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L_p-THEORY FOR THE STOCHASTIC HEAT EQUATION WITH INFINITE-DIMENSIONAL FRACTIONAL NOISE*

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Abstract. In this article, we consider the stochastic heat equation $du = (\Delta u + f(t,x))\mathrm{d}t + \sum_{k=1}^{\infty} g^k(t,x)\delta\beta_t^k, t \in [0,T]$, with random coefficients f and g^k, driven by a sequence $(\beta^k)_k$ of i.i.d. fractional Brownian motions of index $H > 1/2$. Using the Malliavin calculus techniques and a p-th moment maximal inequality for the infinite sum of Skorohod integrals with respect to $(\beta^k)_k$, we prove that the equation has a unique solution (in a Banach space of summability exponent $p \geq 2$), and this solution is Hölder continuous in both time and space.

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1. Introduction

The study of stochastic partial differential equations driven by colored noise has become an active area of research in the recent years, which is viewed as an alternative (with an increased potential for applications) to the classical theory of equations perturbed by space-time white noise (see [5,10,13,26] for fundamental developments – using different approaches – in the white noise case.)

A Gaussian noise is said to be fractional in time, if its temporal covariance structure coincides with that of a fractional Brownian motion (fBm). Recall that a centered Gaussian process $(\beta_t)_{t\in[0,T]}$ is a fBm of index $H \in (0,1)$ if $R_H(t,s) := E(\beta_t\beta_s) = (t^{2H} + s^{2H} - |t-s|^{2H})/2$. The case $H > 1/2$ is referred as the "regular" case, whereas the case $H = 1/2$ corresponds to the Brownian motion. (The survey articles [19] and [9] offer more details on the fBm.)

Since the fBm is not a semimartingale, one cannot use the Itô calculus, which lies at the foundation of the study of equations driven by white noise. Various methods exist in the literature to circumvent this difficulty, based on the Skorokod integral (*e.g.* [1,2,4,6,7]), the pathwise generalized Stieltjes integrals (*e.g.* [21,23,27]), or the "rough paths" analysis (*e.g.* [15,16]).

The present article is dedicated to the study of the stochastic heat equation with (additive) infinite-dimensional fractional noise:

$$\mathrm{d}u(t,x) = (\Delta u(t,x) + f(t,x))\mathrm{d}t + \sum_{k=1}^{\infty} g^k(t,x)\delta\beta_t^k, \quad t \in [0,T], x \in \mathbb{R}^d, \tag{1.1}$$

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where $(\beta^k)_k$ is a sequence of i.i.d. fBm's of index $H > 1/2$, the solution is defined in the weak sense (using integration against test functions $\phi \in C_0^\infty(\mathbb{R}^d)$), and $\delta\beta_t^k$ is a formal way of indicating that the stochastic integrals (which are used for defining the solution) are interpreted in the Skorohod sense.

Let $H_p^n(\mathbb{R}^d)$ ($n \in \mathbb{R}, p \geq 2$) be the Sobolev space of all generalized functions on $\mathbb{R}^d$ whose derivatives of order $k \leq n$ lie in $L_p(\mathbb{R}^d)$. Our main result shows that for suitable initial condition u_0, and Sobolev-space valued random processes $f = \{f(t,\cdot)\}_{t\in[0,T]}$ and $g^k = \{g^k(t,\cdot)\}_{t\in[0,T]}, k \geq 1$, equation (1.1) has a unique $H_p^n(\mathbb{R}^d)$-valued solution $u = \{u(t,\cdot)\}_{t\in[0,T]}$, and $u \in C([0,T], H_p^{n-2}(\mathbb{R}^d))$ a.s., such that

$$E \sup_{t\leq T} \|u(t,\cdot)\|^p_{H_p^{n-2}(\mathbb{R}^d)} < \infty, \quad E\int_0^T \|u(t,\cdot)\|^p_{H_p^n(\mathbb{R}^d)} \mathrm{d}t < \infty.$$

Moreover, u belongs to the Hölder space $C^{\alpha-1/p}([0,T], H_p^{n-2\beta})$, with probability 1, for any $1/2 \geq \beta > \alpha > 1/p$. If in addition, $\gamma := n - 2\beta - d/p > 0$, u is also γ-Hölder continuous in space, since $H_p^{n-2\beta}(\mathbb{R}^d) \subset C^\gamma(\mathbb{R}^d)$. These results provide generalizations to the fractional case of the existing results for the heat equation driven by a sequence $(w^k)_k$ of i.i.d. Brownian motions (see [12,13,22]).

We note that our result cannot be inferred from the results existing in the literature for parabolic equations driven by Hilbert-space valued fractional noise with trace-class covariance operator (*e.g.* [8,17,25]). Nevertheless, we should mention the recent related investigations of [21] and [23], using fractional calculus techniques (as opposed to the Malliavin calculus techniques used here), which establish the existence and Hölder continuity (in time) of a variational/mild $L_2(D)$-valued solution for a parabolic initial-boundary value problem with multiplicative fractional noise, when $D \subset \mathbb{R}^d$ is a bounded open set.

Similarly to the Brownian motion case, at the origin of our developments lie two basic tools: (1) a generalization of the Littlewood-Paley inequality for Banach-space valued functions (Thm. A.2, Appendix); and (2) a suitable p-th moment maximal inequality for the sum of Skorokod integrals with respect to $(\beta^k)_k$ (Thm. 3.6):

$$\begin{aligned} E\sup_{t\leq T}\left|\sum_{k=1}^\infty \int_0^t u_s^k \delta\beta_s^k\right|^p \quad &\leq \quad C_{p,H,T}\left\{ E\left|\int_0^T \sum_{k=1}^\infty |u_s^k|^2 \mathrm{d}s\right|^{p/2}\right. \\ &\left. + E\left|\int_0^T \left[\int_0^T \left(\sum_{k=1}^\infty |D_\theta^{\beta^k} u_s^k|^2\right)^{1/(2H)} \mathrm{d}\theta\right]^{2H} \mathrm{d}s\right|^{p/2}\right\}. \end{aligned} \tag{1.2}$$

Compared to the Burkholder-Davis-Gundy inequality (which was used in the Brownian motion case), inequality (1.2) contains an additional term involving the Malliavin derivative $D^{\beta^k}u^k$ of the process u^k with respect to β^k. It is because of this extra term that our developments deviate significantly from the white noise case, and we require that the multiplication coefficient g^k lie in a suitable space of Malliavin differentiable functions with respect to β^k (which in particular, implies that g^k is measurable with respect to β^k).

This article is organized as follows. In Section 2, we give some preliminaries on the Malliavin calculus for Hilbert-space valued fractional processes, and we develop a maximal inequality for these processes. In Section 3, we convert the inequality obtained in Section 2 (which speaks about the Skorohod integral with respect to a Hilbert-space valued fractional process), into an inequality which speaks about the sum of Skorohod intregrals with respect to a sequence $(\beta^k)_k$ of i.i.d. fBm's. In Section 4, we introduce the stochastic Banach spaces in which we are allowed to select the coefficients f and $(g^k)_k$. Section 5 is dedicated to the main result, as well as the Hölder continuity of the solution. The appendix contains the generalization of the Littlewood-Paley inequality to Banach space valued functions.

2. Malliavin calculus for fractional processes

In this section, we introduce the basic facts about the Malliavin calculus with respect to (Hilbert-space valued) fractional processes. We refer the reader to [18] and [20] for a comprehensive account on this subject. Throughout this work, we let $H \in (1/2, 1)$ be fixed.

We begin by introducing some Banach spaces and Hilbert spaces of deterministic functions, which are used for the Malliavin calculus with respect to fractional processes.

If V is an arbitrary Banach space, we let $\mathcal{E}_V$ be the class of all elementary functions $\phi : [0,T] \to V$ of the form $\phi(t) = \sum_{i=1}^{m} 1_{(t_{i-1},t_i]}(t)\varphi_i$ with $0 \le t_0 < \ldots < t_m \le T$ and $\varphi_i \in V$. Let $|\mathcal{H}_V|$ be the space of all strongly measurable functions $\phi : [0,T] \to V$ with $\|\phi\|_{|\mathcal{H}_V|} < \infty$, where

$$\|\phi\|^2_{|\mathcal{H}_V|} := \alpha_H \int_0^T \int_0^T \|\phi(t)\|_V \ \|\phi(s)\|_V |t-s|^{2H-2} \mathrm{d}t\mathrm{d}s, \quad \alpha_H = H(2H-1).$$

The space $\mathcal{E}_V$ is dense in $|\mathcal{H}_V|$ with respect to the norm $\|\cdot\|_{|\mathcal{H}_V|}$. It is known that there exists a constant $b_H > 0$ such that $\|\phi\|_{|\mathcal{H}_V|} \le b_H \|\phi\|_{L_{1/H}([0,T];V)}$ for any $\phi \in L_{1/H}([0,T];V)$ (see *e.g.* relation (11) of [2]).

In particular, if $V = \mathbb{R}$, we denote $\mathcal{E}_V = \mathcal{E}$ and $|\mathcal{H}_V| = |\mathcal{H}|$.

We let $|\mathcal{H}| \otimes |\mathcal{H}_V|$ be the space of all strongly measurable functions $\phi : [0,T]^2 \to V$ with $\|\phi\|_{|\mathcal{H}|\otimes|\mathcal{H}_V|} < \infty$, where

$$\|\phi\|^2_{|\mathcal{H}|\otimes|\mathcal{H}_V|} := \alpha_H^2 \int_{[0,T]^4} \|\phi(t,\theta)\|_V \ \|\phi(s,\eta)\|_V \ |t-s|^{2H-2} \ |\theta-\eta|^{2H-2} \mathrm{d}\theta\mathrm{d}\eta\mathrm{d}s\mathrm{d}t.$$

If V is a Hilbert space, we let $\mathcal{H}_V$ be the completion of $\mathcal{E}_V$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_V}$ defined by:

$$\langle \phi, \psi \rangle_{\mathcal{H}_V} := \alpha_H \int_0^T \int_0^T \langle \phi(t), \psi(s) \rangle_V |t-s|^{2H-2} \mathrm{d}s\mathrm{d}t.$$

We have:

$$\|\phi\|_{\mathcal{H}_V} \le \|\phi\|_{|\mathcal{H}_V|} \le b_H \|\phi\|_{L_{1/H}([0,T];V)} \le b_H \|\phi\|_{L_2([0,T];V)}, \tag{2.1}$$

and $L_2([0,T];V) \subset L_{1/H}([0,T];V) \subset |\mathcal{H}_V| \subset \mathcal{H}_V$. In particular, if $V = \mathbb{R}$, we denote $\mathcal{H}_V = \mathcal{H}$. The space $\mathcal{H}$ may contain distributions of order $-(2H-1)$. Note that $\mathcal{H}_V$ is isomorphic with $\mathcal{H} \otimes V$, and the inner products in the two spaces are the same.

We let $|\mathcal{H}_V| \otimes |\mathcal{H}_V|$ be the space of all strongly measurable functions $\phi : [0,T]^2 \to V \otimes V$ with $\|\phi\|_{|\mathcal{H}_V|\otimes|\mathcal{H}_V|} < \infty$, where

$$\|\phi\|^2_{|\mathcal{H}_V|\otimes|\mathcal{H}_V|} := \alpha_H^2 \int_{[0,T]^4} \|\phi(t,\theta)\|_{V\otimes V} \ \|\phi(s,\eta)\|_{V\otimes V} \ |t-s|^{2H-2} \ |\theta-\eta|^{2H-2} \mathrm{d}\theta\mathrm{d}\eta\mathrm{d}s\mathrm{d}t,$$

and $\mathcal{H}_V \otimes \mathcal{H}_V$ be the completion of $\mathcal{E}_V \otimes \mathcal{E}_V$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_V\otimes\mathcal{H}_V}$ defined by:

$$\langle \phi, \psi \rangle_{\mathcal{H}_V\otimes\mathcal{H}_V} := \alpha_H^2 \int_{[0,T]^4} \langle \phi(t,\theta), \psi(s,\eta) \rangle_{V\otimes V} |t-s|^{2H-2} |\theta-\eta|^{2H-2} \mathrm{d}\theta\mathrm{d}\eta\mathrm{d}s\mathrm{d}t.$$

We have: (see *e.g.* Lem. 1, [2] for the second inequality below)

$$\|\phi\|_{\mathcal{H}_V\otimes\mathcal{H}_V} \le \|\phi\|_{|\mathcal{H}_V|\otimes|\mathcal{H}_V|} \le b_H \|\phi\|_{L_{1/H}([0,T]^2;V\otimes V)} \le b_H \|\phi\|_{L_2([0,T]^2;V\otimes V)}, \tag{2.2}$$

and $L_2([0,T]^2; V\otimes V) \subset L_{1/H}([0,T]^2; V \otimes V) \subset |\mathcal{H}_V| \otimes |\mathcal{H}_V| \subset \mathcal{H}_V \otimes \mathcal{H}_V$.

We begin now to introduce the main ingredients of the Malliavin calculus with respect to fractional processes.

Let V be an arbitrary Hilbert space and $B = (B(\phi))_{\phi \in \mathcal{H}_V}$ be a centered Gaussian process, defined on a probability space $(\Omega, \mathcal{F}, P)$, with covariance:

$$E(B(\phi)B(\psi)) = \langle \phi, \psi \rangle_{\mathcal{H}_V}, \quad \forall \phi, \psi \in \mathcal{H}_V. \tag{2.3}$$

If we let $B_t(\varphi) := B(1_{[0,t]}\varphi)$ for any $\varphi \in V, t \in [0,T]$, then

$$E(B_t(\varphi)B_s(\eta)) = R_H(t,s)\langle \varphi, \eta \rangle_V, \quad \forall \varphi, \eta \in V, s, t \in [0,T].$$

(In particular, if $V = \mathbb{R}$, then $\beta_t := B(1_{[0,t]}), t \in [0,T]$ is a fBm of index H.)

Let

$$\mathcal{S}_B := \{F = f(B(\phi_1), \dots, B(\phi_n)); f \in C_b^\infty(\mathbb{R}^n), \phi_i \in \mathcal{H}_V, n \geq 1\}$$

be the space of all "smooth cylindrical" random variables, where $C_b^\infty(\mathbb{R}^d)$ denotes the class of all bounded infinitely differentiable functions on $\mathbb{R}^n$, whose partial derivatives are also bounded. Clearly $\mathcal{S}_B \subset L_p(\Omega)$ for any $p \geq 1$.

The **Malliavin derivative** of an element $F = f(B(\phi_1), \dots, B(\phi_n)) \in \mathcal{S}_B$, with respect to B, is defined by:

$$D^B F := \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\phi_1), \dots, B(\phi_n))\phi_i.$$

Note that $D^B F \in L_p(\Omega; \mathcal{H}_V)$ for any $p \geq 1$; by abuse of notation, we write $D^B F = (D_t^B F)_{t \in [0,T]}$ even if $D_t^B F$ is not a function in t. We endow $\mathcal{S}_B$ with the norm:

$$\|F\|^p_{\mathbb{D}_B^{1,p}} := E|F|^p + E\|D^\beta F\|^p_{\mathcal{H}_V},$$

and we let $\mathbb{D}_B^{1,p}$ be the completion of $\mathcal{S}_B$ with respect to this norm. The operator D^B can be extended to $\mathbb{D}_B^{1,p}$. The adjoint

$$\delta^B : \text{Dom } \delta^B \subset L_2(\Omega; \mathcal{H}_V) \to L_2(\Omega)$$

of the operator D^B, is called the **Skorohod integral** with respect to B. The operator δ^B is uniquely defined by the following relation:

$$E(F\delta^B(U)) = E\langle D^B F, U \rangle_{\mathcal{H}}, \quad \forall F \in \mathbb{D}_B^{1,2}.$$

Note that $E(\delta^B(U)) = 0$ for any $u \in \text{Dom } \delta^B$. If $U \in \text{Dom } \delta^B$, we use the notation $U = (U_t)_{t \in [0,T]}$ and $\delta^B(U) = \int_0^T U_s \delta B_s$.

If V' is an arbitrary Hilbert space, we let

$$\mathcal{S}_B(V') := \left\{ U = \sum_{j=1}^m F_j \phi_j; F_j \in \mathcal{S}_B, \phi_j \in V', m \geq 1 \right\}$$

be the class of all "smooth cylindrical" V'-valued random variables. Clearly $\mathcal{S}_B(V') \subset L_p(\Omega; V')$ for any $p \geq 1$.

