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EXTREMAL AND ADDITIVE PROCESSES GENERATED BY PARETO DISTRIBUTED RANDOM VECTORS

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Abstract. Pareto distributions are most popular for modeling heavy tailed data. Here, we obtain weak limits of a sequence of extremal and a sequence of additive processes constructed by a series of Bernoulli point processes with bivariate Pareto space components. For the limiting processes we derive the one dimensional distributions in explicit forms. Some of the main properties of these distributions are also proved.

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1. Introduction

Many processes in real life (especially related to finance, economics, hydrology and telecommunications) have heavy tails. The most popular distributions for modeling heavy tails are the Pareto distributions. Here, we focus on bivariate processes. There are several bivariate Pareto distributions. The earliest one of these due to Mardia [31] has the joint survival function specified by

$$\overline{F}\left(x_1, x_2\right) = \left[\frac{x_1}{c_1} + \frac{x_2}{c_2} - 1\right]^{-\alpha} \tag{1.1}$$

for $x_i > c_i \geq 0$ and $\alpha > 0$. This distribution is known as the *bivariate Pareto distribution of the first kind*.

The bivariate Pareto distribution of the first kind has received widespread applications: joint distribution of drivers' injuries in road accidents [18]; modeling of clustering of sociopathy and hysteria in families [19]; outlier tests [3]; Bayesian inference [1]; modeling of the life lengths of components of a system sharing a common environment [28]; theory of queues [32]; modeling the detection of targets in a combat [55]; modeling of operational risk [37]; modeling of daily exchange rate data for four major currencies [47]; reliability evaluation for a multi-state system under stress-strength setup [11]; risk estimation [22]; drought modeling [38]; modeling of bodily injury liability closed claims [54]; statistical analyses on the tail losses of equity portfolios constructed from the stock indexes of six major global financial markets [27]; to mention just a few.

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In the stated applications and others, one is interested in extremes and sums. For example, suppose (1.1) is used to model two different claim amounts. Then one would be interested in maximum claim amounts as well as total claim amounts. As another example suppose (1.1) is used to model life lengths of two different components. Then one would be interested in maximum life lengths as well as average life lengths.

The aim of this paper is to investigate the weak limits of sequences of extremal and additive processes generated by a series of Bernoulli point processes with components distributed according to (1.1). During the last ten years, Pancheva and her students have investigated the relation between sequences of such processes, see [42–46]. The main result in [46] is the functional extremal criterion. It establishes the equivalence between the convergence of a sequence of extremal processes and a sequence of additive processes in very general settings. On the other hand, for modeling of any real phenomena people use particular probability distributions. It was mentioned above that bivariate Pareto distributions are appropriate choices for modeling extremes and sums simultaneously. In the present paper, we assume that the sequences of extremal and additive processes are generated by random vectors following a bivariate Pareto distribution. Using the general results and especially the functional extremal criterion, we obtain limiting processes, which exhibit additional useful properties. Univariate processes of this kind have been used for modeling of operational risk, see [37].

In the paper, calculations are done in the bivariate case, but results can be readily extended to the multivariate case.

The contents of this paper are organized as follows. In Section 2, we give basic definitions. In Section 3, we define and derive properties of the extremal process. In Section 4, we define the corresponding additive processes and derive their properties. Finally, some discussion and conclusions are noted in Section 5.

2. Definitions

Suppose $\mathbf{X}_k = (X_{1k}, X_{2k})$, $k = 1, 2, 3, \ldots$ are independent random vectors having the bivariate Pareto distributions

$$\begin{aligned} &\mathbf{P}\left\{X_{1k} > x_1, X_{2k} > x_2\right\} = \left[\frac{x_1}{c_1(k)} + \frac{x_2}{c_2(k)} - 1\right]^{-\alpha}, \\ &x_i > c_i(k) \geq 0,\ i = 1, 2, \end{aligned} \tag{2.1}$$

where

$$c_i(k) = C_i k^{\beta_i},\ C_i > 0,\ \beta_i > 0,\quad i = 1, 2,\ \text{and } 0 < \alpha < 1. \tag{2.2}$$

In other words, the left endpoint of the support of the distribution tends to infinity $c_i(k) \uparrow \infty$ as $k \to \infty$ on each coordinate $i = 1, 2$.

The condition $\alpha \in (0, 1)$ provides very heavy tailed distributions. Additionally, we suppose that the lower bound of the support of Pareto distributions tends to infinity. This means that, in cases with finite and infinite mathematical expectations, the possible values of these random variables are very large with positive probability. It is possible to assume different rates of increase of the lower bound. We choose $c_i(k)$ to vary regularly in accordance with the regularly varying tails of Pareto distributions.

Let us consider a series of Bernoulli point processes with non-random time components defined as follows

$$\mathcal{N}_n = \left\{\left(t_{nk}, \mathbf{X}_{nk}\right), k = 1, 2, \ldots\right\} := \left\{\left(\frac{k}{n}, \left[\frac{X_{1k}}{a_1(n)}, \frac{X_{2k}}{a_2(n)}\right]\right), k = 1, 2, \ldots\right\},$$

where $\mathbf{a}(n), n = 1, 2, \ldots$, is defined as follows

$$\mathbf{a}(n) = (a_1(n), a_2(n)) = \left[\left(\frac{C_1^\alpha n^{\alpha\beta_1+1}}{\alpha\beta_1+1}\right)^{1/\alpha}, \left(\frac{C_2^\alpha n^{\alpha\beta_2+1}}{\alpha\beta_2+1}\right)^{1/\alpha}\right]. \tag{2.3}$$

Note that the series of point processes is uniformly zero, because

$$\sup_{k\le n} \mathbf{P}\left\{\frac{\mathbf{X}_k}{\mathbf{a}(n)} > \mathbf{x}\right\} = \sup_{k\le n}\left[\frac{x_1 a_1(n)}{c_1(k)} + \frac{x_1 a_1(n)}{c_2(k)} - 1\right]^{-\alpha} = \left[\frac{x_1 a_1(n)}{c_1(n)} + \frac{x_1 a_1(n)}{c_2(n)} - 1\right]^{-\alpha} \to 0,\ n \to \infty,$$

taking in view that $a_i(n)/c_i(n) \to \infty$, $n \to \infty$.

The series of point processes defines:

(i) The sequence of extremal processes

$$\mathbf{Y}_n(t) = (Y_{1n}(t), Y_{2n}(t)) = \bigvee_{k=1}^{k_n(t)} \mathbf{X}_{nk} = \bigvee_{k=1}^{\lfloor nt\rfloor}\left[\frac{X_{1k}}{a_1(n)}, \frac{X_{2k}}{a_2(n)}\right], \quad t > 0, \tag{2.4}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

(ii) The sequence of additive processes

$$\mathbf{S}_n(t) = (S_{1n}(t), S_{2n}(t)) = \sum_{k=1}^{k_n(t)} \mathbf{X}_{nk} = \sum_{k=1}^{\lfloor nt\rfloor}\left[\frac{X_{1k}}{a_1(n)}, \frac{X_{2k}}{a_2(n)}\right], \quad t > 0. \tag{2.5}$$

Here and later we denote $k_n(t) = \max\{k : t_{nk} \le t\} = \max\{k : k \le nt\} = \lfloor x \rfloor$.

3. Sequence of extremal processes

It is known that the main characteristics of an extremal process are its distribution function (d.f.) and its lower curve. For the process $\mathbf{Y}_n$, defined above, the d.f. is

$$H_n(t, \mathbf{x}) = H_n(t, x_1, x_2) = \Pr(\mathbf{Y}_n(t) \le \mathbf{x}) = \Pr(Y_{1n}(t) \le x_1, Y_{2n}(t) \le x_2).$$

The lower curve of the process $\mathbf{Y}_n$ is

$$\mathbf{C}_n(t) = \bigvee_{k=1}^{\lfloor nt\rfloor} \frac{\mathbf{c}(k)}{\mathbf{a}(n)} = \frac{\mathbf{c}([nt])}{\mathbf{a}(n)}, \quad t > 0.$$

Thus, the process $\mathbf{Y}_n$ is defined in the area $[\mathbf{0}, \mathbf{C}_n]^c$. The closed set $[\mathbf{0}, \mathbf{C}_n]$ is called the explosion area of the process.

