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QUASI-ERGODICITY FOR ABSORBING MARKOV PROCESSES *VIA* DEVIATION INEQUALITY

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Abstract. In this note, taking the killed Brownian motion as an illustrative model, we derive a conditional deviation inequality for $\int_0^t V(X_s)ds$ for certain (unbounded) functions V. Then we apply it to prove a quasi L^1-ergodic theorem for the killed process.

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1. Introduction

Killed Markov processes constitute an important class of non-ergodic Markov processes. For such a process, one of the most important topics is its asymptotic behavior before killing. Quasi-stationarity and quasi-ergodicity are two fundamental problems. It has been shown in [1–3] that for many typical processes, a quasi-stationary distribution is different from a quasi-ergodic distribution. Using different approaches, quasi-L^1-ergodicity for bounded functions was proved in [1–3] respectively. To handle the case of unbounded functions, in this paper we take the killed Brownian motion as an illustrative model. We derive a conditional deviation inequality for (unbounded) functions in the Kato class. Then such inequality is applied to quasi-ergodicity. The arguments applied to more general processes.

We first introduce the some necessary notations and preliminary results. Let $\{X_t,\ t \geq 0\}$ be a standard d-dimensional ($d \geq 1$) Brownian motion (BM) on $\{\Omega, \{\mathscr{F}_t\}, \mathbb{P}\}$, where $\Omega = C([0,+\infty), \mathbb{R}^d)$ and $\mathscr{F}_t = \sigma\{X_s, 0 \leq s \leq t\}$. Let $\{\mathbb{P}_x,\ x \in \mathbb{R}^d\}$ be the corresponding Markov family, and $\mathbb{E}_x$ denote the expectation under $\mathbb{P}_x$. Given D an open, bounded and connected subset of $\mathbb{R}^d$ with boundary ∂D, its closure is $\overline{D} = D \cup \partial D$. We define

$$\tau(\omega) = \inf\{t > 0 : X_t(\omega) \notin D\} \tag{1.1}$$

to be the first exit time of D. The killed Brownian motion we are considering is defined by

$$X_t^D = \begin{cases} X_t, & \text{if} \quad \tau > t, \\ \partial, & \text{if} \quad \tau \leq t. \end{cases} \tag{1.2}$$

where ∂ is an extra point. We call X_t^D the Brownian motion killed outside D. It is well known that the transition function of X_t^D, denoted by $P^D(t;\cdot,\cdot)$, has a density $p^D(t;\cdot,\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ which

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admits an explicit expansion in terms of the Dirichlet eigenvalues $0 < \lambda_1 < \lambda_2 \le \cdots$ and the corresponding eigenfunctions $\{\varphi_n, n \ge 1\}$ of $-\frac{\Delta}{2}$ on $\overline{D}$. To be more precise, we summarize below some well known results crucial for our discussion.

Proposition 1.1.

(i) ([4], p. 33).

$$p^D(t,x,B) = \mathbb{P}_x(X_t \in B; \tau > t) = \int_B p^D(t;x,y)\mathrm{d}y, x \in D, B \in \mathscr{B}(D), t > 0. \tag{1.3}$$

The density function $p^D(t;\cdot,\cdot)$ is symmetric, continuous, strictly positive on $D \times D$, with the following expansion:

$$p^D(t;x,y) = \sum_{n=1}^{\infty} \exp(-\lambda_n t)\varphi_n(x)\varphi_n(y). \tag{1.4}$$

(ii) ([7], p. 123). $\{\varphi_n, n \ge 1\}$ *forms a complete orthonormal system of the Hilbert space $L^2(D, \mathrm{d}x)$ and satisfies*

$$\varphi_n^2(x) \le \exp(\lambda_n \epsilon)\left(\frac{1}{2\pi\epsilon}\right)^{\frac{d}{2}}, \quad x \in D. \tag{1.5}$$

Furthermore for $0 < \epsilon < t$,

$$\sum_{n=1}^{\infty} \exp(-\lambda_n t)|\varphi_n(x)\varphi_n(y)| \le \left(\frac{1}{2\pi\epsilon}\right)^{\frac{d}{2}} \sum_{n=1}^{\infty} \exp(-\lambda_n(t-\epsilon)) < +\infty; \tag{1.6}$$

(iii) ([6], p. 336). λ_1 *is simple, so $\lambda_1 < \lambda_n$, for $n > 1$. Furthermore, φ_1 is strictly positive and infinitely differentiable.*

We introduce the following notations.

