ESAIM: PS 18 (2014) 117-129
DOI: 10.1051/ps/2012029

**ESAIM: Probability and Statistics**
www.esaim-ps.org

# UPPER LARGE DEVIATIONS FOR MAXIMAL FLOWS THROUGH A TILTED CYLINDER

Marie Theret[1]

**Abstract.** We consider the standard first passage percolation model in $\mathbb{Z}^d$ for $d \geq 2$ and we study the maximal flow from the upper half part to the lower half part (respectively from the top to the bottom) of a cylinder whose basis is a hyperrectangle of sidelength proportional to $n$ and whose height is $h(n)$ for a certain height function $h$. We denote this maximal flow by $\tau_n$ (respectively $\phi_n$). We emphasize the fact that the cylinder may be tilted. We look at the probability that these flows, rescaled by the surface of the basis of the cylinder, are greater than $\nu(\boldsymbol{v}) + \varepsilon$ for some positive $\varepsilon$, where $\nu(\boldsymbol{v})$ is the almost sure limit of the rescaled variable $\tau_n$ when $n$ goes to infinity. On one hand, we prove that the speed of decay of this probability in the case of the variable $\tau_n$ depends on the tail of the distribution of the capacities of the edges: it can decay exponentially fast with $n^{d-1}$, or with $n^{d-1}\min(n, h(n))$, or at an intermediate regime. On the other hand, we prove that this probability in the case of the variable $\phi_n$ decays exponentially fast with the volume of the cylinder as soon as the law of the capacity of the edges admits one exponential moment; the importance of this result is however limited by the fact that $\nu(\boldsymbol{v})$ is not in general the almost sure limit of the rescaled maximal flow $\phi_n$, but it is the case at least when the height $h(n)$ of the cylinder is negligible compared to $n$.

**Mathematics Subject Classification.** 60K35, 60F10.

Received July 3, 2009.

## 1. Definitions and main results

Let $d \geq 2$. We consider the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ having for vertices $\mathbb{Z}^d$ and for edges $\mathbb{E}^d$, the set of pairs of nearest neighbours for the standard $L^1$ norm. With each edge $e$ in $\mathbb{E}^d$ we associate a random variable $t(e)$ with values in $[0, +\infty[$. We assume that the family $(t(e), e \in \mathbb{E}^d)$ is independent and identically distributed, with a common distribution function $F$: this is the standard model of first passage percolation on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$. We interpret $t(e)$ as the capacity of the edge $e$; it means that $t(e)$ is the maximal amount of fluid that can go through the edge $e$ per unit of time.

The maximal flow $\phi(F_1 \rightarrow F_2 \text{ in } C)$ from $F_1$ to $F_2$ in $C$, for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d$) can be defined properly this way. We will say that an edge $e = \langle x, y \rangle$ belongs to a subset $A$ of $\mathbb{R}^d$, which we denote by $e \in A$, if the set $\{x + t\,\overrightarrow{xy} \,|\, t \in ]0,1[\}$ is included in $A$. We define $\widetilde{\mathbb{E}}^d$ as the set of all the oriented edges, *i.e.*, an element $\widetilde{e}$ in $\widetilde{\mathbb{E}}^d$ is an ordered pair of vertices which are nearest neighbours. We denote an element $\widetilde{e} \in \widetilde{\mathbb{E}}^d$ by $\langle\langle x, y \rangle\rangle$, where $x$, $y \in \mathbb{Z}^d$ are the endpoints of $\widetilde{e}$ and the edge is oriented from $x$

*Keywords and phrases.* First passage percolation, maximal flow, large deviations.

[1] D.M.A., E.N.S., 45 rue d'Ulm, 75230 Paris Cedex 05, France. marie.theret@ens.fr

towards $y$. We consider the set $\mathcal{S}$ of all pairs of functions $(g,o)$, with $g:\mathbb{E}^d\to[0,+\infty[$ and $o:\mathbb{E}^d\to\widetilde{\mathbb{E}}^d$ such that $o(\langle x,y\rangle)\in\{\langle\langle x,y\rangle\rangle,\langle\langle y,x\rangle\rangle\}$, satisfying:

- for each edge $e$ in $C$ we have
$$0\,\leq\,g(e)\,\leq\,t(e),$$
- for each vertex $v$ in $C\smallsetminus(F_1\cup F_2)$ we have
$$\sum_{e\in C\,:\,o(e)=\langle\langle v,\cdot\rangle\rangle}g(e)\;=\;\sum_{e\in C\,:\,o(e)=\langle\langle\cdot,v\rangle\rangle}g(e),$$

where the notation $o(e)=\langle\langle v,.\rangle\rangle$ (respectively $o(e)=\langle\langle .,v\rangle\rangle$) means that there exists $y\in\mathbb{Z}^d$ such that $e=\langle v,y\rangle$ and $o(e)=\langle\langle v,y\rangle\rangle$ (respectively $o(e)=\langle\langle y,v\rangle\rangle$). A couple $(g,o)\in\mathcal{S}$ is a possible stream in $C$ from $F_1$ to $F_2$: $g(e)$ is the amount of fluid that goes through the edge $e$, and $o(e)$ gives the direction in which the fluid goes through $e$. The two conditions on $(g,o)$ express only the fact that the amount of fluid that can go through an edge is bounded by its capacity, and that there is no loss of fluid in the graph. With each possible stream we associate the corresponding flow

$$\mathrm{flow}(g,o)\;=\;\sum_{u\in F_2,\,v\notin C\,:\,\langle u,v\rangle\in\mathbb{E}^d}g(\langle u,v\rangle)\mathbb{1}_{o(\langle u,v\rangle)=\langle\langle u,v\rangle\rangle}-g(\langle u,v\rangle)\mathbb{1}_{o(\langle u,v\rangle)=\langle\langle v,u\rangle\rangle}.$$

This is the amount of fluid that crosses $C$ from $F_1$ to $F_2$ if the fluid respects the stream $(g,o)$. The maximal flow through $C$ from $F_1$ to $F_2$ is the supremum of this quantity over all possible choices of streams

$$\phi(F_1\to F_2\text{ in }C)\;=\;\sup\{\mathrm{flow}(g,o)\,|\,(g,o)\in\mathcal{S}\}.$$

The maximal flow $\phi(F_1\to F_2\text{ in }C)$ can be expressed differently thanks to the max-flow min-cut theorem (see [1]). We need some definitions to state this result. A path on the graph $\mathbb{Z}^d$ from $v_0$ to $v_m$ is a sequence $(v_0,e_1,v_1,\dots,e_m,v_m)$ of vertices $v_0,\dots,v_m$ alternating with edges $e_1,\dots,e_m$ such that $v_{i-1}$ and $v_i$ are neighbours in the graph, joined by the edge $e_i$, for $i$ in $\{1,\dots,m\}$. A set $E$ of edges in $C$ is said to cut $F_1$ from $F_2$ in $C$ if there is no path from $F_1$ to $F_2$ in $C\smallsetminus E$. We call $E$ an $(F_1,F_2)$-cut if $E$ cuts $F_1$ from $F_2$ in $C$ and if no proper subset of $E$ does. With each set $E$ of edges we associate its capacity which is the random variable

$$V(E)\;=\;\sum_{e\in E}t(e).$$

The max-flow min-cut theorem states that

$$\phi(F_1\to F_2\text{ in }C)\;=\;\min\{\,V(E)\,|\,E\text{ is a }(F_1,F_2)\text{-cut in }C\,\}.$$

