A NOTE ON UPPER-PATCHED GENERATORS FOR ARCHIMEDEAN COPULAS

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Abstract. The class of multivariate Archimedean copulas is defined by using a real-valued function called the generator of the copula. This generator satisfies some properties, including d-monotonicity. We propose here a new basic transformation of this generator, preserving these properties, thus ensuring the validity of the transformed generator and inducing a proper valid copula. This transformation acts only on a specific portion of the generator, it allows both the non-reduction of the likelihood on a given dataset, and the choice of the upper tail dependence coefficient of the transformed copula. Numerical illustrations show the utility of this construction, which can improve the fit of a given copula both on its central part and its tail.

1. Introduction

The class of Archimedean copulas is a well-known class of copulas, indexed by a function $\phi : \mathbb{R}^+ \to [0,1]$ called the generator of the copula (see [30]). In practice, depending on the problem at hand and the data, many procedures are available to choose a suitable generator, either by selecting a parametric one, or by trying non-parametric estimation of the latter (see among many references, [12,15,19–21,25,27] . A problem is that it can be difficult to fit both the global shape of the copula and its tail dependence.

In this article we aim at proposing a basic transformation of a given initial generator such that the transformed generator is guaranteed to be a valid Archimedean generator, such that the upper tail dependence can be chosen, and which ensures that the likelihood of a data-set is not reduced by the transformation. In other words, we are looking for valid distortions that are changing mainly the upper tail dependence behaviour of a given initial copula, *i.e.* that would be able to exhibit any chosen tail dependence coefficient, and that would also preserve the shape of a copula on its central part.

We focus on Archimedean copulas. We consider an initial d-dimensional Archimedean copula with a generator ϕ. This generator is a real function, $\phi : [0, \infty) \to [0, 1]$, non-increasing and continuous, such that $\phi(0) = 1$ and

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$\lim_{x \to +\infty} \phi(x) = 0$. From Theorem 2.2 in [30],

$$C(u_1, \ldots, u_d) := \phi(\phi^{-1}(u_1) + \ldots + \phi^{-1}(u_d))$$

is a $d-$dimensional copula if and only if its generator ϕ is $d-$monotone on $[0, \infty)$, where the d-monotonicity definition is recalled hereafter. In the following, a $d-$ monotone generator, will be called *valid generator* in the dimension d. We write ϕ^{-1} for the inverse function of a given generator ϕ, with the convention $\phi^{-1}(0) = \inf \{x \in \mathbb{R}^+ : \phi(x) = 0\}$. This function $\phi^{-1} : [0, 1] \to \mathbb{R}^+$ will be called an *inverse generator*.

Furthermore, we always write $f^{(k)}(t) = \frac{d^k}{dt^k} f(t)$ the derivative of order k of a $k-$differentiable function f.

Let us recall the definition of the d-monotony, as given in [30].

Definition 1.1 (*d*-monotone function).
A real function f is called $d-$monotone in (a, b), where $a, b \in \bar{\mathbb{R}}$ and $d \geq 2$, if it is differentiable there up to the order $d - 2$ and the derivatives satisfy

$$(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, 1, \ldots, d - 2,$$

for any $x \in (a, b)$ and further if $(-1)^{d-2} f^{(d-2)}$ is non-increasing and convex in (a, b). For $d = 1$, f is called $1-$monotone in (a, b) if it is nonnegative and non-increasing over (a, b).

If f has derivatives of all orders in (a, b) and if $(-1)^k f^{(k)}(x) \geq 0$, for any $x \in (a, b)$, then f is called *completely monotone*.

As announced, the aim of this work is to change a part of a given generator in order to change the tail dependence behaviour of the resulting copula. We thus recall hereafter some classical indicators of the tail dependence of a copula. Among them one can propose, for assessing the tail behaviour of a copula, the so-called tail dependence coefficients (TDC). In the general multivariate case, they can be expressed as follows (as adapted from definitions in [10, 28]).

Definition 1.2 (Multivariate tail dependence coefficients).
Assume that the considered copula C is the distribution of some random vector $\mathbf{U} := (U_1, \ldots, U_d)$. Denote $I = \{1, \ldots, d\}$ and consider two non-empty subsets $I_h \subset I$ and $\bar{I}_h = I \setminus I_h$ of respective cardinality $h \geq 1$ and $d - h \geq 1$. Let us define, for $u \in (0, 1)$,

$$\lambda_L^{I_h, \bar{I}_h}(u) = \mathbb{P}\left[U_i \leq u, i \in I_h \mid U_i \leq u, i \in \bar{I}_h\right]$$
$$\lambda_U^{I_h, \bar{I}_h}(u) = \mathbb{P}\left[U_i \geq u, i \in I_h \mid U_i \geq u, i \in \bar{I}_h\right].$$

A multivariate version of classical bivariate tail dependence coefficients, when the limits exist, is given by

$$\lambda_L^{I_h, \bar{I}_h} = \lim_{u \to 0^+} \lambda_L^{I_h, \bar{I}_h}(u), \quad \lambda_U^{I_h, \bar{I}_h} = \lim_{u \to 1^-} \lambda_U^{I_h, \bar{I}_h}(u).$$

If for all $I_h \subset I$, $\lambda_L^{I_h, \bar{I}_h} = 0$, (*resp.* $\lambda_U^{I_h, \bar{I}_h} = 0$) then we say $\mathbf{U}$ is lower tail independent (*resp.* upper tail independent).

Definition 1.3 (Multivariate tail dependence coefficients for Archimedean copulas). For Archimedean copulas the multivariate lower and upper tail dependence coefficients, when the limit exist, are respectively:

$$\lambda_L^{I_h, \bar{I}_h} = \lambda_L^{(h, d-h)} = \lim_{u \to 0^+} \lambda_L^{(h, d-h)}(u), \quad \lambda_U^{I_h, \bar{I}_h} = \lambda_U^{(h, d-h)} = \lim_{u \to 1^-} \lambda_U^{(h, d-h)}(u),$$

where $\lambda_L^{(h, d-h)}(u) = \frac{\phi(d \phi^{-1}(u))}{\phi((d-h) \phi^{-1}(u))}$ and $\lambda_U^{(h, d-h)}(u) = \frac{\sum_{i=0}^{d} (-1)^i C_d^i \phi(i \phi^{-1}(u))}{\sum_{i=0}^{d-h} (-1)^i C_{d-h}^i \phi(i \phi^{-1}(u))}$, for $u \in (0, 1)$, and where C_d^i and C_{d-h}^i are the binomial coefficients.

One can show that, when it exists, the lower tail dependence coefficient $\lambda_L^{I_h, \bar{I}_h}$ is linked with the asymptotic behaviour of $\phi(x)$ when $x \to +\infty$, and that $\lambda_U^{I_h, \bar{I}_h}$ is linked with the behaviour of $\phi(x)$ when $x \to 0$ (see *e.g.* [6,14]). Thus modifying a generator $\phi(x)$ for small values of x allows to change the upper tail dependence coefficient of the copula.

In the literature, some transformations of copulas correspond to the creation of a new Archimedean generator $T \circ \phi$, by composition of a distortion function T with an initial generator ϕ, see for example [18, 26, 31, 35]. In previous works, we investigated the use of hyperbolic distortions of Archimedean generator, for example in a logit scale, see [11,12] for the estimation of some transformations and [13] for illustrations with specific real data in the dimension $d = 5$. These constructions have the advantage of being readily invertible, *i.e.* an explicit parametric expression of both the generator and its inverse function are available. This eases in particular calculations that are done using the transformed copula. However, the verification of the validity of the transformed generator requires the calculation of the signs of the d first derivatives of the transformed generator (in the case it is d times differentiable), which can be difficult, especially when d is greater than two, when Faa Di Bruno's formula is involved. This is why we propose here another approach, based on a local transformation of an Archimedean copula. Notice that the approach here does not aim at leaving the Archimedean class of copulas: the distorted copula will stay within this class.

Applying a local patch to a copula appears in many constructions of the recent literature. It can be used to preserve an initial copula on a given subset of $[0,1]^d$, and to modify some given portions of the copula, typically boxes of the unit hypercube. Specific constructions involving changes in chosen boxes of $[0,1]^d$ are Shuffles of Min, Ordinal Sums, or Gluing methods (see *e.g.* [34]). A general construction of multivariate patchwork copulas is detailed in [17]. Another specific construction in the Extreme-Value copula framework is given in [1], where a portion of a copula is replaced by a GPD-copula, leading to a so-called Piecing-Together approach. An alternative distortion is also given in [7], where a portion of the copula is replaced by a comonotonic copula.

Here, our approach aims at preserving the Archimedean structure of an initial copula. It can be seen as a contribution to create more valid and flexible generators. Our approach aims at changing smoothly a univariate function, contrary to other patchwork approaches that remain multivariate.

Changing a generator on a interval $[0, t_0]$ will modify in a measurable way the C-measure of any subset of $[0,1]^d \setminus [0, \phi(t_0)]^d$, and one can determine at most $2^d - 1$ successive patches corresponding to this generator change. But conversely, it remains unclear how to construct a valid patchwork copula that would remain Archimedean, without directly patching the generator. As we will see, this Archimedean constraint implies more smoothness conditions (using derivatives up to an order $d - 2$). One drawback of this Archimedean setting is a lower flexibility than patchwork constructions, with boxes corners along the diagonal and differentiability constraints. One advantage is that it ensures smooth changes that are not always ensured by patchwork constructions. It also benefits from a more synthetic representation that depends only on a real function.