The Malliavin derivative of an element $U = \sum_{j=1}^m F_j \phi_j \in \mathcal{S}_B(V')$ is defined by $D^B U := \sum_{j=1}^m (D^B F_j)\phi_j$. We have $D^B U \in L_p(\Omega; \mathcal{H}_V \otimes V')$ for any $p \geq 1$. We endow $\mathcal{S}_B(V')$ with the norm:

$$\|U\|^p_{\mathbb{D}_B^{1,p}(V')} := E\|U\|^p_{V'} + E\|D^B U\|^p_{\mathcal{H}_V \otimes V'},$$

and let $\mathbb{D}_B^{1,p}(V')$ be the completion of $\mathcal{S}_B(V')$ with respect to this norm. The operator D^B can be extended to $\mathbb{D}_B^{1,p}(V')$.

In particular, if $V' = \mathcal{H}_V$, then $\mathbb{D}_{\beta}^{1,2}(\mathcal{H}_V) \subset \text{Dom } \delta^B$. If $U \in \mathbb{D}_B^{1,2}(\mathcal{H}_V)$ then $D^B U \in L_2(\Omega; \mathcal{H}_V \otimes \mathcal{H}_V)$; by abuse of notation, we write $D^B U = (D_t^B U_s)_{s,t \in [0,T]}$.

The space $\mathbb{D}_B^{1,2}(\mathcal{H}_V)$ is viewed as a "suitable" class of Skorohod integrands with respect to B. For any $U \in \mathbb{D}_B^{1,2}(\mathcal{H}_V)$, we have:

$$\begin{aligned} E|\delta^B(U)|^2 &= E\|U\|_{\mathcal{H}_V}^2 + E\left(\langle D^B U, (D^B U)^* \rangle_{\mathcal{H}_V \otimes \mathcal{H}_V}\right) \\ &\leq E\|U\|_{\mathcal{H}_V}^2 + E\|D^B U\|_{\mathcal{H}_V \otimes \mathcal{H}_V}^2 = \|U\|_{\mathbb{D}_B^{1,2}(\mathcal{H}_V)}^2, \end{aligned} \tag{2.4}$$

where $(D^B U)^*$ is the adjoint of $D^B U$ in $\mathcal{H}_V \otimes \mathcal{H}_V$.

The following result is a consequence of Meyer's inequalities.

Proposition 2.1 (Prop. 2.4.4 of [18]). *Let $p > 1$ and $U \in \mathbb{D}_B^{1,p}(\mathcal{H}_V)$. Then U lies in the domain of δ^B in $L_p(\Omega)$ and*

$$E|\delta^B(U)|^p \leq C_{H,p} \left\{ \|E(U)\|_{\mathcal{H}_V}^p + E\|D^B U\|_{\mathcal{H}_V \otimes \mathcal{H}_V}^p \right\},$$

where $C_{H,p}$ is a constant depending on H and p.

As a consequence of Proposition 2.1, (2.1) and (2.2), we obtain:

$$E|\delta^B(U)|^p \leq C_{H,p} b_H \{ \|E(U)\|_{L_{1/H}([0,T];V)}^p + E\|D^B U\|_{L_{1/H}([0,T]^2; V \otimes V)}^p \}. \tag{2.5}$$

We denote by $\mathbb{D}_B^{1,p}(|\mathcal{H}_V|)$ the set of all elements $U \in \mathbb{D}_B^{1,p}(\mathcal{H}_V)$, such that $U \in |\mathcal{H}_V|$ a.s., $D^B U \in |\mathcal{H}_V| \otimes |\mathcal{H}_V|$ a.s., and $\|U\|_{\mathbb{D}_B^{1,p}(|\mathcal{H}_V|)} < \infty$, where

$$\|U\|_{\mathbb{D}_B^{1,p}(|\mathcal{H}_V|)}^p := E\|U\|_{|\mathcal{H}_V|}^p + E\|D^B U\|_{|\mathcal{H}_V| \otimes |\mathcal{H}_V|}^p.$$

The following result generalizes Theorem 4 of [2] to the case of V-valued fractional processes.

Theorem 2.2. *Let $1/2 < H < 1$, $p > 1/H$ and $0 < \varepsilon < H - 1/p$. Then, there exists a constant C depending on H, p, ε and T such that*

$$\begin{aligned} E \sup_{t \leq T} \left| \int_0^t U_s \delta B_s \right|^p &\leq C \left\{ \left(\int_0^T \|E(U_s)\|_V^{1/(H-\varepsilon)} \mathrm{d}s \right)^{p(H-\varepsilon)} \right. \\ &\quad \left. + E \left[\int_0^T \left(\int_0^T \|D_\theta^B U_s\|_{V \otimes V}^{1/H} \mathrm{d}\theta \right)^{\frac{H}{H-\varepsilon}} \mathrm{d}s \right]^{p(H-\varepsilon)} \right\} \end{aligned} \tag{2.6}$$

for any process $U = (U_t)_{t \in [0,T]} \in \mathbb{D}_B^{1,p}(|\mathcal{H}_V|)$ for which the right-hand side of (2.6) is finite.

Proof. The argument is similar to the one used in the proof of Theorem 4 of [2]. We include it for the sake of completeness. Let $\alpha = 1 - 1/p - \varepsilon$.

By writing $\int_0^t U_s \delta B_s = c_\alpha \int_0^t (t-r)^{-\alpha} \left(\int_0^r U_s (r-s)^{\alpha-1} \delta B_s \right) \mathrm{d}r$, and using Hölder's inequality, we obtain:

$$E \sup_{t \leq T} \left| \int_0^t U_s \delta B_s \right|^p \leq c_{\alpha,p} E \int_0^T \left| \int_0^r U_s (r-s)^{\alpha-1} \delta B_s \right|^p \mathrm{d}r,$$

where $c_{\alpha,p}$ is a constant depending on α and p. Using (2.5), we have:

$$E\sup_{t\leq T}\left|\int_0^t U_s\delta B_s\right|^p \leq c_{\alpha,p,H}\left\{\int_0^T\left(\int_0^r \|E(U_s)\|_V^{1/H}(r-s)^{(\alpha-1)/H}\mathrm{d}s\right)^{pH}\mathrm{d}r\right.$$
$$\left.+E\int_0^T\left(\int_0^r\int_0^T \|D_\theta^B U_s\|_{V\otimes V}^{1/H}(r-s)^{(\alpha-1)/H}\mathrm{d}\theta\mathrm{d}s\right)^{pH}\mathrm{d}r\right\},$$

where $c_{\alpha,p,H}$ is a constant which depends on α,p and$H$. The result follows by applying Hardy-Littlewood inequality (p. 119 of [24]). $\square$

When $p\geq 2$, the previous theorem leads to the following result.

Corollary 2.3. *Let $1/2<H<1$ and $p\geq 2$ be arbitrary. Then, there exists a constant C depending on H,p and T such that*

$$E\sup_{t\leq T}\left|\int_0^t U_s\delta B_s\right|^p \leq C\left\{\left(\int_0^T \|E(U_s)\|_V^2\mathrm{d}s\right)^{p/2}\right.$$
$$\left.+E\left[\int_0^T\left(\int_0^T \|D_\theta^B U_s\|_{V\otimes V}^{1/H}\mathrm{d}\theta\right)^{2H}\mathrm{d}s\right]^{p/2}\right\} \tag{2.7}$$

for any process $U=(U_t)_{t\in[0,T]}\in\mathbb{D}_B^{1,p}(|\mathcal{H}_V|)$ for which the right-hand side of (2.7) is finite.

Proof. The result follows by applying Theorem 2.2 with $\varepsilon<H-1/2$ and using the fact that $\|\phi\|_{L_{1/(H-\varepsilon)}([0,T])}\leq C_T\|\phi\|_{L_2([0,T])}$ for any $\phi\in L_2([0,T])$. $\square$

3. The Maximal inequality

The goal of this section is to translate the p-th moment maximal inequality given by Corollary 2.3 into a similar inequality (in the l_2-norm) for a sequence $(u^k)_k$ of Skorohod integrable processes, with respect to a sequence $(\beta^k)_k$ of i.i.d. fBm's. The idea is to recover a Gaussian process B (as in Sect. 2) from $(\beta^k)_k$, and to construct a Skorohod integrable process U (with respect to B) from the sequence $(u^k)_k$, such that $\delta^B(U1_{[0,t]})=\sum_{k=1}^\infty \delta^{\beta^k}(u^k 1_{[0,t]})$ for all $t\in[0,T]$ a.s.

Let $\beta^k=(\beta_t^k)_{t\in[0,T]}, k\geq 1$ be a sequence of i.i.d. fBm's of Hurst index $H>1/2$, defined on the same probability space $(\Omega,\mathcal{F},P)$. Let V be an arbitrary Hilbert space, and $(e_k)_k$ a complete orthonormal system in V.

The first result shows that it is possible to construct a centered Gaussian process B with covariance (2.3), from the sequence $(\beta^k)_k$. This result is probably well-known; we state it for the sake of completeness.

Lemma 3.1. *Let $(\phi^k)_k\subset\mathcal{H}$ be such that $\sum_{k=1}^\infty\|\phi^k\|_{\mathcal{H}}^2<\infty$. Then:*

a) $\varphi^{(N)}:=\sum_{k=1}^N\phi^k e_k\in\mathcal{H}_V$ for all $N\geq 1$, and there exists $\varphi:=\sum_{k=1}^\infty\phi^k e_k\in\mathcal{H}_V$ such that $\|\varphi^{(N)}-\varphi\|_{\mathcal{H}_V}\to 0$ as $N\to\infty$. We have:

$$\|\varphi\|_{\mathcal{H}_V}^2=\sum_{k=1}^\infty\|\phi^k\|_{\mathcal{H}}^2; \tag{3.1}$$

b) $B^{(N)}(\varphi):=\sum_{k=1}^N\beta^k(\phi^k)\in L_2(\Omega)$ for any $N\geq 1$, and there exists $B(\varphi):=\sum_{k=1}^\infty\beta^k(\phi^k)\in L_2(\Omega)$ such that $E|B^{(N)}(\varphi)-B(\varphi)|^2\to 0$ as $N\to\infty$. The process $B=\{B(\varphi)\}_{\varphi\in\mathcal{H}_V}$ is Gaussian with mean zero and *covariance (2.3). In particular, for any $t \in [0,T], \varphi \in V$, we have:*

$$B_t(\varphi) := B(1_{[0,t]}\varphi) = \sum_{k=1}^{\infty} \langle \varphi, e_k \rangle_V \beta_t^k \quad \textit{in } L_2(\Omega). \tag{3.2}$$

Proof. a) The sequence $\{\varphi^{(N)}\}_N$ is Cauchy in $\mathcal{H}_V$, since $(\varphi^{(N)} - \varphi^{(M)})(t) = \sum_{k=M+1}^{N} \phi^k(t) e_k$ for any $N > M \geq 1$, and hence

$$\begin{aligned} \|\varphi^{(N)} - \varphi^{(M)}\|_{\mathcal{H}_V}^2 &= \alpha_H \sum_{k=M+1}^{N} \int_0^T \int_0^T \phi^k(t)\phi^k(s)|t-s|^{2H-2} \mathrm{d}s\mathrm{d}t \\ &= \sum_{k=M+1}^{N} \|\phi^k\|_{\mathcal{H}}^2 \to 0, \text{ as } M, N \to \infty. \end{aligned}$$

In particular, $\|\varphi^{(N)}\|_{\mathcal{H}_V}^2 = \sum_{k=1}^{N} \|\phi^k\|_{\mathcal{H}}^2$. By letting $N \to \infty$, we obtain (3.1).

b) The sequence $\{B^{(N)}(\varphi)\}_N$ is Cauchy in $L_2(\Omega)$, since $B^{(N)}(\varphi) - B^{(M)}(\varphi) = \sum_{k=M+1}^{N} \beta^k(\phi^k)$ for any $N > M \geq 1$, and hence

$$E|B^{(N)}(\varphi) - B^{(M)}(\varphi)|^2 = \sum_{k=M+1}^{N} E|\beta^k(\phi^k)|^2 = \sum_{k=M+1}^{N} \|\phi^k\|_{\mathcal{H}}^2 \to 0, \text{ as } M, N \to \infty.$$

To prove (3.2), note that $1_{[0,t]}\varphi = \sum_{k=1}^{\infty} \phi^k e_k$, where $\phi^k = 1_{[0,t]}\langle \varphi, e_k \rangle_V$. It follows that $B(1_{[0,t]}\varphi) = \sum_{k=1}^{\infty} \beta^k(\phi^k) = \sum_{k=1}^{\infty} \langle \varphi, e_k \rangle_V \beta_t^k$. □

We begin now to explore the relationship between the Malliavin derivatives with respect to $(\beta^k)_k$ and the Malliavin derivative with respect to B.

An immediate consequence of (3.2) is that $\beta_t^k = B(1_{[0,t]} e_k)$ for any $t \in [0,T]$, and hence

$$\beta^k(\phi) = B(\phi e_k), \quad \forall \phi \in \mathcal{H}. \tag{3.3}$$

Let $F = f(\beta^k(\phi_1), \ldots, \beta^k(\phi_n)) \in \mathcal{S}_{\beta^k}$ be arbitrary, with $f \in C_b^{\infty}(\mathbb{R}^n)$ and $\phi_i \in \mathcal{H}$. Then $\varphi_i := \phi_i e_k \in \mathcal{H}_V$, $F = f(B(\varphi_1), \ldots, B(\varphi_n)) \in \mathcal{S}_B$, and

$$\begin{aligned} D_t^B F &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i = \left[\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\beta^k(\phi_1), \ldots, \beta^k(\phi_n))\phi_i\right] e_k \\ &= (D_t^{\beta^k} F) e_k. \end{aligned}$$

From here we conclude that $\mathcal{S}_{\beta^k} \subset \mathcal{S}_B$, and for any $F \in \mathcal{S}_{\beta^k}$,

$$\|D^B F\|_{\mathcal{H}_V} = \|D^{\beta^k} F\|_{\mathcal{H}}, \quad \|F\|_{\mathbb{D}_B^{1,p}} = \|F\|_{\mathbb{D}_{\beta^k}^{1,p}}, \ \forall p \geq 1.$$

It follows that $\mathbb{D}_{\beta^k}^{1,p} \subset \mathbb{D}_B^{1,p}$ for any $p \geq 1$, and

$$D^B F = (D^{\beta^k} F) e_k, \quad \text{for any } F \in \mathbb{D}_{\beta^k}^{1,2}. \tag{3.4}$$

If $u=\sum_{j=1}^m F_j\phi_j \in \mathcal{S}_{\beta^k}(\mathcal{H})$ is arbitrary, with $F_j\in\mathcal{S}_{\beta^k}$ and $\phi_j\in\mathcal{H}$, then $ue_k\in\mathbb{D}_B^{1,2}(\mathcal{H}_V)$ and

$$D^B(ue_k)=\sum_{j=1}^m (D^B F_j)\phi_j e_k=\sum_{j=1}^m (D^{\beta^k}F_j)\phi_j e_k\otimes e_k=(D^{\beta^k}u)e_k\otimes e_k.$$

In general, if $u\in\mathbb{D}_{\beta^k}^{1,2}(\mathcal{H})$, then $ue_k\in\mathbb{D}_B^{1,2}(\mathcal{H}_V)$ and

$$D^B(ue_k)=(D^{\beta^k}u)e_k\otimes e_k. \tag{3.5}$$

Moreover, we have the following result:

Lemma 3.2. *If $u^k\in\mathbb{D}_{\beta^k}^{1,2}(\mathcal{H})$, then $\sum_{k=1}^N u^k e_k\in\mathbb{D}_B^{1,2}(\mathcal{H}_V)$, $D^B(\sum_{k=1}^N u^k e_k)=\sum_{k=1}^N (D^{\beta^k}u^k)e_k\otimes e_k$, and $\|\sum_{k=1}^N u^k e_k\|^2_{\mathbb{D}_B^{1,2}(\mathcal{H}_V)}=\sum_{k=1}^N\|u^k\|^2_{\mathbb{D}_{\beta^k}^{1,2}(\mathcal{H})}$.*

Proof. The result follows from the definitions of the norms in $\mathbb{D}_B^{1,2}(\mathcal{H}_V)$, respectively $\mathbb{D}_{\beta^k}^{1,2}(\mathcal{H})$, and the following two identities:

$$\begin{aligned}
\left\|\sum_{k=1}^N u^k e_k\right\|^2_{\mathcal{H}_V} &= \alpha_H\int_0^T\int_0^T\left\langle \sum_{k=1}^N u_t^k e_k,\sum_{l=1}^N u_s^l e_l\right\rangle_V |t-s|^{2H-2}\mathrm{d}s\mathrm{d}t\\
&= \alpha_H\sum_{k,l=1}^N\int_0^T\int_0^T u_t^k u_s^l\langle e_k,e_l\rangle_V|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t\\
&= \sum_{k=1}^N\|u^k\|^2_{\mathcal{H}}\\
\left\|D^B\left(\sum_{k=1}^N u^k e_k\right)\right\|^2_{\mathcal{H}_V\otimes\mathcal{H}_V} &= \alpha_H^2\int_{[0,T]^4}\left\langle\sum_{k=1}^N D_\theta^B(u_t^k e_k),\sum_{l=1}^N D_\eta^B(u_s^l e_l)\right\rangle_{V\otimes V}\\
&\quad\times|t-s|^{2H-2}|\theta-\eta|^{2H-2}\mathrm{d}\theta\mathrm{d}\eta\mathrm{d}s\mathrm{d}t\\
&= \alpha_H^2\sum_{k,l=1}^N\int_{[0,T]^4}(D_\theta^{\beta^k}u_t^k)(D_\eta^{\beta^k}u_s^l)\langle e_k\otimes e_k,e_l\otimes e_l\rangle_{V\otimes V}\\
&\quad\times|t-s|^{2H-2}|\theta-\eta|^{2H-2}\mathrm{d}\theta\mathrm{d}\eta\mathrm{d}s\mathrm{d}t\\
&= \sum_{k=1}^N\|D^{\beta^k}u^k\|^2_{\mathcal{H}\otimes\mathcal{H}},
\end{aligned}$$

where we used (3.5) for the second-last equality above. □

We need an auxiliary result.