We need now some geometric definitions. For a subset $X$ of $\mathbb{R}^d$, we denote by $\mathcal{H}^s(X)$ the $s$-dimensional Hausdorff measure of $X$ (we will use $s=d-1$ and $s=d-2$). If $X$ is a subset of $\mathbb{R}^d$ included in an hyperplane of $\mathbb{R}^d$ and of co-dimension 1 (for example a non degenerate hyperrectangle), we denote by $\mathrm{hyp}(X)$ the hyperplane spanned by $X$, and we denote by $\mathrm{cyl}(X,h)$ the cylinder of basis $X$ and of height $2h$ defined by

$$\mathrm{cyl}(X,h)\;=\;\{x+t\boldsymbol{v}\,|\,x\in X,\,t\in[-h,h]\},$$

where $\boldsymbol{v}$ is one of the two unit vectors orthogonal to $\mathrm{hyp}(X)$.

Let $A$ be a non degenerate hyperrectangle, *i.e.*, a box of dimension $d-1$ in $\mathbb{R}^d$. All hyperrectangles will be supposed to be closed in $\mathbb{R}^d$. We denote by $\boldsymbol{v}$ one of the two unit vectors orthogonal to $\mathrm{hyp}(A)$. For $h$ a positive real number, we consider the cylinder $\mathrm{cyl}(A,h)$. The set $\mathrm{cyl}(A,h)\smallsetminus\mathrm{hyp}(A)$ has two connected components, which we denote by $\mathcal{C}_1(A,h)$ and $\mathcal{C}_2(A,h)$. For $i=1,2$, let $A_i^h$ be the set of the points in $\mathcal{C}_i(A,h)\cap\mathbb{Z}_n^d$ which have a nearest neighbour in $\mathbb{Z}^d\smallsetminus\mathrm{cyl}(A,h)$:

$$A_i^h\;=\;\{x\in\mathcal{C}_i(A,h)\cap\mathbb{Z}^d\,|\,\exists y\in\mathbb{Z}^d\smallsetminus\mathrm{cyl}(A,h),\,\langle x,y\rangle\in\mathbb{E}^d\}.$$

Let $T(A,h)$ (respectively $B(A,h)$) be the top (respectively the bottom) of $\mathrm{cyl}(A,h)$, *i.e.*,

$$T(A,h) \,=\, \{x \in \mathrm{cyl}(A,h) \,|\, \exists y \notin \mathrm{cyl}(A,h),\ \langle x,y\rangle \in \mathbb{E}^d \text{ and } \langle x,y\rangle \text{ intersects } A+h\boldsymbol{v}\}$$

and

$$B(A,h) \,=\, \{x \in \mathrm{cyl}(A,h) \,|\, \exists y \notin \mathrm{cyl}(A,h),\ \langle x,y\rangle \in \mathbb{E}^d \text{ and } \langle x,y\rangle \text{ intersects } A-h\boldsymbol{v}\}.$$

For a given realization $(t(e), e\in\mathbb{E}^d)$ we define the variable $\tau(A,h)=\tau(\mathrm{cyl}(A,h),\boldsymbol{v})$ by

$$\tau(A,h) \,=\, \tau(\mathrm{cyl}(A,h),\boldsymbol{v}) \,=\, \phi(A_1^h \to A_2^h \text{ in } \mathrm{cyl}(A,h)),$$

and the variable $\phi(A,h)=\phi(\mathrm{cyl}(A,h),\boldsymbol{v})$ by

$$\phi(A,h) \,=\, \phi(\mathrm{cyl}(A,h),\boldsymbol{v}) \,=\, \phi(B(A,h) \to T(A,h) \text{ in } \mathrm{cyl}(A,h)),$$

where $\phi(F_1 \to F_2 \text{ in } C)$ is defined previously.

There exist laws of large numbers concerning these two variables. We summarize the results here. The law of large numbers for $\tau$ and for $\phi$ in flat cylinders is the following:

**Theorem 1.1** (Rossignol and Théret [3])**.** *We suppose that*

$$\int_{[0,+\infty[} x \,\mathrm{d}F(x) \,<\, \infty.$$

*Then for every unit vector $\boldsymbol{v}$, there exists a constant $\nu(\boldsymbol{v})=\nu(\boldsymbol{v},d,F)$ such that for every non degenerate hyperrectangle $A$ orthogonal to $\boldsymbol{v}$, for every function $h:\mathbb{N}\to\mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n)=+\infty$, we have*

$$\lim_{n\to\infty} \frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(\boldsymbol{v}) \quad \textit{in } L^1.$$

*Moreover, if $0\in A$, where $0$ is the origin of the graph, or if*

$$\int_{[0,+\infty[} x^{1+\frac{1}{d-1}} \,\mathrm{d}F(x) \,<\, \infty,$$

*then*

$$\lim_{n\to\infty} \frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(\boldsymbol{v}) \quad \textit{a.s.}$$

*If $\lim_{n\to\infty} h(n)/n = 0$, the same convergences (in $L^1$ and a.s.) hold for $\phi(nA,h(n))$ under the same hypotheses.*

Thanks to the works of Kesten [2] and Zhang [5], we know that $\nu(\boldsymbol{v})>0$ if and only if $F(0)<1-p_c(d)$, where $p_c(d)$ is the critical parameter for the edge percolation on $\mathbb{Z}^d$. Kesten, Zhang, and finally Rossignol and Théret have proved a law of large numbers for the variable $\phi(A,h)$ in straight cylinders, *i.e.*, when $A$ is of the form $\prod_{i=1}^{d-1}[0,k_i]\times\{0\}$ with $k_i>0$ for all $i=1,\dots,d-1$, for large $A$ and $h$. Kesten and Zhang have worked in the general case where the dimensions of the cylinder go to infinity with possibly different speed. We present here the result stated by Rossignol and Théret in [3], with the best conditions on the moments of $F$ and on the height function $h$, but in the more restrictive case where the cylinder we consider is simply $\mathrm{cyl}(nA,h(n))$:

**Theorem 1.2** (Rossignol and Théret [3])**.** *We suppose that*

$$\int_{[0,+\infty[} x \,\mathrm{d}F(x) \,<\, \infty.$$

*Let $\boldsymbol{v}_0=(0,\dots,0,1)$. For every hyperrectangle $A$ of the form $\prod_{i=1}^{d-1}[0,k_i]\times\{0\}$ with $k_i>0$ for all $i=1,..,d-1$, and for every function $h:\mathbb{N}\to\mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n)=+\infty$ and $\lim_{n\to\infty}\log h(n)/n^{d-1}=0$, we have*

$$\lim_{n\to\infty} \frac{\phi(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(\boldsymbol{v}) \quad \textit{a.s. and in } L^1.$$