The structure of the paper is as follows. We give in Section 2 the definition of the proposed upper-patched generator. In particular, in Section 2.1 we show the automatic validity of this generator, which induces a valid distorted copula. In Section 2.2 we investigate the upper tail dependence behaviour of the distorted copula. In Section 2.3 we show that one can choose an upper tail dependence coefficient, without reducing the likelihood of a given data-set. At last, Section 3 gives some illustrations where a copula exhibiting the wrong dependence behaviour is transformed using an upper-patched generator. A comparison with alternative techniques of the recent literature is also given in this Section 3.

2. Upper-patched generator

In this section, we introduce a new construction, that we have chosen to call upper-patched generator (the adjective upper refers to the fact that the transformation will modify the upper tail dependence coefficient of the copula, the word patched is because it makes only a local change on the generator). We show that this construction necessarily yields a valid Archimedean generator in a chosen dimension $d \geq 2$. We also show that

TABLE 1. Some examples of non-strict Archimedean generators from Table 4.1 in [32]. λ_L and λ_U are here the classical bivariate tail dependence coefficients and d_0 the end-point of the generator.

Copula	$\phi_D(t)$	$\phi_D^{-1}(t)$	parameter θ	d_0	λ_L	λ_U
4.1.2	$1 - t^{1/\theta}$	$(1-t)^\theta$	$\theta \in [1, \infty)$	1	0	$2 - 2^{\frac{1}{\theta}}$
4.1.7	$\frac{1}{\theta}(\theta + \exp(-t) - 1)$	$-\ln(\theta t + (1 - \theta))$	$\theta \in (0, 1]$	$-\ln(-\theta + 1)$	0	0
4.1.8	$\frac{t-1}{-\theta t - t - 1}$	$\frac{1-t}{1+(1+\theta) t}$	$\theta \in [1, \infty)$	1	0	0
4.1.11	$(2 - \exp(t))^{\frac{1}{\theta}}$	$\ln(2 - t^\theta)$	$\theta \in (0, 1/2]$	$\ln(2)$	0	0
4.1.15	$(1 - t^{\frac{1}{\theta}})^\theta$	$(1 - t^{1/\theta})^\theta$	$\theta \in [1, \infty)$	1	0	$2 - 2^{\frac{1}{\theta}}$
4.1.18	$\frac{\ln(t) + \theta}{\ln(t)}$	$\exp(\frac{\theta}{t-1})$	$\theta \in [2, \infty)$	$\exp(-\theta)$	0	1
4.1.21	$1 - [-(-t+1)^\theta + 1]^{1/\theta}$	$1 - [1 - (1-t)^\theta]^{1/\theta}$	$\theta \in [1, \infty)$	1	0	$2 - 2^{\frac{1}{\theta}}$
4.1.22	$(-\sin(t) + 1)^{1/\theta}$	$\arcsin(1 - t^\theta)$	$\theta \in (0, 1]$	$\frac{\pi}{2}$	0	0

any upper tail dependence coefficient can be obtained using this construction, without reducing the likelihood of a given data-set.

We will distort the initial Archimedean copula C having generator ϕ. To this aim we use another d-dimensional Archimedean copula having generator ϕ_D. Both generators will be assumed to be valid. We first give some conditions on these generators ϕ and ϕ_D.

Definition 2.1 (Initial and distortion generators). Let $t_0 \in (0, +\infty)$ be a given real value, and $d \geq 2$ be a given integer dimension. The initial generator $\phi : \mathbb{R}^+ \to [0, 1]$ and distortion generator $\phi_D : \mathbb{R}^+ \to [0, 1]$ are such that

- ϕ is a valid d-dimensional Archimedean generator, thus d-monotone on $\mathbb{R}^+$.
- ϕ_D is a valid d-dimensional Archimedean generator, thus d-monotone on $\mathbb{R}^+$, such that kth derivatives $\phi_D^{(k)}(d_0) = 0$, for all $k = 0, \ldots, d - 2$ and for some $d_0 \in (0, t_0]$. This implies in particular that ϕ_D is a non-strict generator with end-point $d_0 = \inf \{t \in \mathbb{R}^+ : \phi_D(t) = 0\} \leq t_0$.

Some examples of non-strict generators from Table 1 in [32] are given in Table 1. Remark that, when a generator ϕ_D^0 has an end-point d_0, $\phi_D(t) = \phi_D^0(\frac{d_0}{t_0} t)$ is a valid d-dimensional generator with end-point t_0. Remark also that, for instance, Copula 4.1.11 in Table 1 satisfies: $\phi_D^{(k)}(d_0) = 0$, for all $k = 0, \ldots, d - 2$ with $d_0 = \ln(2)$.

Definition 2.2 (Patched generator).

Consider an initial generator ϕ and distortion one ϕ_D as in Definition 2.1, where d_0 is the end-point of ϕ_D. Let $t_0 \geq d_0$. A patched generator $\widetilde{\phi}$, built from an initial generator ϕ and using a distortion generator ϕ_D, is given by

$$\widetilde{\phi}(t) = \begin{cases} p_{d-1}(t) + (1 - p_{d-1}(0))\phi_D(t), & \text{if} \quad t < t_0, \\ \phi(t), & \text{if} \quad t \geq t_0; \end{cases} \tag{2.1}$$

for all $t \in \mathbb{R}^+$, where $p_{d-1}(t) = \sum_{i=0}^{d-1} \frac{\phi_+^{(i)}(t_0)}{i!}(t - t_0)^i$ is the Taylor expansion of $\phi(t)$ at order $d - 1$ (with a right derivative at order $d - 1$, since ϕ is $d - 2$ times differentiable and $d - 1$ time right differentiable, by convexity of $\phi^{(d-2)}$).

The validity of this patched generator will be shown in Proposition 2.4 below. Its principle is that the initial generator remains unchanged on $[t_0, +\infty]$, so that the change acts only on the upper tail dependence behavior
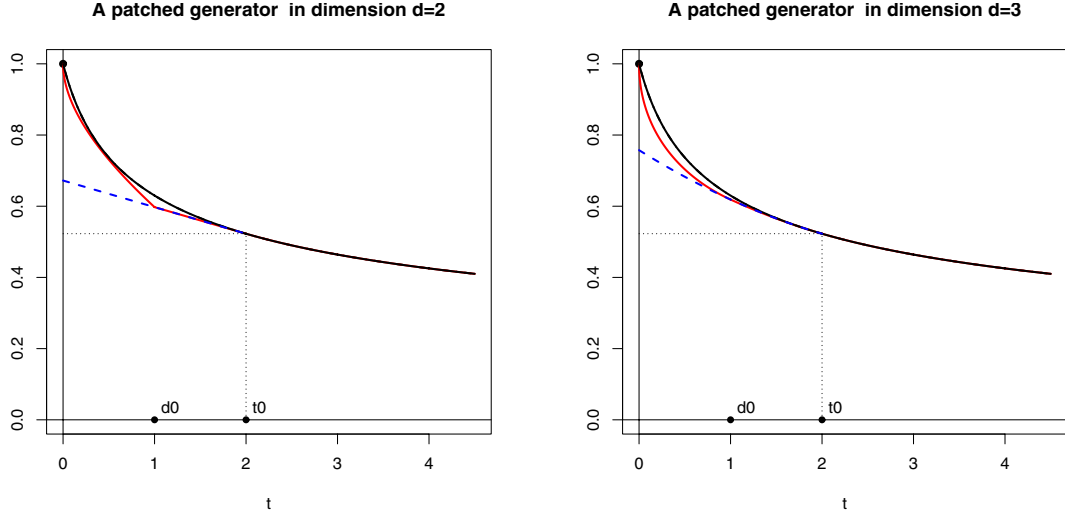
FIGURE 1. *Left panel*: a patched generator $\widetilde{\phi}$ (*red line*) in the case where $d = 2$. We consider here the distortion generator $\phi_D(t) = 1 - t^{\frac{1}{\theta_D}}$ with $\theta_D = 2$ and end-point $d_0 = 1$ (see Copula 4.1.2 in Tab. 1). *Right panel*: a patched generator $\widetilde{\phi}$ (*red line*) in the case where $d = 3$. Here $\phi_D(t) = (1 - t^{\frac{1}{\theta_D}})^{\theta_D}$ with $\theta_D = 2$ and end-point $d_0 = 1$ (see Copula 4.1.15 in Tab. 1). In both plots, the function $p_{d-1}(t)$ for $t \leq t_0$ is the blue dashed tangent line of $\phi(t)$ at abscissa $t_0 = 2$. The considered initial Clayton generator $\phi(t) = (1 + \theta\,t)^{\frac{-1}{\theta}}$ with $\theta = 3$ is represented by the black curve in both panels. (Color online)

the copula. On $[0, t_0]$, due to constraints on derivatives at $t = t_0$ and convexity conditions, the patched generator must be above the Taylor expansion $p_{d-1}(t)$, so that we just add a new distortion generator ϕ_D, with suitable normalisation constant $1 - p_{d-1}(0)$ to ensure that $\widetilde{\phi}(0) = 1$.

