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POISSON SPHERE COUNTING PROCESSES WITH RANDOM RADII

Nicolas Privault[1]

Abstract. We consider a random sphere covering model made of random balls with interacting random radii of the product form $R(r,\omega) = rG(\omega)$, based on a Poisson random measure $\omega(\mathrm{d}y, \mathrm{d}r)$ on $\mathbb{R}^d \times \mathbb{R}_+$. We provide sufficient conditions under which the corresponding random ball counting processes are well-defined, and we study the fractional behavior of the associated random fields. The main results rely on moment formulas for Poisson stochastic integrals with random integrands.

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1. Introduction

Sphere counting processes yield continuous-time processes with heavy tails, and have applications to the modeling of telecommunication networks. Under a suitable renormalization procedure, the counting of Euclidean balls in $\mathbb{R}^d$ whose centers and radii are distributed according to a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+$ is known to define a fractional random field, *cf. e.g.* [2,3] and references therein.

In [2,3] the fractional behavior of such processes has been studied for independent Poisson distributed radii, see also [5,8] for weighted random balls models.

In the modeling of communication networks, point processes can be used to represent the spatial locations of wireless sensors, *cf. e.g.* [7,9] and references therein. In this framework, the random ranges of sensors are modelled using random spheres located according to a Poisson point process. Sphere counting processes then help to estimate the coverage and connectivity of the network of sensors. In practice, the presence of a number of sensors within a restricted area can create interferences under which all sensing ranges are potentially modified *via* the mutual interaction of neighboring sensors. In this paper we address such a situation by extending the results of [2,3] to a framework permitting interactions between the random ball radii, which are allowed to depend on a whole Poisson cloud instead of being given by a single Poisson mark.

Our main results are as follows.

(i) In Section 3 we obtain sufficient conditions for the existence of such generalized sphere counting processes, by extending the arguments of [5] under suitable hypotheses.
(ii) We determine the fractional behaviour of those counting processes, *cf.* Corollary 5.1, based on L^p bounds derived in Proposition 4.1. In order to deal with the new dependence induced in our model we use moment

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[1] Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, 637371 Nanyang, Singapore. nprivault@ntu.edu.sg

identities for Poisson stochastic integrals with random integrands, see (2.2) below, as the classical identities of [1], used in [2,3], do not apply in our interacting setting.

Given a sigma-compact metric space X with Borel sigma-algebra $\mathcal{B}(X)$, the Poisson random measure $\omega(\mathrm{d}x)$ with sigma-finite diffuse intensity measure $\sigma(\mathrm{d}x)$ on X and probability distribution π_σ is built on the space of Radon point measures

$$\Omega^X = \left\{ \omega = \sum_{i=1}^{\omega(X)} \delta_{x_i},\ (x_i)_{i=1}^N \subset X,\ x_i \neq x_j\ \forall i \neq j,\ N \in \mathbb{N} \cup \{\infty\} \right\}$$

without multiple points, where δ_x denotes the Dirac measure at $x \in X$. Each element $\omega = \sum_{i=1}^{\omega(X)} \delta_{x_i}$ of Ω^X is identified with the sequence $(x_i)_{i=1}^{\omega(X)}$, and $\omega(X) \in \mathbb{N} \cup \{\infty\}$ represents the cardinality of ω.

Poisson-based random balls model

Taking $X = \mathbb{R}^d \times \mathbb{R}_+$ and given $z \in \mathbb{R}^d$, consider the number

$$N_z(\omega) := \int_0^\infty \int_{\mathbb{R}^d} \mathbf{1}_{\{ z \in \mathscr{B}(y,r)\}}\, \omega(\mathrm{d}y, \mathrm{d}r) \tag{1.1}$$

of closed balls $\mathscr{B}(y,r)$ containing $z \in \mathbb{R}^d$, centered at y in $\mathbb{R}^d$ and with radius $r > 0$, where $(y,r) \in \omega$. Clearly, $N_z(\omega)$ may be infinite depending on the integrability properties of the intensity measure $\sigma(\mathrm{d}y, \mathrm{d}r)$.

More generally, when $\mu(\mathrm{d}y)$ is a signed measure on $\mathbb{R}^d$, the stochastic integral

$$\int_0^\infty \int_{\mathbb{R}^d} \mu(\mathscr{B}(y,r))\omega(\mathrm{d}y, \mathrm{d}r), \tag{1.2}$$

is proved to exist under certain conditions on $\mu(\mathrm{d}x)$ and $\sigma(\mathrm{d}y, \mathrm{d}r)$, *cf.* Lemmas 3.1 and 3.3 below, which cover $\mu(\mathrm{d}x) = \delta_z(\mathrm{d}x)$ in (1.1) above as a particular case.

When

$$\sigma(\mathrm{d}y, \mathrm{d}r) = r^{2H-d-1}\mathrm{d}r\mathrm{d}y \quad \text{and} \quad \mu_z(\mathrm{d}y) := \delta_z(\mathrm{d}y) - \delta_0(\mathrm{d}y),$$

with $0 < H < 1/2$ the integral (1.2) defines the normalized shot noise process

$$\begin{aligned} F(z) &:= \int_0^\infty \int_{\mathbb{R}^d} \mu_z(\mathscr{B}(y,r))\omega(\mathrm{d}y, \mathrm{d}r) \\ &= \int_0^\infty \int_{\mathbb{R}^d} (\mathbf{1}_{\mathscr{B}(y,r)}(z) - \mathbf{1}_{\mathscr{B}(y,r)}(0))\omega(\mathrm{d}y, \mathrm{d}r), \qquad z \in \mathbb{R}^d, \end{aligned} \tag{1.3}$$

whose fractional behavior has been studied in [2]. In particular it has been shown in [2], *cf.* also [3], that $F(z)$ in (1.3) is well-defined and satisfies the bound

$$C_q|t-s|^{2H} \leq E\left[\int_0^1 |F(z+t\theta) - F(z+s\theta)|^q \mathrm{d}t\right] \leq C'_q|t-s|^{2H}, \tag{1.4}$$

for all $q \in [2, \infty)$, $0 < H < 1/2$, $0 \leq |t-s| \leq \eta_q$, $z \in \mathbb{R}^d$, and for all θ in the $(d-1)$-dimensional unit sphere S^{d-1} in $\mathbb{R}^d$, where $C_q, C'_q, \eta_q > 0$ are constants.

Random balls with interacting radii

In this paper we consider a model in which the radius of the sphere $\mathscr{B}(y_i, r_i)$ centered at y_i is not only given by the Poisson mark r_i, but is possibly depending on the whole Poisson sample $(y_j, r_j)_{j=1}^{\omega(X)}$. Namely, we consider shot noise processes $(F(z))_{z\in\mathbb{R}^d}$ defined by stochastic integrals of the form

$$F(z) = \int_0^\infty \int_{\mathbb{R}^d} \mu_z(\mathscr{B}(y, R(r,\omega)))\omega(\mathrm{d}y, \mathrm{d}r), \qquad z \in \mathbb{R}^d, \tag{1.5}$$

where $(\mu_z)_{z\in\mathbb{R}^d}$ is a family of measures on $\mathbb{R}^d$ and

$$R : \mathbb{R}_+ \times \Omega^X \longrightarrow \mathbb{R}_+$$

is a random radius chosen from a large family of probability distributions, in contrast with the deterministic relation $R(r,\omega) = r$ treated in (1.2).

Existence results for random sums of the form (1.5) are presented in Lemmas 3.1 and 3.3 below when $\sigma(\mathrm{d}y, \mathrm{d}r)$ takes the form $\sigma(\mathrm{d}y, \mathrm{d}r) = \rho(\mathrm{d}r)\nu(\mathrm{d}y)$ and $\nu(\mathrm{d}y)$ is dominated by the Lebesgue measure $\mathrm{d}y$ on $\mathbb{R}^d$.