We investigate the upper large deviations of the variables $\phi$ and $\tau$. We will prove the following theorem concerning $\tau$:

**Theorem 1.3.** *Let $A$ be a non degenerate hyperrectangle, and $\boldsymbol{v}$ one of the two unit vectors normal to $A$. Let $h : \mathbb{N} \to \mathbb{R}^+$ be a height function satisfying $\lim_{n\to\infty} h(n) = +\infty$. The upper large deviations of $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ depend on the tail of the distribution of the capacities. Indeed, we obtain that:*
*i) if the law of the capacity of the edges has bounded support, then for every $\lambda > \nu(\boldsymbol{v})$ we have*

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)\min(h(n), n)} \log \mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda\right] \,>\, 0\,; \tag{1.1}$$

*the upper large deviations are then of volume order for height functions $h$ such that $h(n)/n$ is bounded, and of order $n^d$ if $\lim_{n\to\infty} h(n)/n = +\infty$.*
*ii) if the capacity of the edges follows the exponential law of parameter 1, then there exists $n_0(d, A, h)$, and for every $\lambda > \nu(\boldsymbol{v})$ there exists a positive constant $D$ depending only on $d$ and $\lambda$ such that for all $n \geq n_0$ we have*

$$\frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda\right] \,\leq\, D. \tag{1.2}$$

*iii) if the law of the capacity of the edges satisfies*

$$\forall \theta > 0 \qquad \int_{[0,+\infty[} \mathrm{e}^{\theta x} \mathrm{d}F(x) \,<\, \infty,$$

*then for all $\lambda > \nu(\boldsymbol{v})$ we have*

$$\lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda\right] \,=\, +\infty. \tag{1.3}$$

We also prove the following partial result concerning the variable $\phi$:

**Theorem 1.4.** *Let $A$ be a non degenerate hyperrectangle in $\mathbb{R}^d$, of normal unit vector $\boldsymbol{v}$, and $h : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying $\lim_{n\to\infty} h(n) = +\infty$. We suppose that the law of the capacities of the edges admits an exponential moment:*

$$\exists \theta > 0 \qquad \int_{[0,+\infty[} \mathrm{e}^{\theta x} \,\mathrm{d}F(x) \,<\, \infty.$$

*Then for every $\lambda > \nu(\boldsymbol{v})$, we have*

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)h(n)} \log \mathbb{P}[\phi(nA, h(n)) \geq \lambda \mathcal{H}^{d-1}(nA)] \,>\, 0.$$

**Remark 1.5.** We recall the reader that the asymptotic behaviour of $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ for large $n$ is not known in general. For straight cylinders, *i.e.*, cylinders of basis $A$ of the form $\prod_{i=1}^{d-1}[a_i, b_i] \times \{c\}$ with real numbers $a_i$, $b_i$ and $c$, we know thanks to the works of Kesten [2], Zhang [6] and Rossignol and Théret [3] that $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ converges a.s. towards $\nu((0, \ldots, 0, 1))$ when $n$ goes to infinity, and in this case the upper large deviations of $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ have been studied by Théret in [4]: they are of volume order, and the corresponding large deviation principle was even proved. For tilted cylinders, we do not know the asymptotic behaviour of this variable in general, but looking at the trivial case where $t(e) = 1$ for every edge $e$, we can easily see that $\tau(nA, h(n))$ and $\phi(nA, h(n))$ do not have the same behaviour for large $n$. However, in the case where $\lim_{n\to\infty} h(n)/n = 0$, we also know that $\lim_{n\to\infty} \phi(nA, h(n))/\mathcal{H}^{d-1}(nA) = \nu(\boldsymbol{v})$ almost surely under the same hypotheses as for the variable $\tau(nA, h(n))$, so in this case we really study here the upper large deviations of the variable $\phi(nA, h(n))$.

**Remark 1.6.** We were not able to prove a large deviation principle from above for the variables $\tau$, or $\phi$ in tilted cylinders. The idea used in [4] to prove a large deviation principle for the variable $\phi(nA, h(n))$ in straight cylinders is the following: we pile up cylinders, and we let a large amount of flow cross the cylinders one after each other, using the fact that the top of a cylinder, *i.e.* the area through which the water goes out of this cylinder, is exactly the bottom of the cylinder above, *i.e.* the area through which the water can go into that cylinder. We cannot use the same method to prove a large deviation principle for $\tau(nA, h(n))$, even in straight cylinders, because in this case we cannot glue together the entire area through which the water goes out of a cylinder with the entire area through which the water goes into the cylinder above. In the case of tilted cylinders we even loose the symmetry of the graph with regard to the hyperplanes spanned by the faces of the cylinder. These symmetries were of huge importance in the proof of the large deviation principle from above for $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ in [4].

## 2. Upper large deviations for the rescaled variable $\tau$

### 2.1. Geometric construction

To study these upper large deviations, we will use the same idea as in the proof of the strict positivity of the rate function of the large deviation principle we proved in [4] for the variable $\phi(nA, h(n))$ in straight cylinders. Thus the main tool is the Cramér Theorem in $\mathbb{R}$. We will consider two different scales on the graph, *i.e.*, cylinders of two different sizes indexed by $n$ and $N$, with $N$ very large compared to $n$. We want to divide the cylinder $\mathrm{cyl}(NA, h(N))$ into images of $\mathrm{cyl}(nA, h(n))$ by integer translations, *i.e.*, translations whose vectors have integer coordinates, and to compare the maximal flows through these cylinders. In fact, we will first fill $\mathrm{cyl}(NA, h(N))$ with translates of $\mathrm{cyl}(nA, h(n))$ and then move slightly these translates to obtain integer translates. The problem is that we want to obtain disjoint small cylinders so that the associated flows are independent, therefore we need some extra space between the different images of $\mathrm{cyl}(nA, h(n))$ in order to move them separately and to obtain disjoint cylinders. Then we add some edges to glue together the different cutsets in the small cylinders to obtain a cutset in the big one.

The last remark we have to make before the beginning of the complete proof is that we may not divide the entire cylinder $\mathrm{cyl}(NA, h(N))$ into slabs, but a possibly smaller one, $\mathrm{cyl}(NA, Mh(n))$ with a not too large $M$. Indeed, we will see that the upper large deviations of $\tau(NA, h(N))$ are related to the behaviour of the edges of the cylinder that are "not too far" from $NA$, because the cutset is pinned at the boundary of $NA$ so it cannot explore regions too far away from $NA$ in $\mathrm{cyl}(NA, h(N))$.