Recall that a sequence of extremal processes $\mathbf{Y}_n$ converges weakly to the extremal process $\mathbf{Y}$ if the sequence of d.f.s $H_n(t, \mathbf{x})$ converges to the the d.f. $H(t, \mathbf{x})$ at each of its points of continuity. We will denote weak convergence by $\Rightarrow$. Balkema and Pancheva [2] give a detailed discussion of the properties of extremal processes.

Now we are ready to formulate and to prove the following theorem.

Theorem 3.1. *Assume the conditions (2.1), (2.2), and (2.3). Then as $n \to \infty$,*

$$\mathbf{Y}_n \Rightarrow \mathbf{Y},$$

where $\mathbf{Y}(t)$, $t \ge 0$ is a max-infinitely divisible extremal process with d.f.

$$H(t, \mathbf{x}) = \exp\left[-\left(\frac{1}{x_1}\right)^\alpha t^{\alpha\beta_1+1} - \left(\frac{1}{x_2}\right)^\alpha t^{\alpha\beta_2+1} + \int_0^t \left(\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha} u^{\beta_1}} + \frac{x_2}{(\alpha\beta_2+1)^{1/\alpha} u^{\beta_2}}\right)^{-\alpha} \mathrm{d}u\right]$$

and lower curve $\mathbf{C}(t) \equiv 0$.

Proof. Since the d.f. of the process $\mathbf{Y}_n$ is defined in the area $[\mathbf{0}, \mathbf{C}_n]^c$ but the d.f. of the limiting process $\mathbf{Y}$ is defined on $[0, \mathbf{C}]^c$, we have to check first the lower curve condition (see *e.g.*, [46]), *i.e.*, for every $t > 0$, $\mathbf{C}_n(t) \vee \mathbf{C}(t) \to \mathbf{C}(t)$, as $n \to \infty$. Indeed, because of the definitions above, we easily see that for $t > 0$,

$$\mathbf{C}_n(t) \vee \mathbf{C}(t) = \frac{\mathbf{c}([nt])}{\mathbf{a}(n)} \vee \mathbf{0} \to \mathbf{0} \quad \text{as} \quad n \to \infty.$$

For the d.f.s of the extremal processes $\mathbf{Y}_n$, one has

$$\begin{aligned}
H_n(t, \mathbf{x}) = \mathbf{P}\left\{\mathbf{Y}_n(t) < \mathbf{x}\right\} = \prod_{k/n \le t} \mathbf{P}\left\{\frac{\mathbf{X}_k}{\mathbf{a}(n)} < \mathbf{x}\right\} \\
= \prod_{k/n \le t} \left\{1 - \left(\frac{a_1(n)x_1}{c_1(k)}\right)^{-\alpha} - \left(\frac{a_1(n)x_1}{c_1(k)}\right)^{-\alpha} + \left(\frac{a_1(n)x_1}{c_1(k)} + \frac{a_2(n)x_1}{c_2(k)} - 1\right)^{-\alpha}\right\} \\
= \exp\left\{\sum_{k/n \le t} \log\left\{1 - \left(\frac{a_1(n)x_1}{c_1(k)}\right)^{-\alpha} - \left(\frac{a_1(n)x_1}{c_1(k)}\right)^{-\alpha} + \left(\frac{a_1(n)x_1}{c_1(k)} + \frac{a_2(n)x_1}{c_2(k)} - 1\right)^{-\alpha}\right\}\right\} \\
\sim \exp\left\{-\sum_{k/n \le t} \left(\frac{a_1(n)x_1}{c_1(k)}\right)^{-\alpha} - \sum_{k/n \le t} \left(\frac{a_1(n)x_1}{c_1(k)}\right)^{-\alpha} + \sum_{k/n \le t} \left(\frac{a_1(n)x_1}{c_1(k)} + \frac{a_2(n)x_1}{c_2(k)} - 1\right)^{-\alpha}\right\},
\end{aligned}$$

where $\omega_n \sim \delta_n$ as $n \to \infty$ means that $\omega_n/\delta_n \to 1$ as $n \to \infty$. In the last step, we have used the well known relation $\log(1 - x) \sim -x$, as $x \to 0$, and the relations

$$\left(\frac{a_i(n)x_i}{c_i(k)}\right)^{-\alpha} \to 0, \quad \left(\frac{a_1(n)x_1}{c_1(k)} + \frac{a_2(n)x_1}{c_2(k)} - 1\right)^{-\alpha} \to 0$$

uniformly on $k \le nt$ as $n \to \infty$ (*cf.* (2.2), (2.3)). Let us consider the sum

$$\sum_{k/n \le t} \left[\left(\frac{a_1(n)x_1}{c_1(k)}\right)^{-\alpha} + \left(\frac{a_1(n)x_1}{c_1(k)}\right)^{-\alpha} - \left(\frac{a_1(n)x_1}{c_1(k)} + \frac{a_2(n)x_1}{c_2(k)} - 1\right)^{-\alpha}\right] = \Sigma_1(n, t) + \Sigma_2(n, t) - \Sigma_{12}(n, t).$$

We have for $\Sigma_1(n, t)$ that

$$\begin{aligned}
\Sigma_1(n, t) = \sum_{k/n \le t} \left(\frac{c_1(k)}{a_1(n)x_1}\right)^{\alpha} = \left(\frac{1}{a_1(n)x_1}\right)^{\alpha} \sum_{k/n \le t} c_1(k)^{\alpha} \\
\sim \frac{1}{x_1^{\alpha}} \frac{\alpha\beta_1 + 1}{C_1^{\alpha} n^{\alpha\beta_1 + 1}} \sum_{k/n \le t} C_1^{\alpha} k^{\alpha\beta_1} \\
\sim \frac{C_1^{\alpha}}{x_1^{\alpha}} \frac{\alpha\beta_1 + 1}{C_1^{\alpha} n^{\alpha\beta_1 + 1}} \frac{\lfloor nt \rfloor^{\alpha\beta_1 + 1}}{\alpha\beta_1 + 1} \to \frac{1}{x_1^{\alpha}} t^{\alpha\beta_1 + 1}, \quad n \to \infty
\end{aligned}$$

(see Thm. 1.5.11 in Bingham *et al.* [4]). Similarly,

$$\Sigma_2(n, t) \to \frac{1}{x_2^{\alpha}} t^{\alpha\beta_2 + 1}, \quad n \to \infty.$$

Let us consider now $\Sigma_{12}(n,t)$. We get

$$\begin{aligned}
\Sigma_{12}(n,t) &= \sum_{k=1}^{\lfloor nt\rfloor}\left(\frac{a_1(n)x_1}{c_1(k)}+\frac{a_2(n)x_1}{c_2(k)}-1\right)^{-\alpha}\\
&= \sum_{k=1}^{\lfloor nt\rfloor}\left(\frac{C_1 n^{\beta_1}n^{1/\alpha}x_1}{(\alpha\beta_1+1)^{1/\alpha}\,C_1 k^{\beta_1}}+\frac{C_2 n^{\beta_2}n^{1/\alpha}x_2}{(\alpha\beta_2+1)^{1/\alpha}\,C_2 k^{\beta_2}}-1\right)^{-\alpha}\\
&= \frac{1}{n}\sum_{0\le k/n\le t}\left(\frac{n^{\beta_1}x_1}{(\alpha\beta_1+1)^{1/\alpha}\,k^{\beta_1}}+\frac{n^{\beta_2}x_2}{(\alpha\beta_2+1)^{1/\alpha}\,k^{\beta_2}}-\frac{1}{n^{1/\alpha}}\right)^{-\alpha}\\
&= \frac{1}{n}\sum_{0\le k/n\le t}\left(\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}\,(k/n)^{\beta_1}}+\frac{x_2}{(\alpha\beta_2+1)^{1/\alpha}\,(k/n)^{\beta_2}}-\frac{1}{n^{1/\alpha}}\right)^{-\alpha}\\
&\to \int_0^t\left(\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}\,u^{\beta_1}}+\frac{x_2}{(\alpha\beta_2+1)^{1/\alpha}\,u^{\beta_2}}\right)^{-\alpha}\mathrm{d}u,\quad n\to\infty.
\end{aligned}$$