Using a non-strict generator with $\phi_D(t_0) = 0$ ensures the continuity of $\widetilde{\phi}$, but the supplementary constraints on derivatives of ϕ_D, in Definition 2.1 , are needed in order to ensure the d-monotonicity of $\widetilde{\phi}$, as discussed in Proposition 2.4 below.

We give in Figure 1 an illustration of the patched generator $\widetilde{\phi}$ proposed in Definition 2.2 when $d = 2$ (*left panel*) and $d = 3$ (*right panel*). As one can see in Figure 1 (*left panel*), the patched generator in $d = 2$ is continuous decreasing and convex, so that in particular it can be used as a valid 2-dimensional generator. One can also verify that the proposed patched generator $\widetilde{\phi}$ in Figure 1 (*right panel*) is a valid 3-dimensional generator. The absolute continuity property of the patched copulas induced by the two patched generators in Figure 1 will be studied in Remark 2.6.

Notice that on the domain $[0, t_0]$, as $t \leq t_0$ and as ϕ is d-monotone, $\phi^{(i)}(t)(t - t_0)^i \geq 0$, so that p_{d-1} can be written as a sum of positive terms

$$p_{d-1}(t) = \sum_{i=0}^{d-1} \left| \frac{\phi^{(i)}(t_0)}{i!}(t - t_0)^i \right|.$$

As ϕ is d-monotone, one can also check that

$$p_{d-1}^{(k)}(t) = (-1)^k 1_{\{d-1 \geq k\}} \left| \sum_{j=0}^{d-1-k} \frac{\phi^{(j+k)}(t_0)}{j!}(t - t_0)^j \right|,$$

so that p_{d-1} is also d-monotone and even completely monotone (see Def. 1.1).

The patched generator $\widetilde{\phi}$ was constructed in two steps. Firstly, the shape of the initial generator ϕ was preserved on $[t_0, \infty)$. Secondly, the generator ϕ was replaced by another function, say f, on $[0, t_0)$. The following result analyses the bounds of a candidate function f, in order to preserve the d-monotone shape of the obtained patched generator $\widetilde{\phi}$. As a consequence of this result, it is natural to build $\widetilde{\phi}$ from $p_{d-1}(t)$, as $p_{d-1}(t)$ can be seen as the smallest d-monotone candidate for the prolongation of ϕ on $[0, t_0]$.

Proposition 2.3 (Smallest and largest suitable candidate).

Let ϕ be an initial d-dimensional generator as in Definition 2.1. *Let f be a valid d-dimensional generator such that $f(t) = \phi(t)$ for all $t \geq t_0$ and such that $f(0) = 1$. It holds that*

(i). $f(t) \geq p_{d-1}(t), \quad t \in [0, t_0]$.

(ii). $f(t) \leq p_{d-1}(t) + (1 - p_{d-1}(0)) \cdot \left(\dfrac{t_0 - t}{t_0} \right), \quad t \in [0, t_0]$.

Thus $f(t) \leq p_{d-1}(t) + (1 - p_{d-1}(0)) \cdot \phi_D^{\max}(t)$, where $\phi_D^{\max}(t) = (\frac{t_0-t}{t_0})_+$ is the function associated to the lower Fréchet–Hoeffding bound. In the particular case where $d = 2$, f is necessarily below the line joining the two points $(0, 1)$ and $(t_0, \phi(t_0))$.

Proof. We firstly prove the (i). item. The function f is a valid d-dimensional generator, so that f is continuous, non-increasing and $d-$monotone on $\mathbb{R}^+$, thus $(d - 2)$ times differentiable, in particular at abscissa t_0. As a consequence, $f^{(k)}(t_0) = \phi^{(k)}(t_0)$ for all $k = 0, \ldots, d - 2$.

Let $I = [t_0, t_0 + \varepsilon]$. Remark that, since $\phi^{(d-2)}$ is a convex function, then in particular it is right-differentiable in t_0. As $f(t) = \phi(t)$ for any $t \geq t_0$, then $\lim_{a \to t_0^+, a \in I} f^{(d-1)}(a) = \phi_+^{(d-1)}(t_0)$.

Let us now consider $a \in I$. Then f can be written by Taylor's theorem with the Lagrange form of the remainder, as

$$f(t) = \sum_{k=0}^{d-2} \frac{(t-a)^k}{k!} f^{(k)}(a) + \frac{(t-a)^{d-1}}{(d-1)!} f^{(d-1)}(\xi_t), \quad \text{for } t \in [0, t_0],$$

where $\xi_t \in [t, a]$. As f is d-monotone, then $(-1)^{d-2} f^{(d-2)}$ is non-negative (also non-increasing and convex), which implies $(-1)^{d-1} f^{(d-1)}$ is non-increasing. Thus

$$f(t) \geq \sum_{k=0}^{d-2} \frac{(t-a)^k}{k!} f^{(k)}(a) + \frac{(t-a)^{d-1}}{(d-1)!} f^{(d-1)}(a).$$

Recall that $f^{(k)}(t_0) = \phi^{(k)}(t_0)$, $k = 0, \ldots, d - 2$. Now, by passing to the limit for $a \to t_0^+$ in the last inequality and by recalling that $p_{d-2}(t) = \sum_{k=0}^{d-2} \frac{(t-t_0)^k}{k!} \phi^{(k)}(t_0)$, one gets,

$$f(t) \geq p_{d-2}(t) + \frac{(t-t_0)^{d-1}}{(d-1)!} \phi_+^{(d-1)}(t_0), \quad \text{for } t \in [0, t_0].$$

Hence the first result. We now prove the second statement of this result (see item (ii).). Let $t \in [0, t_0]$. As f is d-monotone and thus convex, $f(t) \leq \frac{t}{t_0} f(t_0) + (1 - \frac{t}{t_0}) f(0)$. As $f(0) = 1$ and $p_0(0) = f(t_0)$, one easily shows that $f(t) \leq p_0(0) + (1 - p_0(0)) \frac{t_0-t}{t_0}$. Now consider a given integer $k \leq d - 2$, so that $p_{k+1}(t)$ exists, and define the assumption H_k: for all $t \in [0, t_0]$, $f(t) \leq p_k(0) + (1 - p_k(0)) \frac{t_0-t}{t_0}$. We have shown that H_0 holds. Now, define the function $g_{k+1}(t)$ such that $f(t) = p_{k+1}(0) + (1 - p_{k+1}(0)) g_{k+1}(t)$, for all $t \in [0, t_0]$. If H_k holds, we easily get $(1 - p_{k+1}(0)) g_{k+1}(t) \leq p_k(0) - p_{k+1}(0) + (1 - p_k(0)) \frac{t_0-t}{t_0}$ for all $t \in [0, t_0]$. Defining $\Delta_k(0) = p_{k+1}(0) - p_k(0)$, we get $(1 - p_{k+1}(0))(g_{k+1}(t) - \frac{t_0-t}{t_0}) \leq -\Delta_k(0) + \frac{t_0-t}{t_0} \Delta_k(0)$, and finally $(1 - p_{k+1}(0))(\frac{t_0-t}{t_0} - g_{k+1}(t)) \geq \Delta_k(0) \frac{t}{t_0}$. One can check that $p_{k+1}(0) \leq 1$ and, by d-monotonicity of ϕ, $\Delta_k(0) \geq 0$. It follows that $\frac{t_0-t}{t_0} - g_{k+1}(t) \geq 0$, so that H_{k+1} holds. By induction, the result holds. $\square$

Notice that in the particular case where $d = 2$, f is necessarily below the line joining the two points $(0, 1)$ and $(t_0, \phi(t_0))$.

From Proposition 2.3, it follows that $p_{d-1}(t)$ is the smallest candidate function for $\widetilde{\phi}(t)$ on $[0, t_0]$, and that it is thus natural to replace $\phi(t)$ on $[0, t_0]$ by the sum of $p_{d-1}(t)$ and another positive d-monotone function (but other constructions could be imagined, for example using a product instead of a sum). Notice that if ϕ_D has an end-point d_0 that tends to 0, the integrated distance $\int_0^{t_0} \left| \widetilde{\phi}(t) - p_{d-1}(t) \right| \mathrm{d}t$ between $\widetilde{\phi}$ and its lower bound can be as small as desired. As ϕ is also a d-monotone function, another consequence of previous Proposition 2.3 is that $p_{d-1}(t) \leq \phi(t)$, so that in particular when $t = 0$ we get $p_{d-1}(0) \leq 1$.

We will see in the further section that, due to Proposition 2.4, the quantity $p_{d-1}(t) + (1 - p_{d-1}(0)) \cdot \phi_D^{\max}(t)$ is a $d-$monotone function (at least in dimension $d = 2$), then the patched generator $\widetilde{\phi}$ can be equal to this upper bound.

2.1. Validity

We give here conditions under which the patched generator is a valid Archimedean copula generator. Furthermore, we study the absolutely continuity of the associated patched copula. Remark that this latter property is crucial for statistical inference.