$\mathscr{P}_1(D) :=$ the class of probability measures on D.

$C_b(D) :=$ the class of real valued bounded continuous functions on D.

$C_0^\infty(D) :=$ the class of functions on D which are infinitely differentiable with compact support.

$L^p(D, \mathrm{d}x) = L^p(D)(1 \le p < \infty)$ be the usual class of real and measurable functions on D, which are pth integrable with respective to the Lebesgue measure.

Let ∇ and $\triangle$ be the divergence and Laplacian on $C_0^\infty(D)$. Let $H_0^1(D)$ be the completion of $C_0^\infty(D)$ with respect to the norm

$$\|f\|_{H_0^1(D)} = \left(\int_D f^2(x)\mathrm{d}x + \frac{1}{2}\int_D \nabla f \cdot \nabla f \mathrm{d}x\right)^{1/2}.$$

The deviation inequality we will derive concerns functions in a Kato class J to be defined as follows (refer to [4], p. 62 for more details). Define

$$g(u) = g(|u|) = \begin{cases} |u|^{2-d}, & \text{if} \quad d \ge 3, \\ -\log|u|, & \text{if} \quad d = 2, \\ |u|, & \text{if} \quad d = 1. \end{cases} \tag{1.7}$$

Let V be a measurable function from $\mathbb{R}^d$ to $[-\infty, +\infty]$, then $V \in J$ iff

$$\lim_{s \downarrow 0}\left[\sup_{x\in\mathbb{R}^d} \int_{|y-x|<s} |g(y-x)V(y)|\mathrm{d}y\right] = 0.$$

In this paper, V is only defined on the domain D, but we can extend it to $\mathbb{R}^d$ by fixing it to be 0 on $\mathbb{R}^d - D$.

For a function in Kato class, we can define the Feynman–Kac semigroup on $L^2(D)$.

Proposition 1.2 ([4], *p.* 82 and 94. Thms. 3.17 and 3.27)**.** *Given $V \in J$, we have the strongly continuous and symmetric Feynman–Kac semigroup $\{P_t^V, t \geq 0\}$ on $L^2(D)$:*

$$P_t^V f(x) = \mathbb{E}_x \left\{ \exp\left[\int_0^t V(X_s)\mathrm{d}s \right] f(X_t); \tau > t \right\}, \quad f \in L^2(D).$$

The domain of the generator A_V of the semigroup is

$$\mathcal{D}(A_V) = \left\{ f \in H_0^1(D) : \Delta f \text{ exits weakly and } \left(\frac{\Delta}{2} + V \right) f \in L^2(D) \right\},$$

and if $f \in \mathcal{D}(A_V)$, then

$$A_V f = \left(\frac{\Delta}{2} + V \right) f.$$

2. The governing functional

In this section, we will introduce the functional which will govern the deviation inequality we are going to derive. We use the Poincare inequality to get an upper bound of the exponential growth rate of the Feynman–Kac semigroup. The Legendre transform of this bound will be shown to be just the functional governing the deviation inequality. By the Fenchel Legendre theorem, this functional can be expressed in a more explicit form. Following a standard variational approach we provide some sufficient conditions for the functional to achieve a unique minimum.

For $\lambda \in \mathbb{R}$, we define

$$\begin{aligned} H_V(\lambda) &= \sup \left\{ \int_D (f \cdot A_{\lambda V} f)\mathrm{d}x : f \in \mathcal{D}(A_{\lambda V}), \int_D f^2 \mathrm{d}x = 1 \right\} \\ &= \sup \left\{ \int_D \left(\lambda V f^2 - \frac{1}{2}|\nabla f|^2 \right) \mathrm{d}x : f \in \mathcal{D}(A_{\lambda V}), \int_D f^2 \mathrm{d}x = 1 \right\}. \end{aligned} \tag{2.1}$$