Lemma 3.3. *Let X be a normed space and $y_N,x_{N,n},x_n,x\in X$ be such that: $\lim_{n\to\infty}\sup_{N\geq1}\|y_N-x_{N,n}\|=0$, $\lim_{N\to\infty}\|x_{N,n}-x_n\|=0$ for all n, and $\lim_{n\to\infty}\|x_n-x\|=0$. Then $\lim_{N\to\infty}\|y_N-x\|=0$.*

Proof. We use $\|y_N-x\|\leq\|y_N-x_{N,n}\|+\|x_{N,n}-x_n\|+\|x_n-x\|$. □

The previous observations allow us to extend Lemma 3.1 to the case of random integrands.

Theorem 3.4. *Let $u^k\in\mathbb{D}_{\beta^k}^{1,2}(\mathcal{H})$ for all $k\geq1$, such that*

$$\sum_{k=1}^\infty\|u^k\|^2_{\mathbb{D}_{\beta^k}^{1,2}(\mathcal{H})}<\infty. \tag{3.6}$$

Then:

a) $U^{(N)} := \sum_{k=1}^{N} u^k e_k \in \mathbb{D}_B^{1,2}(\mathcal{H}_V)$ *for any* $N \geq 1$*, and there exists* $U := \sum_{k=1}^{\infty} u^k e_k \in \mathbb{D}_B^{1,2}(\mathcal{H}_V)$ *such that* $\|U^{(N)} - U\|_{\mathbb{D}_B^{1,2}(\mathcal{H}_V)} \to 0$ *as* $N \to \infty$*. We have:* $D^B U = \sum_{k=1}^{\infty} (D^{\beta^k} u^k) e_k \otimes e_k$ *and*

$$\|U\|^2_{\mathbb{D}_B^{1,2}(\mathcal{H}_V)} = \sum_{k=1}^{\infty} \|u^k\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})}. \tag{3.7}$$

b) the sequence $W^{(N)} := \sum_{k=1}^{N} \delta^{\beta^k}(u^k), N \geq 1$ *has a limit in* $L_2(\Omega)$*, which coincides with* $\delta^B(U)$*. We write*

$$\delta^B(U) = \sum_{k=1}^{\infty} \delta^{\beta^k}(u^k) \quad \textit{in } L_2(\Omega). \tag{3.8}$$

Proof. a) By Lemma 3.2, $\{U^{(N)}\}_N$ is a Cauchy sequence in $\mathbb{D}_B^{1,2}(\mathcal{H}_V)$, since:

$$\|U^{(N)} - U^{(M)}\|^2_{\mathbb{D}_B^{1,2}(\mathcal{H}_V)} = \sum_{k=M+1}^{N} \|u^k\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})} \to 0, \text{ as } M, N \to \infty.$$

Hence, $U := \lim_{N\to\infty} U^{(N)}$ exists in $\mathbb{D}_B^{1,2}(\mathcal{H}_V)$, and $D^B U = \lim_{N\to\infty} D^B U^{(N)}$ in $L_2(\Omega; \mathcal{H}_V \otimes \mathcal{H}_V)$. Also, $\|U^{(N)}\|^2_{\mathbb{D}_B^{1,2}(\mathcal{H}_V)} = \sum_{k=1}^{N} \|u^k\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})}$, and relation (3.7) follows by letting $N \to \infty$.

b) By inequality (2.4) (applied for $V = \mathbb{R}$ and $B = \beta^k$), we have:

$$\sum_{k=M+1}^{N} E|\delta^{\beta^k}(u^k)|^2 \leq \sum_{k=M+1}^{N} \|u^k\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})} \to 0, \text{ as } M, N \to \infty,$$

i.e. the sequence $\{W^{(N)}\}_N$ is Cauchy in $L_2(\Omega)$. We let W be the limit of $\{W^{(N)}\}_N$ in $L_2(\Omega)$. We now prove that $W = \delta^B(U)$ (in $L_2(\Omega)$).

Step 1. Suppose that $u^k \in \mathcal{S}_{\beta^k}(\mathcal{H})$ for all k, *i.e.* $u^k = \sum_{j=1}^{m_k} F_j^k \phi_j^k$ for some $F_j^k \in \mathcal{S}_{\beta^k}$ and $\phi_j^k \in \mathcal{H}$. Since $U^{(N)} \to U$ in $\mathbb{D}_B^{1,2}(\mathcal{H}_V)$, $\delta^B(U^{(N)}) \to \delta^B(U)$ in $L_2(\Omega)$. On the other hand $\sum_{k=1}^{N} \delta^{\beta^k}(u^k) \to W$ in $L_2(\Omega)$. Hence, it suffices to prove that:

$$\delta^B(U^{(N)}) = \sum_{k=1}^{N} \delta^{\beta^k}(u^k). \tag{3.9}$$

Note that $U^{(N)} = \sum_{k=1}^{N} \sum_{k=1}^{m_k} F_j^k \phi_j^k e_k \in \mathcal{S}_B(\mathcal{H}_V)$, since $F_j^k \in \mathcal{S}_{\beta^k} \subset S_B$ and $\phi_j^k e_k \in \mathcal{H}_V$. Relation (3.9) follows from (3.3) and (3.4), since:

$$\begin{aligned}
\delta^B(U^{(N)}) &= \sum_{k=1}^{N} \sum_{j=1}^{m_k} F_j^k B(\phi_j^k e_k) - \sum_{k=1}^{N} \sum_{j=1}^{m_k} \langle D^B F_j^k, \phi_j^k e_k \rangle_{\mathcal{H}_V} \\
\sum_{k=1}^{N} \delta^{\beta^k}(u^k) &= \sum_{k=1}^{N} \sum_{j=1}^{m_k} F_j^k \beta^k(\phi_j^k) - \sum_{k=1}^{N} \sum_{j=1}^{m_k} \langle D^{\beta^k} F_j^k, \phi_j^k \rangle_{\mathcal{H}}.
\end{aligned}$$

(We used relation (1.9) of [18], for the equalities above.)

Step 2. Suppose that $u^k \in \mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})$ for all k. For any $\varepsilon > 0$, there exists $u^k_\varepsilon \in \mathcal{S}_{\beta^k}(\mathcal{H})$ such that $\|u^k_\varepsilon - u^k\|_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})} < \varepsilon/2^k$; hence $\sum_{k=1}^{\infty} \|u^k_\varepsilon\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})} < \infty$. By part a), $U_\varepsilon := \sum_{k=1}^{\infty} u^k_\varepsilon e_k \in \mathbb{D}_B^{1,2}(\mathcal{H}_V)$ and

$\|U_\varepsilon - U\|^2_{\mathbb{D}^{1,2}_B(\mathcal{H}_V)} = \sum_{k=1}^\infty \|u^k_\varepsilon - u^k\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})} \leq \varepsilon^2$. Taking $\varepsilon = 1/n$, we conclude that for any k, there exists a sequence $(u^k_n)_n \subset \mathcal{S}_{\beta^k}(\mathcal{H})$, such that $U_n := \sum_{k=1}^\infty u^k_n e_k \in \mathbb{D}^{1,2}_B(\mathcal{H}_V)$ and

$$\|U_n - U\|^2_{\mathbb{D}^{1,2}_B(\mathcal{H}_V)} = \sum_{k=1}^\infty \|u^k_n - u^k\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})} \to 0, \quad \text{as } n \to \infty.$$

We now invoke Lemma 3.3, with $X = L_2(\Omega)$, and

$$y_N = W^{(N)} = \sum_{k=1}^N \delta^{\beta^k}(u^k), \quad x_{N,n} = \sum_{k=1}^N \delta^{\beta^k}(u^k_n), \quad x_n = \delta^B(U_n), \quad x = \delta^B(U).$$

The hypothesis of the lemma are verified, since $\lim_{N\to\infty} \|x_{N,n} - x_n\|_{L_2(\Omega)} = 0$ for all n (by *Step 1*), $\lim_{n\to\infty} \|x_n - x\|_{L_2(\Omega)} = 0$ (since $U_n \to U$ in $\mathbb{D}^{1,2}_B(\mathcal{H}_V)$),

$$\begin{aligned} \|y_N - x_{N,n}\|^2_{L_2(\Omega)} &= E\left|\sum_{k=1}^N \delta^{\beta^k}(u^k - u^k_n)\right|^2 = \sum_{k=1}^N E|\delta^{\beta^k}(u^k - u^k_n)|^2 \\ &\leq \sum_{k=1}^N \|u^k - u^k_n\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})}, \end{aligned}$$

and hence $\sup_{N\geq 1} \|y_N - x_{N,n}\|^2_{L_2(\Omega)} \leq \sum_{k=1}^\infty \|u^k - u^k_n\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})} \to 0$, as $n \to \infty$. We conclude that $\lim_{N\to\infty} \|y_N - x\|_{L_2(\Omega)} = 0$, *i.e.* $W = \delta^B(U)$. □

In the case $p = 2$, we have the following preliminary result.

Theorem 3.5. *There exists a constant C depending on H and T such that*

$$\begin{aligned} E\sup_{t\leq T}\left|\sum_{k=1}^\infty \int_0^t u^k_s \delta\beta^k_s\right|^2 &\leq C\left\{\sum_{k=1}^\infty E\int_0^T |u^k_s|^2 \mathrm{d}s \right. \\ &\quad \left. + \sum_{k=1}^\infty E\int_0^T \left(\int_0^T |D^{\beta^k}_\theta u^k_s|^{1/H} \mathrm{d}\theta\right)^{2H} \mathrm{d}s\right\} \end{aligned} \tag{3.10}$$

for any process $u = (u^k)_k$ for which $u^k \in \mathbb{D}^{1,2}_{\beta^k}(|\mathcal{H}|)$ for all $k \geq 1$, and the right-hand side of (3.10) is finite.

Proof. Let $0 < \varepsilon < H - 1/2$ and $\alpha = 1/2 - \varepsilon$. As in the proof of Theorem 4, [2], one can show that

$$\sup_{t\leq T}\left|\sum_{k=1}^\infty \int_0^t u^k_s \delta\beta^k_s\right|^2 \leq c'_\alpha \int_0^T \left|\sum_{k=1}^\infty \int_0^r u^k_s (r-s)^{\alpha-1} \delta\beta^k_s\right|^2 \mathrm{d}r.$$

Since the random variables $X_k = \int_0^r u^k_s (r-s)^{\alpha-1} \delta\beta^k_s, k \geq 1$ are independent with zero mean, $E(\sum_{k=1}^n X_k)^2 = \sum_{k=1}^n E(X_k^2)$ for all n. By the Fatou's lemma,

$$E\left|\sum_{k=1}^\infty \int_0^r u^k_s (r-s)^{\alpha-1} \delta\beta^k_s\right|^2 \leq \sum_{k=1}^\infty E\left|\int_0^r u^k_s (r-s)^{\alpha-1} \delta\beta^k_s\right|^2.$$

Using (2.5) and Hölder's inequality we get:

$$\begin{aligned} E\sup_{t\le T}\left|\sum_{k=1}^{\infty}\int_0^t u_s^k\delta\beta_s^k\right|^2 &\le c'_\alpha\sum_{k=1}^{\infty}\int_0^T E\left|\int_0^r u_s^k(r-s)^{\alpha-1}\delta\beta_s^k\right|^2 \mathrm{d}r \\ &\le c_{\alpha,H}\left\{\sum_{k=1}^{\infty}\int_0^T E\left(\int_0^r |u_s^k|^{1/H}(r-s)^{(\alpha-1)/H}\mathrm{d}s\right)^{2H}\mathrm{d}r\right. \\ &\left.+\sum_{k=1}^{\infty}\int_0^T E\left(\int_0^r\int_0^T |D_\theta^{\beta^k}u_s^k|^{1/H}\mathrm{d}\theta(r-s)^{(\alpha-1)/H}\mathrm{d}s\right)^{2H}\mathrm{d}r\right\} \\ &\le c_{\alpha,H}\left\{\sum_{k=1}^{\infty}\int_0^T r^{2(\alpha-1)+2H-1}E\int_0^r |u_s^k|^2\mathrm{d}s\mathrm{d}r\right. \\ &\left.+\sum_{k=1}^{\infty}\int_0^T r^{2(\alpha-1)+2H-1}E\int_0^r\left(\int_0^T |D_\theta^{\beta^k}u_s^k|^{1/H}\mathrm{d}\theta\right)^{2H}\mathrm{d}s\mathrm{d}r\right\}. \qquad \square \end{aligned}$$

Let l_2 be the set of sequences $a=(a^k)_k, a^k\in\mathbb{R}$ with $|a|_{l_2}^2:=\sum_{k=1}^{\infty}|a^k|^2<\infty$. If $u=(u^k)_k$ is such that $u^k\in\mathbb{D}_{\beta^k}^{1,2}(\mathcal{H})$ for all $k\ge 1$, we denote $Du:=(D^{\beta^k}u^k)_k$.

The next theorem is the main result of this section. Its proof is based on Corollary 2.3, the connection between the Skorohod integrals with respect to $(\beta^k)_k$ and the Skorohod integral with respect to B (given by Thm. 3.4), and Theorem 3.5.

Theorem 3.6. *Let $1/2<H<1$ and $p\ge 2$. Then, there exists a constant C depending on H,p and T such that*

$$\begin{aligned} E\sup_{t\le T}\left|\sum_{k=1}^{\infty}\int_0^t u_s^k\delta\beta_s^k\right|^p &\le C\left\{E\left(\int_0^T |u_s|_{l_2}^2\mathrm{d}s\right)^{p/2}\right. \\ &\left.+E\left[\int_0^T\left(\int_0^T |D_\theta u_s|_{l_2}^{1/H}\mathrm{d}\theta\right)^{2H}\mathrm{d}s\right]^{p/2}\right\} \end{aligned} \tag{3.11}$$

for any process $u=(u^k)_k$ for which $u^k\in\mathbb{D}_{\beta^k}^{1,p}(|\mathcal{H}|)$ for all $k\ge 1$, and the right-hand side of (3.11) is finite.

Proof. Let $u=(u^k)_k$ be such that $u^k\in\mathbb{D}_{\beta^k}^{1,p}(|\mathcal{H}|)$ for all $k\ge 1$, and the right-hand side of (3.11) is finite. Since $p\ge 2$, $|EX|^{p/2}\le E|X|^{p/2}$, for any $X\in L_{p/2}(\Omega)$, and hence, $E\int_0^T |u_s|_{l_2}^2\mathrm{d}s<\infty$ and $E\int_0^T\left(\int_0^T |D_\theta u_s|_{l_2}^{1/H}\mathrm{d}\theta\right)^{2H}\mathrm{d}s<\infty$. By Minkowski's inequality, $\sum_{k=1}^{\infty}E\int_0^T\left(\int_0^T |D_\theta^{\beta^k}u_s^k|^{1/H}\mathrm{d}\theta\right)^{2H}\mathrm{d}s<\infty$. From here we conclude that relation (3.6) holds, since:

$$\begin{aligned} \sum_{k=1}^{\infty}\|u^k\|_{\mathbb{D}_{\beta^k}^{1,2}(\mathcal{H})}^2 &= \sum_{k=1}^{\infty}E\|u^k\|_{\mathcal{H}}^2+\sum_{k=1}^{\infty}E\|D^{\beta^k}u^k\|_{\mathcal{H}\otimes\mathcal{H}}^2 \\ &\le \sum_{k=1}^{\infty}E\|u^k\|_{L_2([0,T])}^2+\sum_{k=1}^{\infty}E\|D^{\beta^k}u^k\|_{L_2([0,T];L_{1/H}([0,T]))}^2<\infty. \end{aligned}$$

By Theorem 3.4.(a), there exists $U := \sum_{k=1}^{\infty} u^k e_k \in \mathbb{D}_B^{1,2}(\mathcal{H}_V)$ and $D^B U = \sum_{k=1}^{\infty} (D^{\beta^k} u^k) e_k \otimes e_k$. Similarly, $U 1_{[0,t]} = \sum_{k=1}^{\infty} u^k 1_{[0,t]} e_k \in \mathbb{D}_B^{1,2}(\mathcal{H}_V)$ for any $t \in [0,T]$.

