1))^2\ ) \\
&\quad +2\rho\int_0^t \gamma_s(1)\ \gamma_s(V\ (P^v_{s,T}(f-\eta_T(f))/\gamma_T(1))^2)\ \mathrm{d}s \\
&\quad +2(1-\rho)\int_0^t \gamma_s(V')\ \gamma_s((P^v_{s,T}(f-\eta_T(f))/\gamma_T(1))^2)\ \mathrm{d}s.
\end{aligned}$$

Recalling that

$$\eta_s(P^v_{s,T}(1)) = \gamma_s(P^v_{s,T}(1))/\gamma_s(1) = \gamma_T(1)/\gamma_s(1)$$

the end of the proof of the proposition is now easily completed. □

As in [7] we have not considered central limit theorems, we are not sure of the validity of the previous proposition in its context. Nevertheless the natural conjecture is that it should be true, because the limit variances are indeed well defined for any $f \in \mathcal{B}_b(E)$ (no "carré du champ" is entering in the formulation of the final result).

We end this paper with a discussion on the form of the covariance functions associated to the fluctuations of the N-approximating models η^N_t corresponding to the choice of parameter $\rho = 0$ and $\rho = 1$. We first examine the situation where the potential function V is constant and $V(x) = a$ for any $x \in E$.

In this case η_t is clearly the distribution of X_t and the semigroup $\overline{P}^v$ coincides with the semigroup P. In this simple situation and for $\rho = 1$ we have that (in what follows, a function $f \in D(L)$ with $\|f\| \leq 1$ is assumed to be chosen)

$$\mathbb{E}(W^{\eta}_{t,T}(f)^2) = \eta_t([P_{t,T}(f-\eta_T(f))]^2) + 2\int_0^t\ a\ \eta_s([P_{s,T}(f-\eta_T(f))]^2)\mathrm{d}s. \tag{38}$$

With the choice $V'(x) = a - V(x) = 0$ the N-particle model consists of N independent copies of X and clearly for $\rho = 0$

$$\mathbb{E}(W^{\eta}_{t,T}(f)^2) = \eta_t([P_{t,T}(f-\eta_T(f))]^2).$$

The integral term in the right hand side of the first covariance function (38) comes from the fact that the N-particle approximating model has an interacting jump part. This extra degree of randomness induces a greater covariance function. For instance in the degenerate situation where $L = 0$ we have $X_t = X_0$, $\eta_0 = \eta_t$ and for $\rho = 1$

$$\mathbb{E}(W^{\eta}_{t,T}(f)^2) = (1+2at)\ \eta_0((f-\eta_0(f))^2) \xrightarrow[t\to\infty]{} \infty$$

but for $\rho = 0$

$$\mathbb{E}(W^{\eta}_{t,T}(f)^2) = \eta_0((f-\eta_0(f))^2).$$

In contrast to the latter situation suppose the semigroup P satisfies the mixing condition $(\mathcal{P}_1)$ for some $\varepsilon > 0$. In this situation, for any $s+1 \leq T$ and $x, x' \in E$ we have that

$$
\begin{aligned}
P^v_{s,T}(1)(x) &= \mathbb{E}_{s,x}\left(\exp\left[\int_s^T V(X_r)\mathrm{d}r\right]\right)\\
&= \mathbb{E}_{s,x}\left(\exp\left[\int_s^{s+1} V(X_r)\mathrm{d}r\right]\ \mathbb{E}_{s+1,X_{s+1}}\left(\exp\left[\int_{s+1}^T V(X_r)\mathrm{d}r\right]\right)\right)\\
&\leq \varepsilon^{-1}\ \mathrm{e}^{\|V\|}\ \int_E\ P_1(x',\mathrm{d}y)\ \mathbb{E}_{s+1,y}\left(\exp\left[\int_{s+1}^T V(X_r)\mathrm{d}r\right]\right)\\
&= \varepsilon^{-1}\ \mathrm{e}^{\|V\|}\ \mathbb{E}_{s,x'}\left(\mathbb{E}_{s+1,X_{s+1}}\left(\exp\left[\int_{s+1}^T V(X_r)\mathrm{d}r\right]\right)\right).
\end{aligned}
$$

As a result we get

$$
\begin{aligned}
P^v_{s,T}(1)(x) &\leq \varepsilon^{-1}\ \mathrm{e}^{\mathrm{osc}(V)}\ \mathbb{E}_{s,x'}\left(\exp\left[\int_s^{s+1} V(X_r)\mathrm{d}r\right]\mathbb{E}_{s+1,X_{s+1}}(\exp\left[\int_{s+1}^T V(X_r)\,\mathrm{d}r\right])\right)\\
&= \varepsilon^{-1}\ \mathrm{e}^{\mathrm{osc}(V)}\ P^v_{s,T}(1)(x').
\end{aligned}
$$