The Poincare inequality shows that $H_V(\lambda)$ is an upper bound for the exponential growth rate of the semigroup. More precisely,

$$||P_t^{\lambda V}||_2 \leq \mathrm{e}^{tH_V(\lambda)}. \tag{2.2}$$

By the next Lemma due to Kato, we see that

$$H_V(\lambda) < +\infty. \tag{2.3}$$

Lemma 2.1 ([4], *p.* 91*. Thm.* 3.25)**.** *Given $V \in J$, for any $\beta \in (0,1)$, there exists an $\alpha > 0$, s.t. for any $f \in H_0^1(D)$:*

$$\int_{\mathbb{R}^d} |V(x)| f^2(x)\mathrm{d}x \leq \alpha \int_{\mathbb{R}^d} f^2(x)\mathrm{d}x + \beta \int_{\mathbb{R}^d} |\nabla f|^2 \mathrm{d}x \tag{2.4}$$

In order to apply the Fenchel Legendre theorem, we give other expressions of $H_V(\lambda)$ by changing the space of functions. We first recall the following

Lemma 2.2 ([4], *p.* 100*, Prop.* 3.29)**.**

$$H_V(\lambda) = \sup \left\{ \int_D \left(\lambda V f^2 - \frac{1}{2}|\nabla f|^2 \right) : f \in C_0^\infty(\mathrm{D}), \int_D f^2 \mathrm{d}x = 1 \right\} \tag{2.5}$$

By the definition of $H_0^1(D)$, it is not hard to verify the following

Lemma 2.3.

$$H_V(\lambda) = \sup\left\{\int_D \left(\lambda V f^2 - \frac{1}{2}|\nabla f|^2\right) : f \in H_0^1(D), \int_D f^2 \mathrm{d}x = 1\right\} \tag{2.6}$$

We further express H_V in the form of a Legendre transform of certain function J_V on $\mathbb{R}$.

$$\begin{aligned} H_V(\lambda) &= \sup\left\{\int_D \left(\lambda V f^2 - \frac{1}{2}|\nabla f|^2\right) \mathrm{d}x : f \in H_0^1(D), \int_D f^2 \mathrm{d}x = 1\right\} \\ &= \sup_{z\in\mathbb{R}} \sup_{\int_D V f^2 \mathrm{d}x = z} \left\{\lambda z - \int_D \frac{1}{2}|\nabla f|^2 \mathrm{d}x : f \in H_0^1(D), \int_D f^2 \mathrm{d}x = 1\right\} \\ &= \sup_{z\in\mathbb{R}}\{\lambda z - J_V(z)\}, \end{aligned} \tag{2.7}$$

where

$$J_V(z) = \inf\left\{\int_D \frac{1}{2}|\nabla f|^2 \mathrm{d}x : f \in H_0^1(D), \int_D f^2 \mathrm{d}x = 1, \int_D V f^2 \mathrm{d}x = z\right\}. \tag{2.8}$$

Lemma 2.4. $J_V : \mathbb{R} \to [\lambda_1, \infty]$ *is a convex function and attains its minimum* λ_1 *at*

$$a = \int_D V \mathrm{d}\mu_0, \tag{2.9}$$

where

$$\mathrm{d}\mu_0 = 1_D \varphi_1^2(x) \mathrm{d}x. \tag{2.10}$$

Proof. The convexity of J_V can be verified directly from its definition, or regard it as a standard result in large deviation theory, since $H_0^1(D)$ is the domain of the Dirichlet form corresponding to the killed BM on D, see, for example, ([5], p. 135, Exercise 4.2.63).

The last statement is just the variational principle for the first Dirichlet eigenvalue of the Laplacian operator ([6], p. 336). □

By the convexity of J_V, $\{J_V < +\infty\}^\circ$ is an open interval (l, m), where $-\infty \le l \le m \le +\infty$. We now define the function I_V as the lower semi-continuous regularization of J_V.

$$I_V(x) = \begin{cases} J_V(x), & \text{if} \quad m < x < l, \\ J_V(m+), & \text{if} \quad x = m, \\ J_V(l-), & \text{if} \quad x = l, \\ \infty, & \text{if} \quad x = \infty. \end{cases} \tag{2.11}$$

In the next section, we will see that I_V is actually the governing functional for the deviation inequality. Now we study some further properties of I_V.

Proposition 2.5.

