

NUMERICAL PROCEDURE TO APPROXIMATE A SINGULAR OPTIMAL CONTROL PROBLEM

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Abstract. In this work we deal with the numerical solution of a Hamilton-Jacobi-Bellman (HJB) equation with infinitely many solutions. To compute the maximal solution – the optimal cost of the original optimal control problem – we present a complete discrete method based on the use of some finite elements and penalization techniques.

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1. INTRODUCTION

It is well known that some optimal control problems lead us to a Hamilton-Jacobi-Bellman (HJB) equation with infinitely many solutions, where only one of these solutions is the optimal cost for the optimal control problem. Shape-from-shading problem in image analysis is a classical example of this situation and it has been widely studied in [2, 10, 11]. The lack of uniqueness requires not only the identification of the optimal cost among all the solutions, but also the use of non-classical techniques to find approximate solutions which converge to the solution of the original problem. When the HJB equation has a unique viscosity solution, the results by Barles and Souganidis in [1] ensure that any discretization scheme which satisfies some suitable properties (monotonicity, consistency and stability) produces a sequence of “approximate solutions” which converges to the one of the original problem. Due to the non-uniqueness phenomenon, it is not possible, in this case, to use directly Barles and Souganidis techniques. In this work we focus on the numerical solution to these problems, restricting our analysis to the optimal control problem associated with the eikonal equation

$$\|\nabla U\| = f,$$

when f vanishes somewhere. Some theoretical results for the Hamilton-Jacobi-Bellman equations associated to this degenerate problem can be found in [3, 4, 9]. Although the numerical analysis of this problem was started at [3] by Camilli and Grüne, we present an alternative scheme of approximation which not only brings a sequence of convergent approximations, but also a complete procedure to compute the approximating solution. The solution of the fully discrete problem can be computed using iterative algorithms which converge in a finite number of steps. The convergence requires a suitable relation among the three parameters appearing in the

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discretization process, which not only differs from the proposed one in [3], but also improve it, in some sense, since we use a smaller penalization parameter. Preliminary results may be found in [7, 8].

In the next section we introduce our problem. Section 3 is devoted to its continuous approximation and in Sections 4 and 5 we deal with discrete time and fully discrete approximations, respectively. To conclude, we present numerical examples and a function which shows the need of using penalization techniques.

2. THE PROBLEM

We consider the following control problem with controlled dynamics

$$\begin{cases} \xi'(t) = q(t), & t \geq 0 \\ \xi(0) = x, \end{cases}$$

where $x \in \Omega$ and Ω bounded. We define the exit time τ of the trajectory with initial condition x and velocity $q(\cdot)$ by

$$\tau = \tau(x, q(\cdot)) = \inf\{t > 0 : \xi_{x,q(\cdot)}(t) \notin \Omega\},$$

and we restrict the control policies in the following way: $q(\cdot) \in Q_x$, where

$$Q_x := \{q(\cdot) : (0, \infty) \mapsto \mathbb{R}^N, \text{ measurable with } \|q(t)\| \leq 1 \text{ a.e. } t, \tau < \infty\}.$$

The performance of the control policy $q(\cdot)$ is given by the functional

$$J(x, q) = \int_0^\tau f(\xi(t)) |q(t)| dt + g(\xi(\tau)). \tag{1}$$

For the problems considered here, the instantaneous cost is a function of the current state of the system and is proportional to the absolute value of the velocity.

The optimal cost is

$$U(x) = \inf_{Q_x} J(x, q).$$

Assuming Ω bounded, $f(\cdot)$ and $g(\cdot)$ in $Lip(\overline{\Omega})$, and $f(\cdot) : \Omega \mapsto \mathbb{R}$ a non-negative function, Camilli and Grüne proved in [3] that $U \in Lip(\overline{\Omega})$ and U is a solution (subsolution and singular supersolution) of the HJB equation associated with this problem. The HJB equation and the boundary condition, respectively, are

$$\|\nabla U(x)\| - f(x) = 0, \quad \text{a.e. } x \in \Omega, \tag{2}$$

$$U(x) = g(x), \quad x \notin \Omega.$$

If the singular set, $K := \{x \in \Omega : f(x) = 0\} \neq \emptyset$, (2) may have many solutions. Camilli and Grüne also proved in [3] that U is the maximal subsolution of (2) in the viscosity sense, solving in this way the identification problem produced by the lack of uniqueness of the solution of the equation. The assumptions on the singular set K are in this paper the same as in [3, 4] in order to use the same concepts of solutions for the HJB equation.

Definition 2.1. Specifically, we mean by solution of (2) a maximal subsolution and a singular supersolution.

The function u is a subsolution of (2) if it is a subsolution for the differential equation and $\lim_{y \rightarrow x, y \in \Omega} u(y) \leq g(x), x \in \partial\Omega$.

The function u is a maximal subsolution of (2) if it is a subsolution such that $\lim_{y \rightarrow x, y \in \Omega} u(y) = g(x), x \in \partial\Omega$.

The function u is a singular supersolution of (2) if it is a l.s.c. function such that, for any $x_0 \in \Omega$, it does not admit L -subtangent which is a strict subsolution of (2) in a neighbourhood of $B_L(x_0)$. See [4] for further details.