Proposition 2.4 (Validity of the patched generator). *Consider two generators ϕ and ϕ_D as in Definition 2.1, and let $\widetilde{\phi}$ be a patched distorted generator as in Definition 2.2. Then $\widetilde{\phi}$ is a valid d-dimensional Archimedean generator.*

Proof. Let n be a given order, $2 \leq n \leq d+1$. Assume that ϕ and ϕ_D are n-monotone, and that $\phi_D^{(k)}(t_0) = 0$ for all $k = 0, \ldots, n-2$. One first shows that, up to order $n-2$, $\widetilde{\phi}$ is differentiable on $(0, +\infty)$. The calculation of the k-derivatives of the patched generator, for $k \geq 1$, gives

$$\widetilde{\phi}^{(k)}(t) = 1_{\{t \geq t_0\}} \phi^{(k)}(t) + 1_{\{t < t_0\}} \left(p_{d-1}^{(k)}(t) + (1 - p_{d-1}(0)) \phi_D^{(k)}(t) \right) \tag{2.2}$$

Then, when $t > t_0$, this is obvious as ϕ is n-monotone. When $t < t_0$, this is clear since both p_{d-1} and ϕ_D are n-monotone. Now, as $n - 2 \leq d - 1$, $p_{d-1}^{(k)}(t_0) = \phi^{(k)}(t_0)$ for all $k \leq n - 2$. Furthermore, $\phi_D^{(k)}(t_0) = 0$ for all $k = 0, \ldots, n-2$, so that $\widetilde{\phi}^{(k)}(t_0) = \phi^{(k)}(t_0)$. In particular when $k = 0$, $\widetilde{\phi}$ is clearly continuous. We have shown that for all $t \in (0, +\infty)$, the kth derivative $\widetilde{\phi}^{(k)}(t)$ exists for all $k = 0, \ldots, n-2$.
One then show that $(-1)^k \widetilde{\phi}^{(k)} \geq 0$, for all $k = 0, \ldots, n-2$. Again, this is clear for $t \geq t_0$, and for $t < t_0$ as $\widetilde{\phi}$ is the sum of n-monotone functions, recalling that $1 - p_{d-1}(0)$ is a non-negative coefficient.

It remains to be shown that $(-1)^{n-2} \widetilde{\phi}^{(n-2)}$ is a non-increasing convex function. From n-monotonicity, functions $(-1)^{n-2} p_{d-1}^{(n-2)}$, $(-1)^{n-2} \phi_D^{(n-2)}$ and $(-1)^{n-2} \phi^{(n-2)}$ are all three non-increasing and convex functions. As $\widetilde{\phi}^{(n-2)}$ is continuous, it follows that $(-1)^{n-2} \widetilde{\phi}^{(n-2)}$ is non-increasing. It is also clear that $(-1)^{n-2} \widetilde{\phi}^{(n-2)}$ is convex on $[0, t_0]$ and convex on $[t_0, +\infty)$. At the junction $t = t_0$ of these two convex functions $(-1)^{n-2} \widetilde{\phi}^{(n-2)}(t_0) = (-1)^{n-2} p_{d-1}^{(n-2)}(t_0)$, and one can show that $(-1)^{n-2} \widetilde{\phi}^{(n-2)}(t) \geq (-1)^{n-2} p_{d-1}^{(n-2)}(t)$ for all t (see Prop. 2.3). The two convex portions are bounded below by a convex function which is reached at t_0, and thus $(-1)^{n-2} \widetilde{\phi}^{(n-2)}(t)$ is convex. Finally, we have shown that $\widetilde{\phi}$ is a continuous n-monotone function with $\widetilde{\phi}(0) = 1$ and $\widetilde{\phi}(+\infty) = 0$. In Proposition 2.4, assumptions hold true for $n = d$, hence $\widetilde{\phi}$ is a valid $d-$dimensional Archimedean generator. $\square$

Proposition 2.5 (Absolute Continuity). *Consider two generators ϕ and ϕ_D as in Definition 2.1, and let $\widetilde{\phi}$ be a patched distorted generator as in Definition 2.2. Consider one supplementary order differentiability condition: assume that ϕ and ϕ_D are $(d+1)$-monotone on $\mathbb{R}^+$. Furthermore, assume that $\phi_D^{(k)}(t_0) = 0$ for all $k = 1, \ldots, d-1$, then $\widetilde{\phi}$ is a valid d-dimensional Archimedean generator which induces an absolutely continuous copula.*

Proof. The proof of Proposition 2.4 applies by using $n = d + 1$ instead of $n = d$. Thus, $\widetilde{\phi}$ is valid Archimedean generator which is furthermore $(d + 1)$-monotone. Using Proposition 4.1 (ii) in [30], the corresponding patched copula $\widetilde{C}$ is thus absolutely continuous. □

Remark 2.6 (Singular component)**.** In the dimension $d = 2$, consider the patched generator $\widetilde{\phi}$ of Figure 1, *left panel*. One can check that all conditions in Proposition 2.5 are not fulfilled, so that the copula may contain a singular component. In particular the considered distortion generator $\phi_D(t) = 1 - t^{\frac{1}{\theta_D}}$ with $\theta_D = 2$ and endpoint $d_0 = 1$ (see Copula 4.1.2 in Tab. 1) is not $3-$monotone on $(0, t_0)$. Consider the curve of level $\alpha = \widetilde{\phi}(d_0)$, *i.e.* the set $\partial L_\alpha = \{(u, v) : \widetilde{C}(u, v) = \alpha\}$. One can easily show that the inverse generator $\widetilde{\phi}^{-1}$ is discontinuous at abscissa α, and thus using Theorem 4.3.3. in [32], that the $\widetilde{C}$-measure of the level line ∂L_α is strictly positive. Furthermore, this level line can be shown to have 0 Lebesgue measure. It is thus clear that despite $\widetilde{\phi}$ is a valid bivariate Archimedean generator, the induced copula is not absolutely continuous. The interested reader is also referred to Example 4.6. in [32].

Conversely, the patched generator $\widetilde{\phi}$ presented in Figure 1, *right panel* is a valid trivariate generator which induces an absolutely continuous copula.

2.2. Upper tail dependence

Here, we investigate the upper tail dependence of an upper-patched generator.

In the following, we denote by $f \in \mathcal{RV}_r(x_0)$, $r \in \mathbb{R}$, a measurable function $f : (0, \infty) \to (0, \infty)$ which is regularly varying at x_0 with index r, where typically x_0 stands for 0, 1 or $+\infty$ (see *e.g.* [2]). In particular, we say that f is regularly varying at 0 with index r if

$$f \in \mathcal{RV}_r(0) \quad \Leftrightarrow \quad \lim_{x \to 0^+} \frac{f(s\,x)}{f(x)} = s^r, \ \forall \ s > 0.$$

Now, it is well known that the upper tail behaviour of a copula is directly linked with the behavior of ϕ in the neighbourhood of the attachment point $\phi(0) = 1$ (see *e.g.* [6,14]). The following proposition shows that the regular variation properties of the proposed patched generator $\widetilde{\phi}$ are directly linked to those of the distortion generator ϕ_D.

Proposition 2.7 (Regular index for the patched generator)**.** *Consider two generators ϕ and ϕ_D as in Definition 2.1, and let $\widetilde{\phi}$ be a patched distorted generator as in Definition 2.2. If $1 - \phi_D \in \mathcal{RV}_{1/\rho}(0)$ with $\rho \in [1, +\infty]$, then $1 - \widetilde{\phi} \in \mathcal{RV}_{1/\rho}(0)$.*

Furthermore, the patched upper tail dependence coefficient is given by

$$\widetilde{\lambda}_U^{(h, d-h)} = \begin{cases} 0, & \text{if} \quad \rho = 1, \\ \dfrac{\sum_{i=1}^d (-1)^i \, C_d^i \cdot i^{1/\rho}}{\sum_{i=1}^{d-h} (-1)^i \, C_{d-h}^i \cdot i^{1/\rho}}, & \text{if} \quad \rho \in (1, +\infty), \\ 1, & \text{if} \quad \rho = +\infty. \end{cases} \tag{2.3}$$

Equation (2.3) *in the particular bivariate case (i.e., $d = 2$ and $h = 1$) is given by*

$$\widetilde{\lambda}_U^{(1,1)} = \begin{cases} 0, & \text{if} \quad \rho = 1, \\ 2 - 2^{1/\rho}, & \text{if} \quad \rho \in (1, +\infty), \\ 1, & \text{if} \quad \rho = +\infty. \end{cases} \tag{2.4}$$

Proof. Let $x > 0$ and $x \le t_0$. We prove that $1 - \widetilde{\phi} \circ I \in \mathcal{RV}_{-1/\rho}(+\infty)$. To this aim, we write:

$$
\begin{aligned}
1 - \widetilde{\phi}(x) &= 1 - p_{d-1}(x) - (1 - p_{d-1}(0))\phi_D(x) \\
&= 1 - p_{d-1}(x) - (1 - p_{d-1}(0))\phi_D(x) + (1 - p_{d-1}(0)) - (1 - p_{d-1}(0)) \\
&= 1 - p_{d-1}(x) + (1 - p_{d-1}(0))(1 - \phi_D(x)) - (1 - p_{d-1}(0)) \\
&= p_{d-1}(0) - p_{d-1}(x) + (1 - p_{d-1}(0))(1 - \phi_D(x)).
\end{aligned}
$$