Therefore, as $n\to\infty$,

$$\begin{aligned}
H_n(t,\mathbf{x}) = \mathbf{P}\left\{\mathbf{Y}_n(t)\le x\right\}\to H(t,\mathbf{x}) = \exp\Bigg[&-\left(\frac{1}{x_1}\right)^{\alpha}t^{\alpha\beta_1+1}-\left(\frac{1}{x_2}\right)^{\alpha}t^{\alpha\beta_2+1}\\
&+\int_0^t\left(\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}\,u^{\beta_1}}+\frac{x_2}{(\alpha\beta_2+1)^{1/\alpha}\,u^{\beta_2}}\right)^{-\alpha}\mathrm{d}u\Bigg]
\end{aligned}$$

for $t>0$, $x>0$. Because of the asymptotic negligibility of the series of Bernoulli point processes, we conclude that $\mathbf{Y}_n$ converges weakly to a max-infinitely divisible extremal process $\mathbf{Y}$, generated by a Poisson point process with mean measure (see [46], Thm. 4),

$$\mu\left([0,t]\times[\mathbf{x},\infty)\right) = \left(\frac{1}{x_1}\right)^{\alpha}t^{\alpha\beta_1+1}+\left(\frac{1}{x_2}\right)^{\alpha}t^{\alpha\beta_2+1}-\int_0^t\left(\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}\,u^{\beta_1}}+\frac{x_2}{(\alpha\beta_2+1)^{1/\alpha}\,u^{\beta_2}}\right)^{-\alpha}\mathrm{d}u$$

for $x>0$, $t>0$. The theorem is proved. □

Some properties of the limiting process are proved in the following corollaries.

Corollary 3.2. *The limiting extremal process* $\mathbf{Y}$ *is operator self-similar with exponent*

$$D = \begin{bmatrix}\beta_1+1/\alpha & 0\\ 0 & \beta_2+1/\alpha\end{bmatrix}.$$

Proof. From the d.f. we get

$$\begin{aligned}
\mathbf{P}\left\{Y_1(ct)\le x_1, Y_2(ct)\le x_2\right\} = H(ct,x) = \exp\Bigg[&-\left(\frac{1}{x_1}\right)^{\alpha}(ct)^{\alpha\beta_1+1}-\left(\frac{1}{x_2}\right)^{\alpha}(ct)^{\alpha\beta_2+1}\\
&+\int_0^{tc}\left(\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}\,u^{\beta_1}}+\frac{x_2}{(\alpha\beta_2+1)^{1/\alpha}\,u^{\beta_2}}\right)^{-\alpha}\mathrm{d}u\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \exp\Bigg[-\left(\frac{c^{\beta_1+1/\alpha}}{x_1}\right)^{\alpha} t^{\alpha\beta_1+1} - \left(\frac{c^{\beta_2+1/\alpha}}{x_2}\right)^{\alpha} t^{\alpha\beta_2+1} \\
&\qquad + \int_0^t \left(\frac{c^{-\beta_1-1/\alpha} x_1}{\left(\alpha\beta_1+1\right)^{1/\alpha} u^{\beta_1}} + \frac{c^{-\beta_2-1/\alpha} x_2}{\left(\alpha\beta_2+1\right)^{1/\alpha} u^{\beta_2}} \right)^{-\alpha} \mathrm{d}u \Bigg] \\
&= \mathbf{P}\left\{ Y_1(t) \le \frac{x_1}{c^{\beta_1+1/\alpha}}, Y_2(t) \le \frac{x_2}{c^{\beta_2+1/\alpha}} \right\} \\
&= \mathbf{P}\left\{ c^{\beta_1+1/\alpha} Y_1(t) \le x_1, c^{\beta_2+1/\alpha} Y_2(t) \le x_2 \right\},
\end{aligned}
$$

which completes the proof. □

Corollary 3.3. *Max-increments of the limiting process are independent but not time-homogeneous.*

Proof. The independence of increments follows from the definition. We have to check that they are non-homogeneous. Let $0 < \tau < t$. Then

$$
\mathbf{P}\left\{\mathbf{Y}(t) < \mathbf{x}\right\} = \mathbf{P}\left\{\mathbf{Y}(\tau) \vee \mathbf{U}(\tau, t] < \mathbf{x}\right\} = \mathbf{P}\left\{\mathbf{Y}(\tau) < \mathbf{x}\right\} \mathbf{P}\left\{\mathbf{U}(\tau, t] < \mathbf{x}\right\},
$$

where $\mathbf{U}(\tau, t]$ is the max-increment of $\mathbf{Y}$ in the interval $(\tau, t]$. For its distribution, we have

$$
\begin{aligned}
\mathbf{P}\left\{\mathbf{U}(\tau, t] < \mathbf{x}\right\} = \frac{\mathbf{P}\left\{\mathbf{Y}(t) < \mathbf{x}\right\}}{\mathbf{P}\left\{\mathbf{Y}(\tau) < \mathbf{x}\right\}} = \frac{H(t,\mathbf{x})}{H(\tau,\mathbf{x})} &= \exp\Bigg[-\left(\frac{1}{x_1}\right)^{\alpha} \left(t^{\alpha\beta_1+1} - \tau^{\alpha\beta_1+1}\right) - \left(\frac{1}{x_2}\right)^{\alpha} \left(t^{\alpha\beta_2+1} - \tau^{\alpha\beta_2+1}\right) \\
&\qquad + \int_\tau^t \left(\frac{x_1}{\left(\alpha\beta_1+1\right)^{1/\alpha} u^{\beta_1}} + \frac{x_2}{\left(\alpha\beta_2+1\right)^{1/\alpha} u^{\beta_2}} \right)^{-\alpha} \mathrm{d}u \Bigg],
\end{aligned}
$$

which shows that the increment is not time homogeneous. □

Corollary 3.4. *For every fixed $t > 0$ the random variable $\mathbf{Y}(t)$ has the following distribution*

$$
F_t(\mathbf{x}) = H(t,\mathbf{x}) = \exp\left\{ -\int_0^1 \left[\left(\frac{x_1}{C_1(t)u^{\beta_1}}\right)^{-\alpha} + \left(\frac{x_2}{C_2(t)u^{\beta_2}}\right)^{-\alpha} - \left(\frac{x_1}{C_1(t)u^{\beta_1}} + \frac{x_2}{C_2(t)u^{\beta_2}}\right)^{-\alpha} \right] \mathrm{d}u \right\},
$$

where

$$
C_1(t) = \left(\alpha\beta_1+1\right)^{1/\alpha} t^{\beta_1+1/\alpha}, \quad C_2(t) = \left(\alpha\beta_2+1\right)^{1/\alpha} t^{\beta_2+1/\alpha}.
$$

Proof. The result follows from the representation of the d.f. $H(t,\mathbf{x})$ after some algebraic transforms. □

Corollary 3.5. *The standardized unit Fréchet one dimensional d.f. $F_t^*(\mathbf{x})$ has the following form*

$$
F_t^*\left(y_1, y_2\right) = \exp\left\{ -\frac{1}{y_1} - \frac{1}{y_2} + \int_0^1 \left[\left(\frac{y_1}{\alpha\beta_1+1}\right)^{1/\alpha} u^{-\beta_1} + \left(\frac{y_2}{\alpha\beta_2+1}\right)^{1/\alpha} u^{-\beta_2} \right]^{-\alpha} \mathrm{d}u \right\},
$$

which does not depend on t.

Proof. The marginal distributions of $F_t(\mathbf{x})$ are Fréchet. Indeed, if we let $x_1 \to \infty$ then

$$
\left(\frac{x_1}{C_1(t)u^{\beta_1}}\right)^{-\alpha} \to 0,
$$

and

$$\left(\frac{x_1}{C_1(t)u^{\beta_1}}+\frac{x_2}{C_2(t)u^{\beta_2}}\right)^{-\alpha}\to 0$$

uniformly in $u\in(0,1]$. Therefore, the marginal distribution

$$\begin{aligned}F_{2t}\left(x_2\right)=F_t\left(\infty,x_2\right)&=\exp\left\{-\int_0^1\left(\frac{x_2}{C_2(t)u^{\beta_2}}\right)^{-\alpha}\mathrm{d}u\right\}\\&=\exp\left\{-\left(\frac{x_2}{C_2(t)}\right)^{-\alpha}\int_0^1 u^{\beta_1\alpha}\mathrm{d}u\right\}=\exp\left\{-\left(\frac{x_2}{C_2(t)}\right)^{-\alpha}(\alpha\beta_2+1)\right\}.\end{aligned}$$

Similarly,

$$F_{1t}\left(x_1\right)=F_t\left(x_1,\infty\right)=\exp\left\{-\left(\frac{x_1}{C_1(t)}\right)^{-\alpha}(\alpha\beta_1+1)\right\}.$$