(i) *For any* $x \in \mathbb{R}$, *we have that*

$$I_V(x) = \sup_{\lambda\in\mathbb{R}}\{\lambda x - H_V(\lambda)\}. \tag{2.12}$$

(ii) *H is a convex function, and for any* $b > 0$

$$I_V(a+b) = \sup_{\lambda \geq 0}\{\lambda(a+b) - H_V(\lambda)\} \tag{2.13}$$

and

$$I_V(a-b) = \sup_{\lambda \leq 0}\{\lambda(a-b) - H_V(\lambda)\}. \tag{2.14}$$

Proof.

(i) The result is due to the celebrated Fenchel Legendre theorem.
(ii) We only need to prove (2.13). By the convexity of J_V (Lem. 2.4) and (1.5), we see that H is a convex.

Taking $z = a$ in (1.5), we see that

$$H_V(\lambda) \geq \lambda \cdot a - \lambda_1.$$

Thus for $\lambda < 0$,

$$\lambda(a+b) - H_V(\lambda) \leq \lambda \cdot b + \lambda_1 < \lambda_1.$$

But from (2.11) and Lemma 2.4,

$$\inf_{x \in \mathbb{R}} I_V(x) = \lambda_1,$$

it follows that

$$I_V(a+b) = \sup_{\lambda \in \mathbb{R}}\{\lambda(a+b) - H_V(\lambda)\} = \sup_{\lambda \geq 0}\{\lambda(a+b) - H_V(\lambda)\}.$$ □

The next results are important for the quasi-ergodicity to be discussed in Section 4.

Theorem 2.6. *If* $V \in J$ *is bounded and measurable, or* $V \in J \cap L^{\frac{d}{2}}(D)$ *with* $d \geq 3$*, then* I_V *achieves its minimum* λ_1 *uniquely at* a*, that is*

$$\lambda_1 = I_V(a) = \inf\{I_V(x) : x \in \mathbb{R}\}$$

Proof. From (2.11) we see that the desired assertion for I_V follows from that of J_V.

For J_V, we first note that

$$\lambda_1 = \inf\left\{\int_D \frac{1}{2}|\nabla f|^2 \mathrm{d}x : f \in H_0^1(D), \int_D f^2 \mathrm{d}x = 1\right\},$$

and that the infimum is uniquely achieved at φ_1 ([6], p. 336), which gives $J_V(a) = \lambda_1 = \inf\{J_V(x) : x \in \mathbb{R}\}$.

Secondly, given $z \neq a$, we need to prove that

$$J_V(z) > \lambda_1. \tag{2.15}$$

We can choose a minimizing sequence $\{f_n \in H_0^1(D) : n \geq 1\}$ such that

$$J_V(z) = \lim_{n\to\infty} \int_D \frac{1}{2}|\nabla f_n|^2 \mathrm{d}x, \quad \int_D f_n^2 \mathrm{d}x = 1, \quad \int_D V f_n^2 \mathrm{d}x = z.$$

Since $\{f_n : n \geq 1\}$ are bounded in $H_0^1(D)$, by Banach–Alaoglu theorem we can extract a subsequence (still denoted by $\{f_n : n \geq 1\}$) such that $\lim_{n\to\infty} f_n = f$ weakly in $H_0^1(D)$. Thus

$$\begin{aligned} J_V(z) &= \liminf_{n\to\infty} \int_D |\nabla f_n|^2 \mathrm{d}x \\ &= \liminf_{n\to\infty} \int_D |\nabla f_n - \nabla f + \nabla f|^2 \mathrm{d}x \\ &= \liminf_{n\to\infty} \left\{\int_D |\nabla f_n - \nabla f|^2 \mathrm{d}x + 2\int_D (\nabla f_n - \nabla f)\cdot \nabla f \mathrm{d}x\right\} + \int_D |\nabla f|^2 \mathrm{d}x \\ &\geq \int_D |\nabla f|^2 \mathrm{d}x. \end{aligned} \tag{2.16}$$

Passing to a subsequence if necessary, the Rellich–Kondrashov compact embedding theorem gives that $\lim_{n\to\infty} f_n = f$ strongly in $L^2(D)$, which implies $\int_D f^2 \mathrm{d}x = 1$.