3. APPROXIMATE CONTINUOUS PROBLEMS

3.1. Approximations with finite horizon

In order to develop some numerical approximation techniques, it is necessary to know whether the infinite horizon problem can be approximated by a family of finite horizon problems. Let $T_0 = \max_{x \in \Omega} d(x, \partial\Omega)$. Thus, for every $T > T_0$ and $x \in \Omega$, it is possible to define the non-void set

$$Q_x^T = \{q(\cdot) \in Q_x : \tau(x, q(\cdot)) \leq T\}.$$

Let us also define

$$U^T(x) = \inf_{Q_x^T} J(x, q).$$

Since for every $x \in \Omega$, and $\tilde{T} \geq T \geq T_0$, we have $U(x) \leq U^{\tilde{T}}(x) \leq U^T(x) \leq U^{T_0}(x)$, it is easy to prove that

$$\lim_{T \rightarrow \infty} U^T(x) = U(x), \quad \forall x \in \Omega. \tag{3}$$

3.2. Approximations by penalizations

To obtain convergent approximations, we deal with penalizations of the original problem.

Definition 3.1. Let $\epsilon > 0$. We define the penalized functional J_ϵ and two optimal costs U_ϵ and U_ϵ^T by

$$J_\epsilon(x, q) = \int_0^\tau (\epsilon + f(\xi(t))) |q(t)| dt + g(\xi(\tau)), \tag{4}$$

$$U_\epsilon(x) = \inf_{Q_x} J_\epsilon(x, q) \tag{5}$$

$$U_\epsilon^T(x) = \inf_{Q_x^T} J_\epsilon(x, q).$$

The control problem associated with (4) has strictly positive instantaneous cost. In consequence, the corresponding HJB equation has a unique solution, the optimal cost U_ϵ . Even though U_ϵ differs a little from the penalization proposed by Camilli and Grüne in [3], we agree with the fact that some penalization is necessary. We explain this fact in detail in Section 4.2.

The following properties, whose proofs are easily deduced, enable us to affirm that $U_\epsilon(\cdot) \in Lip(\Omega)$ and

$$\lim_{\epsilon \rightarrow 0} U_\epsilon(x) = U(x), \quad x \in \Omega, \tag{6}$$

the convergence being uniform toward $U(\cdot)$:

- $U(x) \leq U_\epsilon(x)$ for every $\epsilon > 0$;
- $U_\epsilon(x) \leq U_{\tilde{\epsilon}}(x)$, for every $x \in \Omega$, $\epsilon \leq \tilde{\epsilon}$;
- $U^T(x) \leq U_\epsilon^T(x)$, for every $x \in \Omega$, $T \geq T_0$. More precisely, $|U^T(x) - U_\epsilon^T(x)| \leq T\epsilon$;
- $U_\epsilon^T(x) \leq U_{\tilde{\epsilon}}^T(x)$, for every $x \in \Omega$, $T \geq T_0$, $\epsilon \leq \tilde{\epsilon}$;
- $U_\epsilon(x) \leq U_{\tilde{\epsilon}}(x) \leq U_\epsilon^T(x) \leq U_{\tilde{\epsilon}}^T(x) \leq U_\epsilon^{T_0}(x)$, for every $x \in \Omega$, $T \geq T_0$, $\epsilon \leq \tilde{\epsilon}$.

4. THE TIME DISCRETE PROBLEM

4.1. Control policies discretization

The complete discretization procedure consists of two steps: time and space discretization. In this section we analyze the effect of time discretization.

Definition 4.1. For $h > 0$, we define the following elements

$$Q_{x,h} = \{q(\cdot) \in Q_x : q(\cdot) \text{ constant in } [\nu h, (\nu + 1)h], \nu \in \mathbb{N}_0\},$$

$$U_h(x) = \inf_{Q_{x,h}} J(x, q),$$

and for $T > T_0$ a fixed finite horizon, we consider also

$$Q_{x,h}^T = \{q(\cdot) \in Q_{x,h} : \tau(x, q(\cdot)) \leq T\},$$

and

$$U_h^T(x) = \inf_{Q_{x,h}^T} J(x, q).$$

The above approximation has the following properties:

- $U(x) \leq U_h(x)$, for every $x \in \Omega$;
- $U_h(x) \leq U_h^T(x)$, for every x and $T \geq T_0$;
- $U_{h/p}(x) \leq U_h(x)$, for every x and $p \in \mathbb{N}$;
- $U_{h/p}^T(x) \leq U_h^T(x)$, for every x , $T \geq T_0$ and $p \in \mathbb{N}$,

allowing us to prove

Proposition 4.2.

- (1) $\lim_{p \rightarrow \infty} U_{h/p}^{T+h/p}(x) = U^T(x)$, for every $x \in \Omega$;
- (2) $\lim_{p \rightarrow \infty} U_{h/p}(x) = U(x)$, for every $x \in \Omega$.

Proof. Let $\varepsilon > 0$ and $q^T \in Q_x^T$ such that

$$J(x, q^T) \leq U^T(x) + \varepsilon.$$

By definition, $\tau = \tau(x, q^T(\cdot)) \leq T$ and $\xi(\tau) \in \partial\Omega$. Without loss of generality, we assume $\xi(t) \notin \Omega$, for every $t > \tau(x, q^T(\cdot))$.

Let ν_τ such that $\tau(x, q^T(\cdot)) \in [\nu_\tau h, (\nu_\tau + 1)h]$ and define

$$q_{h,p}(s) = (p/h) \int_{\nu h/p}^{(\nu+1)h/p} q^T(\theta) d\theta, \quad s \in [\nu h, (\nu + 1)h].$$

Clearly, $q_{h,p} \in Q_{x,(h/p)}^{T+h/p}$. In fact, being $\xi_{h,p}$ the resulting trajectory, we have

- $q_{h,p}$ is constant in $[\nu h/p, (\nu + 1)h/p]$;
- $\xi(\nu h/p) = \xi_{h,p}(\nu h/p)$;
- $\nu_\tau h \leq \tau(x, q_{h,p}) \leq (\nu_\tau + 1)h$;
- $|\tau(x, q_{h,p}) - \tau| \leq h$ and in particular $|\tau(x, q_{h,p}) - \tau| \rightarrow 0$ as $h \rightarrow 0$.