On the other hand, in Proposition 4.1 and Corollary 5.1 below we show that when $\sigma(\mathrm{d}y, \mathrm{d}r) = r^{2H-d-1}\mathrm{d}r\nu(\mathrm{d}y)$ with $0 < H < 1/2$ and $\nu(\mathrm{d}y)$ the Lebesgue measure, a fractional behavior similar to (1.4) occurs for the normalized random field

$$F(z) := \int_0^\infty \int_{\mathbb{R}^d} \mu_z(\mathscr{B}(y, R(r,\omega)))\omega(\mathrm{d}y, \mathrm{d}r), \qquad z \in \mathbb{R}^d,$$

obtained as in (1.3) by taking $\mu = \mu_z := \delta_z - \delta_0$, when the random radius $R(r,\omega)$ is possibly depending on the marks (r_i) of the Poisson sample (y_i, r_i). Precisely, $R(r,\omega)$ has the form

$$\begin{aligned} R : \mathbb{R}_+ \times \Omega^X &\longrightarrow \mathbb{R}_+ \\ (r,\omega) &\longmapsto R(r,\omega) = rG((r_i)_i), \end{aligned}$$

under condition (4.3) below, where $G(\omega)$ depends on the component $(r_i)_{i=1}^{\omega(X)} \in \Omega^{\mathbb{R}_+}$ of

$$\omega = \sum_{i=1}^{\omega(X)} \delta_{(y_i,r_i)} \in \Omega^{\mathbb{R}^d\times\mathbb{R}_+}.$$

Our results rely on moment formulas for Poisson stochastic integrals with random integrands, *cf.* [11,12], which are recalled in Section 2. Random integrals with interacting sphere radii are constructed in Section 3, and the main bounds are presented in Section 4. The results on the fractional behavior of $(F(z))_{z\in\mathbb{R}^d}$ are presented in Section 5.

2. Nonlinear Mecke identity

The results of this paper rely on the moment formula (2.2) below for Poisson stochastic integrals with random integrands. Recall that the expectation of the Poisson stochastic integral $\int_X u(x,\omega)\omega(\mathrm{d}x)$ of a measurable process $(\omega, x) \longmapsto u(x,\omega)$ can be expressed *via* the Mecke [10] identity, *i.e.*

$$E\left[\int_X u(x,\omega)\omega(\mathrm{d}x)\right] = E\left[\int_X \varepsilon_x^+ u(x,\omega)\sigma(\mathrm{d}x)\right], \tag{2.1}$$

where ε_x^+ is the addition operator that acts on any random variable $F : \Omega^X \longrightarrow \mathbb{R}$ by addition of a point at $x \in X$ to the point measure ω, *i.e.*

$$\varepsilon_x^+ F(\omega) = F(\omega + \delta_x), \qquad \omega \in \Omega^X, \quad x \in X,$$

provided that

$$(\omega, x) \longmapsto \varepsilon_x^+ u(x,\omega) \in L^1(\Omega^X \times X).$$

We will use the nonlinear extension

$$E\left[\left(\int_X u(x,\omega)\omega(\mathrm{d}x)\right)^n\right] = \sum_{k=0}^{n} \sum_{P_1\cup\ldots\cup P_k=\{1,\ldots,n\}} E\left[\int_{X^k} \varepsilon_{\mathfrak{s}_k}^+ (u^{|P_1|}(s_1,\omega)\ldots u^{|P_k|}(s_k,\omega))\,\sigma(\mathrm{d}s_1)\ldots\sigma(\mathrm{d}s_k)\right] \tag{2.2}$$

of (2.1) for the powers of Poisson stochastic integrals of random integrands

$$u : \Omega^X \times X \longrightarrow \mathbb{R},$$

where

$$\mathfrak{s}_k := (s_1,\ldots,s_k) \in X^k, \quad k \geq 1,$$

and the iterated addition operator

$$\varepsilon_{\mathfrak{s}_k}^+ = \varepsilon_{s_1,\ldots,s_k}^+ := \varepsilon_{s_1}^+ \ldots \varepsilon_{s_k}^+$$

is defined on $F : \Omega^X \longrightarrow \mathbb{R}$ by

$$\varepsilon_{\mathfrak{s}_k}^+ F(\omega) = F\left(\omega + \delta_{s_1} + \ldots + \delta_{s_k}\right), \qquad \omega \in \Omega^X,$$

cf. Proposition 3.1 of [11] or Theorem 1 of [12]. The above sum runs over all (disjoint) partitions $P_1 \cup \ldots \cup P_k$ of $\{1,\ldots,n\}$, $k = 1,\ldots,n$, and $|P|$ denotes the cardinality of the subset $P \subset \{1,\ldots,n\}$.

For example when $n = 2$, Relation (2.2) yields

$$\begin{aligned}
E\left[\left(\int_X u(x,\omega)\omega(\mathrm{d}x)\right)^2\right] &= E\left[\int_X \varepsilon_x^+ |u(x,\omega)|^2 \sigma(\mathrm{d}x)\right] \\
&+E\left[\int_{X^2} \varepsilon_{x_2}^+ \varepsilon_{x_1}^+ (u(x_1,\omega)u(x_2,\omega))\,\sigma(\mathrm{d}x_1)\sigma(\mathrm{d}x_2)\right] \\
&= E\left[\int_X |u(x,\omega+\delta_x)|^2 \sigma(\mathrm{d}x)\right] \\
&\quad +E\left[\int_{X^2} u(x_1,\omega+\delta_{x_1}+\delta_{x_2}) u(x_2,\omega+\delta_{x_1}+\delta_{x_2})\,\sigma(\mathrm{d}x_1)\sigma(\mathrm{d}x_2)\right].
\end{aligned}$$

Note that when h is a deterministic function we have $\varepsilon_{\mathfrak{s}_k}^+ h(x) = h(x)$ and (2.2) recovers the classical moment formula

$$E\left[\left(\int_X h(x)\omega(\mathrm{d}x)\right)^n\right] = \sum_{k=0}^{n} \sum_{P_1\cup\ldots\cup P_k=\{1,\ldots,n\}} \int_{X^{|P_1|}} h^{|P_1|}(x_1)\sigma(\mathrm{d}x_1)\ldots \int_{X^{|P_k|}} h^{|P_k|}(x_k)\sigma(\mathrm{d}x_k), \tag{2.3}$$

obtained in [1] using the Lévy−Khintchine representation of the Laplace transform of $\int_X h(x)\omega(\mathrm{d}x)$. This relation rewrites as

$$E\left[\left(\int_X h(x)\omega(\mathrm{d}x)\right)^n\right] = A_n\left(\int_X h(x)\sigma(\mathrm{d}x), \int_X h^2(x)\sigma(\mathrm{d}x), \ldots, \int_X h^n(x)\sigma(\mathrm{d}x)\right),$$

where

$$A_n(y_1,\ldots,y_n) = n! \sum_{\substack{r_1+2r_2+\ldots+nr_n=n \\ r_1,\ldots,r_n\geq 0}} \prod_{l=1}^{n} \left(\frac{1}{r_l!}\left(\frac{y_l}{l!}\right)^{r_l}\right)$$

is the Bell polynomial of degree n, based on the relation between moments and cumulants by the Faà di Bruno formula, *cf.* [12] and references therein for details.

3. Sphere counting with random radii

From now on we let $X := \mathbb{R}^d \times \mathbb{R}_+$, $d \geq 1$, and we consider a Poisson random measure

$$\omega(\mathrm{d}y, \mathrm{d}r) = \sum_{k=1}^{\omega(X)} \delta_{(y_k, r_k)}(\mathrm{d}y, \mathrm{d}r),$$

with intensity of the form

$$\sigma(\mathrm{d}y, \mathrm{d}r) = \nu(\mathrm{d}y)\rho(\mathrm{d}r), \qquad (y, r) \in X = \mathbb{R}^d \times \mathbb{R}_+,$$

where $\rho(\mathrm{d}r)$ is a measure on $\mathbb{R}_+$. In addition, in this section we assume that $\nu(\mathrm{d}y)$ is dominated by the Lebesgue measure, *i.e.*

$$\nu(\mathrm{d}y) \leq \mathrm{d}y, \qquad y \in \mathbb{R}^d. \tag{3.1}$$

Lemmas 3.1 and 3.3 below, which provide sufficient conditions for the existence of the random sum (1.5).