For any $t \in [0,T]$, let

$$X_t := \sum_{k=1}^{\infty} \delta^{\beta^k}(u^k 1_{[0,t]}) \quad \text{and} \quad Y_t := \delta^B(U 1_{[0,t]}).$$

Using the same argument as in Theorem 5 of [2], one can prove that $Y = (Y_t)_{t \in [0,T]}$ has an a.s. continuous modification. We work with this modification.

Also, for each $N \geq 1$, the process $X^{(N)} = (X_t^{(N)})_{t \in [0,T]}$, defined by $X_t^{(N)} := \sum_{k=1}^{N} \delta^{\beta^k}(u^k 1_{[0,t]}), t \in [0,T]$, has an a.s. continuous modification.

By Chebyshev's inequality, Theorem 3.5, and (3.6), the sequence $(X^{(N)})_N$ converges in probability to X, in the sup-norm metric, since for any $\varepsilon > 0$,

$$\begin{aligned} P(\sup_{t \leq T} |X_t^{(N)} - X_t| > \varepsilon) &\leq \frac{1}{\varepsilon^2} E \sup_{t \leq T} \left| \sum_{k=N+1}^{\infty} \int_0^t u_s^k \delta \beta_s^k \right|^2 \\ &\leq \frac{C}{\varepsilon^2} \left\{ \sum_{k=N+1}^{\infty} E \int_0^T |u_s^k|^2 \mathrm{d}s + \sum_{k=N+1}^{\infty} E \int_0^T \left(\int_0^T |D_\theta^{\beta^k} u_s^k|^{1/H} \mathrm{d}\theta \right)^{2H} \mathrm{d}s \right\} \to 0, \end{aligned} \tag{3.12}$$

as $N \to \infty$. Therefore, X has an a.s. continuous modification. We work with this modification.

From Theorem 3.4.(b), we know that $Y_t = X_t$ a.s., for any $t \in [0,T]$. Since both Y and X are a.s. continuous, it follows that $Y_t = X_t$ for all $t \in [0,T]$ a.s. In particular, $E \sup_{t \leq T} |Y_t|^p = E \sup_{t \leq T} |X_t|^p$, *i.e.*

$$E \sup_{t \leq T} \left| \int_0^t U_s \delta B_s \right|^p = E \sup_{t \leq T} \left| \sum_{k=1}^{\infty} \int_0^t u_s^k \delta \beta_s^k \right|^p. \tag{3.13}$$

We now invoke Corollary 2.3. Note that $E(U_s) = \sum_{k=1}^{\infty} E(u_s^k) e_k$. Hence $\|E(U_s)\|_V^2 = \sum_{k=1}^{\infty} |E(u_s^k)|^2 \leq \sum_{k=1}^{\infty} E|u_s^k|^2 = E|u_s|_{l_2}^2$ for any $s \in [0,T]$, and

$$\left(\int_0^T \|E(U_s)\|_V^2 \mathrm{d}s \right)^{p/2} \leq \left(E \int_0^T |u_s|_{l_2}^2 \mathrm{d}s \right)^{p/2} \leq E \left(\int_0^T |u_s|_{l_2}^2 \mathrm{d}s \right)^{p/2}. \tag{3.14}$$

Note also that $D_\theta^B U_s = \sum_{k=1}^{\infty} (D_\theta^{\beta^k} u_s^k) e_k \otimes e_k$, and hence,

$$\|D_\theta^B U_s\|_{V \otimes V}^2 = \sum_{k=1}^{\infty} |D_\theta^{\beta^k} u_s^k|^2 = |D_\theta u|_{l_2}^2. \tag{3.15}$$

Relation (3.11) becomes a consequence of (2.7), combined with (3.13), (3.14) and (3.15). □

The following result is an immediate consequence of Theorem 3.6.

Corollary 3.7. *Let $1/2 < H < 1$ and $p \geq 2$. Then, there exists a constant C depending on H, p and T such that*

$$E \sup_{t \leq T} \left| \sum_{k=1}^{\infty} \int_0^t u_s^k \delta \beta_s^k \right|^p \leq C \left\{ E \int_0^T |u_s|_{l_2}^p \mathrm{d}s + E \int_0^T \left(\int_0^T |D_\theta u_s|_{l_2}^{1/H} \mathrm{d}\theta \right)^{pH} \mathrm{d}s \right\} := C \|u\|_{\mathbb{L}_H^{1,p}(l_2)}^p \tag{3.16}$$

for any process $u = (u^k)_k$ for which $u^k \in \mathbb{D}_{\beta^k}^{1,p}(|\mathcal{H}|)$ for all $k \geq 1$, and the right-hand side of (3.16) is finite.

4. Stochastic banach spaces

In this section, we introduce some Banach spaces of stochastic integrands for the sequence of Skorohod integrals with respect to $(\beta^k)_k$, which are suitable for our analysis. To ease the exposition, we first treat the case of a single fBm (Sect. 4.1), and then the case of a sequence of i.i.d. fBm's (Sect. 4.2).

4.1. **The case of a single fBm**

We begin by recalling some basic facts about fractional Sobolev spaces, using the notation in [13]. We let $C_0^\infty = C_0^\infty(\mathbb{R}^d)$ be the space of infinitely differentiable functions on $\mathbb{R}^d$, with compact support, and $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ be the space of real-valued Schwartz distributions on C_0^∞. For $p \geq 1$, we denote by $L_p = L_p(\mathbb{R}^d)$ the set of all measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ such that $\|u\|_{L_p}^p := \int_{\mathbb{R}^d} |u(x)|^p \mathrm{d}x < \infty$.

For any $p > 1$ and $n \in \mathbb{R}$, we let $H_p^n = H_p^n(\mathbb{R}^d) := \{u \in \mathcal{D}; (1-\Delta)^{n/2} u \in L_p\}$ be the fractional Sobolev space, with the norm $\|u\|_{H_p^n} := \|(1-\Delta)^{n/2}u\|_{L_p}$. For any $u \in H_p^n$ and $\phi \in C_0^\infty$, we define

$$(u, \phi) := \int_{\mathbb{R}^d} [(1-\Delta)^{n/2}u](x) \cdot [(1-\Delta)^{-n/2}\phi](x)\mathrm{d}x.$$

By Hölder's inequality, for any $u \in H_p^n$ and $\phi \in C_0^\infty$, we have:

$$|(u,\phi)|^2 \leq N \|u\|_{H_p^n}^2, \tag{4.1}$$

where $N = \|(1-\Delta)^{-n/2}\phi\|_{L_{p/(p-1)}}^2$ is a constant depending on n, p and ϕ.

Let $\beta = (\beta_t)_{t\in[0,T]}$ be a fBm of index $H > 1/2$, defined on a probability space $(\Omega, \mathcal{F}, P)$. We introduce the following spaces of Banach-space valued integrands for the Skorohod integral with respect to β.

Definition 4.1. Let V be an arbitrary Banach space and $p > 1$.

a) We denote by $\mathbb{D}_\beta^{1,p}(|\mathcal{H}_V|)$ the set of all elements $g \in \mathbb{D}_\beta^{1,p}(\mathcal{H}_V)$ such that $g \in |\mathcal{H}_V|$ a.s., $D^\beta g \in |\mathcal{H}| \otimes |\mathcal{H}_V|$ a.s., and $\|g\|_{\mathbb{D}_\beta^{1,p}(|\mathcal{H}_V|)} < \infty$, where

$$\|g\|_{\mathbb{D}_\beta^{1,p}(|\mathcal{H}_V|)}^p := E\|g\|_{|\mathcal{H}_V|}^p + E\|D^\beta g\|_{|\mathcal{H}|\otimes|\mathcal{H}_V|}^p.$$

b) We denote by $\mathbb{L}_{H,\beta}^{1,p}(V)$ the set of all elements $g \in \mathbb{D}_\beta^{1,p}(|\mathcal{H}_V|)$ such that $\|g\|_{\mathbb{L}_{H,\beta}^{1,p}(V)} < \infty$, where

$$\|g\|_{\mathbb{L}_{H,\beta}^{1,p}(V)}^p := E\int_0^T \|g_s\|_V^p \mathrm{d}s + E\int_0^T \left(\int_0^T \|D_t^\beta g_s\|_V^{1/H} \mathrm{d}t\right)^{pH} \mathrm{d}s.$$

c) We denote by $\widetilde{\mathbb{L}}_{H,\beta}^{1,p}(V)$ the completion of $\mathcal{S}_\beta(\mathcal{E}_V)$ in $\mathbb{D}_\beta^{1,p}(|\mathcal{H}_V|)$, with respect to the norm $\|\cdot\|_{\mathbb{L}_{H,\beta}^{1,p}(V)}$.

Using (2.1) and (2.2), one can prove that:

$$\|g\|_{\mathbb{D}_\beta^{1,p}(|\mathcal{H}_V|)} \leq b_H \|g\|_{\mathbb{L}_{H,\beta}^{1,p}(V)}, \quad \forall u \in \mathbb{L}_{H,\beta}^{1,p}(V). \tag{4.2}$$

Remark 4.2. If $V = \mathbb{R}$, we denote $\mathbb{D}_\beta^{1,p}(|\mathcal{H}_V|) = \mathbb{D}_\beta^{1,p}(|\mathcal{H}|)$, $\mathbb{L}_{H,\beta}^{1,p}(V) = \mathbb{L}_{H,\beta}^{1,p}$, and $\widetilde{\mathbb{L}}_{H,\beta}^{1,p}(V) = \widetilde{\mathbb{L}}_{H,\beta}^{1,p}$.

Note that the space $\mathbb{D}_\beta^{1,p}(|\mathcal{H}_V|$ is *not* the particular instance of the space $\mathbb{D}_B^{1,p}(|\mathcal{H}_V|)$ (introduced in Sect. 2) obtained for $V = \mathbb{R}$. The fundamental difference between the two spaces is that $\mathbb{D}_\beta^{1,p}(|\mathcal{H}_V|)$ contains V-valued random processes $g = \{g(s,\cdot)\}_{s\in[0,T]}$, for an arbitrary Banach space V (which has nothing to do with the underlying Hilbert space $\mathbb{R}$ of the fBm β), whereas the space $\mathbb{D}_B^{1,p}(|\mathcal{H}_V|)$ contains V-valued random processes $U = \{U(s,\cdot)\}_{s\in[0,T]}$, where V *is* the underlying space of the Gaussian process B.

In the present article, we let $V = H_p^n$. Since C_0^∞ is dense in H_p^n, we introduce the set $\mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ of smooth elementary processes of the form

$$g(t,\cdot) = \sum_{i=1}^m F_i 1_{(t_{i-1},t_i]}(t)\phi_i(\cdot), \quad t \in [0,T]$$

with $F_i \in \mathcal{S}_\beta$, $0 \le t_0 < \ldots < t_m \le T$ and $\phi_i \in C_0^\infty$. The set $\mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ is dense in $\mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)$ with respect to the norm $\|\cdot\|_{\mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)}$. The space $\widetilde{\mathbb{L}}_{H,\beta}^{1,p}(H_p^n)$ is the completion of $\mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ in $\mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)$, with respect to the norm $\|\cdot\|_{\mathbb{L}_{H,\beta}^{1,p}(H_p^n)}$. From (4.2), it follows that $\widetilde{\mathbb{L}}_{H,\beta}^{1,p}(H_p^n) \subset \mathbb{L}_{H,\beta}^{1,p}(H_p^n)$.

For any $g \in \mathbb{L}_{H,\beta}^{1,p}(H_p^n)$, we have:

$$\|g\|^p_{\mathbb{L}_{H,\beta}^{1,p}(H_p^n)} = \|g\|^p_{\mathbb{H}_p^n} + \|D^\beta g\|^p_{\mathbb{H}_{p,H}^n}, \tag{4.3}$$

where

$$\begin{aligned}
\mathbb{H}_p^n \quad &:= \quad L_p(\Omega \times [0,T], \mathcal{F} \times \mathcal{B}([0,T]); H_p^n) \\
\mathbb{H}_{p,H}^n \quad &:= \quad L_p(\Omega \times [0,T], \mathcal{F} \times \mathcal{B}([0,T]); L_{1/H}([0,T]; H_p^n)).
\end{aligned}$$

For an arbitrary element $g \in \mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)$, we write $g(*,\cdot) = \{g(s,\cdot)\}_{s\in[0,T]}$. Using (4.1), for any $g \in \mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)$ and $\phi \in C_0^\infty$, we have:

$$E\|(g(*,\cdot),\phi)\|^2_{|\mathcal{H}|} \quad \le \quad NE\|g\|^2_{|\mathcal{H}_{H_p^n}|} \tag{4.4}$$

$$E\|(D^\beta g(*,\cdot),\phi)\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} \quad \le \quad NE\|D^\beta g\|^2_{|\mathcal{H}|\otimes|\mathcal{H}_{H_p^n}|}, \tag{4.5}$$

where N is a constant depending on n, p and ϕ.

Proposition 4.3. *a) If $g \in \mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)$, then for any $\phi \in C_0^\infty$, $(g(*,\cdot),\phi) \in \mathbb{D}_\beta^{1,2}(|\mathcal{H}|)$, $D^\beta(g(*,\cdot),\phi) = (D^\beta g(*,\cdot),\phi)$, and*

$$\|(g(*,\cdot),\phi)\|_{\mathbb{D}_\beta^{1,p}(|\mathcal{H}|)} \le N\|g\|_{\mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)}, \tag{4.6}$$

where N is a constant depending on n, p and ϕ.

b) If $g \in \mathbb{L}_{H,\beta}^{1,p}(H_p^n)$, then for any $\phi \in C_0^\infty$, $(g(*,\cdot),\phi) \in \mathbb{L}_{H,\beta}^{1,p}$, and

$$\|(g(*,\cdot),\phi)\|_{\mathbb{L}_{H,\beta}^{1,p}} \le N\|g\|_{\mathbb{L}_{H,\beta}^{1,p}(H_p^n)}, \tag{4.7}$$

where N is a constant depending on n, p and ϕ.

Proof. a) Using an approximation argument and the completeness of the space $\mathbb{D}_\beta^{1,p}(|\mathcal{H}|)$, it suffices to assume that $g(t,\cdot) = \sum_{i=1}^m F_i 1_{(t_i,t_{i+1}]}(t)\phi_i$ with $F_i \in \mathcal{S}_\beta$, $0 \le t_1 < \ldots < t_{m+1} \le T$ and $\phi_i \in C_0^\infty$. Clearly, $(g(*,\cdot),\phi) = \sum_{i=1}^m F_i(\phi_i,\phi)1_{(t_i,t_{i+1}]} \in \mathcal{S}_\beta(\mathcal{E}) \subset \mathbb{D}_\beta^{1,2}(|\mathcal{H}|)$, and due to the linearity of D^β,

$$D_t^\beta(g(s,\cdot),\phi) = \sum_{i=1}^m (D_t^\beta F_i)(\phi_i,\phi)1_{(t_i,t_{i+1}]}(s) = (D_t^\beta g(s,\cdot),\phi).$$

Using (4.4) and (4.5), we get:

$$\begin{aligned}\|(g(*,\cdot),\phi)\|^p_{\mathbb{D}^{1,p}_\beta(|\mathcal{H}|)} &= E\|(g(*,\cdot),\phi)\|^p_{|\mathcal{H}|} + E\|D^\beta(g(*,\cdot),\phi)\|^p_{|\mathcal{H}|\otimes|\mathcal{H}|} \\ &\leq N\left(E\|g\|^p_{|\mathcal{H}_{H^n_p}|} + E\|D^\beta g\|^p_{|\mathcal{H}|\otimes|\mathcal{H}_{H^n_p}|}\right) = N\|g\|^p_{\mathbb{D}^{1,p}_\beta(|\mathcal{H}_{H^n_p}|)}.\end{aligned}$$

b) By part a), $(g(*,\cdot),\phi)\in\mathbb{D}^{1,p}_\beta(|\mathcal{H}|)$. Using (4.1),

$$\begin{aligned}\|(g(*,\cdot),\phi)\|^p_{\mathbb{L}^{1,p}_{H,\beta}} &= E\int_0^T |(g(s,\cdot),\phi)|^p \mathrm{d}s + E\int_0^T\left(\int_0^T |(D^\beta_t g(s,\cdot),\phi)|^{1/H}\mathrm{d}t\right)^{pH}\mathrm{d}s \\ &\leq NE\int_0^T \|g(s,\cdot)\|^p_{H^n_p}\mathrm{d}s + E\int_0^T\left(\int_0^T \|D^\beta_t g(s,\cdot)\|^{1/H}_{H^n_p}\mathrm{d}t\right)^{pH}\mathrm{d}s \\ &= \|g\|^p_{\mathbb{L}^{1,p}_{H,\beta}(H^n_p)} < \infty.\end{aligned}$$

□

4.2. The case of a sequence of fBm's

For any $p>1$ and $n\in\mathbb{R}$, we let $H^n_p(l_2)$ be the set of all sequences $u=(u^k)_k$ such that $u^k\in H^n_p$ for all k, and $\|u\|_{H^n_p(l_2)} := \|\,|(1-\Delta)^{n/2}u|_{l_2}\|_{L_p}<\infty$. By Minkowski's inequality, $\|u\|^2_{H^n_p(l_2)} \leq \sum_{k=1}^\infty \|u^k\|^2_{H^n_p}$ (with equality if $p=2$). By Hölder's inequality, for any $u\in H^n_p(l_2)$ and $\phi\in C_0^\infty$, we have:

$$\sum_{k=1}^\infty |(u^k,\phi)|^2 \leq N\|u\|^2_{H^n_p(l_2)} \tag{4.8}$$

where N is the same constant as in (4.1).