Let us transform $F_t(\mathbf{x})$ to $F_t^*(\mathbf{y})$ with unit Fréchet marginal distributions by the substitution

$$\begin{aligned}x_1&=\left(\frac{1}{-\log F_{1t}\left(y_1\right)}\right)^{\leftarrow}=C_1(t)\left(\frac{y_1}{\alpha\beta_1+1}\right)^{1/\alpha},\\x_2&=\left(\frac{1}{-\log F_{2t}\left(y_2\right)}\right)^{\leftarrow}=C_2(t)\left(\frac{y_2}{\alpha\beta_2+1}\right)^{1/\alpha}.\end{aligned}$$

In this way, we get

$$\begin{aligned}F_t^*(\mathbf{y})&=F_t\left(C_1(t)\left(\frac{y_1}{\alpha\beta_1+1}\right)^{1/\alpha},C_2(t)\left(\frac{y_2}{\alpha\beta_2+1}\right)^{1/\alpha}\right)\\&=\exp\Bigg\{-\int_0^1\Bigg[\left(\frac{C_1(t)y_1^{1/\alpha}}{C_1(t)\left(\alpha\beta_1+1\right)^{1/\alpha}u^{\beta_1}}\right)^{-\alpha}+\left(\frac{C_2(t)y_2^{1/\alpha}}{C_2(t)\left(\alpha\beta_2+1\right)^{1/\alpha}u^{\beta_2}}\right)^{-\alpha}\\&\quad-\left(\frac{C_1(t)y_1^{1/\alpha}}{C_1(t)\left(\alpha\beta_1+1\right)^{1/\alpha}u^{\beta_1}}+\frac{C_2(t)y_2^{1/\alpha}}{C_2(t)\left(\alpha\beta_2+1\right)^{1/\alpha}u^{\beta_2}}\right)^{-\alpha}\Bigg]\mathrm{d}u\Bigg\}\\&=\exp\Bigg\{-\int_0^1\Bigg[\frac{1}{y_1}\frac{u^{\alpha\beta_1}}{\alpha\beta_1+1}+\frac{1}{y_2}\frac{u^{\alpha\beta_2}}{\alpha\beta_2+1}\\&\quad-\left(\left(\frac{y_1}{\alpha\beta_1+1}\right)^{1/\alpha}u^{-\beta_1}+\left(\frac{y_2}{\alpha\beta_2+1}\right)^{1/\alpha}u^{-\beta_2}\right)^{-\alpha}\Bigg]\mathrm{d}u\Bigg\}\\&=\exp\left\{-\frac{1}{y_1}-\frac{1}{y_2}+\int_0^1\left[\left(\frac{y_1}{\alpha\beta_1+1}\right)^{1/\alpha}u^{-\beta_1}+\left(\frac{y_2}{\alpha\beta_2+1}\right)^{1/\alpha}u^{-\beta_2}\right]^{-\alpha}\mathrm{d}u\right\}.\end{aligned}$$

The proof is complete. □

According to Pickands [48], any bivariate extreme value distribution with unit Fréchet margins can be expressed by the d.f.

$$\exp\left[-\left(\frac{1}{x}+\frac{1}{y}\right)A\left(\frac{y}{x+y}\right)\right]\tag{3.1}$$

for $x > 0$ and $y > 0$, where $A(\cdot) : [0,1] \to (1/2, 1]$ is the *dependence function* satisfying: i) $A(0) = A(1) = 1$; ii) $\max(w, 1-w) \le A(w) \le 1$ for all $w \in [0,1]$; iii) $A(\cdot)$ is convex. Corollary 3.6 derives the dependence function corresponding to the extreme value d.f. $F_t^*(\mathbf{y})$ in Corollary 3.5.

Corollary 3.6. *The dependence function of the d.f.* $F_t^*(\mathbf{y})$ *is*

$$A(w) = 1 - \int_0^1 \left[(1-w)^{-1/\alpha} \frac{u^{-\beta_1}}{(\alpha\beta_1 + 1)^{1/\alpha}} + w^{-1/\alpha} \frac{u^{-\beta_2}}{(\alpha\beta_2 + 1)^{1/\alpha}} \right]^{-\alpha} \mathrm{d}u. \tag{3.2}$$

If $\beta_1 = \beta_2 = \beta$ *then*

$$A(w) = 1 - \left[(1-w)^{-1/\alpha} + w^{-1/\alpha} \right]^{-\alpha},$$

the dependence function due to Galambos [13].

Proof. The proof follows immediately from the representation of $F_t^*(\mathbf{y})$ in Corollary 3.5. □

4. Sequence of additive processes

Let us turn now to the sequence of additive processes defined by (2.5). First of all, we have to note that the construction of the process $\mathbf{Y}_n$ adds points of the Bernoulli point process $\mathcal{N}_n$ over the explosion area $[\mathbf{0}, \mathbf{C}]$ of the corresponding extremal process $\mathbf{Y}_n$. This allows us to use the functional extremal criterion (see [46], Thm. 5) for the proof of the weak convergence of the sequence of additive processes.

Theorem 4.1. *The sequence of additive processes*

$$\mathbf{S}_n(t) = (S_{1n}(t), S_{2n}(t)) = \sum_{k=0}^{\lfloor nt \rfloor} \left(\frac{X_{1k}}{a_1(n)}, \frac{X_{2k}}{a_2(n)} \right)$$

converges weakly to the process $\mathbf{S}(t), t \ge 0$ *with independent increments and Levy measure* $\mu(t, \mathbf{x})$*. The characteristic function of the process* $\mathbf{S}(t)$ *has the following form*

$$\begin{aligned} \psi(t, z_1, z_2) &= \exp\left\{ \int_0^t \int_0^\infty \int_0^\infty \left(\mathrm{e}^{\mathrm{i}(x_1 z_1 + x_2 z_2)} - 1 \right) \mu(\mathrm{d}u, \mathrm{d}x_1, \mathrm{d}x_2) \right\} \\ &= \exp\left\{ - \frac{\alpha(\alpha+1)}{(\alpha\beta_1 + 1)^{1/\alpha} (\alpha\beta_2 + 1)^{1/\alpha}} \int_0^t \int_0^\infty \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}(x_1 z_1 + x_2 z_2)} - 1}{u^{\beta_1} u^{\beta_2}} \right. \\ &\cdot \left. \left[\frac{x_1}{(\alpha\beta_1 + 1)^{1/\alpha} u^{\beta_1}} + \frac{x_2}{(\alpha\beta_2 + 1)^{1/\alpha} u^{\beta_2}} \right]^{-\alpha-2} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}u \right\}. \end{aligned}$$

Proof. In order to prove the theorem we have to check the conditions of Theorem 5 in [46]. The first condition, the weak convergence of the corresponding sequence of extremal processes, is proved in Theorem 3.1. Now we have only to check that for fixed $h > 0$ and $t > 0$,

$$\lim_{n \to \infty} \sum_{k=0}^{\lfloor nt \rfloor} \mathbf{E} \left[\frac{X_{1k}}{a_1(n)} I_{\left\{ \frac{X_{1k}}{a_1(n)} \le h \right\}} \right] = \int_0^t \int_0^h \int_0^h x_1 \mu(\mathrm{d}u, \mathrm{d}x_1, \mathrm{d}x_2), \tag{4.1}$$

and

$$\lim_{n \to \infty} \sum_{k=0}^{\lfloor nt \rfloor} \mathbf{E} \left[\frac{X_{2k}}{a_2(n)} I_{\left\{ \frac{X_{2k}}{a_2(n)} \le h \right\}} \right] = \int_0^t \int_0^h \int_0^h x_2 \mu(\mathrm{d}u, \mathrm{d}x_1, \mathrm{d}x_2). \tag{4.2}$$