Lemma 3.1. *Let μ be a signed measure on $\mathbb{R}^d$. Assume that the random radius*

$$R : \mathbb{R}_+ \times \Omega^X \longrightarrow \mathbb{R}_+$$

satisfies

$$\varepsilon^+_{(y,r)} R(r, \omega) \leq U(r, \omega), \qquad (y, r) \in \mathbb{R}^d \times \mathbb{R}_+, \quad \omega \in \Omega^X, \tag{3.2}$$

where

$$U : \mathbb{R}_+ \times \Omega^X \longrightarrow \mathbb{R}_+$$

is a (non-negative) random process. Under condition (3.1) we have the bound

$$E\left[\int_0^\infty \int_{\mathbb{R}^d} |\mu(\mathscr{B}(y, R(r, \omega)))| \omega(\mathrm{d}y, \mathrm{d}r)\right] \leq v_d |\mu|(\mathbb{R}^d) E\left[\int_0^\infty (U(r, \omega))^d \rho(\mathrm{d}r)\right],$$

where v_d denotes the volume of the unit ball $\mathscr{B}(0, 1)$ in $\mathbb{R}^d$ and $|\mu|$ is the total variation of μ.

Proof. For all $r > 0$ and $\omega \in \Omega^X$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \varepsilon^+_{(y,r)} |\mu(\mathscr{B}(y, R(r, \omega)))| \nu(\mathrm{d}y) &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \mathbf{1}_{\mathscr{B}(y, \varepsilon^+_{(y,r)} R(r, \omega))}(z) \mu(\mathrm{d}z) \right| \nu(\mathrm{d}y) \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathscr{B}(y, \varepsilon^+_{(y,r)} R(r, \omega))}(z) |\mu|(\mathrm{d}z) \mathrm{d}y \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathscr{B}(z, U(r, \omega))}(y) \mathrm{d}y |\mu|(\mathrm{d}z) \\
&= v_d (U(r, \omega))^d \int_{\mathbb{R}^d} |\mu|(\mathrm{d}z) \\
&= v_d |\mu|(\mathbb{R}^d) (U(r, \omega))^d,
\end{aligned}$$

hence

$$\begin{aligned}
&E\left[\int_0^\infty \int_{\mathbb{R}^d} |\mu(\mathscr{B}(y, R(r, \omega)))| \omega(\mathrm{d}y, \mathrm{d}r)\right] \\
&= E\left[\int_0^\infty \int_{\mathbb{R}^d} \varepsilon^+_{(y,r)} |\mu(\mathscr{B}(y, R(r, \omega)))| \nu(\mathrm{d}y) \rho(\mathrm{d}r)\right] \leq v_d |\mu|(\mathbb{R}^d) E\left[\int_0^\infty (U(r, \omega))^d \rho(\mathrm{d}r)\right]. \qquad \square
\end{aligned}$$

In order to find a sufficient existence condition more practicable than that of Lemma 3.1 we consider the next definition, which is inspired by Definition 2.1 of [5].

Definition 3.2. Given $p, q \in \mathbb{R}$, let $\mathcal{M}_{p,q}$ denote the set of signed measures $\mu(\mathrm{d}y)$ with finite total variation on $\mathbb{R}^d$, and such that

$$\int_{\mathbb{R}^d} |\mu(\mathscr{B}(z,r))|\mathrm{d}z \leq C_\mu(r^p \wedge r^q), \qquad r \in \mathbb{R}_+, \tag{3.3}$$

for some constant $C_\mu > 0$.

For example, the measure

$$\mu_z(\mathrm{d}y) := \delta_z(\mathrm{d}y) - \delta_0(\mathrm{d}y)$$

belongs to $\mathcal{M}_{d-1,d}$, *cf.* relation (3) in [2]. We note that Proposition 2.2 (v) of [5], originally stated for $\alpha \in (1,2]$, can be extended to $\alpha = 1$ by the same arguments, under the condition $p < d$. As a consequence, every measure $\mu \in \mathcal{M}_{p,q}$ is centered, *i.e.* it satisfies $\mu(\mathbb{R}^d) = 0$, provided that $p < d$.

Similarly, by restating the proof of Proposition 2.3 (ii) of [5] for $\alpha = 1$, we find that any centered measure $\mu(\mathrm{d}y)$ of the form $\mu(\mathrm{d}y) = \phi(y)\mathrm{d}y$ belongs to $\mathcal{M}_{p,q}$ for some $p, q \in \mathbb{R}$ provided that it has bounded support and

$$\int_{\mathbb{R}^d} |\phi(y)|\mathrm{d}y < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \|y\|_{\mathbb{R}^d}|\phi(y)|\mathrm{d}y < \infty,$$

where $\|y\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Assuming that $\mu \in \mathcal{M}_{p,q}$, the next Lemma 3.3 gives a more precise bound for the expectation of (1.5) compared to that of Lemma 3.1.

In the sequel we fix $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\nu(A) < \infty$, and we will use the canonical projection $\pi_A : \Omega^X \longrightarrow \Omega^{\mathbb{R}_+}$ defined by

$$\omega = ((y_i, r_i))_{i\in\mathbb{N}} \longmapsto \pi_A(\omega) := (r_i)_{\{i\in\mathbb{N} \ \ y_i \in A\}} \tag{3.4}$$

whose image measure defines the Poisson random measure with intensity $\nu(A)\rho(\mathrm{d}r)$ on $\mathbb{R}_+$.

Lemma 3.3. *Let $\mu \in \mathcal{M}_{p,q}$ for some $p, q \in \mathbb{R}$, and assume that $R : \mathbb{R}_+ \times \Omega^X \longrightarrow \mathbb{R}_+$ depends only on the marks $(r_i)_i$ of points in A* via *the relation*

$$R(r,\omega) = G(r, \pi_A(\omega)), \qquad \omega \in \Omega^X,$$

where $G : \mathbb{R}_+ \times \Omega^{\mathbb{R}_+} \longrightarrow \mathbb{R}_+$ is a non-negative random process. Then, under condition (3.1) we have the bound

$$E\left[\int_0^\infty \int_{\mathbb{R}^d} |\mu(\mathscr{B}(y, R(r,\omega)))|\omega(\mathrm{d}y, \mathrm{d}r)\right] \leq 2C_\mu E\left[\int_0^\infty (U^q(r,\omega) \wedge U^p(r,\omega))\rho(\mathrm{d}r)\right],$$

where $C_\mu > 0$ is given in (3.3) and

$$U(r,\omega) := \max(\varepsilon_r^+ G(r, \pi_A(\omega)), G(r, \pi_A(\omega))), \qquad r \in \mathbb{R}_+, \quad \omega \in \Omega^{\mathbb{R}_+}. \tag{3.5}$$

Proof. We note that

$$\varepsilon_{(y,r)}^+ R(r,\omega) = \mathbf{1}_A(y)\varepsilon_r^+ G(r, \pi_A(\omega)) + \mathbf{1}_{A^c}(y) G(r, \pi_A(\omega)),$$