As a consequence we obtain

$$|J(x, q^T) - J(x, q_{h,p})| \leq L_f(h/p)(T + h),$$

thus

$$J(x, q_{h,p}) \leq J(x, q^T) + L_f(h/p)(T + h),$$

where L_f is the Lipschitz constant of f . Therefore,

$$U_{h/p}^{T+h/p}(x) \leq U^T(x) + \varepsilon + L_f(h/p)(T + h).$$

Since ε is arbitrary, we have

$$U_{h/p}^{T+h/p}(x) \leq U^T(x) + L_f(h/p)(T + h)$$

and

$$\lim_{p \rightarrow \infty} U_{h/p}^{T+h/p}(x) \leq U^T(x).$$

Since $U_{h/p}(x) \leq U_{h/p}^{T+h/p}(x)$, we get

$$\lim_{p \rightarrow \infty} U_{h/p}(x) \leq \lim_{p \rightarrow \infty} U_{h/p}^{T+h/p}(x) \leq U^T(x).$$

Consequently, $\lim_{p \rightarrow \infty} U_{h/p}(x) \leq U(x)$. Since $U_{h/p}(x) \geq U(x)$, we conclude that

$$\lim_{p \rightarrow \infty} U_{h/p}(x) = U(x). \quad \square$$

Following classical arguments, we have

Proposition 4.3. *The function U_h satisfies the following dynamical programming equation: for every $x \in \Omega$,*

$$U_h(x) = \min_{q \in B_1(x)} \left(\int_0^{h \wedge \tau(x,q)} f(x + qs) |q| ds + U_h(x + (h \wedge \tau(x,q)) q) \right), \tag{7}$$

with boundary condition $U_h(x) = g(x)$, for every $x \notin \Omega$. Moreover, U_h is the maximal solution of (7).

Definition 4.4. Being $\hat{h} = h \wedge \tau(x, q)$, we define the operator $P : B(\bar{\Omega}) \mapsto B(\bar{\Omega})$ by

$$Pw(x) = \begin{cases} \min_{q \in B_1(x)} \int_0^{\hat{h}} f(x + qs) |q| ds + w(x + \hat{h}q), & x \in \Omega, \\ g(x), & x \in \partial\Omega. \end{cases}$$

Remark 4.5. Clearly, Proposition 4.3 shows that U_h is a fixed point for P .

Proposition 4.6. *P is a non-decreasing operator. Moreover, being w a function such that, for $x \in \Omega$, satisfies $w(x) \geq T_0 \max_{q \in B_1(x), x \in \Omega} f(x) |q| + \max_{x \in \Omega} g(x)$ and $w(x) = g(x)$ for $x \in \partial\Omega$, it follows that*

$$\lim_{\nu \rightarrow \infty} P^\nu w(x) = U_h(x).$$

Remark 4.7. The previous proposition gives a theoretical procedure to compute U_h .

4.2. Functional discretization

Since U_h involves the computations of the integrals appearing in (1), its implementation cannot be done directly. In order to obtain practical methods, those integrals should also be discretized.

Definition 4.8. Let $q(\cdot) \in Q_{x,h}$. Since the exit time of the trajectory ξ generated by q is $\tau(x, q) < \infty$, it is easy to see that there exists $K = K(x, h, q)$ such that $\{\nu \in N : \xi(\nu h) \in \Omega\} = \{0, 1, \dots, K\}$. Then we define

$$J_h(x, q) = \sum_{\nu=0}^{K-1} hf(\xi(\nu h)) |q(\nu h)| + (\tau(x, q) - Kh)f(\xi(Kh)) |q(Kh)| + g(\xi(\tau(x, q))). \tag{8}$$

We also define

$$V_h(x) = \inf_{Q_{x,h}} J_h(x, q). \tag{9}$$

It is worth mentioning that a discretization like (8) above is not suitable since, as it was already mentioned in [3], $\lim_{h \rightarrow 0} V_h(x)$, where V_h is defined in (9), can be strictly minor than $U(x)$. An example with this undesirable effect is presented in the appendix of this work.

To eliminate this pathology, we consider a penalized scheme coming from (4) with the special parametrization $\epsilon = L_f h$, which is large enough to avoid the effect described above and that enables us to show that the convergence of this procedure holds. At the same time, it is smaller than those considered in [3]. In fact, to guarantee convergence, Camilli and Grüne had to ask for $\frac{h}{\epsilon} \rightarrow 0$ when $\epsilon \rightarrow 0$ instead of $\frac{h}{\epsilon} = L_f$ which is what we ask for now. Later, this fact will allow us to construct more accurate approximations to the optimal cost.

Definition 4.9. For $q(\cdot) \in Q_{x,h}$, let $K = K(x, h, q)$, like in the previous definition, such that $\{\nu \in N : \xi(\nu h) \in \Omega\} = \{0, 1, \dots, K\}$. Then we define

$$\begin{aligned} \bar{J}_h(x, q) = & h \sum_{\nu=0}^{K-1} (L_f h + f(\xi(\nu h))) |q(\nu h)| \\ & + (\tau(x, q) - Kh) (L_f h + f(\xi(Kh))) |q(Kh)| \\ & + g(\xi(\tau(x, q))). \end{aligned} \tag{10}$$

Also,

$$\bar{V}_h(x) = \inf_{Q_{x,h}} \bar{J}_h(x, q), \quad \bar{V}_h^T(x) = \inf_{Q_{x,h}^T} \bar{J}_h(x, q).$$

The following properties can be easily proved:

- $\bar{V}_h(x) \geq U(x)$, for every $x \in \Omega$;
- $\bar{V}_h(x) \leq \bar{V}_h^T(x)$, for every x , $T \geq T_0$;
- $\bar{V}_{h/p}(x) \leq \bar{V}_h(x)$, for every x , $p \in \mathbb{N}$;
- $\bar{V}_{h/p}^T(x) \leq \bar{V}_h^T(x)$, for every x , $p \in \mathbb{N}$, $T \geq T_0$,

and they enable us to prove

Proposition 4.10. For $T > T_0$, $\lim_{p \rightarrow \infty} \bar{V}_{h/p}^{T+h/p}(x) = U^T(x)$ and $\lim_{p \rightarrow \infty} \bar{V}_{h/p}(x) = U(x)$, for every $x \in \Omega$.