From (2.1) it follows that the density of the vector $\left(\frac{X_{1k}}{a_1(n)}, \frac{X_{2k}}{a_2(n)}\right)$ is

$$f_{1,2}\left(x_1, x_2\right) = \begin{cases} \alpha(\alpha+1)\frac{a_1(n)a_2(n)}{c_1(k)c_2(k)}\left[\frac{x_1 a_1(n)}{c_1(k)} + \frac{x_2 a_2(n)}{c_2(k)} - 1\right]^{-\alpha-2}, \\ \qquad\qquad \text{if } x_1 > c_1(k),\ x_2 > c_2(k), \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Therefore, one gets

$$\begin{aligned} \mathbf{E}\left[\frac{X_{1k}}{a_1(n)} I_{\left\{\frac{X_{1k}}{a_1(n)} \le h\right\}}\right] &= \int_0^h \int_0^h x_1 \alpha(\alpha+1)\frac{a_1(n)a_2(n)}{c_1(k)c_2(k)}\left[\frac{x_1 a_1(n)}{c_1(k)} + \frac{x_2 a_2(n)}{c_2(k)} - 1\right]^{-\alpha-2} \mathrm{d}x_1 \mathrm{d}x_2 \\ &= -\alpha\frac{a_1(n)}{c_1(k)} \int_0^h x_1 \left[\frac{x_1 a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{-\alpha-1} \mathrm{d}x_1 \\ &= \int_0^h x_1 d\left[\frac{x_1 a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{-\alpha} \\ &= x_1 \left[\frac{x_1 a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{-\alpha}\Bigg|_0^h - \int_0^h \left[\frac{x_1 a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{-\alpha} \mathrm{d}x_1 \\ &= h\left[\frac{h a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{-\alpha} \\ &\quad - \frac{c_1(k)}{a_1(n)(1-\alpha)}\left[\frac{x_1 a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{1-\alpha}\Bigg|_0^h \\ &= h\left[\frac{h a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{-\alpha} \\ &\quad - \frac{c_1(k)}{a_1(n)(1-\alpha)}\left[\frac{h a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{1-\alpha}. \end{aligned}$$

Consider the sum

$$\begin{aligned} \sum_{k=0}^{\lfloor nt \rfloor} \mathbf{E}\left[\frac{X_{1k}}{a_1(n)} I_{\left\{\frac{X_{1k}}{a_1(n)} \le h\right\}}\right] &= \sum_{k=0}^{\lfloor nt \rfloor} h\left[\frac{h a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{-\alpha} \\ &\quad - \sum_{k=0}^{\lfloor nt \rfloor} \frac{c_1(k)}{a_1(n)(1-\alpha)}\left[\frac{h a_1(n)}{c_1(k)} + \frac{h a_2(n)}{c_2(k)} - 1\right]^{1-\alpha} = \mathit{\Sigma}_1(n,t) - \mathit{\Sigma}_2(n,t). \end{aligned}$$

For the first sum, taking in view the formulas for $a_i(n)$ and $c_i(k)$, we obtain

$$
\begin{aligned}
\Sigma_1(n,t) &= \sum_{k=0}^{\lfloor nt \rfloor} h \left[\frac{ha_1(n)}{c_1(k)} + \frac{ha_2(n)}{c_2(k)} - 1\right]^{-\alpha} \\
&= \sum_{k=0}^{\lfloor nt \rfloor} h \left[\frac{hC_1 n^{\beta_1} n^{1/\alpha}}{C_1 \left(\alpha\beta_1+1\right)^{1/\alpha} k^{\beta_1}} + \frac{hC_2 n^{\beta_2} n^{1/\alpha}}{C_2 \left(\alpha\beta_2+1\right)^{1/\alpha} k^{\beta_2}} - 1\right]^{-\alpha} \\
&= \frac{h}{n} \sum_{k=0}^{\lfloor nt \rfloor} \left[\frac{h}{\left(\alpha\beta_1+1\right)^{1/\alpha}} \left(\frac{k}{n}\right)^{-\beta_1} + \frac{h}{\left(\alpha\beta_2+1\right)^{1/\alpha}} \left(\frac{k}{n}\right)^{-\beta_2} - \frac{1}{n^{1/\alpha}}\right]^{-\alpha} \\
&\to h \int_0^t \left[\frac{hu^{-\beta_1}}{\left(\alpha\beta_1+1\right)^{1/\alpha}} + \frac{hu^{-\beta_2}}{\left(\alpha\beta_2+1\right)^{1/\alpha}}\right]^{-\alpha} \mathrm{d}u \\
&= h^{1-\alpha} \int_0^t \left[\frac{1}{\left(\alpha\beta_1+1\right)^{1/\alpha} u^{\beta_1}} + \frac{1}{\left(\alpha\beta_2+1\right)^{1/\alpha} u^{\beta_2}}\right]^{-\alpha} \mathrm{d}u
\end{aligned}
$$

as $n \to \infty$.

Following the same way, one has for the second sum that

$$
\begin{aligned}
\Sigma_2(n,t) &= \sum_{k=0}^{\lfloor nt \rfloor} \frac{c_1(k)}{a_1(n)(1-\alpha)} \left[\frac{ha_1(n)}{c_1(k)} + \frac{ha_2(n)}{c_2(k)} - 1\right]^{1-\alpha} \\
&= \sum_{k=0}^{\lfloor nt \rfloor} \frac{C_1 \left(\alpha\beta_1+1\right)^{1/\alpha} k^{\beta_1}}{C_1 n^{\beta_1} n^{1/\alpha}(1-\alpha)} \left[\frac{hC_1 n^{\beta_1} n^{1/\alpha}}{C_1 \left(\alpha\beta_1+1\right)^{1/\alpha} k^{\beta_1}} + \frac{hC_2 n^{\beta_2} n^{1/\alpha}}{C_2 \left(\alpha\beta_2+1\right)^{1/\alpha} k^{\beta_2}} - 1\right]^{1-\alpha} \\
&= \frac{n^{(1-\alpha)/\alpha} \left(\alpha\beta_1+1\right)^{1/\alpha}}{n^{1/\alpha}(1-\alpha)} \sum_{k=0}^{\lfloor nt \rfloor} \frac{k^{\beta_1}}{n^{\beta_1}} \\
&\quad \cdot \left[\frac{hn^{\beta_1}}{\left(\alpha\beta_1+1\right)^{1/\alpha} k^{\beta_1}} + \frac{hn^{\beta_2}}{\left(\alpha\beta_2+1\right)^{1/\alpha} k^{\beta_2}} - \frac{1}{n^{1/\alpha}}\right]^{1-\alpha} \\
&= \frac{\left(\alpha\beta_1+1\right)^{1/\alpha}}{1-\alpha} \frac{1}{n} \sum_{k=0}^{\lfloor nt \rfloor} \left(\frac{k}{n}\right)^{\beta_1} \\
&\quad \cdot \left[\frac{h}{\left(\alpha\beta_1+1\right)^{1/\alpha}} \left(\frac{k}{n}\right)^{-\beta_1} + \frac{h}{\left(\alpha\beta_2+1\right)^{1/\alpha}} \left(\frac{k}{n}\right)^{-\beta_2}\right]^{1-\alpha} \\
&\to \frac{\left(\alpha\beta_1+1\right)^{1/\alpha}}{1-\alpha} \int_0^t u^{\beta_1} \left(\frac{h}{\left(\alpha\beta_1+1\right)^{1/\alpha}} u^{-\beta_1} + \frac{h}{\left(\alpha\beta_2+1\right)^{1/\alpha}} u^{-\beta_2}\right)^{1-\alpha} \mathrm{d}u, \quad n \to \infty.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{n\to\infty} \sum_{k=0}^{\lfloor nt \rfloor} \mathbf{E}\left[\frac{X_{1k}}{a_1(n)} I_{\left\{\frac{X_{1k}}{a_1(n)} \le h\right\}}\right] &= h^{1-\alpha} \int_0^t \left[\frac{1}{\left(\alpha\beta_1+1\right)^{1/\alpha} u^{\beta_1}} + \frac{1}{\left(\alpha\beta_2+1\right)^{1/\alpha} u^{\beta_2}}\right]^{-\alpha} \\
&\quad - \frac{\left(\alpha\beta_1+1\right)^{1/\alpha}}{1-\alpha} \int_0^t u^{\beta_1} \left[\frac{h}{\left(\alpha\beta_1+1\right)^{1/\alpha} u^{\beta_1}} + \frac{h}{\left(\alpha\beta_2+1\right)^{1/\alpha} u^{\beta_2}}\right]^{1-\alpha} \mathrm{d}u.
\end{aligned}
$$

Similarly, one obtains

$$\lim_{n\to\infty}\sum_{k=0}^{\lfloor nt\rfloor}\mathbf{E}\left[\frac{X_{2k}}{a_2(n)}I_{\left\{\frac{X_{2k}}{a_2(n)}\le h\right\}}\right]=h^{1-\alpha}\int_0^t\left[\frac{1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{1}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha}$$
$$-\frac{(\alpha\beta_2+1)^{1/\alpha}}{1-\alpha}\int_0^t u^{\beta_2}\left[\frac{h}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{1-\alpha}\mathrm{d}u.$$