$(y,r) \in \mathbb{R}^d \times \mathbb{R}_+$. Hence, since $\mu \in \mathcal{M}_{p,q}$, by (3.2) and (3.3), for all $r > 0$ we have

$$\begin{aligned}
\int_A |\mu(\mathscr{B}(y, \varepsilon_{(y,r)}^+ R(r,\omega)))|\nu(\mathrm{d}y) &= \int_A \varepsilon_{(y,r)}^+ |\mu(\mathscr{B}(y, R(r,\omega)))|\nu(\mathrm{d}y) \\
&\leq \int_A |\mu(\mathscr{B}(y, \varepsilon_r^+ G(r, \pi_A(\omega))))|\mathrm{d}y \\
&\leq \int_{\mathbb{R}^d} |\mu(\mathscr{B}(y, \varepsilon_r^+ G(r, \pi_A(\omega))))|\mathrm{d}y \\
&\leq C_\mu(\varepsilon_r^+ G^q(r, \pi_A(\omega)) \wedge \varepsilon_r^+ G^p(r, \pi_A(\omega))) \\
&\leq C_\mu(U^q(r,\omega) \wedge U^p(r,\omega)),
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\int_{A^c} \varepsilon^+_{(y,r)} |\mu(\mathscr{B}(y, R(r,\omega)))| \nu(\mathrm{d}y) &\leq \int_{\mathbb{R}^d} |\mu(\mathscr{B}(y, G(r, \pi_A(\omega))))| \mathrm{d}y \\ &\leq C_\mu (G^q(r, \pi_A(\omega)) \wedge G^p(r, \pi_A(\omega))) \\ &\leq C_\mu (U^q(r,\omega) \wedge U^p(r,\omega)), \end{aligned} \tag{3.7}$$

which yields, by (2.1),

$$\begin{aligned} E\left[\int_0^\infty \int_{\mathbb{R}^d} |\mu(\mathscr{B}(y, R(r,\omega)))| \omega(\mathrm{d}y, \mathrm{d}r)\right] &= E\left[\int_0^\infty \int_{\mathbb{R}^d} \varepsilon^+_{(y,r)} |\mu(\mathscr{B}(y, R(r,\omega)))| \nu(\mathrm{d}y)\rho(\mathrm{d}r)\right] \\ &\leq 2C_\mu E\left[\int_0^\infty U^q(r,\omega) \wedge U^p(r,\omega) \rho(\mathrm{d}r)\right]. \end{aligned}$$

□

In the case where $\rho(\mathrm{d}r)$ has the density

$$\rho(r) = r^{2H-d-1}, \qquad r > 0, \tag{3.8}$$

for some $H \in \mathbb{R}$, Lemma 3.3 yields the following result.

Proposition 3.4. *Let $p < q \in \mathbb{R}$ and $\mu \in \mathcal{M}_{p,q}$. Assume that $\rho(r)$ takes the form (3.8) and that U in (3.5) satisfies*

$$U(r,\omega) \leq rU(\omega), \qquad \omega \in \Omega^{\mathbb{R}_+}, \quad r \in \mathbb{R}_+, \tag{3.9}$$

for $U : \Omega^X \longrightarrow \mathbb{R}_+$ a non-negative random variable. Then under condition (3.1), for some constant $C_{p,q} > 0$ we have the bound

$$E\left[\int_0^\infty \int_{\mathbb{R}^d} |\mu(\mathscr{B}(y, R(r,\omega)))|\, \omega(\mathrm{d}y, \mathrm{d}r)\right] \leq C_{p,q} E\left[U^{d-2H}\right] \int_0^\infty (r^q \wedge r^p) r^{2H-d-1} \mathrm{d}r, \tag{3.10}$$

which is finite provided that

$$\frac{d-q}{2} < H < \frac{d-p}{2} \tag{3.11}$$

and $E\left[U^{d-2H}\right] < \infty$, with $\rho(r)$ given by (3.8).

Proof. By Lemma 3.3 and the inequality (3.9) we have

$$\begin{aligned} E\left[\int_0^\infty \int_{\mathbb{R}^d} |\mu(\mathscr{B}(y, R(r,\omega)))| \omega(\mathrm{d}y, \mathrm{d}r)\right] &\leq 2C_\mu E\left[\int_0^\infty (U^q(r,\omega) \wedge U^p(r,\omega))\rho(r)\mathrm{d}r\right] \\ &\leq 2C_\mu E\left[\int_0^\infty ((rU)^q \wedge (rU)^p) r^{2H-d-1} \mathrm{d}r\right] \\ &= 2C_\mu E\left[U^{d-2H}\right] \int_0^\infty (r^q \wedge r^p) r^{2H-d-1} \mathrm{d}r, \end{aligned}$$

and to conclude the proof we check that

$$\begin{aligned} \int_0^\infty \left(r^{q-p} \wedge 1\right) r^{2H-d-1+p} \mathrm{d}r &= \int_0^1 r^{2H-d-1+q} \mathrm{d}r + \int_1^{+\infty} r^{2H-d-1+p} \mathrm{d}r \\ &= \frac{q-p}{(2H+q-d)(d-2H-p)} < +\infty, \end{aligned}$$

provided that (3.11) holds. □

Proposition 3.4 also covers the deterministic case where $R(r) = r$ and $U = 1$, *cf. e.g.* Section 1.2 of [4], in particular with

$$\mu_z(\mathrm{d}y) := \delta_z(\mathrm{d}y) - \delta_0(\mathrm{d}y),$$

which belongs to $\mathcal{M}_{d-1,d}$, *cf.* [2].

We close this section with some product extensions of Lemma 3.3 and Proposition 3.4, which will be needed in the proof of Proposition 4.1 below.

In the sequel we will use the notation

$$\mathfrak{y}_a := (y_1, \ldots, y_a) \in \mathbb{R}^d, \qquad \mathfrak{r}_a := (r_1, \ldots, r_a) \in \mathbb{R}_+^a.$$

1. Let $\mu \in \mathcal{M}_{p,q}$ and assume that

$$\max(\varepsilon^+_{\mathfrak{r}_a} G(r, \pi_A(\omega)), G(r, \pi_A(\omega))) \leq U_a(r, \omega) := rU_a(\omega), \quad \mathfrak{r}_a \in \mathbb{R}_+^a, \quad \omega \in \Omega^X, \tag{3.12}$$

   where $U_a : \Omega^X \longrightarrow \mathbb{R}_+$ is a non-negative random variable for all $a \geq 1$ and

$$\varepsilon^+_{\mathfrak{y}_a, \mathfrak{r}_a} R(r, \omega) = R\left(r, \omega + \delta_{(y_1,r_1)} + \ldots + \delta_{(y_a,r_a)}\right),$$

   $\mathrm{d}\mathfrak{y}_a \mathrm{d}\mathfrak{r}_a$-a.e., with

$$\nu^{\otimes a}(\mathrm{d}\mathfrak{y}_a) := \nu(\mathrm{d}y_1) \ldots \nu(\mathrm{d}y_a).$$

   Reasoning as in (3.6) and (3.7), for all $z \in \mathbb{R}^d$ we obtain the product estimate

$$\begin{aligned}
&E\left[\int_{(\mathbb{R}^d \times \mathbb{R}_+)^a} \prod_{k=1}^a |\mu(\mathscr{B}(y_k, \varepsilon^+_{\|z\|\mathfrak{y}_a, \|z\|\mathfrak{r}_a} R(r_k, \omega)))| \rho(r_k) \nu^{\otimes a}(\mathrm{d}\mathfrak{y}_a) \mathrm{d}\mathfrak{r}_a\right] \\
&= \sum_{b=0}^a \binom{a}{b} E\left[\int_{(A \times \mathbb{R}_+)^b \times (A^c \times \mathbb{R}_+)^{a-b}} \prod_{k=1}^a |\mu(\mathscr{B}(y_k, \varepsilon^+_{\|z\|\mathfrak{y}_a, \|z\|\mathfrak{r}_a} R(r_k, \omega)))| \rho(r_k) \nu^{\otimes a}(\mathrm{d}\mathfrak{y}_a) \mathrm{d}\mathfrak{r}_a\right] \\
&\leq (2C_\mu)^a E\left[\int_{\mathbb{R}_+^a} \prod_{k=1}^a ((r_k U_a)^p \wedge (r_k U_a)^q) \rho(r_k) \mathrm{d}\mathfrak{r}_a\right],
\end{aligned} \tag{3.13}$$

   where

$$\|z\|\mathfrak{y}_a = (\|z\|y_1, \ldots, \|z\|y_a), \quad \text{and} \quad \|z\|\mathfrak{r}_a = (\|z\|r_1, \ldots, \|z\|r_a).$$