Denote $\Delta(x) := p_{d-1}(0) - p_{d-1}(x)$. Remark that $\Delta(x)$ is a positive function. Furthermore, if $\phi'(t_0) \ne 0$, $\Delta \in \mathcal{RV}_1(0)$. Let $I(x) := \frac{1}{x}$. Notice that $1 - \widetilde{\phi} \circ I$ can be written as a sum of positive functions, *i.e.*, $1 - \widetilde{\phi}\left(\frac{1}{x}\right) = f_1(x) + f_2(x)$, with $f_1(x) = p_{d-1}(0) - p_{d-1}\left(\frac{1}{x}\right)$ and $f_2(x) = (1 - p_{d-1}(0))(1 - \phi_D\left(\frac{1}{x}\right))$. Furthermore, as provided before, $f_1 \in \mathcal{RV}_{-1}(+\infty)$ and by assumption $f_2 \in \mathcal{RV}_{-1/\rho}(+\infty)$. Then, by using Proposition B.1.9 in [9], we obtain that $1 - \widetilde{\phi} \circ I \in \mathcal{RV}_{\max(-1, -1/\rho)}(+\infty)$. Hence the first result. As a consequence, by using [6,14] we get the upper tail dependence coefficient for the patched generator. The bivariate case comes down from the application of Theorem 4.4 in [24]. Hence the results. $\qquad\square$

2.3. Likelihood

We now show, on a given data-set, that it is possible to choose a given upper tail dependence behaviour, without reducing the likelihood of the copula on pseudo-observations.

The patched generator $\widetilde{\phi}$ is equal to ϕ on a set $[t_0, +\infty)$, so that the resulting copula is unchanged on a domain D_0 that the following result gives explicitly. As a consequence of previous Proposition 2.7, one can easily choose any upper tail coefficient of a copula $\widetilde{C}$. This can be done without modifying the generator ϕ on a set $[t_0, +\infty)$, *i.e.* without modifying the initial copula C on a domain $D_0 = [0, \phi(t_0)]^d$. It follows that whatever the pseudo-observations in $(0,1)^d$, one can find t_0 such that all pseudo-observations are included in D_0. For such a value t_0 the likelihood of the data using copula $\widetilde{C}$ is equal to the likelihood using copula C. The following result indicates that it is thus possible to transform copula C into $\widetilde{C}$, in order to choose its upper tail coefficient, without reducing the likelihood function on a given data-set.

Proposition 2.8 (Likelihood improvement)**.** *Let C be the Archimedean copula associated to an initial generator ϕ. Consider a patched generator $\widetilde{\phi}$, build from the initial generator ϕ and the distortion one ϕ_D as in Definition 2.2. Denote $\widetilde{\phi}^{-1}$ the inverse function of $\widetilde{\phi}$ and let $\widetilde{C}(u_1, \dots, u_d) = \widetilde{\phi}(\widetilde{\phi}^{-1}(u_1) + \dots + \widetilde{\phi}^{-1}(u_d))$, then*

$$
\widetilde{C}(u_1, \dots, u_d) = C(u_1, \dots, u_d), \quad \text{for all } (u_1, \dots, u_d) \in [0, \phi(t_0)]^d. \tag{2.5}
$$

Let $\lambda_U^{(h,d-h)}$ the upper tail coefficient of C and $\mathcal{L}$ the likelihood function under C on given (pseudo-)observations $\mathbf{U}_1, \dots, \mathbf{U}_n \in (0,1)^d$. Let $\widetilde{\lambda}_U$ and $\widetilde{\mathcal{L}}$ be corresponding quantities using the patched generator $\widetilde{\phi}$. Let $\lambda_0 \in [0,1]$ be a chosen upper tail coefficient, one can find $t_0 \in \mathbb{R}^{+*}$ and an associated non-strict generator ϕ_D such that

$$
\widetilde{\lambda}_U^{(h,d-h)} = \lambda_0 \quad \text{and} \quad \widetilde{\mathcal{L}} \ge \mathcal{L}. \tag{2.6}
$$

Proof. The generator $\widetilde{\phi}$ is equal to ϕ on a set $[t_0, +\infty)$. Equivalently, the inverse generator $\widetilde{\phi}^{-1}$ is equal to ϕ^{-1} on $[0, \phi(t_0)]$, where t_0 can be close to 0 and $\phi(t_0)$ close to 1. A sufficient condition for the copula to be unchanged is that all $u_i \in [0, \phi(t_0)]$, so that all $\widetilde{\phi}^{-1}(u_i) = \phi^{-1}(u_i)$. As ϕ^{-1} is decreasing, the minimal value of $\phi^{-1}(u_i)$ when $u_i \le \phi(t_0)$ is t_0, so that in all cases $\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d) \ge d\,t_0 \ge t_0$, ensuring that $\widetilde{\phi}(\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d)) = \phi(\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d))$. One can find a point t_0 such that $\widetilde{\mathcal{L}} = \mathcal{L}$ on a domain $D_0 = [0, \phi(t_0)]^d$. Since equation (2.3) gives a bijection between $\rho \in [0, +\infty]$ and $\widetilde{\lambda}_U^{(h,d-h)} \in [0,1]$, we can find ϕ_D such that $\widetilde{\lambda}_U^{(h,d-h)} = \lambda_0$, whatever the choice of t_0. Furthermore, from equation (2.5), we can choose t_0 such that $\widetilde{\mathcal{L}} = \mathcal{L}$: the likelihood is ensured to be not reduced, but other choices of t_0 can possibly improve the likelihood. Hence the result. $\qquad\square$

3. Numerical illustrations

3.1. Proposed Patched generator

In the following example we build some valid patched generators $\widetilde{\phi}$ under assumptions of Definition 2.1 (see Prop. 2.4) and we investigate the tail properties of the obtained distorted Archimedean copulas by using the tail concentration functions (TCF).

Example 3.1 (Tail concentration function for some valid patched generators)**.**
We consider the tail concentration function (TCF), for $u \in [0, 1]$, $h \geq 1$ and $d - h \geq 1$,

$$\lambda_{LU}^{(h,d-h)}(u) = \mathbf{1}_{\{u \leq 1/2\}} \lambda_L^{(h,d-h)}(u) + \mathbf{1}_{\{u > 1/2\}} \lambda_U^{(h,d-h)}(u), \tag{3.1}$$

where $\lambda_L^{(h,d-h)}$ and $\lambda_U^{(h,d-h)}$ are as in Definition 1.3 (see for instance [16, 36]).
We study both the bivariate ($d = 2$) and the trivariate ($d = 3$) case by choosing distortion generators ϕ_D satisfying the assumption of Definition 2.1. Then we obtain valid $d-$Archimedean patched generators (see Prop. 2.4).
In both dimensional situations, we start from an initial Clayton generator $\phi(t) = (1 + \theta\, t)^{\frac{-1}{\theta}}$, with $\theta = 3$.

– In the case $d = 2$ and $h = 1$ we choose the distortion generator $\phi_D(t) = 1 - t^{\frac{1}{\theta_D}}$ with $\theta_D = 2$ and end-point $d_0 = 1$ (see Copula 4.1.2 in Tab. 1). The obtained tail concentration function is displayed in Figure 2 (first panel).

– In the case $d = 3$ we take the distortion generator $\phi_D = (1 - t^{\frac{1}{\theta_D}})^{\theta_D}$ with $\theta_D = 2$ and end-point $d_0 = 1$ (see Copula 4.1.15 in Tab. 1). The obtained tail concentration function for $d = 3$ and $h = 1$ (*resp.* $d = 3$ and $h = 2$) is displayed in the second panel (*resp.* third panel) of Figure 2.

As one can see in Figure 2, the tail concentration function is firstly identical to the one of the initial Clayton generator ϕ, then shifts toward the one of the distortion generator ϕ_D. This is the desired feature where one aims at changing an initial generator in order to choose the final tail dependence behavior. It is also noticeable that the patched generator $\widetilde{\phi}$, despite its validity, generates a large variety of shapes of the tail concentration function, contrary to most classical copulas where the concentration function is first increasing and then decreasing. For a detailed discussion of this aspect, the interested reader is referred to Section 3 in ([14]).