We consider a fixed non degenerate hyperrectangle $A$. Let $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{d-1})$ be an orthonormal basis of $\mathrm{hyp}(A)$ such that the sides of $A$ are parallel to these vectors. Thus $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{d-1}, \boldsymbol{v})$ is an orthonormal basis of $\mathbb{R}^d$. We denote by $d_\infty$ the $l^\infty$ distance according to this basis. The $r$-neighbourhood $\mathcal{V}(X, r)$ of a subset $X$ of $\mathbb{R}^d$ for the the distance $d_\infty$ is defined by

$$\mathcal{V}(X, r) \,=\, \{y \in \mathbb{R}^d \,|\, d_\infty(y, X) < r\}.$$

Let $\lambda > \nu(\boldsymbol{v})$ and $\varepsilon > 0$ such that $\lambda > \nu(\boldsymbol{v}) + 3\varepsilon$. We take an $h$ as in Theorem 1.3, a large $N$, and a smaller $n$. We fix $\zeta = 2d$. We define $\mathrm{cyl}'(nA, h(n))$ as

$$\mathrm{cyl}'(nA, h(n)) \,=\, \mathcal{V}(\mathrm{cyl}(nA, h(n)), \zeta/2).$$

We fix an $M = M(n, N)$ such that $M(h(n) + \zeta/2) \leq h(N)$. We divide $\mathrm{cyl}(NA, M(h(n) + \zeta/2))$ into slabs $S_i$, $i = 1, \dots, M(n, N)$, of the form

$$S_i \,=\, \{x + t\boldsymbol{v} \,|\, x \in NA,\ t \in \mathcal{T}_i\}$$

where

$$\mathcal{T}_i \,=\, [-M(h(n) + \zeta/2) + (i-1)(2h(n) + \zeta), -M(h(n) + \zeta/2) + i(2h(n) + \zeta)]$$

(see Fig. 1). By a Euclidean division of the dimensions of $S_i$, we divide then each $S_i$ into $m$ translates of $\mathrm{cyl}'(nA, h(n))$, which we denote by $S'_{i,j}$, $j = 1, \dots, m$, plus a remaining part $S'_{i,m+1}$. Here $m$ is smaller than

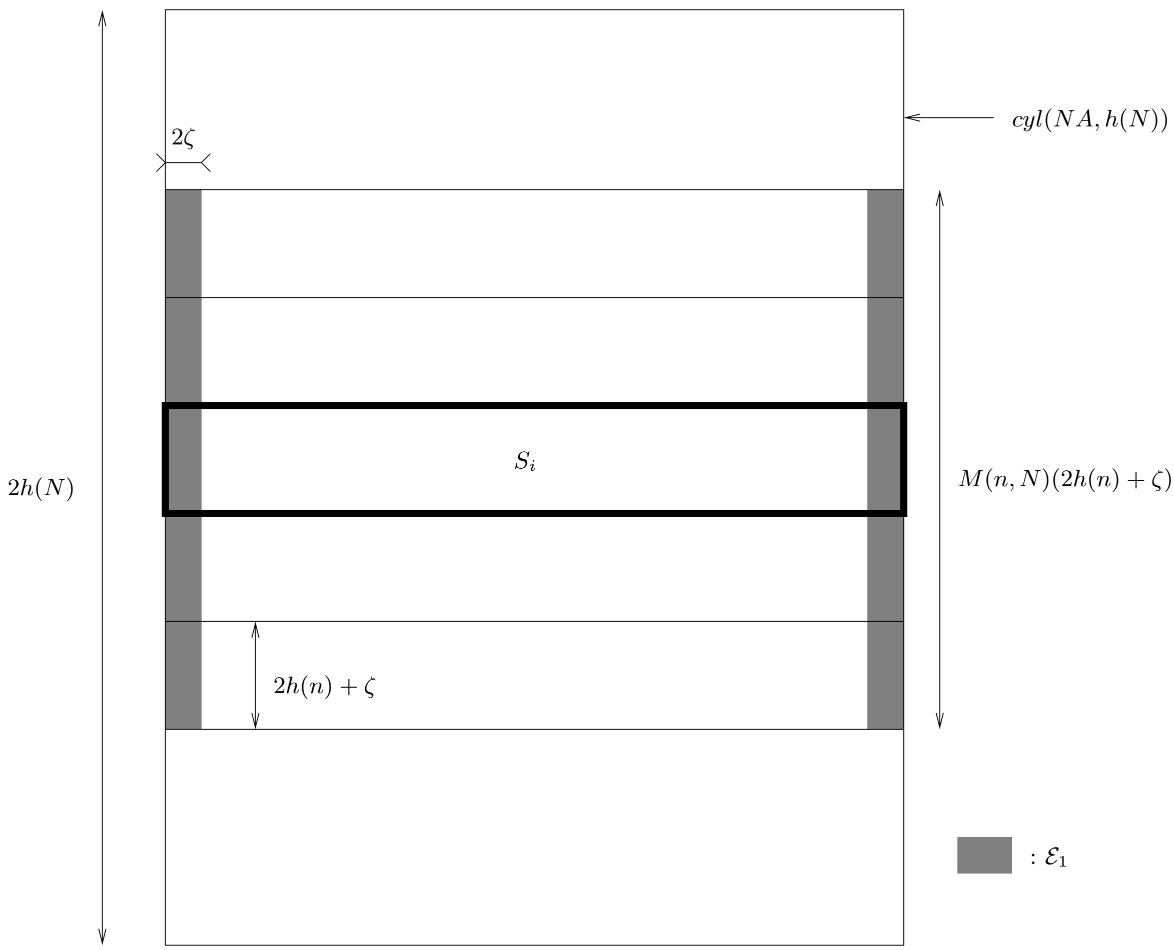


Figure 1. $\mathrm{cyl}(NA, h(N))$ and $S_i$.

$\mathcal{M}(n,N) = \lfloor \mathcal{H}^{d-1}(NA)/\mathcal{H}^{d-1}(nA) \rfloor$, where $\lfloor x \rfloor$ is the integer part of $x$. Each $S'_{i,j}$ is a translate of $\mathrm{cyl}'(nA, h(n))$, which contains $\mathrm{cyl}(nA, h(n))$, and so we denote by $D_{i,j}$ the corresponding translate of $\mathrm{cyl}(nA, h(n))$ by the same translation ($D_{i,j} \subset S'_{i,j}$). See Figure 2 which illustrates these definitions.

For all $(i,j)$ there exists a vector $\boldsymbol{w}_{i,j}$ in $\mathbb{R}^d$ such that $\|\boldsymbol{w}_{i,j}\|_\infty < 1$ (where the norm $\|\cdot\|_\infty$ is taken according to the basis $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{d-1}, \boldsymbol{v})$) and $B_{i,j} = D_{i,j} + \boldsymbol{w}_{i,j}$ is the image of $\mathrm{cyl}(nA, h(n))$ by an integer translation, *i.e.*, a translation whose vector has integer coordinates; moreover since $\zeta/2 \geq 1$ we have $B_{i,j} \subset S'_{i,j}$, so the $B_{i,j}$ are disjoint. We define $\tau_i = \tau(S_i, \boldsymbol{v})$ and $\tau_{i,j} = \tau(B_{i,j}, \boldsymbol{v})$. We denote by $E_1$ the set of the edges which belong to $\mathcal{E}_1 \subset \mathbb{R}^d$ defined by

$$\mathcal{E}_1 \;=\; \{x + t\boldsymbol{v} \,|\, x \in NA,\; d_\infty(x, \partial(NA)) \leq 2\zeta \;\text{ and }\; t \in [-M(h(n)+\zeta/2), M(h(n)+\zeta/2)]\}.$$