2. Assuming in addition to (3.12) that $\rho(r)$ takes the form (3.8), from (3.13) and under condition (3.12) as in the proof of Proposition 3.4 we get the product estimate

$$\begin{aligned}
&E\left[\int_{(\mathbb{R}^d \times \mathbb{R}_+)^a} \prod_{k=1}^a |\mu(\mathscr{B}(y_k, \varepsilon^+_{\|z\|\mathfrak{y}_a, \|z\|\mathfrak{r}_a} R(r_k, \omega)))| r_k^{2H-d-1} \nu^{\otimes a}(\mathrm{d}\mathfrak{y}_a) \mathrm{d}\mathfrak{r}_a\right] \\
&\quad \leq (2C_\mu)^a E\left[\int_{\mathbb{R}_+^a} \prod_{k=1}^a ((r_k U_a)^p \wedge (r_k U_a)^q) r_k^{2H-d-1} \mathrm{d}\mathfrak{r}_a\right] \\
&\quad = (2C_\mu)^a E[U_a^{a(d-2H)}] \left(\int_0^\infty (r^p \wedge r^q) r^{2H-d-1} \mathrm{d}r\right)^a,
\end{aligned} \tag{3.14}$$

   which is finite provided that

$$\frac{d-q}{2} < H < \frac{d-p}{2}$$

   and $E\left[U_a^{a(d-2H)}\right] < \infty$. This recovers the bound (3.10) by taking $a = 1$.

4. L^q BOUNDS ON THE SHOT NOISE PROCESS

In this section we consider the random field $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by

$$F(z) := \int_0^\infty \int_{\mathbb{R}^d} \mu_z(\mathscr{B}(y, R(r,\omega)))\omega(\mathrm{d}y, \mathrm{d}r), \qquad z \in \mathbb{R}^d, \tag{4.1}$$

where $\mu_z(\mathrm{d}y) := \delta_z(\mathrm{d}y) - \delta_0(\mathrm{d}y)$, *i.e.*

$$\mu_z(\mathscr{B}(y, R(r,\omega))) = \mathbf{1}_{\mathscr{B}(y,R(r,\omega))}(z) - \mathbf{1}_{\mathscr{B}(y,R(r,\omega))}(0),$$

counts the number of balls containing z, minus the number of balls containing 0 for normalization purposes, *cf.* Section 1.1 of [6] for an interpretation in terms of piling of elementary slices.

In the next proposition we take $\sigma(\mathrm{d}y, \mathrm{d}r) = \rho(r)\mathrm{d}r\nu(\mathrm{d}y)$ with $\rho(r) = r^{2H-d-1}$, $r > 0$, as in Proposition 3.4, $\nu(\mathrm{d}y) = \mathrm{d}y$ is the Lebesgue measure, and π_A is defined in (3.4).

Proposition 4.1. *Let $\sigma(\mathrm{d}y, \mathrm{d}r) = r^{2H-d-1}\mathrm{d}r\mathrm{d}y$ with $0 < H < 1/2$, and let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $\nu(A) < \infty$. Assume that $R(r,\omega)$ has the form*

$$R(r,\omega) = G(r, \pi_A(\omega)) = rG(\pi_A(\omega)), \tag{4.2}$$

$r \in \mathbb{R}_+$, $\omega \in \Omega^X$, as in (3.5), where $G : \Omega^{\mathbb{R}_+} \longrightarrow (0,\infty)$ is a positive random variable with

$$0 < c \leq \varepsilon^+_{\mathfrak{r}_a} G(\omega) \leq U_a(\omega), \qquad \mathfrak{r}_a = (r_1, \ldots, r_a) \in \mathbb{R}^a_+, \quad \omega \in \Omega^{\mathbb{R}_+}, \tag{4.3}$$

and $E[U_a^{a(d-2H)}] < \infty$, $a = 1, \ldots, \lceil q \rceil$, for some $q \in [2, \infty)$. Then the random field (4.1) is well-defined in $L^q(\Omega)$ and it satisfies the bound

$$c_H \|z\|^{2H} \leq E\left[\left|\int_0^\infty \int_{\mathbb{R}^d} \mu_z(\mathscr{B}(y, rG(\pi_A(\omega))))\omega(\mathrm{d}y, \mathrm{d}r)\right|^q\right] \leq \|z\|^{2H} u(z), \quad z \in \mathbb{R}^d,$$

where $c_H > 0$ and $z \mapsto u(z)$ is a continuous, non-vanishing function on $\mathbb{R}^d$.

Proof. First, we observe that for every integer $p \geq 1$ we have

$$|\mu_z(\mathscr{B}(y,r))|^{2p} = |\mu_z(\mathscr{B}(y,r))| = \mathbf{1}_{\mathscr{B}(z,r)\Delta\mathscr{B}(0,r)}(y),$$

$y, z \in \mathbb{R}^d$, $r \in \mathbb{R}_+$, and

$$(\mu_z(\mathscr{B}(y,r)))^{2p+1} = \mu_z(\mathscr{B}(y,r)), \qquad y, z \in \mathbb{R}^d, \quad r \in \mathbb{R}_+,$$

hence for all sequences of even integers $l_1, \ldots, l_a$ we find

$$E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a} \prod_{k=1}^a \left|\mu_z(\mathscr{B}(y_k, r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega)))\right|^{l_k} \sigma^{\otimes a}(\mathrm{d}\mathfrak{y}_a, \mathrm{d}\mathfrak{r}_a)\right]$$
$$= E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a} \left|\prod_{k=1}^a \mu_z(\mathscr{B}(y_k, r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega)))\right| \sigma^{\otimes a}(\mathrm{d}\mathfrak{y}_a, \mathrm{d}\mathfrak{r}_a)\right].$$

On the other hand if a sequence $l_1,\ldots,l_a$ contains at least one odd integer (say l_a is odd for example), we find

$$
\begin{aligned}
&E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\varepsilon^+_{\mathfrak{y}_a,\mathfrak{r}_a}\prod_{k=1}^a(\mu_z(\mathscr{B}(y_k,R(r_k,\omega))))^{l_k}\sigma^{\otimes a}(\mathrm{d}\mathfrak{y}_a,\mathrm{d}\mathfrak{r}_a)\right]\\
&=E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\mu_z(\mathscr{B}(y_a,r_a\varepsilon^+_{\mathfrak{r}_a}G(\omega)))\prod_{k=1}^{a-1}(\mu_z(\mathscr{B}(y_k,r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega))))^{l_k}\mathrm{d}\mathfrak{y}_a\prod_{k=1}^a r_k^{2H-d-1}\mathrm{d}\mathfrak{r}_a\right]\\
&=E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\int_{\mathbb{R}^d}\mathbf{1}_{\mathscr{B}(y_a,r_a\varepsilon^+_{\mathfrak{r}_a}G(\omega))}(x)\mu_z(\mathrm{d}x)\prod_{k=1}^{a-1}(\mu_z(\mathscr{B}(y_k,r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega))))^{l_k}\mathrm{d}\mathfrak{y}_a\prod_{k=1}^a r_k^{2H-d-1}\mathrm{d}\mathfrak{r}_a\right]\\
&=E\left[\int_{(\mathbb{R}_+)^a}\int_{(\mathbb{R}^d)^2}\mathbf{1}_{\mathscr{B}(x,r_a\varepsilon^+_{\mathfrak{r}_a}G(\omega))}(y_a)dy_a\mu_z(\mathrm{d}x)\right.\\
&\qquad\left.\times\int_{(\mathbb{R}^d)^{a-1}}\prod_{k=1}^{a-1}(\mu_z(\mathscr{B}(y_k,r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega))))^{l_k}\mathrm{d}\mathfrak{y}_{a-1}\prod_{k=1}^a r_k^{2H-d-1}\mathrm{d}\mathfrak{r}_a\right]\\
&=E\left[\int_{(\mathbb{R}_+)^a}v_d(r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega))^d\right.\\
&\qquad\left.\times\int_{\mathbb{R}^d}\mu_z(\mathrm{d}x)\int_{(\mathbb{R}^d)^{a-1}}\prod_{k=1}^{a-1}(\mu_z(\mathscr{B}(y_k,r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega))))^{l_k}\mathrm{d}\mathfrak{y}_{a-1}\prod_{k=1}^a r_k^{2H-d-1}\mathrm{d}\mathfrak{r}_a\right]\\
&=v_d\mu_z(\mathbb{R}^d)E\left[\int_{(\mathbb{R}_+)^a}(r_a\varepsilon^+_{\mathfrak{r}_a}G(\omega))^d\int_{(\mathbb{R}^d)^{a-1}}\prod_{k=1}^{a-1}(\mu_z(\mathscr{B}(y_k,r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega))))^{l_k}\mathrm{d}\mathfrak{y}_{a-1}\prod_{k=1}^a r_k^{2H-d-1}\mathrm{d}\mathfrak{r}_a\right]\\
&=0,
\end{aligned}
$$

since $\mu_z(\mathbb{R}^d)=0$.