Example 3.2 (Tail improvement)**.**
In practical applications, the proposed initial model never fits perfectly to the reality, it is thus natural to try improving this initial model by using both the log-likelihood and the estimated tail behavior on the data. This can be particularly useful in the case of model misspecification. In this example we generate a data-set from a Gumbel model and we have deliberately chosen to improve an initial misspecified Frank model. We show hereafter that empirically, one can improve both classical criterions (as log-likelihood, AIC or BIC) and estimated tail dependence on the data. As we will see, it is noticeable that the results obtained by the patched model are quite good compared to the true (supposed unknown) Gumbel model.

i) Initial context. We present here the initial data and the classical estimation procedure for the chosen initial model. We sample 2000 observations from a bivariate Gumbel copula with parameter $\theta = 2$. We fit an initial Frank copula on this data-set based on the ML estimator. We estimate the tail concentration function (TCF) in (3.1) by using the empirical rank based estimator of $\lambda_L^{(h,d-h)}(u)$ and $\lambda_U^{(h,d-h)}(u)$, with $d = 2$ and $h = 1$ (the interested reader is referred to [33]).
In Figure 3, one illustrates the empirical estimated TCF based on the data, the theoretical TCF from the initial Frank generator $\phi_{\widehat{\theta}}$ and from the Gumbel copula generator with ML estimated parameters. Furthermore we represent in Figure 3, the Monte Carlo confidence intervals of 300 empirical TCFs from the initial Frank copula with generator $\phi_{\widehat{\theta}}$. In particular, we empirically estimate the upper tail dependence coefficient $\widehat{\lambda}_U^{(h,d-h)}$, with $d = 2$ and $h = 1$.
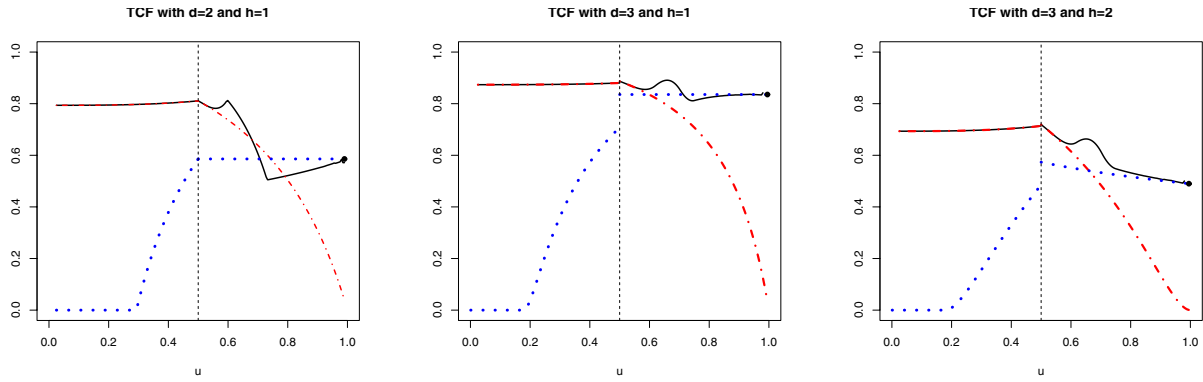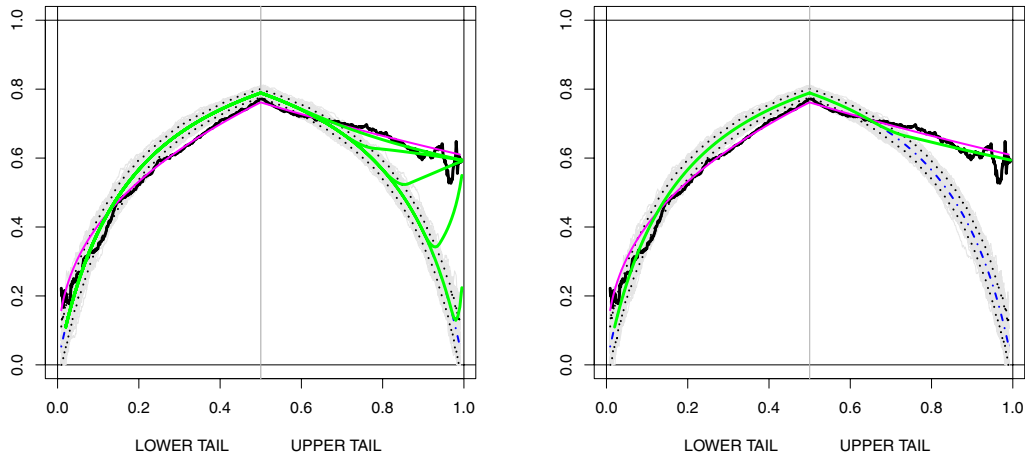
FIGURE 2. The tail concentration functions (TCF) are represented respectively in full black line for the patched generator $\widetilde{\phi}$, in blue dotted line for the distortion generator ϕ_D and in red dashed line for initial Clayton generator ϕ. *First panel*: $d = 2$, $h = 1$; *Second panel*: $d = 3$, $h = 1$; third panel $d = 3$, $h = 2$. (Color online)

(a) with arbitrary chosen u^*'s values.
From the top $u^* = 0.55, 0.7, 0.8, 0.9, 0.97$

(b) with max. likelihood estimation.
$\widehat{u}^* = 0.65$

FIGURE 3. *Patched model*: tail Concentration Function for different u^* in green lines. *Bold black line:* Empirical TCF based on bivariate data with sample size 2000 from Gumbel copula with parameter $\theta = 2$. *Dashed blue line:* Theoretical TCF from the initial Frank generator $\phi_{\widehat{\theta}}$ with ML estimated parameter. *Grey lines:* 300 empirical TCFs from the initial Frank copula with generator $\phi_{\widehat{\theta}}$. *Magenta line:* Theoretical TCF from Gumbel copula with ML estimated parameter. (Color online)

ii) Patched generator. We now try to improve the initial Frank copula fit proposed in the previous paragraph. As one can see in Figure 3 (*bold black and dashed blue lines*) the fit is roughly suited to the central part of the copula, but not to the upper tail of this data-set.

Creating a patched generator is reduced to the choice of the break-point t_0 and the choice of the distortion generator ϕ_D. The break-point is chosen large enough in order to change mainly the tail dependence of the
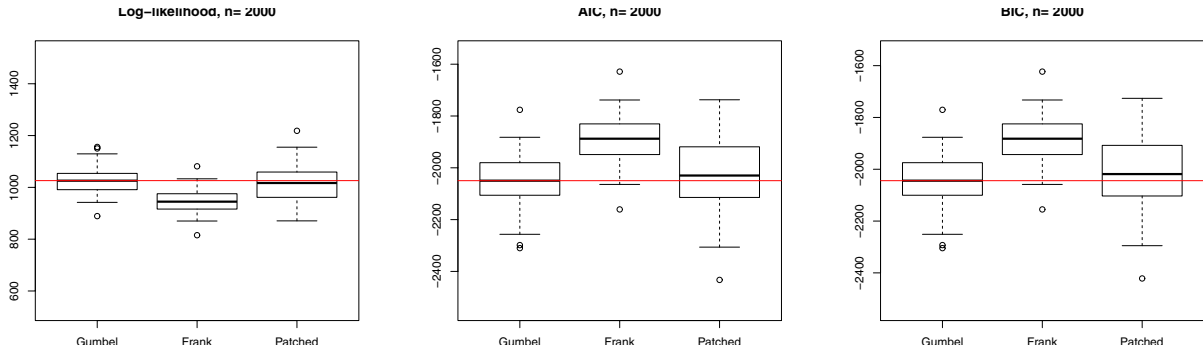
FIGURE 4. Boxplots for the log-likelihood, AIC and BIC criteria for the 3 considered models (true Gumbel model, initial Frank model and patched distorted model). In the red horizontal line we display the median values for the Gumbel model. We consider $M = 100$ Monte Carlo simulations. (Color online)

fitted copula (see Prop. 2.8), and the distortion generator ϕ_D will be chosen in order to ensure that the patched copula is a valid copula which exhibits the target upper tail dependence coefficient, *i.e.*, $\widehat{\lambda}_U^{(h,d-h)}$.

More precisely, here is the detailed procedure to patch the initial Frank generator $\phi_{\widehat{\theta}}$:

(1). *Tail improvement*: we choose a distortion generator $\phi_D(t)$ satisfying admissibility conditions in Definition 2.1 and exhibiting desired tail behavior, *i.e.* such that $1 - \phi_D \in \mathcal{RV}_{1/\hat{\rho}}(0)$ with $\hat{\rho} = \ln(2)/\ln(-\widehat{\lambda}_U^{(h,d-h)} + 2)$, see equation (2.4) when $d = 2$ and $h = 1$. Here, we have chosen $\phi_D(t) = (1 - t^{\frac{1}{\hat{\rho}}})^{\hat{\rho}}$, *i.e.* generator 4.1.15 in Table 1.

(2). *Cutting point choice*: the patched generator is then directly given by Definition 2.2, but depends on a cutting point parameter t_0. We choose the maximum likelihood estimator for this parameter t_0. It corresponds to a level $u^* = \phi_{\widehat{\theta}}(t_0)$ on tail concentration functions. The initial likelihood is thus necessarily improved.

The TCF for the obtained patched generator is displayed in Figure 3 (*full red line*). One can see on this figure that the patched generator fits nicely the empirical tail concentration function, and that it fits quite well the theoretical concentration function for large values of u. This fit is however obtained on one only sample. In the following, we check the statistical procedure described in paragraph *i.* and *ii.* on $M = 100$ Monte Carlo simulations.

In Figure 4, we represent the obtained boxplots for the log-likelihood, AIC and BIC criteria for the 3 considered models (true Gumbel model, initial Frank model and patched distorted model).

To avoid the unnecessary calculation of the density of the patched copula, the log-likelihood has been computed numerically in all three models, using finite differences for the second order partial derivatives. We have checked that it was giving exactly the same results when using theoretical densities for Gumbel and Frank models.