Let $\beta^k=(\beta^k_t)_{t\in[0,T]}, k\geq 1$ be a sequence of i.i.d. fBm's with Hurst index $H>1/2$, defined on the same probability space $(\Omega,\mathcal{F},P)$. We first define the l_2-analogue of the space $\mathbb{L}^{1,p}_{H,\beta}$, introduced in Section 4.1.

Definition 4.4. For any $p>1$, we denote by $\mathbb{L}^{1,p}_H(l_2)$ the set of all elements $u=(u^k)_k$ such that $u^k\in\mathbb{D}^{1,p}_{\beta^k}(|\mathcal{H}|)$ for all k, and $\|u\|_{\mathbb{L}^{1,p}_H(l_2)}<\infty$, where

$$\|u\|^p_{\mathbb{L}^{1,p}_H(l_2)} := E\int_0^T |u_s|^p_{l_2}\mathrm{d}s + E\int_0^T\left(\int_0^T |D_\theta u_s|^{1/H}_{l_2}\mathrm{d}\theta\right)^{pH}\mathrm{d}s.$$

The next lemma shows that condition (3.6) in Theorem 3.4 is satisfied for any $u=(u^k)_k\in\mathbb{L}^{1,p}_H(l_2)$.

Lemma 4.5. *If $p\geq 2$ and $u=(u^k)_k\in\mathbb{L}^{1,p}_H(l_2)$, then $\sum_{k=1}^\infty \|u^k\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})}<\infty$.*

Proof. Note that $\mathbb{D}^{1,p}_{\beta^k}(|\mathcal{H}|)\subset\mathbb{D}^{1,2}_{\beta^k}(|\mathcal{H}|)$. For any $u\in\mathbb{L}^{1,p}_H(l_2)$, we have:

$$\begin{aligned}\sum_{k=1}^\infty \|u^k\|^2_{\mathbb{D}^{1,2}_{\beta^k}(\mathcal{H})} &\leq \sum_{k=1}^\infty\left\{E\int_0^T |u^k_s|^2\mathrm{d}s + E\int_0^T\left(\int_0^T |D^{\beta^k}_\theta u^k_s|^{1/H}\mathrm{d}\theta\right)^{2H}\mathrm{d}s\right\} \\ &\leq E\int_0^T |u_s|^2_{l_2}\mathrm{d}s + E\int_0^T\left(\int_0^T |D_\theta u_s|^{1/H}_{l_2}\mathrm{d}\theta\right)^{2H}\mathrm{d}s \\ &\leq C_{p,H,T}\|u\|^p_{\mathbb{L}^{1,p}_H(l_2)}<\infty,\end{aligned}$$

where $C_{p,H,T}$ is a constant depending on p, H and T. The first inequality above is due to (2.1) and (2.2), the second is due to Minkowski's inequality, and the third is due to Hölder's inequality. □

We now introduce the definition of the space $\widetilde{\mathbb{L}}^{1,p}_H(H^n_p, l_2)$, in which we are allowed to select the coefficients $(g^k)_k$ multiplying the noise in the stochastic heat equation.

Definition 4.6. Let $p > 1$ be arbitrary.

a) We denote by $\mathbb{L}^{1,p}_H(H^n_p, l_2)$ the set of all elements $g = (g^k)_k$ such that $g^k \in \mathbb{D}^{1,p}_{\beta^k}(|\mathcal{H}_{H^n_p}|)$ for all k, and $\|g\|_{\mathbb{L}^{1,p}_H(H^n_p,l_2)} < \infty$, where

$$\|g\|^p_{\mathbb{L}^{1,p}_H(H^n_p,l_2)} := E\int_0^T |g(s,\cdot)|^p_{H^n_p(l_2)}\mathrm{d}s + E\int_0^T \left(\int_0^T |D_\theta g(s,\cdot)|^{1/H}_{H^n_p(l_2)}\mathrm{d}\theta\right)^{pH}\mathrm{d}s.$$

b) We let $\widetilde{\mathbb{L}}^{1,p}_H(H^n_p, l_2)$ be the set of all $g \in \mathbb{L}^{1,p}_H(H^n_p, l_2)$ for which there exists a sequence $(g_j)_j \subset \mathbb{L}^{1,p}_H(H^n_p, l_2)$ such that $\|g_j - g\|_{\mathbb{L}^{1,p}_H(H^n_p,l_2)} \to 0$ as $j \to \infty$, $g^k_j = 0$ for $k > K_j$, and $g^k_j \in \mathcal{S}_{\beta^k}(\mathcal{E}_{C_0^\infty})$ for $k \le K_j$, *i.e.*

$$g^k_j(t,\cdot) = \sum_{i=1}^{m_{jk}} F^{jk}_i 1_{(t^{jk}_{i-1},t^{jk}_i]}(t)\phi^{jk}_i(\cdot), \quad t \in [0,T],$$

with $F^{jk}_i \in \mathcal{S}_{\beta^k}$, $0 \le t^{jk}_0 < \ldots < t^{jk}_{m_{jk}} \le T$ (non-random) and $\phi^{jk}_i \in C_0^\infty$.

Note that, for any $g \in \mathbb{L}^{1,p}_H(H^n_p, l_2)$,

$$\|g\|^p_{\mathbb{L}^{1,p}_H(H^n_p,l_2)} = \|g\|^p_{\mathbb{H}^n_p(l_2)} + \|Dg\|^p_{\mathbb{H}^n_{p,H}(l_2)}, \tag{4.9}$$

where

$$\begin{aligned}\mathbb{H}^n_p(l_2) &:= L_p(\Omega\times[0,T], \mathcal{F}\times\mathcal{B}([0,T]); H^n_p(l_2))\\ \mathbb{H}^n_{p,H}(l_2) &:= L_p(\Omega\times[0,T], \mathcal{F}\times\mathcal{B}([0,T]); L_{1/H}([0,T]; H^n_p(l_2))).\end{aligned}$$

Lemma 4.7. *If $g = (g^k)_k \in \mathbb{L}^{1,p}_H(H^n_p, l_2)$, then $g^k \in \mathbb{L}^{1,p}_{H,\beta^k}(H^n_p)$ for all k, and*

$$\|g^k\|_{\mathbb{L}^{1,p}_{H,\beta^k}(H^n_p)} \le \|g\|_{\mathbb{L}^{1,p}_H(H^n_p,l_2)} \quad \textit{for all } k.$$

In particular, if $g = (g^k)_k \in \widetilde{\mathbb{L}}^{1,p}_H(H^n_p, l_2)$, then $g^k \in \widetilde{\mathbb{L}}^{1,p}_{H,\beta^k}(H^n_p)$ for all k.

Proof. We have:

$$\begin{aligned}\|g^k\|^p_{\mathbb{L}^{1,p}_{H,\beta^k}(H^n_p)} &= E\int_0^T \|g^k(s,\cdot)\|^p_{H^n_p}\mathrm{d}s + E\int_0^T\left(\int_0^T \|D^{\beta^k}_\theta g^k(s,\cdot)\|^{1/H}_{H^n_p}\right)^{pH}\mathrm{d}s\\ &= E\int_0^T \|(1-\Delta)^{n/2}g^k(s,\cdot)\|^p_{L_p}\mathrm{d}s + E\int_0^T\left(\int_0^T \|D^{\beta^k}_\theta[(1-\Delta)^{n/2}g^k(s,\cdot)]\,\|^{1/H}_{H^n_p}\right)^{pH}\mathrm{d}s\\ &\le E\int_0^T \|\,|(1-\Delta)^{n/2}g(s,\cdot)|_{l_2}\|^p_{L_p}\mathrm{d}s + E\int_0^T\left(\int_0^T \|\,|D_\theta[(1-\Delta)^{n/2}g(s,\cdot)]\,|_{l_2}\|^{1/H}_{H^n_p}\right)^{pH}\mathrm{d}s\\ &= \|g\|^p_{\mathbb{L}^{1,p}_H(H^n_p,l_2)}.\end{aligned}$$

The second statement follows from the definitions of spaces $\widetilde{\mathbb{L}}^{1,p}_H(H^n_p, l_2)$ and $\widetilde{\mathbb{L}}^{1,p}_{H,\beta^k}(H^n_p)$. □

5. The main result

The following definition introduces the solution space (see Def. 3.1 of [13]).

Definition 5.1. Let $p \geq 2$ be arbitrary.

Let $u = \{u(t,\cdot)\}_{t\in[0,T]}$ be a $\mathcal{D}$-valued random process defined on the probability space $(\Omega, \mathcal{F}, P)$. We write $u \in \mathcal{H}^n_{p,H}$ if:

(i) $u(0,\cdot) \in L_p(\Omega, \mathcal{F}, H_p^{n-2/p})$;
(ii) $u \in \mathbb{H}^n_p$, $u_{xx} \in \mathbb{H}^{n-2}_p$;
(iii) there exist $f \in \mathbb{H}^{n-2}_p$ and $g \in \widetilde{\mathbb{L}}^{1,p}_H(H^{n-1}_p, l_2)$ such that for any $\phi \in C_0^\infty$, the equality

$$(u(t,\cdot),\phi) = (u(0,\cdot),\phi) + \int_0^t (f(s,\cdot),\phi)\mathrm{d}s + \sum_{k=1}^{\infty}\int_0^t (g^k(s,\cdot),\phi)\delta\beta^k_s \tag{5.1}$$

holds for any $t \in [0,T]$ a.s. We define

$$\|u\|_{\mathcal{H}^n_{p,H}} = \left(E\|u(0,\cdot)\|^p_{H_p^{n-2/p}}\right)^{1/p} + \|u_{xx}\|_{\mathbb{H}^{n-2}_p} + \|f\|_{\mathbb{H}^{n-2}_p} + \|g\|_{\mathbb{L}^{1,p}_H(H^{n-1}_p,l_2)}. \tag{5.2}$$

If $u \in \mathcal{H}^n_{p,H}$, we write $\mathbf{D}u := f$, $\mathbf{S}u := g$ and $\mathrm{d}u = f\mathrm{d}t + \sum_{k=1}^\infty g^k\delta\beta^k_t$, $t \in [0,T]$.

We say that $u \in \mathcal{H}^n_{p,H}$ is a *solution* of (1.1) if $\mathbf{D}u = \Delta u + f$ and $\mathbf{S}u = g$.

Remark 5.2. The series of stochastic integrals in (5.1) converges uniformly in t, in probability. More precisely, if $g \in \mathbb{L}^{1,p}_H(H^n_p, l_2), \phi \in C_0^\infty$ are arbitrary, and we let $u^k_t = (g^k(t,\cdot),\phi), t \in [0,T]$, then

$$u \in \mathbb{L}^{1,p}_H(l_2).$$

(To see this, note that by Lem. 4.7, $g^k \in \mathbb{L}^{1,p}_{H,\beta^k}(H^n_p)$ for all k). By Proposition 4.3, $u^k \in \mathbb{L}^{1,p}_{H,\beta^k}$ for all k. Since by (4.8), $|u_s|_{l_2} \leq N\|g(s,\cdot)\|_{H^n_p(l_2)}$ and $|D_\theta u_s|_{l_2} \leq N\|D_\theta g(s,\cdot)\|_{H^n_p(l_2)}$, we get: $\|u\|_{\mathbb{L}^{1,p}_H(l_2)} \leq N\|g\|_{\mathbb{L}^{1,p}_H(H^n_p,l_2)} < \infty$). By Lemma 4.5, $\sum_{k=1}^\infty \|u^k\|^2_{\mathbb{D}^{1,2}_{\beta^k}(|\mathcal{H}|)} < \infty$. Denoting $X^{(N)}_t := \sum_{k=1}^N \int_0^t u^k_s\delta\beta^k_s$ and $X_t := \sum_{k=1}^\infty \int_0^t u^k_s\delta\beta^k_s$, relation (3.12) shows that

$$\lim_{N\to\infty} P(\sup_{t\leq T}|X^{(N)}_t - X_t| \geq \varepsilon) = 0, \quad \text{for any } \varepsilon > 0.$$

In what follows, we work with an a.s. continuous modification of $X = (X_t)_{t\in[0,T]}$.

Remark 5.3. By the definition of the norm in $\mathcal{H}^n_{p,H}$, the operators $\mathbf{D}: \mathcal{H}^n_{p,H} \to \mathbb{H}^{n-2}_p([0,T])$ and $\mathbf{S}: \mathcal{H}^n_{p,H} \to \widetilde{\mathbb{L}}^{1,p}_H(H^{n-1}_p, l_2)$ are continuous.

Proposition 5.4. *(a) The operator $(1-\Delta)^{m/2}$ maps isometrically $\widetilde{\mathbb{L}}^{1,p}_H(H^n_p,l_2)$ onto $\widetilde{\mathbb{L}}^{1,p}_H(H^{n-m}_p,l_2)$.*

(b) The operator $(1-\Delta)^{m/2}$ maps isometrically $\mathcal{H}^n_{p,H}$ onto $\mathcal{H}^{n-m}_{p,H}$.

Proof. (a) By the definition of $\widetilde{\mathbb{L}}^{1,p}_H(H^n_p,l_2)$, it suffices to prove that $(1-\Delta)^{m/2}$ maps isometrically $\widetilde{\mathbb{L}}^{1,p}_{H,\beta}(H^n_p)$ onto $\widetilde{\mathbb{L}}^{1,p}_{H,\beta}(H^{n-m}_p)$, for a fixed fBm $\beta = (\beta_t)_{t\in[0,T]}$.

Let $g \in \mathbb{L}^{1,p}_{H,\beta}(H^n_p)$ be arbitrary. By Proposition 4.3,

$$\begin{aligned}(D^\beta[(1-\Delta)^{m/2}g(*,\cdot)],\phi) &= D^\beta((1-\Delta)^{m/2}g(*,\cdot),\phi) = D^\beta(g(*,\cdot),(1-\Delta)^{m/2}\phi)\\ &= (D^\beta g(*,\cdot),(1-\Delta)^{m/2}\phi) = ((1-\Delta)^{m/2}[D^\beta g(*,\cdot)],\phi),\end{aligned}$$

for any $\phi \in C_0^\infty$, *i.e.*

$$D_t^\beta[(1-\Delta)^{m/2}g(s,\cdot)] = (1-\Delta)^{m/2}[D_t^\beta g(s,\cdot)], \quad \forall s,t \in [0,T].$$

Using an approximation argument and the fact that $\|u\|_{H_p^n} = \|(1-\Delta)^{m/2}u\|_{H_p^{n-m}}$ for any $u \in H_p^n$, we conclude that $(1-\Delta)^{m/2}g \in \mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^{n-m}}|)$ and

$$\begin{aligned}\|(1-\Delta)^{m/2}g\|^p_{\mathbb{L}^{1,p}_{H,\beta}(H_p^{n-m})} &= \|(1-\Delta)^{m/2}g\|^p_{\mathbb{H}_p^{n-m}} + \|D^\beta[(1-\Delta)^{m/2}g]\|^p_{\mathbb{H}^{n-m}_{p,H}} \\ &= \|g\|^p_{\mathbb{H}_p^n} + \|D^\beta g\|^p_{\mathbb{H}^n_{p,H}} = \|g\|^p_{\mathbb{L}^{1,p}_{H,\beta}(H_p^n)} < \infty.\end{aligned}$$

This proves that $(1-\Delta)^{m/2}g \in \mathbb{L}^{1,p}_{H,\beta}(H_p^{n-m})$. Finally, if $g \in \widetilde{\mathbb{L}}^{1,p}_{H,\beta}(H_p^n)$, then an approximation argument shows that $(1-\Delta)^{m/2}g \in \widetilde{\mathbb{L}}^{1,p}_{H,\beta}(H_p^{n-m})$.