Consequently, the moment formula (2.2) for Poisson random integrals shows that for any integer $p\geq 1$ we have

$$
\begin{aligned}
E[|F(z)|^{2p}]&=\sum_{a=0}^n\sum_{P_1\cup\ldots\cup P_a=\{1,\ldots,2p\}}E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\prod_{k=1}^a(\mu_z(\mathscr{B}(y_k,\varepsilon^+_{\mathfrak{y}_a,\mathfrak{r}_a}R(r_k,\omega))))^{\#P_k}\sigma^{\otimes a}(\mathrm{d}\mathfrak{y}_a,\mathrm{d}\mathfrak{r}_a)\right]\\
&=\sum_{a=0}^p\sum_{\substack{l_1+\ldots+l_a=2p\\ l_1,\ldots,l_a\geq 1}}\mathcal{N}_{\mathfrak{L}_a}E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\prod_{k=1}^a(\mu_z(\mathscr{B}(y_k,r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega))))^{l_k}\sigma^{\otimes a}(\mathrm{d}\mathfrak{y}_a,\mathrm{d}\mathfrak{r}_a)\right]\\
&=\sum_{a=0}^p\sum_{\substack{l_1+\ldots+l_a=2p\\ l_1,\ldots,l_a\geq 2\ \text{even}}}\mathcal{N}_{\mathfrak{L}_a}E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\prod_{k=1}^a|\mu_z(\mathscr{B}(y_k,r_k\varepsilon^+_{\mathfrak{r}_a}G(\omega)))|\sigma^{\otimes a}(\mathrm{d}\mathfrak{y}_a,\mathrm{d}\mathfrak{r}_a)\right],
\end{aligned}
\tag{4.4}
$$

with the notation of Section 2, where in the above summation, $\mathcal{N}_{\mathfrak{L}_a}$ is the number of partitions of a set of $l_1+\ldots+l_a=2p$ elements into a subsets of even cardinalities $l_1,\ldots,l_a\geq 2$. By the rotational invariance of the Lebesgue measure ν, using the notation

$$(\mathfrak{y}_a,\mathfrak{r}_a)=((y_1,r_1),\ldots,(y_a,r_a))\in(\mathbb{R}^d\times\mathbb{R}_+)^a,$$

for all $z \in \mathbb{R}^d$, letting e denote the unit vector $e := z/\|z\|$ in $\mathbb{R}^d$, we have

$$\begin{aligned}
&E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\left|\varepsilon^+_{\mathfrak{y}_a,\mathfrak{r}_a}\prod_{k=1}^a \mu_z(\mathscr{B}(y_k,R(r_k,\omega)))r_k^{2H-d-1}\right|\nu^{\otimes a}(\mathrm{d}\mathfrak{y}_a)\mathrm{d}\mathfrak{r}_a\right]\\
&= E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\left|\varepsilon^+_{\mathfrak{r}_a}\prod_{k=1}^a \mu_z(\mathscr{B}(y_k,r_kG(\pi_A(\omega))))r_k^{2H-d-1}\right|\mathrm{d}\mathfrak{y}_a\mathrm{d}\mathfrak{r}_a\right]\\
&= \|z\|^{ad}E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\left|\varepsilon^+_{\mathfrak{r}_a}\prod_{k=1}^a \mu_z(\mathscr{B}(\|z\|y_k,r_kG(\pi_A(\omega))))r_k^{2H-d-1}\right|\mathrm{d}\mathfrak{y}_a\mathrm{d}\mathfrak{r}_a\right]\\
&= \|z\|^{ad}E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\left|\varepsilon^+_{\mathfrak{r}_a}\prod_{k=1}^a \mu_e(\mathscr{B}(y_k,\|z\|^{-1}r_kG(\pi_A(\omega))))r_k^{2H-d-1}\right|\mathrm{d}\mathfrak{y}_a\mathrm{d}\mathfrak{r}_a\right]\\
&= \|z\|^{ad+a}E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\left|\varepsilon^+_{\|z\|\mathfrak{r}_a}\prod_{k=1}^a \mu_e(\mathscr{B}(y_k,r_kG(\pi_A(\omega))))(\|z\|r_k)^{2H-d-1}\right|\mathrm{d}\mathfrak{y}_a\mathrm{d}\mathfrak{r}_a\right]\\
&= \|z\|^{2aH}E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\left|\varepsilon^+_{\|z\|\mathfrak{r}_a}\prod_{k=1}^a \mu_e(\mathscr{B}(y_k,r_kG(\pi_A(\omega))))r_k^{2H-d-1}\right|\mathrm{d}\mathfrak{y}_a\mathrm{d}\mathfrak{r}_a\right]\\
&= \|z\|^{2aH}E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\left|\prod_{k=1}^a \mu_e(\mathscr{B}(y_k,r_k\varepsilon^+_{\|z\|\mathfrak{r}_a}G(\pi_A(\omega))))r_k^{2H-d-1}\right|\mathrm{d}\mathfrak{y}_a\mathrm{d}\mathfrak{r}_a\right],
\end{aligned}\tag{4.5}$$

and

$$\varepsilon^+_{\mathfrak{y}_n,\mathfrak{r}_n}F(\omega) = F\left(\omega+\delta_{(y_1,r_1)}+\ldots+\delta_{(y_n,r_n)}\right) \qquad \omega\in\Omega^X.$$

Next, by (3.14) applied to $\mu_e \in \mathcal{M}_{d-1,d}$, we get

$$\begin{aligned}
E&\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\left|\prod_{k=1}^a \mu_e\left(\mathscr{B}\left(y_k,r_k\varepsilon^+_{\|z\|\mathfrak{r}_a}G(\omega)\right)\right)r_k^{2H-d-1}\right|\mathrm{d}\mathfrak{y}_a\mathrm{d}\mathfrak{r}_a\right]\\
&\le C^a_{\mu_e}\left(\int_0^\infty (r\wedge 1)r^{2H-2}\mathrm{d}r\right)^a E\left[U_a^{a(d-2H)}\right]\\
&=: (c'_H)^a\\
&< \infty.
\end{aligned}$$