On Figure 4, one can check that in average the patched model performs as well as the true Gumbel model, even on penalized criterions like AIC or BIC. Due to the estimation of the empirical upper tail dependence coefficient $\widehat{\lambda}_U^{(h,d-h)}$, the log-likelihood is more widespread. We clearly check here on Figures 3 and 4, that empirically, the patched generator improve both the likelihood and the tail dependence compared to the initial unsuited Frank model. This is consistent with theoretical results of Proposition 2.8 which state that one can choose any target upper tail dependence coefficient for the patched copula without lowering the likelihood.

To conclude, we represent in Figure 5 the boxplots on the $M = 100$ Monte Carlo simulations of the difference between the theoretical upper tail dependence coefficient for a Gumbel copula with parameter 2, *i.e.*, $\lambda_U = 0.58$, and the estimated values under the 3 considered models (Gumbel, Frank, Patched). Trivially the Frank model
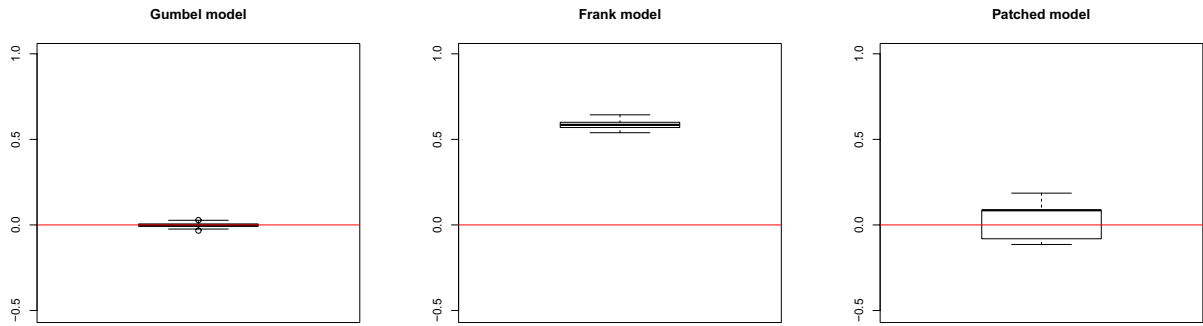
FIGURE 5. Boxplot of the difference between the theoretical $\lambda_U = 0.58$ and the estimated upper tail dependence coefficients under the 3 considered models (true Gumbel model, initial Frank model and patched distorted model).

is not able to capture the considered tail dependency (Fig. 5, *second panel*). Conversely, as remarked in Figures 3 and 4, in average the patched model performs as well as the true Gumbel model (see Fig. 5, *first and third panels*). However, as before, the patched case is more dispersed than the (true) Gumbel one, due to the supplementary empirical estimation of the upper tail dependence coefficient.

3.2. Alternative upper-comonotonic distortions

We investigate here the impact of a first alternative distortion of the literature. This distortion is given in [7]. Let $\widetilde{\mathbf{V}} = (\widetilde{V}_1, \ldots, \widetilde{V}_d)$ be defined as

$$\widetilde{V}_i = U 1_{\{U > \beta\}} + \beta V_i 1_{\{U \leq \beta\}}$$

where $(V_1, \ldots, V_d)$ is a given random vector, $\beta \in (0, 1)$, and where U is a uniformly distributed random variable, independent from $V_1, \ldots, V_d$ (see Eq. (13) in [7]). It is shown in the aforementioned article that $\widetilde{\mathbf{V}}$ has an upper-comonotonic copula with bivariate tail dependence coefficients $\lambda_U = 1$. When a strong tail dependence corresponds to a dangerous situation, it may be a conservative strategy to change an initial copula C of a vector $\mathbf{V}$ into the copula $\widetilde{C}$ of the vector $\widetilde{\mathbf{V}}$, which exhibits perfect (comonotonic) upper tail dependence. It may be especially suited when the sample size is small, which leads to a large uncertainty on the estimation of the empirical upper tail dependence coefficient. One easily shows that this corresponds to the transformation

$$\widetilde{C}(u_1, \ldots, u_d) = \left(\min_{i \in I} u_i - \beta \right)_+ + \beta C \left(\min \left(\frac{u_1}{\beta}, 1 \right), \ldots, \min \left(\frac{u_d}{\beta}, 1 \right) \right).$$

In Figure 6 (*left panel*), one can see the impact of the choice of parameter β. The concentration function reaches the value 1 sooner for small values of β. For higher values, it shifts more abruptly from small values to the target 1. In Figure 6 (*right panel*), one can see the TCF for the value of $\widehat{\beta}$ maximizing the likelihood. While the upper-comonotonic change may be a conservative strategy, one can see that the shift toward the comonotonicity seems here unnatural and distant from the true theoretical Gumbel copula TCF. Furthermore, on the right panel, the strategy is conservative after a given threshold, but overconfident before this threshold.

3.3. Alternative power distortion

Two power transformations are presented in [32] (see Thm. 5.4.4). One distortion $\varphi_{\alpha,1}$ allows to change the lower tail dependence coefficient only, and will not be considered here. The other one $\varphi_{1,\beta}$ allows to change both lower and upper tail dependence coefficients. With our notation, this distortion is equivalent to change an initial generator ϕ into $\widetilde{\phi}$ with, for all $x \in [0, \infty)$,

$$\widetilde{\phi}(x) = \phi(x^{1/\beta}) \tag{3.2}$$
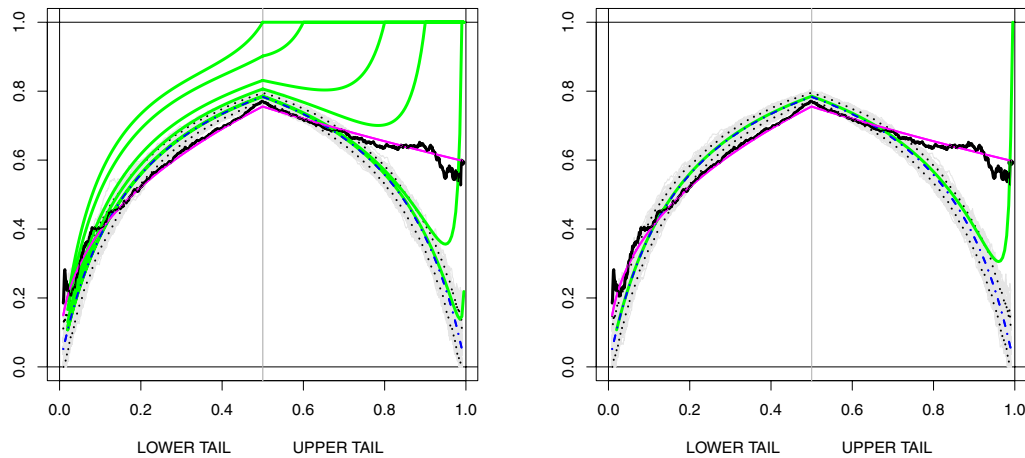
(a) with arbitrary chosen β's values,
from the top $\beta = 0.5, 0.6, 0.8, 0.9, 0.99, 0.999$

(b) with maximum likelihood estimation
$\widehat{\beta} = 0.9942$

FIGURE 6.   *Upper-comonotonic model* (*UC*): tail Concentration Functions for different β in green lines. *Bold black line:* empirical TCF based on bivariate data with sample size 2000 from Gumbel copula with parameter $\theta = 2$. *Dashed blue line:* Theoretical TCF from the initial Frank generator $\phi_{\widehat{\theta}}$ with ML estimated parameter. *Grey lines:* 300 empirical TCFs from the initial Frank copula with generator $\phi_{\widehat{\theta}}$. *Magenta line:* Theoretical TCF from Gumbel copula with ML estimated parameter. (Color online)

In the bivariate case, this leads to the following change of both tail dependence coefficients $\widetilde{\lambda}_L = \lambda_L^{1/\beta}$ and $\widetilde{\lambda}_U = 2 - (2 - \lambda_U)^{1/\beta}$.

In the Figure 7, one can see that the power-type distortion modifies both upper tail dependence and lower tail dependence. By construction, the resulting copula is still Archimedean. However, contrary to the proposed upper patched generator, as seen on the left panel, the distortion affects largely the whole copula. On the right panel, one can see that choosing the parameter β in order to fit the estimated TDC leads to a correct extreme tail dependence behavior, but drastically diminish the global fit of the copula. Choosing another β maximizing the likelihood would lead to an almost unchanged initial Frank copula, with almost zero upper tail dependence coefficient.

It shall be noted that by composition of the two distortions in [32] (Thm. 5.4.4), it is possible to reduce the impact of the considered distortion on the lower tail. The distortion however change the global shape of the copula, contrary to the proposed patched generator in the present paper.

3.4. Discussion

In Figure 8 we illustrate the comparative performance of our patched generator with other simple alternatives considered in Sections 3.2−3.3 in terms of log-likelihood and Akaike information criterion (AIC) for different values of sample size n. As one can see, in average on the 100 Monte Carlo simulations, the patched model performs as well as the true Gumbel model in terms of log-likelihood and AIC.