From (3) it follows that

$$\mu\left(\mathrm{d}t,\mathrm{d}x_1,\mathrm{d}x_2\right)=\frac{\alpha(\alpha+1)}{(\alpha\beta_1+1)^{1/\alpha}t^{\beta_1}(\alpha\beta_2+1)^{1/\alpha}t^{\beta_2}}$$
$$\cdot\left[\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}t^{\beta_1}}+\frac{x_2}{(\alpha\beta_2+1)^{1/\alpha}t^{\beta_2}}\right]^{-\alpha-2}\mathrm{d}t\mathrm{d}x_1\mathrm{d}x_2.$$

Therefore,

$$\begin{aligned}
&\int_0^t\int_0^h\int_0^h x_1\mu\left(\mathrm{d}u,\mathrm{d}x_1,\mathrm{d}x_1\right)\\
&=\int_0^t\int_0^h\int_0^h\frac{\alpha(\alpha+1)x_1u^{-\beta_1}u^{-\beta_2}}{(\alpha\beta_1+1)^{1/\alpha}(\alpha\beta_2+1)^{1/\alpha}}\\
&\quad\cdot\left[\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{x_2}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha-2}\mathrm{d}u\mathrm{d}x_1\mathrm{d}x_2\\
&=\int_0^t\int_0^h\int_0^h\frac{\alpha(\alpha+1)x_1u^{-\beta_1}u^{-\beta_2}}{(\alpha\beta_1+1)^{1/\alpha}(\alpha\beta_2+1)^{1/\alpha}}\\
&\quad\cdot\left[\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{x_2}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha-2}\mathrm{d}x_2\mathrm{d}x_1\mathrm{d}u\\
&=\int_0^t\left\{\int_0^h\frac{-\alpha x_1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}\left[\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha-1}\mathrm{d}x_1\right\}\mathrm{d}u\\
&=\int_0^t\left\{\int_0^h x_1d\left[\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha}\right\}\mathrm{d}u\\
&=\int_0^t\left\{x_1\left[\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha}\Bigg|_0^h\right.\\
&\quad\left.-\int_0^h\left[\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha}\mathrm{d}x_1\right\}\mathrm{d}u\\
&=\int_0^t h\left[\frac{h}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha}\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^t\int_0^h\left[\frac{x_1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha}\mathrm{d}x_1\mathrm{d}u\\
=&\int_0^t h\left[\frac{h}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha}\mathrm{d}u\\
&-\int_0^t\frac{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}{1-\alpha}\left[\frac{h}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{1-\alpha}\mathrm{d}u\\
=&h^{1-\alpha}\int_0^t\left[\frac{1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{1}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha}\mathrm{d}u\\
&-\frac{h^{1-\alpha}(\alpha\beta_1+1)^{1/\alpha}}{1-\alpha}\int_0^t u^{\beta_1}\left[\frac{h}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{1-\alpha}\mathrm{d}u.
\end{aligned}
$$

In the same way, we obtain

$$
\begin{aligned}
\int_0^t\int_0^h\int_0^h x_2\mu(\mathrm{d}u,\mathrm{d}x_1,\mathrm{d}x_1)=&h^{1-\alpha}\int_0^t\left[\frac{1}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{1}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{-\alpha}\mathrm{d}u\\
&-\frac{h^{1-\alpha}(\alpha\beta_2+1)^{1/\alpha}}{1-\alpha}\int_0^t u^{\beta_2}\left[\frac{h}{(\alpha\beta_1+1)^{1/\alpha}u^{\beta_1}}+\frac{h}{(\alpha\beta_2+1)^{1/\alpha}u^{\beta_2}}\right]^{1-\alpha}\mathrm{d}u.
\end{aligned}
$$

Therefore, the relations (4.1) and (4.2) are fulfilled. Using the functional extremal criterion (see Thm. 5, [46]), we complete the proof of the theorem. □

Some properties of the limiting process are given without proofs in the following corollaries.

Corollary 4.2. *The characteristic functions of the marginal distributions are*

$$
\begin{aligned}
\psi(t,z_1)&=\exp\left[-t^{\alpha\beta_1+1}\int_0^\infty\left(\mathrm{e}^{\mathrm{i}x_1z_1}-1\right)x_1^{-\alpha-1}\mathrm{d}x_1\right],\\
\psi(t,z_2)&=\exp\left[-t^{\alpha\beta_2+1}\int_0^\infty\left(\mathrm{e}^{\mathrm{i}x_2z_2}-1\right)x_2^{-\alpha-1}\mathrm{d}x_2\right].
\end{aligned}
$$

Corollary 4.3. *The process* $\mathbf{S}(t)$ *is operator self-similar with exponent*

$$
D=\begin{bmatrix}\beta_1+1/\alpha & 0\\ 0 & \beta_2+1/\alpha\end{bmatrix}.
$$

Remark 4.4. In the particular case $\beta_1=\beta_2=\beta>0$,

$$
\psi(t,z_1,z_2)=\exp\left[-\alpha(\alpha+1)t^{\alpha\beta+1}\int_0^\infty\int_0^\infty\frac{\mathrm{e}^{\mathrm{i}(x_1z_1+x_2z_2)}-1}{(x_1+x_2)^{\alpha+2}}\mathrm{d}x_1\mathrm{d}x_2\right].
$$

For $\mathbf{S}(1)$, we have

$$
\psi(1,z_1,z_2)=\exp\left[-\alpha(\alpha+1)\int_0^\infty\int_0^\infty\frac{\mathrm{e}^{\mathrm{i}(x_1z_1+x_2z_2)}-1}{(x_1+x_2)^{\alpha+2}}\mathrm{d}x_1\mathrm{d}x_2\right].
$$

In this case, the exponent of self-similarity is

$$
D=\begin{bmatrix}\beta+1/\alpha & 0\\ 0 & \beta+1/\alpha\end{bmatrix}.
$$

5. Discussion and conclusions

We have derived limiting laws and exact expressions for sequences of extremal and additive processes made up of the same series of Bernoulli point processes. The space components of the point processes are assumed to follow the earliest known bivariate Pareto distribution with wide ranging applications.

The distributions of the limiting extremal process are described by Theorem 3.1, Corollary 3.4, Corollaries 3.5, and 3.6. The limiting distribution given by each of these results is in closed form except for an integral of the form

$$\int_0^t \left[Au^{-\beta_1} + Bu^{-\beta_1}\right]^{-\alpha} \mathrm{d}u = I\left(\beta_1, \beta_2\right) \tag{5.1}$$

say, where A and B are certain functions of (t, x_1, x_2). For instance, $A = x_1\left(\alpha\beta_1+1\right)^{-1/\alpha}$ and $B = x_2\left(\alpha\beta_2+1\right)^{-1/\alpha}$ in Theorem 3.1, $A = x_1/C_1(t)$ and $B = x_2/C_2(t)$ in Corollary 3.4, $A = y_1^{1/\alpha}\left(\alpha\beta_1+1\right)^{-1/\alpha}$ and $B = y_2^{1/\alpha}\left(\alpha\beta_2+1\right)^{-1/\alpha}$ in Corollary 3.5, and $A = (1-w)^{-1/\alpha}\left(\alpha\beta_1+1\right)^{-1/\alpha}$ and $B = w^{-1/\alpha}\left(\alpha\beta_2+1\right)^{-1/\alpha}$ in Corollary 3.6.