On the other hand, since

$$|\mu_z(\mathscr{B}(y,r))| = \mathbf{1}_{\mathscr{B}(z,r)\Delta\mathscr{B}(0,r)}(y), \qquad y,z\in\mathbb{R}^d, \quad r\in\mathbb{R}_+,$$

where $A\Delta B$ stands for the symmetric difference between $A, B \subset \mathbb{R}^d$, we have

$$\begin{aligned}
&E\left[\int_{(\mathbb{R}^d\times\mathbb{R}_+)^a}\left|\prod_{k=1}^a \mu_e\left(\mathscr{B}\left(y_k,r_k\varepsilon^+_{\|z\|\mathfrak{r}_a}G(\omega)\right)\right)r_k^{2H-d-1}\right|\mathrm{d}\mathfrak{y}_a\mathrm{d}\mathfrak{r}_a\right]\\
&= E\left[\int_{([0,\infty))^a}\left(\int_{\mathbb{R}^d}\mathbf{1}_{\mathscr{B}(e,r_k\varepsilon^+_{\|z\|\mathfrak{r}_a}G(\omega))\Delta\mathscr{B}(0,r_k\varepsilon^+_{\|z\|\mathfrak{r}_a}G(\omega))}(y)\mathrm{d}y\right)^a\prod_{k=1}^a r_k^{2H-d-1}\mathrm{d}\mathfrak{r}_a\right]\\
&\ge E\left[\int_{([1/c,\infty))^a}\left(\int_{\mathbb{R}^d}\mathbf{1}_{\mathscr{B}(e,r_k\varepsilon^+_{\|z\|\mathfrak{r}_a}G(\omega))\Delta\mathscr{B}(0,r_k\varepsilon^+_{\|z\|\mathfrak{r}_a}G(\omega))}(y)\mathrm{d}y\right)^a\prod_{k=1}^a r_k^{2H-d-1}\mathrm{d}\mathfrak{r}_a\right]\\
&\ge c_H^a,
\end{aligned}$$

for some constant $c_H^a > 0$, since

$$\int_{\mathbb{R}^d} \mathbf{1}_{\mathscr{B}\left(e, r_k \varepsilon^+_{\|z\|\mathfrak{r}_a} G(\omega)\right) \Delta \mathscr{B}\left(0, r_k \varepsilon^+_{\|z\|\mathfrak{r}_a} G(\omega)\right)}(y) \mathrm{d}y \geq \int_{\mathbb{R}^d} \mathbf{1}_{\mathscr{B}(e,1)\Delta\mathscr{B}(0,1)}(y)\mathrm{d}y,$$

because $r_k \varepsilon^+_{\|z\|\mathfrak{r}_a} G(\omega) \geq 1$ for $r_k \geq 1/c$, by (4.3), and the volume of the symmetric difference is increasing with the radius.

Therefore there exists $c_H, c'_H \in (0, +\infty)$ such that

$$c_H^a \leq E\left[\int_{(\mathbb{R}^d \times \mathbb{R}_+)^a} \left| \prod_{k=1}^a \mu_e(\mathscr{B}(y_k, r_k \varepsilon^+_{\|z\|\mathfrak{r}_a} G(\omega))) r_k^{2H-d-1} \right| \mathrm{d}\mathfrak{y}_a \mathrm{d}\mathfrak{r}_a\right] \leq (c'_H)^a$$

and (4.5) yields

$$c_H^a \|z\|^{2aH} \leq E\left[\int_{(\mathbb{R}^d \times \mathbb{R}_+)^a} \left| \prod_{k=1}^a \mu_z(\mathscr{B}(y_k, r_k \varepsilon^+_{\mathfrak{r}_a} G(\omega))) r_k^{2H-d-1} \right| \mathrm{d}\mathfrak{y}_a \mathrm{d}\mathfrak{r}_a\right] \leq (c'_H)^a \|z\|^{2aH}. \tag{4.6}$$

By (4.4) and (4.6) we find

$$\sum_{a=1}^p (c_H)^a \|z\|^{2aH} \sum_{\substack{l_1+\ldots+l_a=2p \\ l_1,\ldots,l_a \geq 1 \text{ even}}} \mathcal{N}_{\mathfrak{L}_a} \leq E[|F(z)|^{2p}] \leq \sum_{b=1}^p (c'_H)^b \|z\|^{2bH} \sum_{\substack{l_1+\ldots+l_b=2p \\ l_1,\ldots,l_b \geq 1 \text{ even}}} \mathcal{N}_{\mathfrak{L}_b}, \tag{4.7}$$

$p \geq 1$. Next, we extend (4.7) to all $q \in [2, \infty)$ as in [2], using Hölder interpolation. Given $q \in [2, \infty)$, choose $p \geq 1$ integer such that $2p \leq q < 2p+2$. By the Lyapunov–Hölder inequality applied with $\alpha := p(2p+2-q)/q \in (0,1]$ we have

$$\begin{aligned}
E[|F(z)|^q] &\leq (E[|F(z)|^{2p}])^{\alpha q/(2p)} (E[|F(z)|^{2p+2}])^{(1-\alpha)q/(2p+2)} \\
&\leq u(z)\|z\|^{H(\alpha q/p + (1-\alpha)q/(p+1))} \\
&= u(z)\|z\|^{2H},
\end{aligned}$$

where the function

$$u(z) := \left(\sum_{a=1}^p (c'_H)^a \|z\|^{2(a-1)H} \sum_{\substack{l_1+\ldots+l_a=2p \\ l_1,\ldots,l_a \geq 1 \text{ even}}} \mathcal{N}_{\mathfrak{L}_a}\right)^{\alpha q/(2p)} \left(\sum_{b=1}^{p+1} (c'_H)^b \|z\|^{2(b-1)H} \sum_{\substack{l_1+\ldots+l_b=2p+2 \\ l_1,\ldots,l_b \geq 1 \text{ even}}} \mathcal{N}_{\mathfrak{L}_b}\right)^{(1-\alpha)q/(p+1)}$$

is continuous, non-negative, and non-vanishing on $\mathbb{R}^d$. Similarly, for the lower bound, given $q \in [2, \infty)$, choose $p \geq 2$ integer such that $q < 2p$. By the Lyapunov–Hölder inequality applied with $\alpha := q/(p(2p+2-q)) \in (0,1)$ we have

$$\begin{aligned}
(E[|F(z)|^q])^{2\alpha p/q} &\geq E[|F(z)|^{2p}](E[|F(z)|^{2p+2}])^{-(1-\alpha)p/(p+1)} \\
&\geq v(z)(\|z\|^{2H})^{1-(1-\alpha)p/(p+1)} \\
&= v(z)(\|z\|^{2H})^{2\alpha p/q}, \qquad z \in \mathbb{R}^d,
\end{aligned}$$

where

$$v(z) := \frac{\displaystyle\sum_{a=1}^p (c_H)^a \|z\|^{2(a-1)H} \sum_{\substack{l_1+\ldots+l_a=2p \\ l_1,\ldots,l_a \geq 1 \text{ even}}} \mathcal{N}_{\mathfrak{L}_a}}{\left(\displaystyle\sum_{b=1}^{p+1} (c'_H)^b \|z\|^{2(b-1)H} \sum_{\substack{l_1+\ldots+l_b=2p+2 \\ l_1,\ldots,l_b \geq 1 \text{ even}}} \mathcal{N}_{\mathfrak{L}_b}\right)^{(1-\alpha)p/(p+1)}}, \qquad z \in \mathbb{R}^d.$$

The function $z \mapsto v(z)$ is continuous, non-negative, non-vanishing in $z \in \mathbb{R}^d$, and its leading term as $\|z\|$ tends to infinity has a power of order

$$2\left(p-1-(1-\alpha)\frac{p^2}{p+1}\right)H = 2\left(p-1-p\left(1-\frac{2\alpha p}{q}\right)\right)H = \frac{2(q-2)H}{2p+2-q} \geq 0.$$

Consequently, there exists a constant $C_q > 0$ such that

$$E[|F(z)|^q] \geq C_q \|z\|^{2H}, \qquad z \in \mathbb{R}^d.$$

We conclude that for all $q \in [2, \infty)$ there exists $c_H > 0$ such that

$$c_H \|z\|^{2H} \leq E\left[|F(z)|^q\right] \leq u(z)\|z\|^{2H}, \qquad z \in \mathbb{R}^d. \qquad \square$$