We denote also by $E_{0,i}$ the set of the edges which belong to $\mathcal{E}_{0,i} \subset \mathbb{R}^d$ defined by

$$\mathcal{E}_{0,i} \;=\; \{x + t\boldsymbol{v} \,|\, x \in NA,\; t \in \mathcal{T}'_i\} \cap \left( \bigcup_{j=1}^{m} \mathcal{V}(\partial S'_{i,j}, 3\zeta) \cup S'_{i,m+1} \right),$$

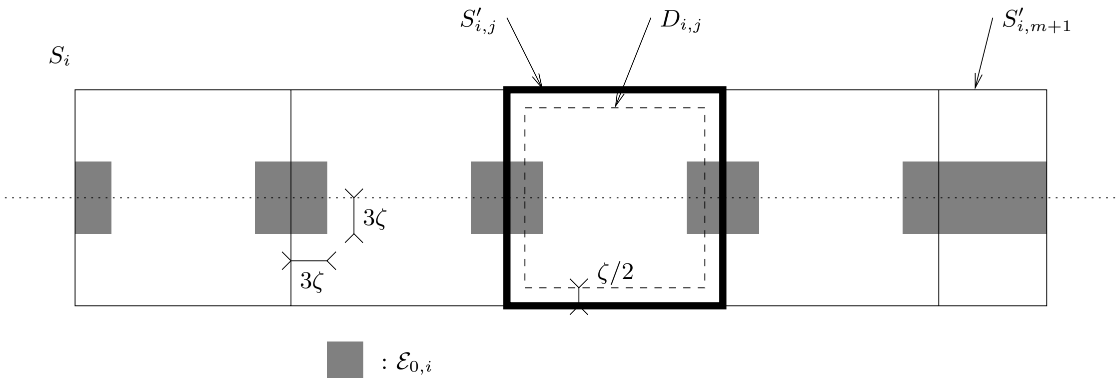


FIGURE 2. The slab $S_i$.

where

$$\mathcal{T}'_i \;=\; [-h(N) + (i-1/2)(2h(n)+\zeta) - 3\zeta, -h(N) + (i-1/2)(2h(n)+\zeta) + 3\zeta].$$

For all $i \in \{1, \dots, M(n,N)\}$, if we denote by $\mathcal{F}_{i,j}$ a set of edges that cuts the lower half part from the upper half part of the cylinder $B_{i,j}$ $(j \in \{1, \dots, m\})$, then $\cup_{j=1}^m \mathcal{F}_{i,j} \cup E_{0,i} \cup E_1$ separates the lower half part from the upper half part of $\mathrm{cyl}(NA, h(N))$. Thus we obtain that

$$\forall i \in \{1, \dots, M(n,N)\}, \qquad \tau(NA, h(N)) \;\leq\; \sum_{j=1}^m \tau_{i,j} + V(E_1 \cup E_{0,i}),$$

so

$$\begin{aligned}
&\mathbb{P}\Big[\tau(NA,h(N)) \;\geq \lambda \mathcal{H}^{d-1}(NA)\Big] \\
&\qquad\qquad \leq\; \mathbb{P}\left[\forall i \in \{1, \dots, M(n,N)\},\; \sum_{j=1}^m \tau_{i,j} + V(E_1 \cup E_{0,i}) \geq \lambda \mathcal{H}^{d-1}(NA)\right] \\
&\qquad\qquad \leq\; \mathbb{P}\left[\forall i \in \{1, \dots, M(n,N)\},\; \sum_{j=1}^m \tau_{i,j} \geq (\lambda - \varepsilon)\mathcal{H}^{d-1}(NA)\right] \\
&\qquad\qquad\quad + \mathbb{P}\left[V(E_1) \geq \varepsilon \mathcal{H}^{d-1}(NA)/2\right] \\
&\qquad\qquad\quad + \mathbb{P}\left[\exists i \in \{1, \dots, M(n,N)\},\, V(E_{0,i}) \geq \varepsilon \mathcal{H}^{d-1}(NA)/2\right]. \qquad (2.1)
\end{aligned}$$

We study the different probabilities appearing here separately.

• Let

$$\alpha(n,N) \;=\; \mathbb{P}\left[\forall i \in \{1, \dots, M(n,N)\},\; \sum_{j=1}^m \tau_{i,j} \geq (\lambda - \varepsilon)\mathcal{H}^{d-1}(NA)\right].$$

Since the families $(\tau_{i,j}, j = 1, \dots, m)$ for $i \in \{1, \dots, M(n,N)\}$ are i.i.d. we have

$$\begin{aligned}\alpha(n,N) \,&=\, \mathbb{P}\left[\sum_{j=1}^{m} \tau_{1,j} \geq (\lambda-\varepsilon)\mathcal{H}^{d-1}(NA)\right]^{M(n,N)}\\ &\leq \mathbb{P}\left[\sum_{j=1}^{\mathcal{M}(n,N)} \tau_n(j) \geq (\lambda-\varepsilon)\mathcal{H}^{d-1}(NA)\right]^{M(n,N)}\\ &\leq \mathbb{P}\left[\frac{1}{\mathcal{M}(n,N)} \sum_{j=1}^{\mathcal{M}(n,N)} \frac{\tau_n^{(j)}}{\mathcal{H}^{d-1}(nA)} \geq \lambda-\varepsilon\right]^{M(n,N)},\end{aligned}$$

where we remember that

$$\mathcal{M}(n,N) \,=\, \lfloor \mathcal{H}^{d-1}(NA)/\mathcal{H}^{d-1}(nA)\rfloor,$$

and $(\tau_n^{(j)}, j \in \mathbb{N})$ is a family of independent and identically distributed variables with $\tau_n^{(j)} = \tau(nA, h(n))$ in law. We know that $\mathbb{E}(\tau(nA,h(n)))/\mathcal{H}^{d-1}(nA)$ converges to $\nu(\boldsymbol{v})$ when $n$ goes to infinity as soon as $\mathbb{E}[t(e)] < \infty$, so there exists $n_0$ large enough to have for all $n \geq n_0$

$$\frac{\mathbb{E}(\tau(nA,h(n)))}{\mathcal{H}^{d-1}(nA)} \,\leq\, \nu(\boldsymbol{v}) + \varepsilon \,<\, \lambda - \varepsilon.$$

In the three cases presented in Theorem 1.3, the law of the capacity of the edges admits at least one exponential moment, and by an easy comparison between $\tau(nA,h(n))$ and the capacity of a fixed flat cutset in $\mathrm{cyl}(nA,h(n))$, we obtain that $\tau(nA,h(n))$ admits an exponential moment. We can then apply the Cramér theorem to obtain that for fixed $n \geq n_0$ and $\lambda$ there exists a constant $c$ (depending on the law of $\tau(nA,h(n))$, $\lambda$ and $\varepsilon$) such that

$$\limsup_{N\to\infty} \frac{1}{\mathcal{M}(n,N)} \log \mathbb{P}\left[\frac{1}{\mathcal{M}(n,N)} \sum_{j=1}^{\mathcal{M}(n,N)} \frac{\tau_n^{(j)}}{\mathcal{H}^{d-1}(nA)} \geq \lambda-\varepsilon\right] \,\leq\, c \,<\, 0,$$

and so for all $n \geq n_0$ and $\lambda$ there exists a constant $c'$ (depending on the law of $\tau(nA,h(n))$, $\lambda$ and $\varepsilon$) such that

$$\limsup_{N\to\infty} \frac{1}{M(n,N)\mathcal{H}^{d-1}(NA)} \log \alpha(n,N) \,<\, c' < 0. \tag{2.2}$$