Proof. Let us consider $q(\cdot) \in Q_{x,h/p}^T$, then $\tau(x, q) < T$. Thus

$$\begin{aligned} 0 &\leq \bar{J}_{h/p}(x, q) - J(x, q) \\ &\leq L_f (h/p)^2 + \sum_{\nu=0}^{K-1} \int_{\nu h/p}^{(\nu+1)(h/p)} |f(\xi(t)) |q(t)| - f(\xi(\nu h/p)) |q(\nu h/p)|| dt. \end{aligned} \tag{11}$$

Since, f is Lipschitz continuous, with constant L_f , and $q(t) = q(\nu h/p)$ in $[\nu h/p, (\nu + 1)(h/p)]$, we have that

$$\begin{aligned} &\sum_{\nu=0}^{K-1} \int_{\nu h/p}^{(\nu+1)(h/p)} |f(\xi(t)) |q(t)| - f(\xi(\nu h/p)) |q(\nu h/p)|| dt \\ &\leq \sum_{\nu=0}^{K-1} \int_{\nu h/p}^{(\nu+1)h/p} L_f \|\xi(t) - \xi(\nu h)\| |q(\nu h)| dt \\ &\leq \sum_{\nu=0}^{K-1} \int_{\nu h/p}^{(\nu+1)h/p} L_f |t - \nu h| |q(\nu h)| dt. \end{aligned} \tag{12}$$

Since $Kh \leq T$, it follows that

$$\bar{J}_{h/p}(x, q) - J(x, q) \leq L_f (h/p)(T + h/p). \tag{13}$$

By considering the infimum over the policies, it is clear that

$$U_{h/p}^{T+h/p}(x) \leq \bar{V}_{h/p}^{T+h/p}(x) \leq U_{h/p}^{T+h/p}(x) + L_f(h/p)(T + h/p),$$

and consequently

$$\lim_{p \rightarrow \infty} \bar{V}_{h/p}^{T+h/p}(x) = \lim_{p \rightarrow \infty} U_{h/p}^{T+h/p}(x).$$

From Proposition 4.2, we have $\lim_{p \rightarrow \infty} U_{h/p}^{T+h/p}(x) = U^T(x)$. Hence,

$$\lim_{p \rightarrow \infty} \bar{V}_{h/p}^{T+h/p}(x) = U^T(x). \tag{14}$$

Since for every x and $T \geq T_0$, we have $\bar{V}_{h/p}(x) \leq \bar{V}_{h/p}^{T+h/p}(x)$, taking into account the properties of \bar{V}_h , we obtain

$$\lim_{p \rightarrow \infty} \bar{V}_{h/p}(x) \leq \lim_{T \rightarrow \infty} U^T(x) = U(x),$$

and

$$\lim_{p \rightarrow \infty} \bar{V}_{h/p}(x) \geq U(x),$$

and thus

$$\lim_{p \rightarrow \infty} \bar{V}_{h/p}(x) = U(x). \tag{□}$$

Proposition 4.10 shows that when we consider the cost functional like in (10), the sequence of optimal costs converges to U .

In what follows we give a theoretical procedure to compute \bar{V}_h .

Proposition 4.11. \bar{V}_h satisfies the following dynamical programming equation:
 For $x \in \Omega$

$$\bar{V}_h(x) = \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{f(x) |q| h + L_f h^2 + \bar{V}_h(x + hq)\}, \tag{15}$$

with boundary condition $\bar{V}_h(x) = g(x)$, for every $x \notin \Omega$.

Definition 4.12. We define the following operators on $C(\bar{\Omega})$:

$$P_h \Phi(x) = \begin{cases} \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{f(x) |q| h + L_f h^2 + \Phi(x + hq)\}, & x \in \Omega \\ g(x), & x \in \partial\Omega \end{cases} \tag{16}$$

and

$$\Pi_h \Psi(x) = \begin{cases} \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{\varphi(q)\Psi(x + hq) + (1 - \varphi(q))\}, & x \in \Omega \\ 1 - \exp(-g(x)), & x \in \partial\Omega, \end{cases} \tag{17}$$

where $\varphi(q) = \exp(- (L_f h^2 + f(x) |q| h))$.

Clearly, \bar{V}_h is a fixed point for the operator P_h . If we could assure the uniqueness of fixed point for P_h , we would have a procedure to compute \bar{V}_h . To prove that, we present P_h as the transformation of a contractive operator and \bar{V}_h as the transformation of the unique fixed point of the other operator.

Definition 4.13. Let K denote the Kruzhkov transformation of functions of $C(\bar{\Omega})$ with inverse K^{-1} , i.e.

$$\begin{cases} z_h(x) = K[v_h](x) = 1 - \exp(-v_h(x)) \\ v_h(x) = K^{-1}[z_h](x) = -\ln(1 - z_h(x)). \end{cases} \tag{18}$$

Lemma 4.14. The operator Π_h defined in (17) is contractive. In addition, $\Pi_h \circ K = K \circ P_h$ and $\Pi_h \circ K^{-1} = K^{-1} \circ P_h$.