It remains to check that

$$\int_D V f^2 \mathrm{d}x = z. \tag{2.17}$$

If V is bounded and measurable, then

$$z = \lim_{n\to\infty} \int_D V f_n^2 \mathrm{d}x = \int_D V f^2 \mathrm{d}x.$$

Thus by the definition of J_V and (2.16),

$$J_V(z) = \int_D \frac{1}{2} |\nabla f|^2 \mathrm{d}x > \lambda_1.$$

If $d \geq 3$, the Sobolev inequalities ([6], p. 265, Thm. 3) give that $\{f_n : n \geq 1\}$ are bounded in $L^{2^*}(D)$, where $2^* = \frac{2d}{d-2}$. If $V \in L^{\frac{d}{2}}(D)$, passing to a subsequence if necessary, we see that

$$z = \lim_{n\to\infty} \int_D V f_n^2 \mathrm{d}x = \int_D V f^2 \mathrm{d}x,$$

which proves the theorem. □

Theorem 2.7. *If $V \in J$ is such that I_V attends its infimum λ_1 at a unique z, then for any $\epsilon > 0$, there exists a $\gamma > 0$, such that $\forall b \geq 0$,*

$$I_V(a + \epsilon + b) \geq \lambda_1 + \gamma b,$$

and

$$I_V(a - \epsilon - b) \geq \lambda_1 + \gamma b,$$

Proof. It is easy to see that H_V is lower semicontinuous and $H_V(0) = -\lambda_1$. Thus

$$\limsup_{\lambda\to 0+} [\lambda(a + \epsilon) - H_V(\lambda)] \leq \lambda_1.$$

From this, Proposition 2.1 and the assumption we see that there is a $\gamma > 0$ such that

$$I_V(a + \epsilon) = \sup_{\lambda \geq \gamma} [\lambda(a + \epsilon) - H_V(\lambda)] > \lambda_1.$$

Now it follows that for each $b \geq 0$,

$$I_V(a + \epsilon + b) = \sup_{\lambda \geq 0} [\lambda(a + \epsilon + b) - H_V(\lambda)] \geq \sup_{\lambda \geq \gamma} [\lambda(a + \epsilon + b) - H_V(\lambda)] \geq \lambda_1 + \gamma b.$$ □

3. The deviation inequality and quasi-ergodicity

In this section, we first derive a deviation inequality for $\frac{1}{t}\int_0^t V(X_s)\mathrm{d}s$ governed by the functional I_V, $v \in J$. Then we apply such inequality to quasi-ergodicity.

Let $|D|$= the Lebesgue measure of D and $\mathrm{d}\mu_1 = \frac{1_D \mathrm{d}x}{|D|}$, the main result of this section is the following

Theorem 3.1. *Given $V \in J$ and $\nu \in \mathscr{P}_1(D)$ with $\nu << \mu_1$ and $\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}\|_2 < \infty$, for any number $t > 0$ and $b > 0$, we have that:*

$$P_\nu\left[\frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a > b, \tau > t\right] \le \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}\right\|_2 \exp[-tI_V(a+b)], \tag{3.1}$$

$$P_\nu\left[\frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a < -b, \tau > t\right] \le \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}\right\|_2 \exp[-tI_V(a-b)]. \tag{3.2}$$

Proof. For any number $t,\ b > 0$, by Chebychev's inequality and (2.2),

$$\begin{aligned}
P_\nu\left[\frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a > b, \tau > t\right] &\le \inf_{\lambda>0} \exp[-\lambda t(a+b)] E_\nu\left\{\exp\left[\int_0^t \lambda V(X_s)\mathrm{d}s\right], \tau > t\right\} \\
&\le \inf_{\lambda>0} \exp[-\lambda t(a+b)] \int_D E_x\left\{\exp\left[\int_0^t \lambda V(X_s)\mathrm{d}s\right], \tau > t\right\} \frac{\mathrm{d}\nu}{\mathrm{d}x}\mathrm{d}x \\
&\le \left\|\frac{\mathrm{d}\nu}{\mathrm{d}x}\right\|_2 \cdot \inf_{\lambda>0} \exp[-\lambda t(a+b)] \cdot ||P_t^{\lambda V} 1||_2 \\
&\le \sqrt{|D|} \cdot \left\|\frac{\mathrm{d}\nu}{\mathrm{d}x}\right\|_2 \cdot \inf_{\lambda>0} \left\{\exp[-\lambda t(a+b)] \cdot \mathrm{e}^{tH_V(\lambda)}\right\} \\
&= \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}\right\|_2 \exp\left\{-t \sup_{\lambda>0} [\lambda(a+b) - H_V(\lambda)]\right\}.
\end{aligned} \tag{3.3}$$

It follows from Proposition 2.5 that

$$\sup_{\lambda>0}\{\lambda(a+b) - H_V(\lambda)\} = I_V(a+b),$$

The first assertion follows. The second one follows by replacing V with $-V$. □

Remark 3.2. The theorem is motivated by the paper [8], which deals with general conservative Markov processes.