• To study the two other terms, we can study more generally the behaviour of

$$\gamma(n,N) \,=\, \mathbb{P}\left[\sum_{i=1}^{l(n,N)} t_i \geq \varepsilon \mathcal{H}^{d-1}(NA)/2\right],$$

where $(t_i, i \in \mathbb{N})$ is a family of i.i.d. random variables of common distribution function $F$. We know that there exists a positive constant $C$ depending on $d$, $A$ and $\zeta$ such that

$$\mathrm{card}(E_{0,i}) \,\leq\, C\left(\frac{N^{d-1}}{n} + N^{d-2}n\right) \tag{2.3}$$

and

$$\mathrm{card}(E_1) \,\leq\, C N^{d-2} M(n,N) h(n). \tag{2.4}$$

Thus the values of $l(n,N)$ we have to consider are

$$l_0(n,N) \,=\, C(N^{d-1}n^{-1} + N^{d-2}n) \quad \text{and} \quad l_1(n,N) \,=\, C N^{d-2} M(n,N) h(n),$$

and we denote by $\gamma_i(n,N)$, $i=0,1$, the corresponding versions of $\gamma(n,N)$. Inequality (2.1) implies that

$$\mathbb{P}[\tau(NA,h(N)) \geq \lambda \mathcal{H}^{d-1}(NA)] \,\leq\, \alpha(n,N) + \gamma_1(n,N) + M(n,N)\,\gamma_0(n,N). \tag{2.5}$$

The behaviour of the quantities $\gamma_i(n,N)$ depends on the law of the capacity of the edges.

## 2.2. Bounded capacities

We suppose that the capacity of the edges is bounded by a constant $K$. Then as soon as

$$2Kl(n,N) \,<\, \varepsilon \mathcal{H}^{d-1}(NA), \tag{2.6}$$

we know that $\gamma(n,N)=0$. It is obvious that there exists a $n_1$ such that for all fixed $n \geq n_1$, for all large $N$ (how large depending on $n$), equation (2.6) is satisfied by $l_0(n,N)$. Moreover, for all $n$ there exists a constant $\kappa(n,A,d,F)$ such that if $M(n,N) \leq \kappa N$, then equation (2.6) is also satisfied by $l_1(n,N)$. We choose $M(n,N)$ to be as large as possible according to the condition we have just mentioned, and the fact that $M(n,N) \leq h(N)(h(n)+\zeta/2)^{-1}$; we define $\kappa'(n) = (h(n)+\zeta/2)^{-1}$ and we choose

$$M(n,N) \,=\, \min(\lfloor \kappa(n)N \rfloor, \lfloor \kappa'(n)h(N) \rfloor).$$

Thus, for a fixed $n \geq n_1$, for all $N$ large enough, we obtain that

$$\gamma_1(n,N) + M(n,N)\,\gamma_0(n,N) \,=\, 0$$

and then thanks to equations (2.2) and (2.5) we obtain that

$$\limsup_{N\to\infty} \frac{1}{\mathcal{H}^{d-1}(NA)\min(N,h(N))} \log \mathbb{P}\left[\frac{\tau(NA,h(N))}{\mathcal{H}^{d-1}(NA)} \geq \lambda\right] \,<\, 0,$$

so equation (1.1) is proved.

**Remark 2.1.** The term $\mathcal{H}^{d-1}(nA)\min(n,h(n))$ can seem strange in (1.1). It is in fact the right order of the upper large deviations in the case of bounded capacities. We try here to explain where it comes from. From the point of view of minimal cutsets, the heuristic is that a cutset in $\mathrm{cyl}(nA,h(n))$ separating the two half cylinders is pinned along the boundary of $nA$, so it cannot explore domains of $\mathrm{cyl}(nA,h(n))$ that are too far away from $nA$, *i.e.*, at distance of order larger than $n$. Thus it is located in a box of volume of order $n^{d-1}\min(n,h(n))$. We think it is this point of view that gives the best intuitive idea of how things work, but actually it is very difficult to study the position of a minimal cutset in the cylinder. From the point of view of the maximal flow, we can also understand why this term appears. In fact, we can find of the order of $n^{d-1}$ disjoint paths (*i.e.*, with no common edge) that cross $\mathrm{cyl}(nA,h(n))$ from its upper half part to its lower half part using only the edges located at distance smaller than $Kn$ of $nA$ for some constant $K$ (thus all the edges of the box if $h(n)/n$ is bounded). If $h(n)/n$ is bounded, we can consider paths that cross the cylinder from its top to its bottom, and if $h(n) \geq n$, we can consider paths that form a part of a loop around a point of $\partial(nA)$ - so they join two points of $\mathrm{cyl}(\partial(nA),Kn)$ that are on the same side of $\mathrm{cyl}(nA,h(n))$ and that are symmetric one to each other by the reflexion of axis the intersection of $\partial(nA)$ with this side (see Fig. 3 that shows these paths in dimension 2). Thus, if all the edges at distance smaller that $Kn$ of $nA$ in the cylinder have a big capacity, then the variable $\tau(nA,h(n))/\mathcal{H}^{d-1}(nA)$ is abnormally big. The number of such edges is of order $n^{d-1}\min(n,h(n))$. We emphasize here the fact that $\phi(nA,h(n))$ does not have these properties, this is the reason why we expect for this variable upper large deviations of volume order for all functions $h$.

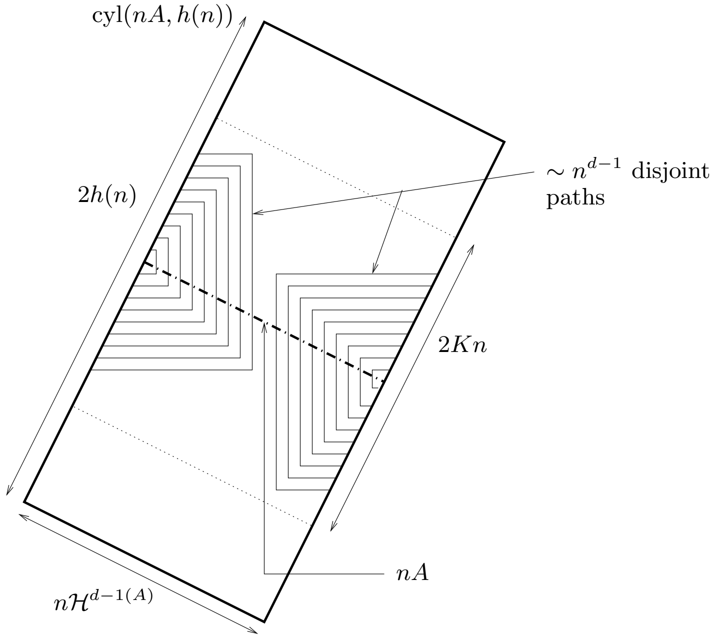


FIGURE 3. Disjoint paths near $nA$ in dimension two.