Proof. Being $q_m \in \arg \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{\varphi(q)\Phi(x + hq) + (1 - \varphi(q))\}$, we have

$$\begin{aligned} |\Pi_h \Psi(x) - \Pi_h \Phi(x)| &\leq \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{\varphi(q)\Psi(x + hq) + (1 - \varphi(q))\} \\ &\quad - \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{\varphi(q)\Phi(x + hq) + (1 - \varphi(q))\} \\ &\leq \varphi(q_m)\Psi(x + hq_m) + (1 - \varphi(q_m)) - \varphi(q_m)\Phi(x + hq_m) + (1 - \varphi(q_m)) \\ &\leq |\varphi(q_m)| |\Psi(x + hq_m) - \Phi(x + hq_m)|. \end{aligned} \tag{19}$$

Since

$$\begin{aligned} |\varphi(q_m)| &= |\exp(- (L_f h^2 + f(x) |q_m| h))| \\ &\leq |\exp(- (L_f h^2))| = L < 1, \end{aligned}$$

inequality (19) becomes

$$\|\Pi_h \Psi(\cdot) - \Pi_h \Phi(\cdot)\| \leq L \|\Psi(\cdot) - \Phi(\cdot)\|$$

and therefore, the operator Π_h is contractive. □

Lemma 4.15. *Let Z_h be the unique fixed point for the operator Π_h . The function \bar{V}_h defined in (15) can also be written as*

$$\bar{V}_h(x) = -\ln(1 - Z_h(x))$$

and is the unique fixed point for (16).

Proof.

$$\begin{aligned} \bar{V}_h(x) &= P_h \bar{V}_h(x) = \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{f(x) |q| h + L_f h^2 + \bar{V}_h(x + hq)\} \\ &= \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{-\ln(\varphi(q)) + \bar{V}_h(x + hq)\}. \end{aligned}$$

Since $\bar{V}_h(\cdot) = -\ln(1 - z_h(\cdot))$ for some z_h , we have

$$\begin{aligned} -\ln(1 - z_h(x)) &= \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{-\ln(\varphi(q)) - \ln(1 - z_h(x + hq))\} \\ &= \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{-\ln \varphi(q) (1 - z_h(x + hq))\} \\ &= -\ln \left[\max_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{\varphi(q) (1 - z_h(x + hq))\} \right]. \end{aligned}$$

Due to the monotony of $\ln(\cdot)$,

$$1 - z_h(x) = \max_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{\varphi(q) (1 - z_h(x + hq))\}$$

and thus,

$$\begin{aligned} z_h(x) &= 1 - \max_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{\varphi(q) (1 - z_h(x + hq))\} \\ &= 1 + \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{-\varphi(q) (1 - z_h(x + hq))\} \\ &= \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{1 - \varphi(q) (1 - z_h(x + hq))\}. \end{aligned}$$

This means that z_h is a fixed point for Π_h , the unique fixed point for Π_h . Therefore $z_h = Z_h$. □

This allows us to prove

Proposition 4.16. $P_h^\nu \Phi \rightarrow \bar{V}_h$ as $\nu \rightarrow \infty$, for every $\Phi \in C(\bar{\Omega})$.

5. A FULLY DISCRETE PROBLEM AND ITS SOLUTION

In order to obtain a complete discrete procedure, we must introduce a space discretization. Let us consider a regular mesh of size k for the set Ω . S_k will denote the set of mesh nodes, Ω_k the polyhedron with vertices S_k and $N_k = \text{card}(S_k)$. We use linear finite elements. We consider the set W_k of functions $w : \Omega_k \mapsto \mathfrak{R}$, $w \in W^{1,\infty}(\Omega_k)$, such that $\partial w / \partial x$ is constant in the interior of each simplex of Ω_k , i.e., the functions w are linear finite elements and they are characterized by their values on S_k .

In particular, $x + hq$ can be written as a convex combination of the vertices $x_{i,k}$ of the simplex to which $x + hq$ belongs. That is to say, there exists $\{\lambda_i\}_{i=1}^{N_k}$, $\lambda_i \geq 0$, $\sum_{i=1}^{N_k} \lambda_i = 1$ such that

$$x + hq = \sum_{i=1}^{N_k} \lambda_i x_{i,k}$$

and for $\Phi \in W_k$, it follows that

$$\Phi(x + hq) = \sum_{i=1}^{N_k} \lambda_i \Phi(x_{i,k}). \tag{20}$$

5.1. An approximating function

Let us introduce the operator $P_{h,k} : W_k \mapsto W_k$ defined by

$$P_{h,k}\Phi(x) = \begin{cases} \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{f(x) |q| h + L_f h^2 + \Phi(x + hq)\}, & x \in S_k \cap \Omega, \\ g(x), & x \in S_k \cap \partial\Omega. \end{cases}$$

Proposition 5.1. *The operator $P_{h,k}$ has a unique fixed point $V_{h,k}$. This function $V_{h,k}$, called the fully discrete approximation, can be obtained solving the equation*

$$V_{h,k}(x) = \begin{cases} \min_{\substack{q \in B_1(x) \\ x + hq \in \bar{\Omega}}} \{f(x) |q| h + L_f h^2 + V_{h,k}(x + hq)\}, \\ g(x), & x \in S_k \cap \partial\Omega. \end{cases} \tag{21}$$

Remark 5.2. Before proving that, note that (21) can be seen as a Bellman equation on a Markov chain. In fact (20), let us consider the coefficients λ_i as transition probabilities.

Proof. The proof consists of the following steps:

- (1) $P_{h,k}$ is monotone, i.e. $\Phi \leq \Psi$ implies that $P_{h,k}\Phi \leq P_{h,k}\Psi$.

This fact is obvious from the definition of $P_{h,k}$.