This leads to the estimates

$$
\varepsilon\mathrm{e}^{-\mathrm{osc}(V)} \quad \leq \quad P^v_{s,T}(1)(x)/P^v_{s,T}(1)(x') \leq \varepsilon^{-1}\ \mathrm{e}^{\mathrm{osc}(V)}. \tag{39}
$$

From Proposition 2.5 we also know there exists some strictly positive constant $\alpha > 0$ (depending on the parameter ε and on the potential V) such that for any $t \geq 2$ and $\mu, \nu \in \mathcal{P}(E)$

$$
\|\Phi_{s,s+t}(\mu) - \Phi_{s,s+t}(\nu)\|_{\mathrm{tv}} \leq\ \mathrm{e}^{-\alpha t}. \tag{40}
$$

At this point it is convenient to notice that for $\mu \in \mathcal{P}(E)$, $x \in E$ and $s \leq T$

$$
\frac{P^v_{s,T}}{\mu P^v_{s,T}(1)}(f - \eta_T(f))(x) = \int_E \eta_s(\mathrm{d}x')\ \frac{P^v_{s,T}(1)(x)}{\mu P^v_{s,T}(1)}\ \frac{P^v_{s,T}(1)(x')}{\eta_s P^v_{s,T}(1)}\ \left[\frac{P^v_{s,T}(f)}{P^v_{s,T}(1)}(x) - \frac{P^v_{s,T}(f)}{P^v_{s,T}(1)}(x')\right].
$$

It also follows from (40) that for $s+2 \leq T$,

$$
\left|\frac{P^v_{s,T}(f)}{P^v_{s,T}(1)}(x) - \frac{P^v_{s,T}(f)}{P^v_{s,T}(1)}(x')\right| \leq 2\ \|\Phi_{s,T}(\delta_x) - \Phi_{s,T}(\delta_{x'})\|_{\mathrm{tv}} \leq\ 2\ \mathrm{e}^{-\alpha(T-s)}
$$

and for any $\mu \in \mathcal{P}(E)$, equation (39) implies that

$$
\left\|\frac{P^v_{s,T}}{\mu P^v_{s,T}(1)}(f - \eta_T(f))\right\| \leq 2\ \varepsilon^{-1}\ \mathrm{e}^{\mathrm{osc}(V)}\ \mathrm{e}^{-\alpha(T-s)}. \tag{41}
$$

.

From the fluctuations presented in Proposition 3.7 we have for $\rho = 1$

$$
\begin{aligned}
\mathbb{E}(W^\eta_{t,T}(f)^2) &\leq \left[4\ \varepsilon^{-2}\ \mathrm{e}^{2\mathrm{osc}(V)}\right]\ \left[\mathrm{e}^{-2\alpha(T-t)} + 2\,\|V\| \int_0^t\ \mathrm{e}^{-2\alpha(T-s)}\mathrm{d}s\right]\\
&\leq 4\ \mathrm{e}^{-2\alpha(T-t)}\ [1 + \|V\|/\alpha]\ \left[\mathrm{e}^{2\mathrm{osc}(V)}/\varepsilon^2\right].
\end{aligned}
$$

This leads for $\rho = 1$ to the following uniform bound with respect to the time parameter

$$\sup_{t\leq T} \mathbb{E}(W^{\eta}_{t,T}(f)^2)^{1/2} \leq 2\ [1 + \|V\| /\alpha]^{1/2}\ [\ \mathrm{e}^{\mathrm{osc}(V)}/\varepsilon].$$

In much the same way for $\rho = 0$ we have that

$$\sup_{t\leq T} \mathbb{E}(W^{\eta}_{t,T}(f)^2)^{1/2} \leq 2\ [1 + \|V'\| /\alpha]^{1/2}\ [\ \mathrm{e}^{\mathrm{osc}(V)}/\varepsilon].$$

Conjecture 3.8. From these observations, when the semigroup P satisfies the mixing condition $(\mathcal{P}_1)$, it is natural to conjecture that the right exponent in Theorem 3.6 is $\beta = 1$. This conjecture has been proved in the discrete time case in [4] (p. 36) (see also Cor. 3.8 in [8] for the empirical process version). The proof is essentially based on an particular decomposition of the errors and it seems difficult to find its analog in the continuous time case.

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