The integral in (5.1) can be expressed in terms of the well known Gauss hypergeometric function defined by

$${}_2F_1\left(a, b; c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. In fact, using equation (3.194.1) in Gradshteyn and Ryzhik [15], we have

$$I\left(\beta_1, \beta_2\right) = \begin{cases} \dfrac{t^{\alpha\beta_1+1}}{A^{\alpha}\left(\alpha\beta_1+1\right)} {}_2F_1\left(\alpha, \dfrac{\alpha\beta_1+1}{\beta_1-\beta_2}; \dfrac{\alpha\beta_1+1}{\beta_1-\beta_2}+1; -\dfrac{B}{A} t^{\beta_1-\beta_2}\right), \\ \qquad\qquad \text{if } \beta_1 > \beta_2, \\ \dfrac{t^{\alpha\beta_2+1}}{B^{\alpha}\left(\alpha\beta_2+1\right)} {}_2F_1\left(\alpha, \dfrac{\alpha\beta_2+1}{\beta_2-\beta_1}; \dfrac{\alpha\beta_2+1}{\beta_2-\beta_1}+1; -\dfrac{A}{B} t^{\beta_2-\beta_1}\right), \\ \qquad\qquad \text{if } \beta_2 > \beta_1, \\ \dfrac{t^{\alpha\beta+1}}{(A+B)^{\alpha}\left(\alpha\beta+1\right)}, \\ \qquad\qquad \text{if } \beta_1 = \beta_2 = \beta. \end{cases}$$

The exact expressions for the distributions of the limiting additive process are described by Theorem 4.1, Corollary 4.2, and Corollary 3.6. The expressions involve triple integrals with respect to x_1, x_2 and u. The integral with respect to u takes the form

$$\int_0^t u^{-\beta_1-\beta_2} \left[Au^{-\beta_1} + Bu^{-\beta_1}\right]^{-\alpha} \mathrm{d}u = J\left(\beta_1, \beta_2\right) \tag{5.2}$$

say, where $A = x_1\left(\alpha\beta_1+1\right)^{-1/\alpha}$ and $B = x_2\left(\alpha\beta_2+1\right)^{-1/\alpha}$.

The integral in (5.2) can also be expressed in terms of the Gauss hypergeometric function. Using equation (3.194.1) in Gradshteyn and Ryzhik [15], we have

$$J\left(\beta_1,\beta_2\right)=\begin{cases}\dfrac{t^{\alpha\beta_1-\beta_1-\beta_2+1}}{A^\alpha\left(\alpha\beta_1-\beta_1-\beta_2+1\right)} \\ \quad \cdot {}_2F_1\left(\alpha,\dfrac{\alpha\beta_1-\beta_1-\beta_2+1}{\beta_1-\beta_2};\dfrac{\alpha\beta_1-\beta_1-\beta_2+1}{\beta_1-\beta_2}+1;-\dfrac{B}{A}t^{\beta_1-\beta_2}\right), \\ \qquad\qquad \text{if } \beta_1>\beta_2, \\ \dfrac{t^{\alpha\beta_2-\beta_1-\beta_2+1}}{B^\alpha\left(\alpha\beta_2-\beta_1-\beta_2+1\right)} \\ \quad \cdot {}_2F_1\left(\alpha,\dfrac{\alpha\beta_2-\beta_1-\beta_2+1}{\beta_2-\beta_1};\dfrac{\alpha\beta_2-\beta_1-\beta_2+1}{\beta_2-\beta_1}+1;-\dfrac{A}{B}t^{\beta_2-\beta_1}\right), \\ \qquad\qquad \text{if } \beta_2>\beta_1, \\ \dfrac{t^{(\alpha-2)\beta+1}}{(A+B)^\alpha\left[(\alpha-2)\beta+1\right]}, \\ \qquad\qquad \text{if } \beta_1=\beta_2=\beta.\end{cases}$$

The Gauss hypergeometric function is well known and well established in the mathematics literature, see Prudnikov *et al.* [49] and Gradshteyn and Ryzhik [15] for detailed properties. In-built numerical routines for computing the Gauss hypergeometric function are available in most mathematical packages, for example, Maple, Mathematica and Matlab.

Corollaries 3.2 and 4.3 gave results involving self-similarity. Self-similar processes have a scaling property, that is the process $X(t)$, $t\geq 0$ is self-similar if $X(at)$ has the same distribution as $a^HX(t)$ for any positive constant a. In the multivariate case (when $\mathbf{X}(t)$, $t\geq 0$ is a vector-valued process) the scaling factors could be different in each coordinate. In this case, the process is said to be operator self-similar. This property is useful for modeling of many real phenomena. Let us mention a few applications in financial modeling: Rachev and Mittnik [50] show that the scaling index will vary between elements of a portfolio containing different stocks; Similar results were obtained by Meerschaert and Scheffler [35] for exchange rates; Jansen and de Vries [20] use these models to explain the 1987 stock market crash. In the analysis of financial data, it is useful to consider the waiting time between trades and the resulting price change as a two dimensional random vector. Meerschaert and Scalas [33] show that different indices apply to price jumps and waiting times. We refer the readers to Chapter 9 in [10] and Chapter 11 in [34] for detailed results and further references on operator self-similar processes.

Next, we discuss how the dependence function in (3.2) compares to known dependence functions in the literature. Some of the most well known dependence functions are:

$$A(w)=1-(\theta+\phi)w+\theta w^2+\phi w^3 \tag{5.3}$$

due to Tawn [52], where $\theta\geq 0$, $\theta+3\phi\geq 0$, $\theta+\phi\leq 1$ and $\theta+2\phi\leq 1$;

$$A(w)=\left(1-\phi_1\right)(1-w)+\left(1-\phi_2\right)w+\left[\left(\phi_1 w\right)^{1/\theta}+\left(\phi_2(1-w)\right)^{1/\theta}\right]^\theta \tag{5.4}$$

due to Tawn [52], where $0<\theta\leq 1$ and $0\leq\phi_1,\phi_2\leq 1$;

$$A(w)=w\Phi\left(\frac{1}{\theta}+\frac{\theta}{2}\log\frac{w}{1-w}\right)+(1-w)\Phi\left(\frac{1}{\theta}-\frac{\theta}{2}\log\frac{w}{1-w}\right) \tag{5.5}$$

due to Hüsler and Reiss [17], where $\theta \geq 0$ and $\Phi(\cdot)$ denotes the d.f. of a standard normal random variable;

$$A(w) = \int_0^1 \max\left[(1-\beta)(1-w)t^{-\beta}, (1-\delta)w(1-t)^{-\delta}\right] \mathrm{d}t \tag{5.6}$$

due to Joe *et al.* [23] and Coles and Tawn [7], where $(\beta, \delta) \in (0,1)^2 \cup (-\infty, 0)^2$; and,

$$A(w) = w t_{\xi+1}\left(\sqrt{\frac{1+\xi}{1-\rho^2}}\left[\left(\frac{w}{1-w}\right)^{1/\xi} - \rho\right]\right) + (1-w) t_{\xi+1}\left(\sqrt{\frac{1+\xi}{1-\rho^2}}\left[\left(\frac{1-w}{w}\right)^{1/\xi} - \rho\right]\right) \tag{5.7}$$

due to Demarta and McNeil [8], where $-1 < \rho < 1$, $\xi > 0$ and $t_\nu(\cdot)$ denotes the d.f. of a Student's t random variable with ν degrees of freedom.

There are no well developed methods to compare dependence functions. Here, we suggest one new method to compare dependence functions. A known measure of dependence due to Tawn [52] is

$$D_1 = 2 - 2A\left(\frac{1}{2}\right) \cdot \tag{5.8}$$

A measure of asymmetry proposed recently by Rosco and Joe [51] is

$$D_2 = \sup_{0<u<1, 0<v<1} \left|C(u,v) - C_R(u,v)\right|, \tag{5.9}$$

where $C(u,v)$ denotes the copula corresponding to (3.1) and $C_R(u,v) = 1-u-v+C(1-u,1-v)$. The possible values of D_1 are $0 \leq D_1 \leq 1$. Values of D_1 close to 1 correspond to high degrees of dependence and values of D_1 close to 0 correspond to low degrees of dependence. The possible values of D_2 are $0 \leq D_2 \leq 1/3$. Values of D_2 close to $1/3$ correspond to high degrees of asymmetry and values of D_2 close to 0 correspond to low degrees of asymmetry.

The method we suggest for comparing dependence functions is a plot of D_1 *versus* D_2 for all possible parameter values. The area covered by the plot will be a measure of flexibility of $A(w)$. The bigger the area the greater the flexibility.

The (D_1, D_2) plots for $A(w)$ given by (3.2), (5.3)–(5.7) are shown in Figure 1. We can see that (3.2) is more flexible than all five of the dependence functions but (5.4), which appears to be the most flexible model. The $A(w)$ in (5.6) appears to be the third most flexible model. The $A(w)$ in (5.3) appears to be the fourth most flexible model. The $A(w)$ in (5.7) appears to be the fifth most flexible model. The $A(w)$ in (5.5) appears to be the least flexible model.