Defining the finite difference operator $D_{\mathfrak{r}_a} = D_{r_1,\dots,r_a}$ as

$$D_{r_1,\dots,r_a} G(\omega) := \varepsilon^+_{r_1,\dots,r_a} G(\omega) - G(\omega), \qquad r_1, \dots, r_a \in \mathbb{R}^a_+,$$

for any random variable $G : \Omega^{\mathbb{R}_+} \longrightarrow \mathbb{R}$, we note that condition (4.3) is satisfied with $U_a = G + aK$ under the uniform bound

$$D_r G(\omega) \leq K, \qquad r \in \mathbb{R}_+, \quad \omega \in \Omega^{\mathbb{R}_+},$$

since by induction on $a \geq 1$ we have

$$\begin{aligned} D_{r_1,\dots,r_a} G(\omega) &= \varepsilon^+_{r_1,\dots,r_a} G(\omega) - G(\omega) \\ &= \varepsilon^+_{r_a} \varepsilon^+_{r_1,\dots,r_{a-1}} G(\omega) - G(\omega) \\ &= \varepsilon^+_{r_a} \varepsilon^+_{r_1,\dots,r_{a-1}} G(\omega) - \varepsilon^+_{r_1,\dots,r_{a-1}} G(\omega) + \varepsilon^+_{r_1,\dots,r_{a-1}} G(\omega) - G(\omega) \\ &= D_{r_a} \varepsilon^+_{r_1,\dots,r_{a-1}} G(\omega) + D_{r_1,\dots,r_{a-1}} G(\omega) \\ &\leq K + (a-1)K = aK, \end{aligned}$$

which implies

$$\varepsilon^+_{r_1,\dots,r_a} G(\omega) \leq G(\omega) + aK,$$

for all $\mathfrak{r}_a = (r_1, \dots, r_a) \in \mathbb{R}^a_+$ and $\omega \in \Omega^{\mathbb{R}_+}$.

5. Fractional behavior of the shot noise process

In this section, as a consequence of Proposition 4.1, in Corollary 5.1 below we investigate the fractional behavior in terms of the index $H \in (0, 1/2)$ of the random field $F : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ defined in (4.1).

By (4.1) and the translation invariance of the Lebesgue measure ν we have the equality in distribution

$$\begin{aligned} F(y+z) - F(y) &= \int_0^\infty \int_{\mathbb{R}^d} (\mathbf{1}_{\mathscr{B}(x, rG(\pi_A(\omega)))}(y+z) - \mathbf{1}_{\mathscr{B}(x, rG(\pi_A(\omega)))}(y))\omega(\mathrm{d}x, \mathrm{d}r) \\ &= \int_0^\infty \int_{\mathbb{R}^d} (\mathbf{1}_{\mathscr{B}(x-y-z, rG(\pi_A(\omega)))}(0) - \mathbf{1}_{\mathscr{B}(x-y, rG(\pi_A(\omega)))}(0))\omega(\mathrm{d}x, \mathrm{d}r) \\ &\overset{\mathrm{d}}{\simeq} \int_0^\infty \int_{\mathbb{R}^d} (\mathbf{1}_{\mathscr{B}(x-z, rG(\pi_A(\omega)))}(0) - \mathbf{1}_{\mathscr{B}(x, rG(\pi_A(\omega)))}(0))\omega(\mathrm{d}x, \mathrm{d}r) \\ &= \int_0^\infty \int_{\mathbb{R}^d} \mu_z(\mathscr{B}(x, rG(\pi_A(\omega))))\omega(\mathrm{d}x, \mathrm{d}r) \\ &= F(z), \end{aligned} \tag{5.1}$$

$y, z \in \mathbb{R}^d$, which shows that the random field $(F(z))_{z\in\mathbb{R}^d}$ has stationary increments by the translation invariance of the Lebesgue measure.

Here, $F(y+z) - F(y)$ counts the difference between the number of balls containing $y+z$ and the number of balls containing y. This random variable is a.s. finite under the conditions of the previous sections. The next result is a consequence of Proposition 4.1 with $\rho(r) = r^{2H-d-1}r$, $r > 0$, and $\nu(\mathrm{d}y) = \mathrm{d}y$ is the Lebesgue measure.

Corollary 5.1. *Let $\sigma(\mathrm{d}y, \mathrm{d}r) = r^{2H-d-1}\mathrm{d}r\mathrm{d}y$ with $0 < H < 1/2$, and let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $\nu(A) < \infty$. Assume that (4.2) holds with $G(r,\omega)$ of the form*

$$G(r, \omega) = rG(\omega), \qquad r \in \mathbb{R}_+, \quad \omega \in \Omega^{\mathbb{R}_+},$$

where

$$c \le \varepsilon^+_{\mathfrak{r}_a} G(\omega) \le U_a(\omega), \qquad \omega \in \Omega^{\mathbb{R}_+}, \quad \mathfrak{r}_a = (r_1, \ldots, r_a) \in \mathbb{R}^a_+, \tag{5.2}$$

and $E[U_a^{a(d-2H)}] < \infty$, for all $a = 1, \ldots, \lceil q \rceil$, for some $q \in [2, \infty)$. Then there exists $C_q, C'_q, \eta_q > 0$ such that the random field

$$F(z) = \int_0^\infty \int_{\mathbb{R}^d} \mu_z(\mathscr{B}(y, R(r,\omega)))\omega(\mathrm{d}y, \mathrm{d}r), \qquad z \in \mathbb{R}^d,$$

defined by (4.1) satisfies

$$C_q|t-s|^{2H} \le E\left[\int_0^1 |F(z+t\theta) - F(z+s\theta)|^q \mathrm{d}t\right] \le C'_q|t-s|^{2H},$$

$z \in \mathbb{R}^d$, $\theta \in S^{d-1}$, $0 \le |t-s| \le \eta_q$, where S^{d-1} is the unit sphere in $\mathbb{R}^d$.

Proof. By Proposition 4.1, for some $\eta_q > 0$ and all $\eta \in [0, \eta_q]$ we have

$$C_q \eta^{2H} \le E[|F(\eta\theta)|^q] \le C'_q \eta^{2H},$$

hence, using the equality in distribution (5.1) we get

$$C_q(t-s)^{2H} \le E[|F(z+t\theta) - F(z+s\theta)|^q] \le C'_q(t-s)^{2H}, \tag{5.3}$$

by the identity in distribution

$$F(z+t\theta) - F(z+s\theta) \overset{\mathrm{d}}{\simeq} F((t-s)\theta) = \int_0^\infty \int_{\mathbb{R}^d} \mu_{(t-s)\theta}(\mathscr{B}(y,r))\omega(\mathrm{d}y, \mathrm{d}r),$$

as in (5.1), $z \in \mathbb{R}^d$, $\theta \in S^{d-1}$, $0 \le t - s \le \eta_q$. The conclusion is obtained by integration with respect to $t \in [0,1]$ in (5.3). □

Finally we consider an example of a random radius G satisfying condition (5.2).

Example 5.2. Let $c, K > 0$, consider $f : \mathbb{R}_+ \longrightarrow [0, K]$ a bounded non-negative function such that

$$\int_0^\infty f^{a(d-2H)}(r) r^{2H-d-1} \mathrm{d}r < \infty, \tag{5.4}$$

$a = 1, \ldots, \lceil q \rceil$, and assume that

$$G(\omega) = c + \int_0^\infty f(r)\omega(\mathrm{d}r) = c + \sum_{r\in\omega} f(r), \qquad \omega \in \Omega^{\mathbb{R}_+},$$

provided that $\nu(\mathbb{R}^d) < \infty$.

Then we have

$$\begin{aligned} 0 < c &\leq \varepsilon^+_{\mathfrak{r}_a} G(\omega) \\ &= G(\omega) + f(r_1) + \ldots + f(r_a) \\ &\leq G(\omega) + aK, \qquad \omega \in \Omega^X, \quad \mathfrak{r}_a = (r_1, \ldots, r_a) \in \mathbb{R}^a_+, \end{aligned}$$

which shows that (5.2) holds with $U_a = G + aK$.

The conditions $E[G^{q(d-2H)}] < \infty$ and $E[U_a^{q(d-2H)}] < \infty$ in Corollary 5.1 are satisfied from (2.3) and (5.4), under the bounds

$$\int_0^\infty |f(r)|^{a(d-2H)} \rho(r) \mathrm{d}r < \infty, \qquad a = 1, \ldots, \lceil q \rceil.$$

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