(b) This is a consequence of part a). See Remark 3.8 of [13]. □

Theorem 5.5. *(a) If $u \in \mathcal{H}^n_{p,H}$, then $u \in C([0,T],H_p^{n-2})$ a.s.,*

$$E\sup_{t\le T}\|u(t,\cdot)\|^p_{H_p^{n-2}} \le N\|u\|^p_{\mathcal{H}^n_{p,H}} \quad \text{and} \quad \|u\|_{\mathbb{H}_p^n} \le N\|u\|_{\mathcal{H}^n_{p,H}},$$

where N is a constant which depends on p,H,T and d.

(b) $\mathcal{H}^n_{p,H}$ is a Banach space with the norm (5.2).

Proof. (a) By Proposition 5.4, it suffices to take $n=0$. We use the same argument as in the proof of Theorem 3.7 of [13]. We refer the reader to this proof for the notation. In our case, we only need to justify that:

$$E\sup_{t\le T}\left\|\sum_{k=1}^\infty\int_0^t g^{(\varepsilon)k}(s,\cdot)\delta\beta_s^k\right\|^p_{L_p} \le C\|u\|^p_{\mathcal{H}^2_{p,H}},$$

where C is a constant which depends on p,H and T.

Using Corollary 3.7, for any $x\in\mathbb{R}^d$, we have:

$$\begin{aligned}E\sup_{t\le T}\left|\sum_{k=1}^\infty\int_0^t g^{(\varepsilon)k}(s,x)\delta\beta_s^k\right|^p &\le C\left\{E\int_0^T|g^{(\varepsilon)}(s,x)|^p_{l_2}\mathrm{d}s\right.\\ &\left.+E\int_0^T\left(\int_0^T|D_\theta g^{(\varepsilon)}(s,x)|^{1/H}_{l_2}\mathrm{d}\theta\right)^{pH}\mathrm{d}s\right\},\end{aligned}$$

where C is a constant depending on p, H and T. We integrate with respect to x. Using Minkowski's inequality and the fact that $\|h^{(\varepsilon)}\|_{L_2} \leq \|h\|_{L_2}$ for any $h \in L_2$, we get:

$$\begin{aligned}
E \sup_{t \leq T} \left\| \sum_{k=1}^{\infty} \int_0^t g^{(\varepsilon)k}(s,\cdot)\delta\beta_s^k \right\|_{L_p}^p &\leq C \Bigg\{ E\int_0^T \int_{\mathbb{R}^d} |g^{(\varepsilon)}(s,x)|_{l_2}^p \mathrm{d}x\mathrm{d}s \\
&\quad + E\int_0^T \int_{\mathbb{R}^d} \left(\int_0^T |D_\theta^{\beta^k} g^{(\varepsilon)}(s,x)|_{l_2}^{1/H} \mathrm{d}\theta \right)^{pH} \mathrm{d}x\mathrm{d}s \Bigg\} \\
&\leq C \left\{ E\int_0^T \| \, |g^{(\varepsilon)}(s,\cdot)|_{l_2}\|_{L_p}^p \mathrm{d}s + E\int_0^T \left(\int_0^T \| \, |D_\theta g^{(\varepsilon)}(s,\cdot)|_{l_2}\|_{L_p}^{1/H} \mathrm{d}\theta \right)^{pH} \mathrm{d}s \right\} \\
&\leq C \left\{ E\int_0^T \| \, |g(s,\cdot)|_{l_2}\|_{L_p}^p \mathrm{d}s + E\int_0^T \left(\int_0^T \| \, |D_\theta g(s,\cdot)|_{l_2}\|_{L_p}^{1/H} \mathrm{d}\theta \right)^{pH} \mathrm{d}s \right\} \\
&= C \left\{ E\int_0^T \|g(s,\cdot)\|_{L_p(l_2)}^p \mathrm{d}s + E\int_0^T \left(\int_0^T \|D_\theta g(s,\cdot)\|_{L_p(l_2)}^{1/H} \mathrm{d}\theta \right)^{pH} \mathrm{d}s \right\} \\
&= C \, \|g\|_{\mathbb{L}_H^{1,p}(L_p,l_2)}^p \leq C\|g\|_{\mathbb{L}_H^{1,p}(H_p^1,l_2)}^p \leq C\|u\|_{\mathcal{H}_{p,H}^2}^p.
\end{aligned}$$

(b) Let $\{u_j\}_j$ be a Cauchy sequence in $\mathcal{H}_{p,H}^n$. By (a), $\{u_j\}_j$ is a Cauchy sequence in $\mathbb{H}_p^n$. Hence, there exists $u \in \mathbb{H}_p^n$ such that $\|u_j - u\|_{\mathbb{H}_p^n} \to 0$. Moreover, $u_{xx} \in \mathbb{H}_p^{n-2}$ and $\|u_{jxx} - u_{xx}\|_{\mathbb{H}_p^{n-2}} \to 0$.

Say u_j satisfies (5.1) for $f_j \in \mathbb{H}_p^{n-2}$, $g_j \in \widetilde{\mathbb{L}}_H^{1,p}(H_p^{n-1}, l_2)$: for any $\phi \in C_0^\infty$,

$$(u_j(t,\cdot),\phi) = (u_j(0,\cdot),\phi) + \int_0^t (f_j(s,\cdot),\phi)\mathrm{d}s + \sum_{k=1}^{\infty}\int_0^t (g_j^k(s,\cdot),\phi)\delta\beta_s^k \tag{5.3}$$

for any $t \in [0,T]$ a.s. Then $\{u_j(0,\cdot)\}_j, \{f_j\}_j$ and $\{g_j\}_j$ are Cauchy in the (complete) spaces $L_p(\Omega,\mathcal{F};H_p^{n-2/p})$, $\mathbb{H}_p^{n-2}$ and $\widetilde{\mathbb{L}}_H^{1,p}(H_p^{n-1},l_2)$, respectively. Hence, there exist $u(0,\cdot) \in L_p(\Omega,\mathcal{F},H_p^{n-2/p})$, $f \in \mathbb{H}_p^{n-2}$, $g \in \widetilde{\mathbb{L}}_H^{1,p}(H_p^{n-1},l_2)$ such that $E\|u_j(0,\cdot) - u(0,\cdot)\|_{H_p^{n-2/p}} \to 0$, $\|f_j - f\|_{\mathbb{H}_p^{n-2}} \to 0$ and $\|g_j - g\|_{\mathbb{L}_H^{1,p}(H_p^{n-1},l_2)} \to 0$.

Since $\|u_j - u\|_{\mathbb{H}_p^n} \to 0$, there exists a subsequence of indices j such that $\|u_j(t,\cdot) - u(t,\cdot)\|_{H_p^n} \to 0$ a.e. in (ω,t). Say that this happens for $\omega \in \Omega\backslash\Gamma$ and $t \in [0,T]\backslash U$, where Γ, U are negligible sets.

Fix $t \in [0,T]\backslash U$. We are now passing to the limit in (5.3). On the left hand side, $|(u_j(t,\cdot) - u(t,\cdot),\phi)| \leq N\|u_j(t,\cdot) - u(t,\cdot)\|_{H_p^n} \to 0$ a.s. On the right hand side of (5.3), the first two terms clearly converge to $(u(0,\cdot),\phi)$, respectively $\int_0^t (f(s,\cdot),\phi)\mathrm{d}s$. For the third term, we invoke Corollary 3.7 and (4.7):

$$\begin{aligned}
E\left|\sum_{k=1}^{\infty}\int_0^t (g_j^k(s,\cdot) - g^k(s,\cdot),\phi)\delta\beta_s^k\right|^p &\leq N\|(g_j^k(*,\cdot) - g^k(*,\cdot),\phi)\|_{\mathbb{L}_H^{1,p}(l_2)}^p \\
&\leq N\|g_j^k - g^k\|_{\mathbb{L}_H^{1,p}(H_p^{n-1},l_2)}^p \to 0, \text{ as } j \to \infty.
\end{aligned}$$

Therefore, $\sum_{k=1}^{\infty}\int_0^t (g_j^k(s,\cdot) - g^k(s,\cdot),\phi)\delta\beta_s^k \to 0$ a.s. (for a subsequence of indices j). We infer that for every $\phi \in C_0^\infty$ and for any $t \in [0,T]\backslash U$, equality (5.1) holds almost surely (with the negligible set depending on t).

To conclude that $u \in \mathcal{H}^n_{p,H}$, it remains to show that equality (5.1) holds for any $t \le T$ a.s. (*i.e.* the negligible set does not depend on t). For this, it suffices to note that the process $(u(*,\cdot),\phi)$ is continuous a.s. This follows from the a.s. continuity of processes $(u_j(*,\cdot),\phi)$, by noting that $(u_j(t,\cdot),\phi)$ converges to $(u(t,\cdot),\phi)$ uniformly in t, in probability. □

The next theorem is the main result of the present article.

Theorem 5.6. *Let $p \ge 2$ and $n \in \mathbb{R}$ be arbitrary. Let*

$$f \in \mathbb{H}^{n-2}_p, \quad g \in \widetilde{\mathbb{L}}^{1,p}_H(H^{n-1}_p, l_2) \quad \textit{and} \quad u_0 \in L_p(\Omega, \mathcal{F}, H^{n-2/p}_p).$$

Then the Cauchy problem for equation (1.1) with initial condition $u(0,\cdot) = u_0$ has a unique solution $u \in \mathcal{H}^n_{p,H}$. For this solution, we have

$$\|u\|_{\mathcal{H}^n_{p,H}} \le N \left\{ \|f\|_{\mathbb{H}^{n-2}_p([0,T])} + \|g\|_{\mathbb{L}^{1,p}_H(H^{n-1}_p, l_2)} + \left(E\|u_0\|^p_{H^{n-2/p}_p} \right)^{1/p} \right\}, \tag{5.4}$$

where N is a constant depending on p, d, T and H.

Proof. We first prove that it suffices to take $u_0 = 0$. To see this, we assume without loss of generality that $n = 2$ (using Prop. 5.4). By Theorem 2.1 of [13], for every $\omega \in \Omega$ fixed, the equation $du = \Delta u \, dt$ with initial condition u_0 has a unique solution $\bar{u} \in H^{1,2}_p$, and $\|\bar{u}\|_{H^{1,2}_p} \le N\|u_0\|_{H^{2-2/p}_p}$ and $\|\bar{u}_{xx}\|_{L_p((0,T)\times\mathbb{R}^d)} \le N\|u_0\|_{H^{2-2/p}_p}$. From here, one can show that $\bar{u} \in \mathcal{H}^2_{p,H}$ and $\|\bar{u}\|_{\mathcal{H}^2_{p,H}} \le N\|u_0\|_{H^{2-2/p}_p}$. Suppose that equation (1.1) with zero initial condition has a unique solution $v \in \mathcal{H}^2_{p,H}$, and $\|v\|_{\mathcal{H}^2_{p,H}} \le N(\|f\|_{\mathbb{H}^0_p} + \|g\|_{\mathbb{L}^{1,p}_H(H^1_p, l_2)})$. Then $u := v + \bar{u} \in \mathcal{H}^2_{p,H}$ is a solution of (1.1) with initial condition u_0, and (5.4) holds.

For the remaining part of the proof, we assume that $u_0 = 0$. By Proposition 5.4, it is enough to consider only one particular value of n. We take $n = 1$.

Case 1. Suppose that $g^k = 0$ for $k > K$, and

$$g^k(t,\cdot) = \sum_{i=1}^{m_k} F^k_i 1_{(t^k_{i-1}, t^k_i]}(t) g^k_i(\cdot), \quad t \in [0,T], \ k \le K,$$

where $F^k_i \in \mathcal{S}_{\beta^k}$, $0 \le t^k_0 < \ldots < t^k_{m_k} \le T$, and $g^k_i \in C^\infty_0$.

Let $v(t,x) = \sum_{k=1}^\infty \int_0^t g^k(s,x) \delta\beta^k_s$ and $z(t,x) = \int_0^t T_{t-s}(\Delta v + f)(s,\cdot)(x) \mathrm{d}s$. One can show that $u = v + z$ is a solution of (1.1).

Let $u_1(t,x) = \int_0^t T_{t-s}[f(s,\cdot)](x)\mathrm{d}s$. We first show that

$$\|u - u_1\|_{\mathbb{H}^0_p([0,T])} \le N\|g\|_{\mathbb{L}^{1,p}_H(L_p, l_2)}, \quad \|u_x - u_{1x}\|_{\mathbb{H}^0_p([0,T])} \le N\|g\|_{\mathbb{L}^{1,p}_H(L_p, l_2)}, \tag{5.5}$$

where N is a constant depending on p, d, T and H.

By definition, $u(t,x)-u_1(t,x)=v(t,x)+\int_0^t T_{t-s}(\Delta v)(s,\cdot)(x)\mathrm{d}s$. Note that $v(s,x)=\sum_{k=1}^{\infty}\sum_{i=1}^{m_k} g_i^k(x)$ $\int_0^s F_i^k 1_{(t_{i-1}^k,t_i^k]}(r)\delta\beta_r^k$. Using the stochastic Fubini's theorem and the fact that $\int_r^t T_{t-s}(\Delta g_i^k)(x)\mathrm{d}s=T_{t-r}g_i^k(x)-g_i^k(x)$, we get:

$$u(t,x)-u_1(t,x)=\sum_{k=1}^{\infty}\sum_{i=1}^{m_k}\int_0^t F_i^k 1_{(t_{i-1}^k,t_i^k]}(r)T_{t-r}g_i^k(x)\delta\beta_r^k=\sum_{k=1}^{\infty}\int_0^t T_{t-r}g^k(r,\cdot)(x)\delta\beta_r^k. \tag{5.6}$$

By Corollary 3.7,

$$\begin{aligned}\|u-u_1\|_{\mathbb{H}_p^0}^p &= \int_0^T\int_{\mathbb{R}^d}E\left|\sum_{k=1}^{\infty}\int_0^t T_{t-s}g^k(s,\cdot)(x)\delta\beta_s^k\right|^p \mathrm{d}x\mathrm{d}t\\ &\le C\left\{\int_0^T\int_{\mathbb{R}^d}E\int_0^t\left(\sum_{k=1}^{\infty}|T_{t-s}g^k(s,\cdot)(x)|^2\right)^{p/2}\mathrm{d}s\,\mathrm{d}x\mathrm{d}t\right.\\ &\quad\left.+\int_0^T\int_{\mathbb{R}^d}E\int_0^t\left[\int_0^T\left(\sum_{k=1}^{\infty}|D_\theta^{\beta^k}[T_{t-s}g^k(s,\cdot)(x)]|^2\right)^{1/(2H)}\mathrm{d}\theta\right]^{pH}\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\right\}\\ &:= C(I_1+I_2).\end{aligned}\tag{5.7}$$

By Theorem 3.6,

$$\begin{aligned}\|u_x-u_{1x}\|_{\mathbb{H}_p^0}^p &= \int_0^T\int_{\mathbb{R}^d}E\left|\sum_{k=1}^{\infty}\int_0^t T_{t-s}g_x^k(s,\cdot)(x)\delta\beta_s^k\right|^p \mathrm{d}x\,\mathrm{d}t\\ &\le C\left\{\int_0^T\int_{\mathbb{R}^d}E\left(\int_0^t\sum_{k=1}^{\infty}|T_{t-s}g_x^k(s,\cdot)(x)|^2\mathrm{d}s\right)^{p/2}\mathrm{d}x\mathrm{d}t\right.\\ &\quad\left.+\int_0^T\int_{\mathbb{R}^d}E\left\{\int_0^t\left[\int_0^T\left(\sum_{k=1}^{\infty}|D_\theta^{\beta^k}[T_{t-s}g_x^k(s,\cdot)(x)]|^2\right)^{1/(2H)}\mathrm{d}\theta\right]^{2H}\mathrm{d}s\right\}^{p/2}\mathrm{d}x\mathrm{d}t\right\}\\ &:= C(J_1+J_2).\end{aligned}\tag{5.8}$$

For evaluating the terms I_2 and J_2 above, we need to observe that:

$$D_\theta^{\beta^k}[T_{t-s}g^k(s,\cdot)(x)]=T_{t-s}[D_\theta^{\beta^k}g^k(s,\cdot)](x). \tag{5.9}$$

(This is a consequence of Prop. 4.3.(a), and the fact that $T_{t-s}g^k(s,\cdot)(x)=(g^k(s,\cdot)*G_{t-s})(x)=(g^k(s,\cdot),G_{t-s}(x-\cdot))$.)