Hence, we can expect (3.2) to be a better model to all areas where the models given by (5.6), (5.3), (5.7) and (5.5) have been applied.

Some recent applications of (5.3) have included storm frequency analysis [56], financial risk assessment [36], modeling of SOA medical large claims database [6], estimation of the probability of two dependent catastrophic events [25], and estimation of risk measures in energy portfolios [21].

Some recent applications of (5.5) have included portfolio risk measurement [5], fitting joint cumulative returns between a market index and a single stock to daily data [16], risk management for Jamaican equity and foreign exchange markets [24], extreme dependence of multivariate catastrophic losses [26], estimation of risk measures in energy portfolios [21], and probabilistic landslide hazard assessment [39].

Some applications of (5.6) have included models for structural design [7] and models for extreme wind speeds [12].

Some recent applications of (5.7) have included models for multivariate high-frequency data in finance [9], risk management [29], spatio-temporal variations of precipitation extremes in Xinjiang, China [57], modeling of multivariate drought characteristics [30], performance assessment of reconstructed watersheds [40], and regional air quality conformity in transportation networks with stochastic dependencies [41].

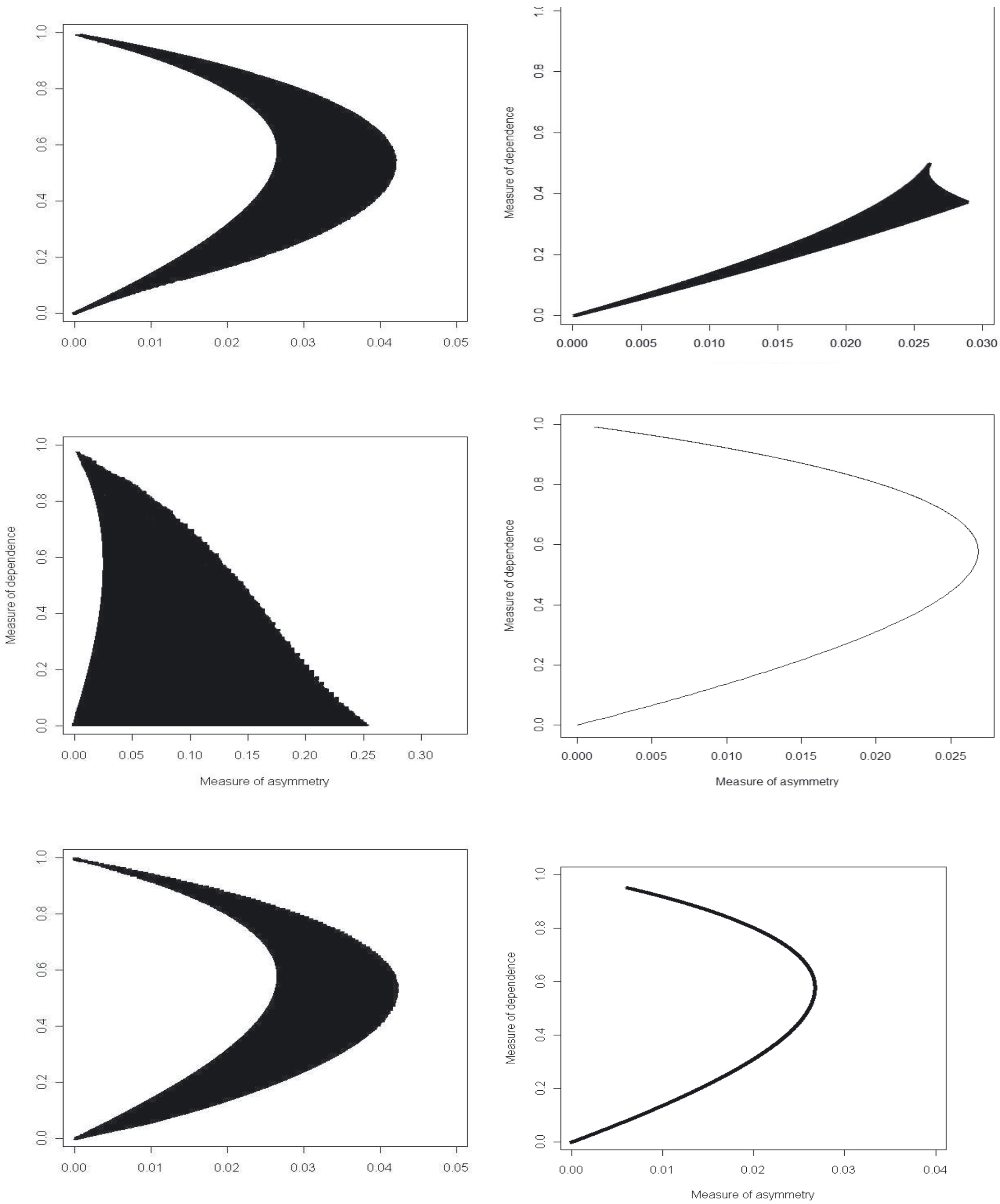

FIGURE 1. (D_1, D_2) plots for $A(w)$ given by (3.2) (top left), (5.3) (top right), (5.4) (middle left), (5.5) (middle right), (5.6) (bottom left) and (5.7) (bottom right).

The dependence functions in (5.3)−(5.7) have also been applied to areas where the marginals are assumed to be Pareto distributed, consistent with the assumptions in Section 2. Examples include temporal and spatial variability of drought in mountain catchments of the Nysa Klodzka basin [53] and estimation of portfolio value at risk [14].

The effect of the parameters, α, β_1 and β_2, of (3.2) on the degree of dependence can be described as follows: small values of α (*i.e.*, the values of α close to 0) correspond to large values of D_1 (*i.e.*, the values of D_1 close to 1); large values of α correspond to small values of D_1 (*i.e.*, the values of D_1 close to 0); for small values of α (*i.e.*, the values of α close to 0) larger values of β_1 and β_2 correspond to larger values of D_2 (*i.e.*, the values of D_2 closer to 1/3); for large values of α larger values of β_1 and β_2 correspond to smaller values of D_2 (*i.e.*, the values of D_2 closer to 0).

Finally, we state the multivariate case of the problem considered in this paper. Suppose $\mathbf{X}_k = (X_{1k}, X_{2k}, \ldots, X_{pk})$, $k = 1, 2, 3, \ldots$ are independent random vectors having the p-variate Pareto distributions

$$\mathbf{P}\{X_{1k} > x_1, X_{2k} > x_2, \ldots, X_{pk} > x_p\} = \left[\sum_{j=1}^{p} \frac{x_j}{c_j(k)} - p + 1\right]^{-\alpha},$$
$$x_i > c_i(k) \geq 0,\ i = 1, 2, \ldots, p,$$

where

$$c_i(k) = C_i k^{\beta_i},\ C_i > 0,\ \beta_i > 0,\quad i = 1, 2, \ldots, p \text{ and } 0 < \alpha < 1.$$

Define $\mathbf{a}(n), n = 1, 2, \ldots$ as follows

$$\mathbf{a}(n) = (a_1(n), a_2(n), \ldots, a_p(n)) = \left[\left(\frac{C_1^\alpha n^{\alpha\beta_1+1}}{\alpha\beta_1+1}\right)^{1/\alpha}, \left(\frac{C_2^\alpha n^{\alpha\beta_2+1}}{\alpha\beta_2+1}\right)^{1/\alpha}, \ldots, \left(\frac{C_p^\alpha n^{\alpha\beta_p+1}}{\alpha\beta_p+1}\right)^{1/\alpha}\right].$$

The sequence of extremal processes and the sequence of additive processes defined by (2.4) and (2.5) generalize to

$$\mathbf{Y}_n(t) = \bigvee_{k=1}^{\lfloor nt \rfloor} \left[\frac{X_{1k}}{a_1(n)}, \frac{X_{2k}}{a_2(n)}, \ldots, \frac{X_{pk}}{a_p(n)}\right], \quad t > 0$$

and

$$\mathbf{S}_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \left[\frac{X_{1k}}{a_1(n)}, \frac{X_{2k}}{a_2(n)}, \ldots, \frac{X_{pk}}{a_p(n)}\right], \quad t > 0,$$

respectively. Under this set up, we seek to generalize Theorems 1 and 2. That is, the problem is to determine the limiting process of $\mathbf{Y}_n(t)$ and the limiting process of $\mathbf{S}_n(t)$ as $n \to \infty$.

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