Conversely, upper-comonotonic and power models do not allow to improve the performance, even when maximizing the likelihood. Indeed here, the tail concentration function with associated parameters maximizing the log-likelihood are very close to the one of the initial misspecified Frank model (see Figs. 6 and 7). In this sense, results gathered in the boxplots of Figure 8 are not surprising.
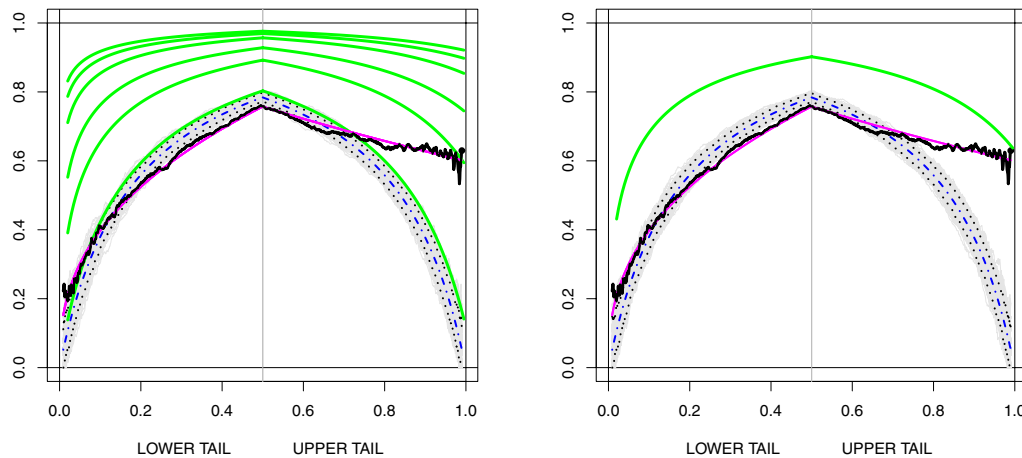
(a) with arbitrary chosen β's values from the top $\beta = 9, 7, 5, 3, 2$ and $\widehat{\beta}_{ML} = 1.095$

(b) with $\widehat{\beta} = 2.199901$ such that the $\lambda_\beta = \widehat{\lambda}_U = 0.63$

FIGURE 7. *Power distortion model*: tail Concentration Functions for different β in green lines. *Bold black line*: empirical TCF based on bivariate data with sample size 2000 from Gumbel copula with parameter $\theta = 2$. *Dashed blue line:* theoretical TCF from the initial Frank generator $\phi_{\widehat{\theta}}$ with ML estimated parameter. *Grey lines:* 300 empirical TCFs from the initial Frank copula with generator $\phi_{\widehat{\theta}}$. *Magenta line:* theoretical TCF from Gumbel copula with ML estimated parameter. (Color online)

Finally, we do not present here a comparison in terms of λ_U since the upper-comonotonic (UC) transformation clearly leads to $\lambda_U = 1$, and in the power transformation one can not set a target $\hat{\lambda}_U$ without causing a large loss in terms of log-likelihood, see, *e.g.*, Figure 7, *right panel* (power transformation is a global copula transformation).

Here is a short summary of advantages and drawbacks of the proposed patched generator, compared to other studied alternatives. Pros: the distorted copula is still Archimedean, the tail behaviour can be easily chosen, proposed patched distortion is only local and, as a consequence, the likelihood is necessarily improved, even for a specifically chosen tail dependence. Among drawbacks: the Archimedean setting is not always suited in the real-life applications. In this sense some efforts to leave the Archimedean framework have be done in the recent literature, as for instance for the nested or hierarchical copulas (see *e.g.* [4, 22, 23, 29]). Furthermore, in our strategy, the tail behaviour must be estimated, which in practice may be sensitive to the size of the considered data-set. For alternative methods, the tail dependence coefficient can also be approached or exceeded (in a conservative strategy). However, once tail behaviour chosen, these alternatives hardly improve the likelihood or approach the empirical tail concentration function.

We conclude this section by remarking that our approach is complectly based on Archimedean copulas. This Archimedean assumption allows to get copulas depending only on one univariate real function: the (transformed) generator. This is interesting for obtaining some analytical expressions for the level curves of the copula, and thus for calculating risk measures relying on level curves. It is also used when some properties are desirable (symmetry, convex level curves, zero-set, associativity, *etc.*). These properties are used for example in [3, 8]. Furthermore, the lower and upper tail dependence behaviours of the copula are directly linked with the regular variation indexes of the generator (see *e.g.* [5]). At last, in the Archimedean transformed setting, one can easily verify the validity and the absolute continuity of the obtained patched copula since they are only related to monotonicity properties of the patched generator (see Props. 2.4 and 2.5).
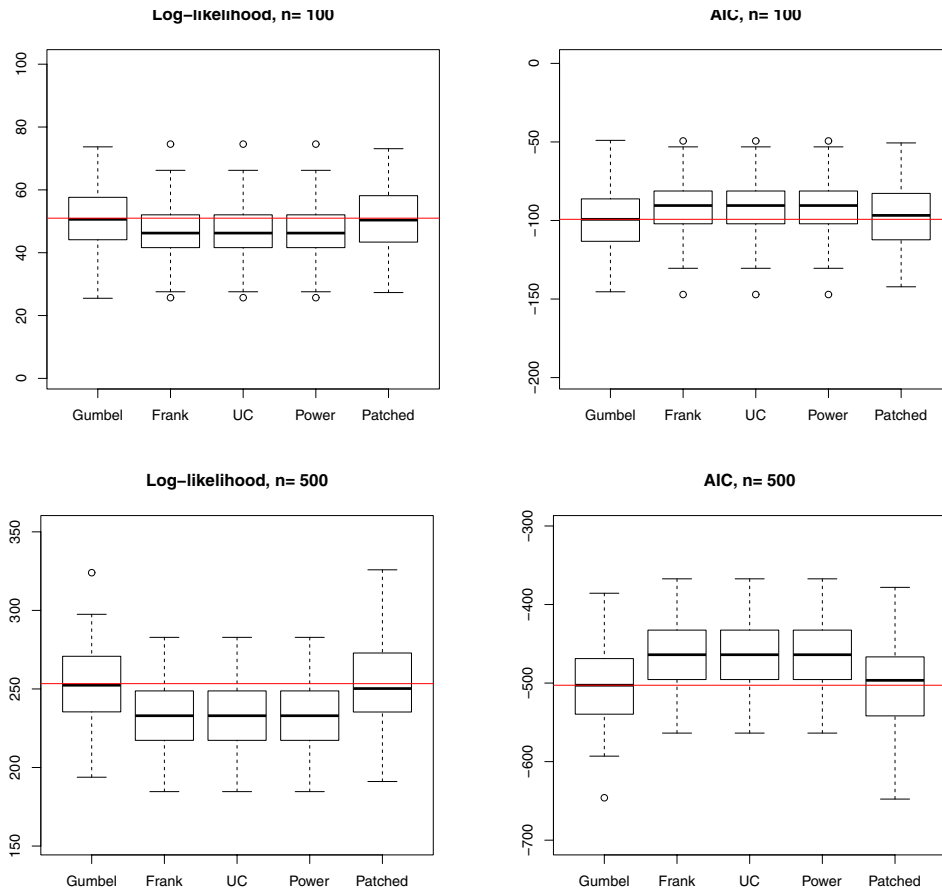
FIGURE 8. Boxplots for the log-likelihood and Akaike information criterion (AIC) for true Gumbel, initial Frank, upper-comonotonic, power distortion and patched model. Sample sizes $n = 100$ (*first row*) and $n = 500$ (*second row*). In the red horizontal line we display the median values for the Gumbel model. We consider $M = 100$ Monte Carlo simulations.

4. CONCLUSION

We have presented a specific transformation of a given Archimedean generator, creating what we called an upper-patched generator. One advantage of such a construction is the simplicity of the method and its straightforward applicability. Contrary to general methods like patchwork copulas, the transformation focuses on the Archimedean case. It thus ensures a synthetic representation, simple expressions for related quantities as level curves or radial part, smoothness conditions and absolute continuity of the transformed copula under some simple conditions. It also opens the perspective of generalizations using nested copulas or hierarchical Kendall copulas. Compared to existing methods that focus on Archimedean copulas, another important advantage is the fact that the patched generator is necessarily valid in the chosen dimension $d \geq 2$. At last, another important advantage is that the upper tail dependence coefficient of the patched generator can be easily chosen, without likelihood reduction on a given data-set.

One drawback of the method is that the inverse function of a patched generator is obtained in practice by numerical inversion: it is not given by a direct explicit closed-form formula and depends on the choice of both the

initial generator and the distortion generator. Another drawback of the proposed methodology is that it remains suited only to Archimedean dependence structures, as the patched copula remains within the Archimedean class.

Concerning perspectives, we have changed the upper tail dependence of an Archimedean copula by modifying its generator ϕ on an interval $[0, t_0]$; a very natural perspective for modifying the lower tail dependence of an Archimedean copula is to change its generator ϕ on an interval $[t_0, +\infty)$. More generally, one can imagine creating an interval-patched generator, where an initial generator is modified on an interval I. Among difficulties on the interval $[t_0, +\infty)$, Taylor expansions of ϕ at different orders are alternatively greater or lower than ϕ, and both diverging toward $-\infty$ or $+\infty$. Creating an upper-patched inverse generator from ϕ^{-1} is also possible, but it is then not straightforward to get the inverse function of this patched inverse, in order to check the d-monotonicity. Thus, it seems to us that the techniques involved for patching a generator on the other (lower) side are probably different from the one treated in this paper; they are let to future work.

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