### 2.3. Capacities of exponential law

The goal of this short study is to emphasize the fact that the condition of having one exponential moment for the law of the capacity of the edges is not sufficient to obtain the speed of decay that we have with bounded capacities. We will consider a particular law, namely the exponential law of parameter 1, and show that we do not have upper deviations of volume order in this case.

We suppose that the law of the capacity of the edges is the exponential law of parameter 1. We know that $\mathbb{E}(\exp(\theta t)) < \infty$ for all $\theta < 1$. Let $x_0$ be a fixed point of the boundary $\partial(nA)$. We know that there exists a path from the lower half cylinder $(nA)_2^{h(n)}$ to the upper half cylinder $(nA)_1^{h(n)}$ in $\mathrm{cyl}(nA, h(n))$ that is included in the neighbourhood of $x_0$ of diameter $\zeta = 2d$ for the euclidean distance, as soon as $n \geq n_0(d, A, h)$, where $n_0(d, A, h)$ is the infimum of the $n$ such that all the sidelengths of the cylinder $\mathrm{cyl}(nA, h(n))$ are larger than $\zeta$ (see Fig. 4). Thus for all $n \geq n_0$, every set of edges that cuts the upper half cylinder $(nA)_1^{h(n)}$ from the lower half cylinder $(nA)_2^{h(n)}$ in $\mathrm{cyl}(nA, h(n))$ must contain at least one of the edges of this neighbourhood of $x_0$. The number of such edges is at most $K$, where $K$ is a constant depending only on $d$. Thus the probability that all of them have a capacity bigger than $\lambda\mathcal{H}^{d-1}(nA)$ for a $\lambda > \nu(\boldsymbol{v})$ is greater than $\exp(-K\lambda\mathcal{H}^{d-1}(nA))$. We obtain that for all $n \geq n_0(d, A, h)$,

$$\mathbb{P}\left[\tau(nA, h(n)) \geq \lambda\mathcal{H}^{d-1}(nA)\right] \geq \exp(-K\lambda\mathcal{H}^{d-1}(nA)),$$

thus equation (1.2) is proved with $D = K\lambda$.

### 2.4. Capacities with exponential moments of all orders

We suppose that the capacity of the edges admits exponential moments of all order, *i.e.*, for all $\theta > 0$ we have $\mathbb{E}(\exp(\theta t(e))) < \infty$. We start again from equation (2.5). If the following hypothesis (**H1**) is satisfied:

$$(\mathbf{H1}) \qquad \forall n \in \mathbb{N}, \quad \lim_{N\to\infty} M(n, N) = 0,$$

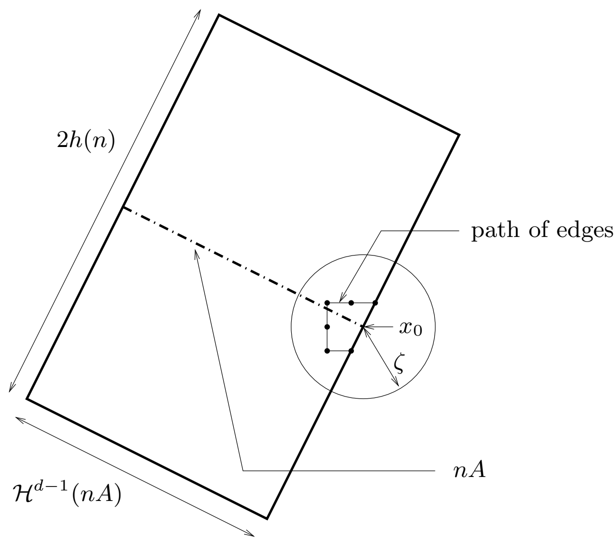


FIGURE 4. Path of edges included in a neighbourhood of $x_0$.

then by equation (2.2) we obtain that there exists $n_0$ such that for all $n \geq n_0$, we have

$$\limsup_{N\to\infty} \frac{1}{\mathcal{H}^{d-1}(NA)} \log \alpha(n,N) \;=\; -\infty. \tag{2.7}$$

We have to deal with the terms $\gamma_1$ and $\gamma_0$ in equation (2.5). By a simple application of the Markov's inequality, we obtain that

$$\gamma(n,N) \;\leq\; \exp\left[-\mathcal{H}^{d-1}(NA)\left(\frac{\theta\varepsilon}{2} - \frac{l(n,N)\log \mathbb{E}(\exp(\theta t(e)))}{\mathcal{H}^{d-1}(NA)}\right)\right] \tag{2.8}$$

(where $\gamma$ stands for $\gamma_i$ and $l$ for $l_i$, $i=0,1$). We want to be able to choose the term

$$\frac{\theta\varepsilon}{2} - \frac{l(n,N)\log \mathbb{E}(\exp(\theta t(e)))}{\mathcal{H}^{d-1}(NA)}$$

as big as we want. For a fixed $R>0$, we can take $\theta>0$ large enough to have $\theta\varepsilon \geq 4R$. If there exists $n_2$ such that for all fixed $n \geq n_2$, for all $N$ sufficiently large (how large depends on $n$), we have

$$\frac{l(n,N)}{\mathcal{H}^{d-1}(NA)} \log \mathbb{E}(\mathrm{e}^{\theta t(e)}) \;\leq\; R, \tag{2.9}$$

then for all fixed $n \geq n_2$, for all large $N$, we would obtain

$$\gamma(n,N) \;\leq\; \exp\left(-R\mathcal{H}^{d-1}(NA)\right),$$

thus for all $n \geq n_2$,

$$\limsup_{N\to\infty} \frac{1}{\mathcal{H}^{d-1}(NA)} \log \gamma(n,N) \;=\; -\infty. \tag{2.10}$$

Concerning $\gamma_0(n,N)$, it is obvious that for a given $R$, there exists $n_2$ such that for all fixed $n \geq n_2$, for all large $N$, the condition (2.9) is satisfied by $l_0(n,N)$, thus for all $n \geq n_2$, equation (2.10) holds for $\gamma=\gamma_0$. Looking at $\gamma_1(n,N)$, we realize that we have to impose an other condition on $M(n,N)$. If the following hypothesis (**H2**) is satisfied:

$$(\mathbf{H2}) \qquad \forall n \in \mathbb{N}, \quad \lim_{N\to\infty} \frac{M(n,N)}{N} \;=\; 0,$$

then for all $n \in \mathbb{N}$, for all large $N$, $l_1(n,N)$ satisfies the condition (2.9) and thus for all $n \in \mathbb{N}$, equation (2.10) holds for $\gamma = \gamma_1$. Moreover, hypothesis (**H2**) implies that for all $n \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{\log M(n,N)}{\mathcal{H}^{d-1}(NA)} \;=\; 0. \tag{2.11}$$

Combining equations (2.5)–(2.10) for $\gamma_1$ and $\gamma_0$, and (2.11), we obtain that if (**H1**) and (**H2**) are satisfied, for all $n \geq \max(n_0, n_2)$,

$$\limsup_{N\to\infty} \frac{1}{\mathcal{H}^{d-1}(NA)} \log \mathbb{P}\left[\frac{\tau(NA,h(N))}{\mathcal{H}^{d-1}(NA)} \geq \lambda\right] \;=\; -\infty,$$

thus equation (1.3) is proved. The last thing we have to do is to prove that we can choose $M(n,N)$ smaller than or equal to $\left\lfloor \frac{h(N)}{h(n)+\zeta/2} \right\rfloor$ and that satisfies (**H1**) and (**H2**). We can simply choose

$$M(n,N) = \min\left(\sqrt{N}, \left\lfloor \frac{h(N)}{h(n)+\zeta/2} \right\rfloor\right).$$

This ends the proof of Theorem 1.3.