By (A.1) (see Appendix A) and Minkowski's inequality, we have:

$$\begin{aligned} I_1 &= E\int_0^T\int_0^t\int_{\mathbb{R}^d}|T_{t-s}g(s,\cdot)(x)|_{l_2}^p \mathrm{d}x\mathrm{d}s\mathrm{d}t = E\int_0^T\int_0^t \|T_{t-s}g(s,\cdot)\|_{L_p(l_2)}^p \mathrm{d}s\mathrm{d}t \\ &\leq E\int_0^T\int_0^t \|g(s,\cdot)\|_{L_p(l_2)}^p \mathrm{d}s\mathrm{d}t \leq T\|g\|_{\mathbb{H}_p^0(l_2)}^p \qquad (5.10) \\ I_2 &= E\int_0^T\int_0^t\int_{\mathbb{R}^d}\left(\int_0^T |T_{t-s}[D_\theta g(s,\cdot)](x)|_{l_2}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}x\mathrm{d}s\mathrm{d}t \\ &\leq E\int_0^T\int_0^t\left[\int_0^T\left(\int_{\mathbb{R}^d}|T_{t-s}[D_\theta g(s,\cdot)](x)|_{l_2}^p\mathrm{d}x\right)^{1/(pH)}\mathrm{d}\theta\right]^{pH}\mathrm{d}s\mathrm{d}t \\ &= E\int_0^T\int_0^t\left(\int_0^T \|T_{t-s}[D_\theta g(s,\cdot)]\ \|_{L_p(l_2)}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}s\mathrm{d}t \\ &\leq E\int_0^T\int_0^t\left(\int_0^T \|D_\theta g(s,\cdot)\|_{L_p(l_2)}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}s\mathrm{d}t \\ &\leq TE\int_0^T\left(\int_0^T \|D_\theta g(s,\cdot)\|_{L_p(l_2)}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}s = T\|Dg\|_{\mathbb{H}_{p,H}^0(l_2)}^p. \qquad (5.11) \end{aligned}$$

From (5.7), (5.10) and (5.11), we conclude that:

$$\|u-u_1\|_{\mathbb{H}_p^0}^p \leq CT\left(\|g\|_{\mathbb{H}_p^0(l_2)}^p + \|Dg\|_{\mathbb{H}_{p,H}^0(l_2)}^p\right) = CT\|g\|_{\mathbb{L}_H^{1,p}(L_p,l_2)}^p.$$

Using Theorem A.1 (Appendix A) and Minkowski's inequality, we have:

$$\begin{aligned} J_1 &= E\int_{\mathbb{R}^d}\int_0^T\left(\int_0^t |T_{t-s}g_x(s,\cdot)(x)|_{l_2}^2\mathrm{d}s\right)^{p/2}\mathrm{d}t\mathrm{d}x \\ &\leq NE\int_{\mathbb{R}^d}\int_0^T |g(s,x)|_{l_2}^p\mathrm{d}s\,\mathrm{d}x = N\|g\|_{\mathbb{H}_p^0(l_2)}^p \qquad (5.12) \end{aligned}$$

Using Theorem A.2 (Appendix A), we have:

$$\begin{aligned} J_2 &= E\int_{\mathbb{R}^d}\int_0^T\left[\int_0^t\left(\int_0^T |T_{t-s}[D_\theta g_x(s,\cdot)](x)|_{l_2}^{1/H}\mathrm{d}\theta\right)^{2H}\mathrm{d}s\right]^{p/2}\mathrm{d}t\mathrm{d}x \\ &\leq NE\int_0^T\left[\int_0^T\left(\int_{\mathbb{R}^d}|D_\theta g(s,x)|_{l_2}^p\mathrm{d}x\right)^{1/(pH)}\mathrm{d}\theta\right]^{pH}\mathrm{d}s \\ &= NE\int_0^T\left(\int_0^T \|D_\theta g(s,x)\|_{L_p(l_2)}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}s = N\|Dg\|_{\mathbb{H}_{p,H}^0(l_2)}^p. \qquad (5.13) \end{aligned}$$

From (5.8), (5.12) and (5.13), we infer that:

$$\|u_x-u_{1x}\|_{\mathbb{H}_p^0}^p \leq CN\left(\|g\|_{\mathbb{H}_p^0(l_2)}^p + \|Dg\|_{\mathbb{H}_{p,H}^0(l_2)}^p\right) = CN\|g\|_{\mathbb{L}_H^{1,p}(L_p,l_2)}^p.$$

This concludes the proof of (5.5).

It remains to prove that $u \in \mathcal{H}^1_{p,H}$. Using (5.5), we have:

$$\begin{aligned} \|u\|_{\mathbb{H}^0_p} &\leq \|u_1\|_{\mathbb{H}^0_p} + \|u-u_1\|_{\mathbb{H}^0_p} \leq N\left(\|f\|_{\mathbb{H}^{-1}_p} + \|g\|_{\mathbb{L}^{1,p}_H(L_p,l_2)}\right) && (5.14)\\ \|u_{xx}\|_{\mathbb{H}^{-1}_p} &\leq \|u_{1xx}\|_{\mathbb{H}^{-1}_p} + \|u_{xx}-u_{1xx}\|_{\mathbb{H}^{-1}_p} \leq \|u_{1x}\|_{\mathbb{H}^0_p} + \|u_x - u_{1x}\|_{\mathbb{H}^0_p} \\ &\leq N\left(\|f\|_{\mathbb{H}^{-1}_p} + \|g\|_{\mathbb{L}^{1,p}_H(L_p,l_2)}\right). && (5.15)\end{aligned}$$

Using the fact that $\|\phi\|_{H^1_p} \leq \|\phi\|_{L_p} + \|\phi_{xx}\|_{H^{-1}_p}$, (5.14) and (5.15), we get:

$$\|u\|_{\mathbb{H}^1_p} \leq \|u\|_{\mathbb{H}^0_p} + \|u_{xx}\|_{\mathbb{H}^{-1}_p} \leq N\left(\|f\|_{\mathbb{H}^{-1}_p} + \|g\|_{\mathbb{L}^{1,p}_H(L_p,l_2)}\right).$$

We conclude that $u \in \mathbb{H}^1_p$ and $u_{xx} \in \mathbb{H}^{-1}_p$, and hence $u \in \mathcal{H}^1_{p,H}$. Since $\mathbf{D}u = \Delta u + f$, we also infer that $\|u\|_{\mathcal{H}^1_{p,H}} \leq N(\|f\|_{\mathbb{H}^{-1}_p} + \|g\|_{\mathbb{L}^{1,p}_H(L_p,l_2)})$.

Case 2. The case of arbitrary $g = (g^k)_k \in \widetilde{\mathbb{L}}^{1,p}_H(L_p,l_2)$ follows as in the proof of Theorem 4.2 of [13], using an approximation argument. This is based on the validity of the result in *Case 1* and the completeness of the spaces $\mathbb{H}^{n-2}_p$, $\widetilde{\mathbb{L}}^{1,p}_H(H^{n-1}_p,l_2)$ and $\mathcal{H}^n_{p,H}$ (Thm. 5.5.(b)) □

Recall that, if V is a Banach space and $\sigma \in (0,1)$, the Hölder space $C^\sigma([0,T],V)$ is defined as the class of all continuous functions $u:[0,T] \to V$ with

$$\|u\|_{C^\sigma([0,T],V)} := \sup_{t\in[0,T]} \|u(t)\|_V + \sup_{0\leq s<t\leq T} \frac{\|u(t)-u(s)\|_V}{(t-s)^\sigma} < \infty.$$

Our final result is an embedding theorem for the space $\mathcal{H}^n_{p,H}$, similar to Theorem 7.2 of [13].

Theorem 5.7. *Let $p>2$, $n \in \mathbb{R}$ and $1/2 \geq \beta > \alpha > 1/p$. If $u \in \mathcal{H}^n_{p,H}$ then $u \in C^{\alpha-1/p}([0,T],H^{n-2\beta}_p)$ a.s. and*

$$E\|u(t,\cdot)-u(s,\cdot)\|^p_{H^{n-2\beta}_p} \leq N(d,\beta,p,T)(t-s)^{\beta p-1}\|u\|^p_{\mathcal{H}^n_{p,H}}, \quad \forall 0\leq s<t\leq T;$$

$$E\|u\|^p_{C^{\alpha-1/p}([0,T],H^{n-2\beta}_p)} \leq N(d,\beta,\alpha,p,T)\|u\|^p_{\mathcal{H}^n_{p,H}}.$$

Proof. We define $f = \mathbf{D}u - \Delta u$, $g = \mathbf{S}u$ and $u_0 = u(0,\cdot)$. Then u satisfies the equation $\mathrm{d}v = (\Delta v + f)\mathrm{d}t + \sum_k g^k \delta\beta^k_t$, with initial condition $v(0,\cdot) = u_0$. By Theorem 5.6, this equation has a unique solution $v \in \mathcal{H}^n_{p,H}$. It follows that $u(t,\cdot) = v(t,\cdot)$ for all $t \in [0,T]$, and it suffices to prove the theorem for v in place of u. By Proposition 5.4, without loss of generality, we take $n = 2\beta$. The theorem will be proved once we show that

$$E\|u(t,\cdot)-u(s,\cdot)\|^p_{L_p} \leq N(t-s)^{\alpha p-1}\left\{\|f\|^p_{\mathbb{H}^{n-2}_p} + \|g\|^p_{\mathbb{L}^{1,p}_H(H^{n-1}_p,l_2)} + E\|u_0\|^p_{H^{n-2/p}_p}\right\} \quad (5.16)$$

$$E\sup_{0\leq s<t\leq T}\frac{\|u(t,\cdot)-u(s,\cdot)\|^p_{L_p}}{(t-s)^{\alpha p-1}} \leq N\left\{\|f\|^p_{\mathbb{H}^{n-2}_p} + \|g\|^p_{\mathbb{L}^{1,p}_H(H^{n-1}_p,l_2)} + E\|u_0\|^p_{H^{n-2/p}_p}\right\}. \quad (5.17)$$

Using an approximation argument and Theorem 5.5, it is enough to assume that $u_0(\cdot) = 1_{A_0}\phi(\cdot)$ with $A_0 \in \mathcal{F}, \phi \in C^\infty_0$,

$$f(t,\cdot) = \sum_{i=1}^m\sum_{j=1}^{m'} 1_{A_j}1_{(t_{i-1},t_i]}(t)f_{ij}(\cdot) \quad \text{and} \quad g^k(t,\cdot) = \sum_{i=1}^{m_k} F^k_i 1_{(t^k_{i-1},t^k_i]}(t)g^k_i(\cdot) \quad (5.18)$$

where $A_j \in \mathcal{F}$, $0 \leq t_1 < \ldots < t_m \leq T$ (non-random), $f_{ij} \in C^\infty_0$, $F^k_i \in \mathcal{S}_{\beta^k}$, $0 \leq t^k_1 < \ldots < t^k_{m_k} \leq T$ (non-random), $g^k_i \in C^\infty_0$, and $g^k_i = 0$ for $k > K$.

Clearly, $u_0 \in L_p(\Omega, \mathcal{F}, H_p^{2-2/p})$, $f \in \mathbb{H}_p^0$ and $g \in \mathbb{L}_H^{1,p}(H_p^1, l_2)$. By Theorem 5.6, it follows that $u \in \mathcal{H}_{p,H}^2$. By Theorem 5.5.(a), $u \in C([0,T], L_p)$ a.s.

Let $u_1(t,x) = T_t u_0(x) + \int_0^t T_{t-s} f(s,\cdot)(x) \mathrm{d}s$ and $u_2(t,x) = u(t,x) - u_1(t,x)$. Relations (5.16) and (5.17) for u_1 follow as in the proof of Theorem 7.2 of [13].

Hence, it suffices to prove (5.16) and (5.17) for u_2. Using (5.6), it follows that

$$u_2(r+\gamma, x) - u_2(r,x) = (T_\gamma - 1)u_2(r,\cdot)(x) + \sum_{k=1}^{\infty} \int_r^{r+\gamma} T_{r+\gamma-\rho} g^k(\rho,\cdot)(x) \delta\beta_\rho^k$$

and hence $E\|u_2(r+\gamma,\cdot) - u_2(r,\cdot)\|_{L_p}^p \leq N(A_2(r,\gamma) + B_2(r,\gamma))$, where

$$\begin{aligned} A_2(r,\gamma) &:= E\int_{\mathbb{R}^d} |(T_\gamma - 1)u_2(r,\cdot)(x)|^p \mathrm{d}x \\ B_2(r,\gamma) &:= E\int_{\mathbb{R}^d} \left| \sum_{k=1}^{\infty} \int_r^{r+\gamma} T_{r+\gamma-\rho} g^k(\rho,\cdot)(x) \delta\beta_\rho^k \right|^p \mathrm{d}x. \end{aligned}$$

We now apply Lemma 7.4 of [13] to the continuous function $u_2 : [0,T] \to L_p$:

$$E\|u_2(t,\cdot) - u_2(s,\cdot)\|_{L_p}^p \leq N(t-s)^{\alpha p - 1}(I_2(t,s) + J_2(t,s))$$

$$E \sup_{0 \leq s < t \leq T} \frac{\|u_2(t,\cdot) - u_2(s,\cdot)\|_{L_p}^p}{(t-s)^{\alpha p - 1}} \leq N(I_2(t,s) + J_2(t,s)),$$

with

$$I_2(t,s) = \int_0^{t-s} \frac{\mathrm{d}\gamma}{\gamma^{1+\alpha p}} \int_s^{t-\gamma} A_2(r,\gamma)\mathrm{d}r, \;\; J_2(t,s) = \int_0^{t-s} \frac{\mathrm{d}\gamma}{\gamma^{1+\alpha p}} \int_s^{t-\gamma} B_2(r,\gamma)\mathrm{d}r.$$

The term $I_2(t,s)$ is estimated as in [13], using Theorem 5.6:

$$I_2(t,s) \leq N(t-s)^{(\beta-\alpha)p} \|g\|_{\mathbb{L}_H^{1,p}(H_p^{n-1}, l_2)}^p. \tag{5.19}$$

It remains to estimate $J_2(t,s)$. Using Theorem 3.6, we have:

$$\begin{aligned} B_2(r,\gamma) &\leq N \left\{ \int_{\mathbb{R}^d} E\left(\int_r^{r+\gamma} |T_{r+\gamma-\rho} g(\rho,\cdot)(x)|_{l_2}^2 \mathrm{d}\rho \right)^{p/2} \mathrm{d}x \right. \\ &\quad \left. + \int_{\mathbb{R}^d} E\left[\int_r^{r+\gamma} \left(\int_0^T |D_\theta[T_{r+\gamma-\rho} g(\rho,\cdot)(x)]|_{l_2}^{1/H} \mathrm{d}\theta \right)^{2H} \mathrm{d}\rho \right]^{p/2} \mathrm{d}x \right\} \\ &:= N(B_2'(r,\gamma) + B_2''(r,\gamma)). \end{aligned} \tag{5.20}$$

The term $B_2'(r,\gamma)$ is treated as in [13]:

$$B_2'(r,\gamma) \leq N\gamma^{\beta p - 1} E\int_0^\gamma \|g(r+\rho,\cdot)\|_{H_p^{n+1}(l_2)}^p \mathrm{d}\rho. \tag{5.21}$$

For the term $B_2''(r,\gamma)$, we use (5.9), Hölder's inequality with $q = p/(p-2)$, Minkowski's inequality, and Lemma 7.3 of [13]:

$$\begin{aligned}
B_2''(r,\gamma) &= E\left[\int_r^{r+\gamma}\left(\int_0^T |T_{r+\gamma-\rho}[D_\theta g(\rho,\cdot)](x)|_{l_2}^{1/H}\mathrm{d}\theta\right)^{2H}\mathrm{d}\rho\right]^{p/2}\mathrm{d}x \\
&= E\left[\int_0^\gamma \rho^{2\beta-1}\rho^{1-2\beta}\left(\int_0^T |T_\rho[D_\theta g(r+\gamma-\rho,\cdot)](x)|_{l_2}^{1/H}\mathrm{d}\theta\right)^{2H}\mathrm{d}\rho\right]^{p/2}\mathrm{d}x \\
&\leq N\gamma^{\beta p-1}E\int_0^\gamma \rho^{(1-2\beta)p/2}\int_{\mathbb{R}^d}\left(\int_0^T |T_\rho[D_\theta g(r+\gamma-\rho,\cdot)](x)|_{l_2}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}x\mathrm{d}\rho \\
&\leq N\gamma^{\beta p-1}E\int_0^\gamma \rho^{(1-2\beta)p/2}\left[\int_0^T\left(\int_{\mathbb{R}^d} |T_\rho[D_\theta g(r+\gamma-\rho,\cdot)](x)|_{l_2}^{p}\mathrm{d}x\right)^{1/(pH)}\mathrm{d}\theta\right]^{pH}\mathrm{d}\rho \\
&\leq N\gamma^{\beta p-1}E\int_0^\gamma \rho^{(1-2\beta)p/2}\left(\frac{\mathrm{e}^\rho}{\rho^{1/2-\beta}}\right)^p\left(\int_0^T \|D_\theta g(r+\gamma-\rho,\cdot)\|_{H_p^{n-1}(l_2)}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}\rho \\
&= N\gamma^{\beta p-1}E\int_0^\gamma\left(\int_0^T \|D_\theta g(r+\gamma-\rho,\cdot)\|_{H_p^{n-1}(l_2)}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}\rho. \qquad (5.22)
\end{aligned}$$