- (2) There exists a finite supersolution $S \geq 0$ of $P_{h,k}$, i.e. $P_{h,k}S \leq S$.

To prove this, let us consider the function

$$S(x) = \begin{cases} 3M_f d(x, \partial\Omega) + M_g & x \in S_k \cap \Omega, \\ g(x) & x \in S_k \cap \partial\Omega, \end{cases} \tag{22}$$

where

$$M_g = \max_{x \notin \partial\Omega} g(x), \quad M_f = \max_{x \in \Omega} f(x)$$

and h small enough. Items (a)–(c) below prove that (22) is a supersolution of $P_{h,k}$.

(a) If $x \in \partial\Omega$, $P_{h,k}(S)(x) = g(x) = S(x)$.

(b) Let $x \in \Omega$, $x + hq \in \Omega$ and $h = d(x, \partial\Omega) - d(x + hq, \partial\Omega) \leq \frac{2M_f}{L_f}$. It follows that

$$\begin{aligned} P_{h,k}(S)(x) &\leq f(x)|q|h + L_f h^2 + (3M_f d(x + hq, \partial\Omega) + M_g) \\ &= f(x)|q|h + L_f h^2 + (3M_f(-h + d(x, \partial\Omega)) + M_g) \\ &= f(x)|q|h + L_f h^2 - 3M_f h + 3M_f d(x, \partial\Omega) + M_g \\ &= S(x) + f(x)|q|h + L_f h^2 - 3M_f h \\ &\leq S(x) + L_f h^2 - 2M_f h \\ &\leq S(x). \end{aligned}$$

(c) Let $x \in \Omega$, $x + hq \in \partial\Omega$ and $h \leq \min\left\{\frac{3}{2}d(x, \partial\Omega), \frac{M_f}{L_f}\right\}$.

Notice that $f(x)|q|h + L_f h^2 \leq M_f h + L_f h^2$. Since $h \leq \frac{M_f}{L_f}$, we have $M_f h + L_f h^2 \leq 2M_f h$. Moreover, since $h < \frac{3}{2}d(x, \partial\Omega)$, we have $2M_f h < 3M_f d(x, \partial\Omega)$; then

$$\begin{aligned} P_{h,k}(S)(x) &\leq f(x)|q|h + L_f h^2 + g(x + hq) \\ &\leq f(x)|q|h + L_f h^2 + M_g \\ &\leq 3M_f d(x, \partial\Omega) + M_g = S(x). \end{aligned}$$

In consequence, $P_{h,k}(S)(x) \leq S(x)$.

(3) $s = 0$ is a finite subsolution of $P_{h,k}$, i.e. $s \leq P_{h,k}s$.

(4) By the monotony of the operator $P_{h,k}$, we have

$$\lim_{\mu \rightarrow \infty} (P_{h,k})^\mu S = \bar{S} \geq \underline{s} = \lim_{\mu \rightarrow \infty} (P_{h,k})^\mu s.$$

(5) Clearly,

$$P_{h,k}\bar{S} = \bar{S}, \quad \underline{s} = P_{h,k}\underline{s}. \tag{23}$$

(6) $\bar{S} = \underline{s}$.

To prove this equality, let us suppose that $(\bar{S} - \underline{s})(x) > 0$. Then, for some \hat{x} , the difference $(\bar{S} - \underline{s})$ is maximal. From (23),

$$\begin{aligned} (\bar{S} - \underline{s})(\hat{x}) &= (P_{h,k}\bar{S} - P_{h,k}\underline{s})(\hat{x}) \\ &= \min_{q \in B_1(\hat{x}), \hat{x} + hq \in \bar{\Omega}} (f(\hat{x})|q|h + L_f h^2 + \bar{S}(\hat{x} + hq)) \\ &\quad - \min_{q \in B_1(\hat{x}), \hat{x} + hq \in \bar{\Omega}} (f(\hat{x})|q|h + L_f h^2 + \underline{s}(\hat{x} + hq)) \\ &\leq (\bar{S} - \underline{s})(\hat{x} + hq_0), \end{aligned}$$

where q_0 is a control which realizes the minimum in

$$\min_{q \in B_1(\hat{x}), \hat{x} + hq \in \bar{\Omega}} (f(\hat{x})|q|h + L_f h^2 + \underline{s}(\hat{x} + hq)).$$

From the way \hat{x} was chosen, the last inequality implies that

$$(\bar{S} - \underline{s})(\hat{x}) = (\bar{S} - \underline{s})(\hat{x} + hq_0),$$

and, in consequence, $\hat{x} + hq_0$ is a convex combination of points belonging to

$$L := \{x \in S_k : (\bar{S} - \underline{s})(x) = (\bar{S} - \underline{s})(\hat{x})\}.$$

Let $\bar{x} \in L$ such that $\bar{S}(\bar{x}) = \min_L \bar{S}(x)$. We have

$$\begin{aligned} \bar{S}(\bar{x}) &= P_{h,k}\bar{S}(\bar{x}) \\ &= \min_{q \in B_1(\bar{x}), \bar{x} + hq \in \bar{\Omega}} (f(\bar{x})|q|h + L_f h^2 + \bar{S}(\bar{x} + hq)) \\ &= f(\bar{x})|\tilde{q}|h + L_f h^2 + \sum_{x_j \in L} \lambda_j \bar{S}(x_j) \\ &\geq f(\bar{x})|\tilde{q}|h + L_f h^2 + \bar{S}(\bar{x}), \end{aligned} \tag{24}$$

where \tilde{q} is the control which realizes the minimum in (24), $\lambda_j \geq 0$, and $\sum_j \lambda_j = 1$. From the inequality in (24), we have $f(\bar{x})|\tilde{q}|h + L_f h^2 \leq 0$, a contradiction.