**Remark 2.2.** This result is used in [3] in the proof of the lower large deviation principle for the variable $\tau(nA, h(n))$.

## 3. Partial result concerning the upper large deviations for $\phi$ through a tilted cylinder

We have already written the main part of the proof of Theorem 1.4 in the previous section. We keep all the notations introduced previously. The proof of Theorem 1.3 was based on the following inequality:

$$\forall i \in \{1,\ldots,M(n,N)\}, \qquad \tau(NA,h(N)) \;\leq\; \sum_{j=1}^{m} \tau_{i,j} + V(E_1 \cup E_{0,i}).$$

We recall that this inequality was obtained by noticing that if $\mathcal{F}_{i,j}$ is a cutset that separates the upper half part from the lower half part of $B_{i,j}$, then $\cup_{j=1}^{m}\mathcal{F}_{i,j} \cup E_{0,i} \cup E_1$ separates the upper half part from the lower half part of $\mathrm{cyl}(NA, h(N))$. Here we want to construct a cutset that separates the bottom from the top of $\mathrm{cyl}(NA, h(N))$. We have no need to add the set of edges $E_1$ in this context because we do not need to obtain a cutset that is pinned at $\partial(NA)$. Thus for all $i$, $\cup_{j=1}^{m}\mathcal{F}_{i,j} \cup E_{0,i}$ cuts the top from the bottom of $\mathrm{cyl}(NA, h(N))$, and we have

$$\forall i \in \{1,\ldots,M(n,N)\}, \qquad \phi(NA,h(N)) \;\leq\; \sum_{j=1}^{m} \tau_{i,j} + V(E_{0,i}).$$

We obtain that for a fixed $\lambda > \nu(\boldsymbol{v})$, and $\varepsilon$ such that $\lambda \geq \nu(\boldsymbol{v}) + 3\varepsilon$, we have by independence

$$\begin{aligned}
\mathbb{P}[\phi(NA,h(N)) &\geq \lambda\mathcal{H}^{d-1}(NA)] \\
&\leq \mathbb{P}\left[\bigcap_{i=1}^{M(n,N)} \left\{\sum_{j=1}^{m} \tau_{i,j} + V(E_{0,i}) \geq \lambda\mathcal{H}^{d-1}(NA)\right\}\right] \\
&\leq \prod_{i=1}^{M(n,N)} \left(\mathbb{P}\left[\sum_{j=1}^{m} \tau_{i,j} \geq (\lambda-\varepsilon)\mathcal{H}^{d-1}(NA)\right] + \mathbb{P}\left[V(E_{0,i}) \geq \varepsilon\mathcal{H}^{d-1}(NA)\right]\right).
\end{aligned} \tag{3.1}$$

We consider here the maximal $M(n,N)$, *i.e.*,

$$M(n,N) \;=\; \left\lfloor \frac{h(N)}{h(n)+\zeta/2} \right\rfloor.$$

Indeed, we do not need to make any restriction on $M(n,N)$ because we do not have to consider the set of edges $E_1$ whose cardinality depends on $M(n,N)$.

From now on we suppose that the capacity of the edges admits an exponential moment. Thanks to the application of the Cramér theorem we have already done to obtain (2.2), we know that for all $n \geq n_0$ there exists a positive $c'$ (depending on the law of $\tau(nA, h(n))$, $\lambda$ and $\varepsilon$) such that

$$\limsup_{N\to\infty} \frac{1}{\mathcal{H}^{d-1}(NA)} \log \mathbb{P}\left[\sum_{j=1}^{m} \tau_{i,j} \geq (\lambda-\varepsilon)\mathcal{H}^{d-1}(NA)\right] \;\leq\; c' \;<\; 0. \tag{3.2}$$

On the other hand, let $\theta > 0$ be such that $\mathbb{E}(\exp(\theta t(e))) < \infty$. Thanks to equation (2.3) and (2.8), obtained by the Chebyshev inequality, we have for this fixed $\theta$:

$$\begin{aligned}\mathbb{P}[V(E_{0,i}) \geq \varepsilon \mathcal{H}^{d-1}(NA)] \;&\leq\; \gamma_0(n,N)\\ &\leq\; \exp\left[-\mathcal{H}^{d-1}(NA)\left(\frac{\theta\varepsilon}{2} - \frac{l_0(n,N)\log \mathbb{E}(\exp(\theta t(e)))}{\mathcal{H}^{d-1}(NA)}\right)\right].\end{aligned}$$

Since $l_0(n,N) \leq C(N^{d-1}n^{-1} + N^{d-2}n)$, we know that there exists $n_3$ such that for all $n \geq n_3$, for all $N$ large enough (how large depending on $n$), we have

$$\frac{l_0(n,N)\log \mathbb{E}(\exp(\theta t(e)))}{\mathcal{H}^{d-1}(NA)} \;\leq\; \frac{\theta\varepsilon}{4},$$

and then

$$\mathbb{P}[V(E_{0,i}) \geq \varepsilon \mathcal{H}^{d-1}(NA)] \;\leq\; \exp\left(-\mathcal{H}^{d-1}(NA)\frac{\theta\varepsilon}{4}\right). \tag{3.3}$$

Combining equations (3.1), (3.2) and (3.3), since $\lim_{N\to\infty} M(n,N)/h(N)$ is a constant for all fixed $n$, Theorem 1.4 is proved.

*Acknowledgements.* The author wants to thank Raphaël Cerf and Raphaël Rossignol, who asked the questions that have motivated this work.

## References

[1] B. Bollobás, *Graph theory*, *Graduate Texts in Mathematics*. Springer-Verlag, New York **63** (1979).

[2] H. Kesten, Surfaces with minimal random weights and maximal flows: a higher dimensional version of first-passage percolation. *Illinois J. Math.* **31** (1987) 99–166.

[3] R. Rossignol and M. Théret, Lower large deviations and laws of large numbers for maximal flows through a box in first passage percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** (2010) 1093–1131.

[4] M. Théret, Upper large deviations for the maximal flow in first-passage percolation. *Stochastic Process. Appl.* **117** (2007) 1208–1233.

[5] Y. Zhang, Critical behavior for maximal flows on the cubic lattice. *J. Stat. Phys.* **98** (2000) 799–811.

[6] Y. Zhang, Limit theorems for maximum flows on a lattice. Available from `arxiv.org/abs/0710.4589` (2007).