The next lemma is to study the absorbing probability. The first Dirichlet eigenvalue characterizes the absorbing rate for the killed BM.

Lemma 3.3. *As $t \to \infty$, $\sum_{n=2}^{\infty} \exp(-(\lambda_n - \lambda_1)t)\varphi_n(x)\varphi_n(y)$ converges to 0 absolutely and uniformly for $(x, y) \in D \times D$.*

Proof. From Proposition 1.1, for $0 < \epsilon < t$ and $x, y \in D$,

$$\sum_{n=2}^{\infty} \exp(-(\lambda_n - \lambda_1)t)|\varphi_n(x)\varphi_n(y)| \le \left(\frac{1}{2\pi\epsilon}\right)^{\frac{d}{2}} \exp(\lambda_1\epsilon) \sum_{n=2}^{\infty} \exp(-(\lambda_n - \lambda_1)(t-\epsilon)) < +\infty.$$

Observing that $\lim_{t\to\infty} \exp(-(\lambda_n - \lambda_1)(t-\epsilon)) = 0$ monotonically and applying the dominated convergence theorem, we see that

$$\begin{aligned}
&\lim_{t\to\infty} \sup_{x,y\in D} \left\{\sum_{n=2}^{\infty} \exp(-(\lambda_n - \lambda_1)t)|\varphi_n(x)\varphi_n(y)|\right\} \\
&\quad\le \lim_{t\to\infty} \left(\frac{1}{2\pi\epsilon}\right)^{\frac{d}{2}} \exp(\lambda_1\epsilon) \sum_{n=2}^{\infty} \exp(-(\lambda_n - \lambda_1)(t-\epsilon)) = 0.
\end{aligned}$$

□

Proposition 3.4. *Given $\nu \in \mathscr{P}_1(D)$, for any number $t > 0$*

$$\mathbb{P}_\nu(\tau > t) = C_\nu(t)\mathrm{e}^{-\lambda_1 t}. \tag{3.4}$$

where $C_\nu(t) : (0, +\infty) \to (0, +\infty)$ is a continuous function such that

$$\lim_{t\to\infty} C_\nu(t) = \int \varphi_1(x)\nu(\mathrm{d}x)\int \varphi_1(y)\mathrm{d}y > 0 \quad \text{and} \quad \lim_{t\to 0} C_\nu(t) = 1.$$

Proof. It follows from Proposition 1.1 that

$$\begin{aligned}\mathbb{P}_\nu(\tau > t) &= \int\int p^D(t; x, y)\mathrm{d}y\nu(\mathrm{d}x)\\ &= \int\int \sum_{n=1}^{\infty} \exp(-\lambda_n t)\varphi_n(x)\varphi_n(y)\mathrm{d}y\nu(\mathrm{d}x)\\ &= \mathrm{e}^{-\lambda_1 t}\left[\int \varphi_1(x)\nu(\mathrm{d}x)\int \varphi_1(y)\mathrm{d}y + \sum_{n=2}^{\infty} \mathrm{e}^{-(\lambda_n-\lambda_1)t}\int \varphi_n(x)\nu(\mathrm{d}x)\int \varphi_n(y)\mathrm{d}y\right].\end{aligned} \tag{3.5}$$

Thus if we define

$$C_\nu(t) = \int \varphi_1(x)\nu(\mathrm{d}x)\int \varphi_1(y)\mathrm{d}y + \sum_{n=2}^{\infty} \mathrm{e}^{-(\lambda_n-\lambda_1)t}\int \varphi_n(x)\nu(\mathrm{d}x)\int \varphi_n(y)\mathrm{d}y, \tag{3.6}$$

Then $C_\nu(t) > 0$ since $\mathbb{P}_x(\tau > t) > 0$. The continuity of $C_\nu(t)$ on $(0, +\infty)$ is guaranteed by formula (1.6). And Lemma 3.3 gives

$$\lim_{t\to\infty} C(t) = \int \varphi_1(x)\nu(\mathrm{d}x)\int \varphi_1(y)\mathrm{d}y > 0.$$