(7) For $\Phi \in \mathbb{R}^{N_k}$,

$$\lim_{\mu \rightarrow \infty} (P_{h,k})^\mu \Phi = \bar{S} = \underline{s}. \tag{25}$$

Following the reasoning in step (6), it is possible to prove that, for M large enough, S_M is a supersolution ($-S_M$ a subsolution) and $|\Phi(x)| \leq S_M(x)$, where

$$S_M(x) = 3M d(x, \partial\Omega) + M \quad x \in S_k \cap \Omega \text{ and } S_M(x) = M, \quad x \in S_k \cap \partial\Omega. \quad \square$$

The following result establishes the convergence of the fully discrete solutions.

Proposition 5.3. $V_{h,k}$ converges to \bar{V}_h as $k \rightarrow 0$ and then,

$$\lim_{h \rightarrow 0} \lim_{k \rightarrow 0} V_{h,k}(x) = U(x), \quad \text{for every } x \in \bar{\Omega}.$$

Proof. It essentially comprises some suitable modifications of the arguments used in [6]. □

Remark 5.4. From the theoretical viewpoint we have a procedure to compute the approximated solution to our problem. However, the convergence in Proposition 5.3 could be too slow. By eliminating some admissible directions, we obtain in the next subsection, an improvement for that convergence.

5.2. Eliminating admissible directions

We define now a control problem on Ω_k . We consider as an admissible controlled path, any finite sequence of points $\{x_0, x_1, \dots, x_\rho\}$ that satisfies the restrictions

$$\begin{cases} x_\mu \in S_k \cap \Omega & \mu = 0, 1, \dots, \rho - 1, \\ x_\rho \in S_k \cap \partial\Omega, \\ \|x_\mu - x_{\mu-1}\| \leq k^{2/3} & \mu = 1, \dots, \rho. \end{cases} \tag{26}$$

Taking into account that $\|q(t)\| \leq 1$ for every t , and being x_μ and $x_{\mu+1}$ two successive switching points of a piecewise linear admissible trajectory, *i.e.* $x_{\mu+1} = x_\mu + hq(h\mu)$, we have that

$$\|x_{\mu+1} - x_\mu\| = h \|q(h\mu)\| \leq h.$$

Remark 5.5. In (26) we ask for $h \leq k^{2/3}$. In this way, given a fixed current point x_μ , the cardinal of the set of nodes in S_k which can be considered a possible $x_{\mu+1}$ is about $k^{-1/3}$. These points are the only ones involved in the next iteration.

This idea is related to the *Fast marching methods* described by Sethian in [11], and by Cristiani and Falcone in [5] in the sense that, in both methods, only the nodes which are nearer to the current one, are involved in the computation. Besides, it is clear that a similar work can be done by considering $h = k^\gamma$ with $\gamma \in (0, 1)$, but the number of neighbours increase when γ decreases. The relation we propose between h and k is as good as the one proposed by Camilli and Grüne in [3].

Given the initial position x_0 , the cost of a trajectory that ends at x_ρ is:

$$F_k(x_0, x_1, \dots, x_\rho) = g(x_\rho) + \sum_{\varsigma=1}^{\rho} \left(L_f k^{2/3} + f(x_{\varsigma-1}) \right) \|x_\varsigma - x_{\varsigma-1}\|.$$

We define $w_k(x_0)$ as the optimal cost when the process starts at the initial position x_0 , *i.e.*

$$w_k(x_0) = \min_{x_1, \dots, x_\rho} F_k(x_0, x_1, \dots, x_\rho). \tag{27}$$

We also define the operator P_k by

$$P_k\Phi(x) = \begin{cases} \min \{ \Phi(y) + (L_f k^{2/3} + f(x)) \|y - x\| : \\ y \in S_k, \|y - x\| \leq k^{2/3} \} & x \in S_k \cap \Omega, \\ g(x), & x \in S_k \cap \partial\Omega. \end{cases} \tag{28}$$

The following proposition relates w_k and P_k .

Proposition 5.6. w_k is the unique solution to the equation

$$\Phi = P_k\Phi. \tag{29}$$

Proof. Similar to the proof of uniqueness of fixed point for P_h . □

Remark 5.7. $\Phi = P_k \Phi$ is the Bellman dynamical programming equation associated with the optimal control of a deterministic Markov chain.

Corollary 5.8. $w_k = \lim_{\mu \rightarrow \infty} (P_k)^\mu \Phi(x)$ for every $\Phi \in \mathbb{R}^{N_k}$.

Proof. Once we introduce the Kruskov transformation, the proof easily holds. □

Remark 5.9. The previous result gives an iterative procedure to get w_k . In fact, it converges in a finite number of iterations. In the numerical applications we note that the algorithm converges faster when the chosen Φ is a supersolution for the operator.

Proposition 5.10. $\lim_{k \rightarrow 0} w_k(x) = U(x)$.

Proof. Let w_k be the unique solution to the equation (29). Given $x \in \Omega$ and $\varepsilon > 0$, there exists an ε -suboptimal control $q_\varepsilon(\cdot) \in Q_x$ with exit time $T(q_\varepsilon)$ such that

$$J(x, q_\varepsilon(\cdot)) \leq U(x) + \varepsilon.$$

If $\xi_{x,\varepsilon}(\cdot)$ is the trajectory generated by $q_\varepsilon(\cdot)$, then $\xi_{x,\varepsilon}(0) = x$.