Using (5.20), (5.21) and (5.22), we obtain:

$$\begin{aligned}
J_2(t,s) &\leq N\left\{E\int_0^{t-s}\frac{1}{\gamma^{2+(\alpha-\beta)p}}\int_s^{t-\gamma}\int_0^\gamma \|g(r+\rho,\cdot)\|_{H_p^{n-1}(l_2)}^p\mathrm{d}r\mathrm{d}\rho\mathrm{d}\gamma\right. \\
&\quad\left.+E\int_0^{t-s}\frac{1}{\gamma^{2+(\alpha-\beta)p}}\int_s^{t-\gamma}\int_0^\gamma\left(\int_0^T \|D_\theta g(r+\rho,\cdot)\|_{H_p^{n-1}(l_2)}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}r\mathrm{d}\rho\mathrm{d}\gamma\right\} \\
&\leq N(t-s)^{(\beta-\alpha)p}\left\{E\int_0^t \|g(r,\cdot)\|_{H_p^{n-1}(l_2)}^p\mathrm{d}r + E\int_0^t\left(\int_0^T \|D_\theta g(r,\cdot)\|_{H_p^{n-1}(l_2)}^{1/H}\mathrm{d}\theta\right)^{pH}\mathrm{d}r\right\} \\
&\leq N(t-s)^{(\beta-\alpha)p}\|g\|_{\mathbb{L}_H^{1,p}(H_p^{n-1},l_2)}^p. \qquad (5.23)
\end{aligned}$$

Relations (5.16) and (5.17) for u_2 follow from (5.19) and (5.23). □

Appendix A. A Banach-space generalization of Littlewood-Paley inequality

Let V be an arbitrary Hilbert space. For any $f \in L_p(V) = L_p(\mathbb{R}^d, V)$, $p \geq 1$, we let

$$T_t f(x) := \int_{\mathbb{R}^d} f(x-y)G_t(y)\mathrm{d}y,$$

where $G_t(x) = (4\pi t)^{-d/2}\exp\{-|x|^2/(4t)\}, t > 0, x \in \mathbb{R}^d$ is the heat kernel.

First, notice that:

$$\|T_t f\|_{L_p(V)} \leq \|f\|_{L_p(V)}. \qquad (A.1)$$

To see this, note that $|T_t f(x)|_V \leq \int_{\mathbb{R}^d} |f(x-y)|_V G_t(y)\mathrm{d}y$ for any $x \in \mathbb{R}^d$. Using Minkowski's inequality for integrals, we have:

$$\begin{aligned}\|T_t f\|_{L_p(V)} &\leq \left[\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(x-y)|_V G_t(y)\mathrm{d}y\right)^p \mathrm{d}x\right]^{1/p} \\ &\leq \int_{\mathbb{R}^d} G_t(y)\left(\int_{\mathbb{R}^d} |f(x-y)|_V^p \mathrm{d}x\right)^{1/p} \mathrm{d}y = \|f\|_{L_p(V)}\|G_t\|_{L_1}.\end{aligned}$$

The following result is a generalization of the Littlewood-Paley inequality, due to [11] (see Thm. 1.1 of [11], and [14]).

Theorem A.1. *Let $p \in [2,\infty)$ and $f \in C_0^\infty((a,b)\times\mathbb{R}^d, V)$, where $-\infty \leq a < b \leq \infty$. Then*

$$\int_{\mathbb{R}^d}\int_a^b \left[\int_a^t |\nabla T_{t-s} f(s,\cdot)(x)|_V^2 \mathrm{d}s\right]^{p/2} \mathrm{d}t\,\mathrm{d}x \leq N \int_{\mathbb{R}^d}\int_a^b |f(t,x)|_V^p \mathrm{d}t\,\mathrm{d}x,$$

where N is a constant depending only on d and p.

In the present article, we need the following generalization of Theorem A.1 to the case of U-valued functions, where $U = L_{1/H}((\alpha,\beta),V)$ is a Banach space.

Theorem A.2. *Let $p \in [2,\infty)$ and $f \in C_0^\infty((a,b)\times\mathbb{R}^d, U)$, where $-\infty \leq a < b \leq \infty$ and $U = L_{1/H}((\alpha,\beta),V)$, with $-\infty \leq \alpha < \beta \leq \infty$ and $1/2 < H < 1$. Then*

$$\begin{aligned}\int_{\mathbb{R}^d}\int_a^b \left[\int_a^t \left(\int_\alpha^\beta |\nabla T_{t-s} f(s,\cdot,\theta)(x)|_V^{1/H} \mathrm{d}\theta\right)^{2H} \mathrm{d}s\right]^{p/2} \mathrm{d}t\,\mathrm{d}x \;\leq \\ N\int_a^b \left[\int_\alpha^\beta \left(\int_{\mathbb{R}^d} |f(t,x,\theta)|_V^p \mathrm{d}x\right)^{1/pH} \mathrm{d}\theta\right]^{pH} \mathrm{d}t, \qquad \text{(A.2)}\end{aligned}$$

where N is a constant depending only on d and p.

The remaining part of this section is dedicated to the proof of Theorem A.2. We follow the lines of the proof of Theorem 16.1 of [14]. It is enough to assume that $a = -\infty$ and $b = \infty$. We first treat the case $p = 2$.

Lemma A.3. *Relation (A.2) holds for $p = 2$.*

Proof. Due to Minkowski's inequality, the left-hand side of (A.2) is smaller than

$$\int_{-\infty}^\infty \left[\int_\alpha^\beta \left(\int_s^\infty \int_{\mathbb{R}^d} |\nabla T_{t-s} f(s,\cdot,\theta)(x)|_V^2 \mathrm{d}x\mathrm{d}t\right)^{1/(2H)} \mathrm{d}\theta\right]^{2H} \mathrm{d}s.$$

Using the Fourier transform, the inner integral equals

$$\begin{aligned}\int_s^\infty \int_{\mathbb{R}^d} |\xi|^2 \mathrm{e}^{-(t-s)|\xi|^2} |\mathcal{F}f(s,\xi,\theta)|_V^2 \mathrm{d}\xi\mathrm{d}t &= \int_{\mathbb{R}^d} |\mathcal{F}f(s,\xi,\theta)|_V^2 |\xi|^2 \left(\int_s^\infty \mathrm{e}^{-(t-s)|\xi|^2}\mathrm{d}t\right)\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} |\mathcal{F}f(s,\xi,\theta)|_V^2 \mathrm{d}\xi,\end{aligned}$$

which proves (A.2) for $p = 2$. □

Assume now that $p > 2$. Note that $\nabla T_t h(x) = t^{-1/2}\Psi_t h(x)$, where $\Psi_t h(x) = t^{-d/2}\phi(x/\sqrt{t}) * h(x)$ and $\phi(x) = -(4\pi)^{-d/2} x \mathrm{e}^{-|x|^2/4}$. Set

$$\begin{aligned} u(t,x) = \mathcal{G}f(t,x) &= \left[\int_{-\infty}^t \left(\int_\alpha^\beta |\Psi_{t-s} f(s,\cdot,\theta)(x)|^{1/H} \mathrm{d}\theta\right)^{2H} \frac{1}{t-s}\mathrm{d}s\right]^{1/2}. \\ &= \left(\int_{-\infty}^t |\Psi_{t-s} f(s,\cdot,*)(x)|_U^2 \ \frac{1}{t-s}\mathrm{d}s\right)^{1/2}. \end{aligned}$$

We want to prove that:

$$\int_{\mathbb{R}^d}\int_{-\infty}^{\infty} |u(t,x)|^p \mathrm{d}t\, \mathrm{d}x \le N\int_{-\infty}^{\infty}\left[\left(\int_{\mathbb{R}^d} |f|_V^p(t,x,\theta)\mathrm{d}x\right)^{1/(pH)} \mathrm{d}\theta\right]^{pH} \mathrm{d}t. \tag{A.3}$$

Recall that the **maximal function** of $g : \mathbb{R}^d \to \mathbb{R}$ is defined by:

$$\mathbb{M}_x g(x) = \sup_{r>0} \frac{1}{B_r}\int_{B_r(x)} |g(y)|\mathrm{d}y,$$

where $B_r(x) = \{y; |y-x| < r\}$ and $B_r = B_r(0)$. If $h : \mathbb{R}^{d+1} \to \mathbb{R}$, we define $\mathbb{M}_x h(t,x) = \mathbb{M}_x h(t,\cdot)(x)$. Let $Q_0 = [-4,0] \times [-1,1]^d$.

Lemma A.4. *Assume that $f(t,x,\theta) = 0$ for $(t,x) \notin (-12,12) \times B_{3d}$. Then for any $(t,x) \in Q_0$*

$$\int_{Q_0} |u(s,y)|^2 \mathrm{d}s\mathrm{d}y \le N\mathbb{M}_t \|\mathbb{M}_x |f|_V^2(t,x,*)\|_{U_0}, \tag{A.4}$$

where $U_0 = L_{1/(2H)}((\alpha,\beta))$ and N depends only on d.

Proof. Using Lemma A.3, the left-hand side of (A.4) is smaller than:

$$\begin{aligned} N\int_{-\infty}^0 \left[\int_\alpha^\beta \left(\int_{\mathbb{R}^d} |f|_V^2(s,y,\theta)\mathrm{d}y\right)^{1/(2H)} \mathrm{d}\theta\right]^{2H} \mathrm{d}s &\le N\int_{-12}^0 \left[\int_\alpha^\beta (\mathbb{M}_x|f|_V^2(s,x,\theta))^{1/(2H)}\mathrm{d}\theta\right]^{2H} \mathrm{d}s \\ &= N\int_{-12}^0 \|\mathbb{M}_x|f|_V^2(s,x,*)\|_{U_0}\mathrm{d}s \le N\mathbb{M}_t\|\mathbb{M}_x|f|_V^2(t,x,*)\|_{U_0}. \end{aligned}$$

□

Lemma A.5. *Assume that $f(t,x,\theta) = 0$ for $t \notin (-12,12)$. Then (A.4) holds for any $(t,x) \in Q_0$.*

Proof. Let $\zeta \in C_0^\infty(\mathbb{R}^d)$ be such that $\zeta = 1$ in B_{2d}, $\zeta = 0$ outside B_{3d}, and $\zeta(x) \in (0,1)$ for $x \in B_{3d}\backslash B_{2d}$. Let $\alpha = \zeta f$ and $\beta = (1-\zeta)f$. Then

$$\begin{aligned} \mathcal{G}f(t,x) &= \mathcal{G}(\alpha+\beta)(t,x) = \left(\int_{-\infty}^t |\Psi_{t-s}(\alpha+\beta)(s,\cdot,*)(x)|_U^2 \frac{1}{t-s}\mathrm{d}s\right)^{1/2} \\ &\le \mathcal{G}\alpha(t,x) + \mathcal{G}\beta(t,x), \end{aligned}$$

using Minkowski's inequality in $L_2(\mathbb{R},U)$, which in turn relies on Minkowski's inequality in the Banach space U. Since α satisfies the conditions of Lemma A.4 and $|\alpha|_V \le |f|_V$, for any $(t,x) \in Q_0$

$$\int_{Q_0} |\mathcal{G}\alpha(s,y)|^2 \mathrm{d}s\mathrm{d}y \le N\mathbb{M}_t\|\mathbb{M}_x|\alpha|_V^2(t,x,*)\|_{U_0} \le N\mathbb{M}_t\|\mathbb{M}_x|f|_V^2(t,x,*)\|_{U_0}.$$

Therefore, it suffices to prove that (A.4) holds for any function f such that $f(t,x,\theta)=0$ if $t \notin (-12,12)$ or $x \in B_{2d}$ (in particular for β). This follows as in the proof of Lemma 16.5 of [14], using Minkowski's inequality for integrals. $\square$

Lemma A.6. *Assume that $f(t,x,\theta)=0$ for $t \geq -8$. Then for any $(t,x) \in Q_0$*

$$\int_{Q_0} |u(s,y)-u(t,x)|^2 \mathrm{d}s\mathrm{d}y \leq N\mathbb{M}_t \|\mathbb{M}_x |f|_V^2(t,x,*)\|_{U_0}.$$

Proof. The argument is similar to the one used in the proof of Lemma 16.6 of [14], with some minor modifications (as above). $\square$

We introduce now the filtration $\mathbb{Q}_n, n \in \mathbb{Z}$ of partitions $\mathbb{Q}_n = \{Q_n(i_0,i_1,\ldots,i_d); i_0,i_1\ldots,i_d \in \mathbb{Z}\}$ of $\mathbb{R}^{d+1}$, as in [14]. For any $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}$, we denote by $Q_n(x)$ the unique $Q \in \mathbb{Q}_n$ containing x. The **sharp function** of $g \in L_{1,loc}(\mathbb{R}^d)$ is defined by:

$$g^{\#}(x) = \sup_{n\in\mathbb{Z}} \frac{1}{|Q_n(x)|} \int_{Q_n(x)} |g(y)-g_{|n}(x)|\mathrm{d}y,$$

where $g_{|n}(x) = |Q_n(x)|^{-1} \int_{Q_n(x)} g(y)\mathrm{d}y$. If $p \in (1,\infty)$, then by the Fefferman-Stein theorem, for any $g \in L_p(\mathbb{R}^d)$, $\|g\|_{L_p(\mathbb{R}^d)} \leq N\|g^{\#}\|_{L_p(\mathbb{R}^d)}$.

Lemma A.7. *Let $f \in C_0^\infty(\mathbb{R}^{d+1},U)$ be arbitrary. For any $(t,x) \in \mathbb{R}^{d+1}$,*

$$(\mathcal{G}f)^{\#}(t,x) \leq N(\mathbb{M}_t\|\mathbb{M}_x|f|_V^2(t,x,*)\|_{U_0})^{1/2}.$$

Proof. The argument is based on Lemma A.5 and Lemma A.6, and is similar to the one used for proving relation (16.20) of [14]. $\square$

Proof of Theorem A.2. Assume that $p>2$. We use the Fefferman-Stein theorem, Lemma A.7, the boundedness of the operators $\mathbb{M}_t$ and $\mathbb{M}_x$ ($p>2$), and Minkowski's inequality for integrals ($pH>1$):

$$\begin{aligned}
\|u\|^p_{L_p(\mathbb{R}^{d+1})} &\leq N\|(\mathcal{G}f)^{\#}\|^p_{L_p(\mathbb{R}^{d+1})} \leq N\int_{\mathbb{R}^d}\int_{\mathbb{R}} (\mathbb{M}_t\|\mathbb{M}_x|f|_V^2(t,x,*)\|_{U_0})^{p/2}\mathrm{d}t\,\mathrm{d}x \\
&= N\int_{\mathbb{R}^d} \|\mathbb{M}_t\|\mathbb{M}_x|f|_V^2(t,x,*)\|_{U_0}\|^{p/2}_{L_{p/2}(\mathbb{R})}\mathrm{d}x \\
&\leq N\int_{\mathbb{R}^d}\int_{\mathbb{R}} \|\mathbb{M}_x|f|_V^2(t,x,*)\|_{U_0}\mathrm{d}t\,\mathrm{d}x \\
&= N\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left[\int_\alpha^\beta (\mathbb{M}_x|f|_V^2(t,x,\theta))^{1/(2H)}\mathrm{d}\theta\right]^{pH} \mathrm{d}x\mathrm{d}t \\
&\leq N\int_{\mathbb{R}}\left[\int_\alpha^\beta\left(\int_{\mathbb{R}^d}(\mathbb{M}_x|f|_V^2(t,x,\theta))^{p/2}\mathrm{d}x\right)^{1/(pH)}\mathrm{d}\theta\right]^{pH}\mathrm{d}t \\
&\leq N\int_{\mathbb{R}}\left[\int_\alpha^\beta\left(\int_{\mathbb{R}^d}|f|_V^p(t,x,\theta))\mathrm{d}x\right)^{1/(pH)}\mathrm{d}\theta\right]^{pH}\mathrm{d}t,
\end{aligned}$$

i.e. (A.3) holds. $\square$

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