By the continuity of the paths of $\{X_t\}_{t\geq 0}$, we have that

$$\lim_{t\to 0} \mathbb{P}_x(\tau > t) = 1 - \lim_{t\to 0} \mathbb{P}_x(\tau \leq t) = 1,$$

which gives that

$$\lim_{t\to 0} \mathbb{P}_\nu(\tau > t) = 1 \quad \text{and} \quad \lim_{t\to 0} C_\nu(t) = 1.$$ □

As a easy consequence, we have the following conditional exponential convergence for $\frac{1}{t}\int_0^t V(X_s)\mathrm{d}s$.

Corollary 3.5. *Under the same hypotheses on V and μ in the Theorem 3.1, for any number $t > 0$ and $b > 0$, we have that*

$$P_\nu\left[\frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a > b\middle|\tau > t\right] \leq \frac{\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}\right\|_2 \exp[-tI_V(a+b)]}{C_\nu(t)\mathrm{e}^{-\lambda_1 t}}, \tag{3.7}$$

$$P_\nu\left[\frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a < -b\middle|\tau > t\right] \leq \frac{\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}\right\|_2 \exp[-tI_V(a-b)]}{C_\nu(t)\mathrm{e}^{-\lambda_1 t}}. \tag{3.8}$$

Furthermore, adding the hypotheses on V and μ in the Theorem 2.6,

$$\lim_{t\to\infty} \frac{1}{t}\log P_\nu\left[\frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a > b\middle|\tau > t\right] \leq \lambda_1 - I_V(a+b) < 0, \tag{3.9}$$

$$\lim_{t\to\infty} \frac{1}{t}\log P_\nu\left[\frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a < -b\middle|\tau > t\right] \leq \lambda_1 - I_V(a-b) < 0. \tag{3.10}$$

Proof. A direct application of Theorem 3.1 and Proposition 3.1. □

The following is the quasi-ergodic theorem.

Theorem 3.6. *Let $V \in J$ be as in Theorem 2.2, and $a = \int V \mathrm{d}\mu_0$. Then for $\nu \in \mathscr{P}_1(D)$ with $\nu << \mu_1$ and $\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}\|_2 < \infty$,*

$$\lim_{t\to\infty} E_\nu \left[\left| \frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a \right| |\tau > t \right] = 0.$$

Proof. Given $\epsilon > 0$, let $\gamma > 0$ be as in Theorem 2.2. Denote

$$\Delta_t = \left| \frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a \right|.$$

Then from Theorems 3.1,2.2 we see that

$$\begin{aligned} E_\nu \left[\left| \frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a \right| |\tau > t \right] &= E_\nu \left[\Delta_t, ; \Delta_t \le 2\epsilon | \tau > t\right] + E_\nu \left[\Delta_t, ; \Delta_t > 2\epsilon, \tau > t\right] \\ &\le 2\epsilon + E_\nu \left[\Delta_t, ; \Delta_t > 2\epsilon, \tau > t\right] P_\nu^{-1}(\tau > t) \\ &\le 2\epsilon + \sum_{k=2}^{\infty} E_\nu \left[\Delta_t, k\epsilon \le \Delta_t < (k+1)\epsilon, \tau > t\right] P_\nu^{-1}(\tau > t) \\ &\le 2\epsilon + \sum_{k=2}^{\infty} (k+1)\epsilon P_\nu(\Delta_t \ge k\epsilon, \tau > t) P_\nu^{-1}(\tau > t) \\ &\le 2\epsilon + 2\epsilon \left\| \frac{\mathrm{d}\nu}{\mathrm{d}\mu_1} \right\|_2 \sum_{k=2}^{\infty} (k+1)\mathrm{e}^{-\lambda_1 t}\mathrm{e}^{-\gamma(k-1)\epsilon t} P_\nu^{-1}(\tau > t). \end{aligned}$$

Letting $t \to \infty$ and applying Proposition 3.1 we obtain that

$$\limsup_{t\to\infty} E_\nu \left[\left| \frac{1}{t}\int_0^t V(X_s)\mathrm{d}s - a \right| |\tau > t \right] \le 2\epsilon,$$

completing the proof. □

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