We can assume w.l.g. that the control we choose is piecewise constant and then the trajectory is piecewise linear. Let t_ν be the switching times of $q_\varepsilon(\cdot)$, for $\nu = 1, \dots, \bar{\nu}_\varepsilon$

$$t_0 = 0, \quad t_{\bar{\nu}+1} = T(q_\varepsilon).$$

We define $\hat{p} = \lceil 2T(q_\varepsilon)k^{-2/3} \rceil + 1$, when $2T(q_\varepsilon)k^{-2/3}$ is not an integer and $\hat{p} = 2T(q_\varepsilon)k^{-2/3}$ otherwise. Clearly, $\xi_{x,\varepsilon}(\frac{p}{2}k^{2/3}) \in \Omega$, for every $p = 0, 1, \dots, \hat{p} - 1$.

Let us construct $\xi_{x,\varepsilon}^k(\cdot)$, a trajectory close to $\xi_{x,\varepsilon}(\cdot)$, which joins nodes of Ω_k .

For this purpose, let us define

$$x_p = \arg \min \left\{ \left\| \xi_{x,\varepsilon} \left(\frac{p}{2} k^{2/3} \right) - y \right\| : y \in S_k \cap \Omega \right\} \quad \text{for } p = 0, 1, \dots, \hat{p} - 1,$$

$$x_{\hat{p}} = \arg \min \{ \|\xi_{x,\varepsilon}(T(q_\varepsilon)) - y\| : y \in S_k \cap \partial\Omega \}.$$

Since the mesh size is k , it is clear that

$$\left\| \xi_{x,\varepsilon} \left(\frac{p}{2} k^{2/3} \right) - x_p \right\| \leq k, \quad \text{for } p = 0, 1, \dots, \hat{p}.$$

Then, for $p = 1, \dots, \hat{p}$

$$\begin{aligned} \|x_p - x_{p-1}\| &\leq \left\| \xi_{x,\varepsilon} \left(\frac{p}{2} k^{2/3} \right) - \xi_{x,\varepsilon} \left(\frac{p-1}{2} k^{2/3} \right) \right\| + 2k \\ &\leq q \frac{k^{2/3}}{2} + 2k, \end{aligned}$$

where q is the constant value of $q_\varepsilon(t)$ for $t \in [\frac{p-1}{2}k^{2/3}, \frac{p}{2}k^{2/3}]$. Moreover,

$$\left\| \xi_{x,\varepsilon} \left(\frac{p}{2} k^{2/3} \right) - \xi_{x,\varepsilon} \left(\frac{p-1}{2} k^{2/3} \right) \right\| \leq \|x_p - \xi_{x,\varepsilon} \left(\frac{p}{2} k^{2/3} \right)\| + \left\| \xi_{x,\varepsilon} \left(\frac{p-1}{2} k^{2/3} \right) - x_{p-1} \right\| + \|x_p - x_{p-1}\|$$

and thus,

$$\|x_p - x_{p-1}\| \geq q \frac{k^{2/3}}{2} - 2k.$$

For any interval $\left(\frac{p-1}{2} k^{2/3}, \frac{p}{2} k^{2/3}\right)$ which does not contain some t_ν , we have that the cost associated with the continuous trajectory becomes smaller than

$$\int_{\frac{p-1}{2} k^{2/3}}^{\frac{p}{2} k^{2/3}} f(\xi_{x,\varepsilon}(t))q \, dt$$

and the term associated with the discrete trajectory is

$$\left(L_f k^{2/3} + f(x_{p-1})\right) \|x_p - x_{p-1}\|.$$

Hence,

$$\begin{aligned} & \left| \int_{\frac{p-1}{2} k^{2/3}}^{\frac{p}{2} k^{2/3}} f(\xi_{x,\varepsilon}(t))q \, dt - (L_f k^{2/3} + f(x_{p-1})) \|x_p - x_{p-1}\| \right| \\ & \leq \int_{\frac{p-1}{2} k^{2/3}}^{\frac{p}{2} k^{2/3}} |f(\xi_{x,\varepsilon}(t)) - f(x_{p-1})| q \, dt \\ & \quad + \left| f(x_{p-1})q \frac{k^{2/3}}{2} - (L_f k^{2/3} + f(x_{p-1})) \|x_p - x_{p-1}\| \right| \\ & \leq \int_{\frac{p-1}{2} k^{2/3}}^{\frac{p}{2} k^{2/3}} L_f \|\xi_{x,\varepsilon}(t) - x_{p-1}\| q \, dt \\ & \quad + f(x_{p-1}) \left| q \frac{k^{2/3}}{2} - \|x_p - x_{p-1}\| \right| + L_f k^{2/3} \|x_p - x_{p-1}\| \\ & \leq \int_{\frac{p-1}{2} k^{2/3}}^{\frac{p}{2} k^{2/3}} L_f |k + t| q \, dt + M_f 2k + L_f k^{2/3} \left(\frac{qk^{2/3}}{2} + 2k \right) \\ & \leq L_f \left(k + \frac{k^{2/3}}{4} \right) \frac{k^{2/3}}{2} + M_f 2k + L_f \left(\frac{k^{2/3}}{2} + 2k \right) k^{2/3}. \end{aligned}$$

Then, for k small enough this last bound becomes not greater than $3M_f k$.

For any interval $\left(\frac{p-1}{2} k^{2/3}, \frac{p}{2} k^{2/3}\right)$ which contains one or more switching points, we get $2M_f k^{2/3}$ as a bound for the difference between the continuous and the discrete trajectory. The same bound holds for the final interval $\left(\frac{\hat{p}-1}{2} k^{2/3}, T(q_\varepsilon)\right)$.

Considering the inequality

$$\|\xi_{x,\varepsilon}(T(q_\varepsilon)) - x_{\hat{p}}\| \leq k,$$

we have that the difference between the final costs is bounded by $L_g k$.

Finally we get

$$\begin{aligned}
 & |J(x, q_\varepsilon(\cdot)) - F(x_0, x_1, \dots, x_{\hat{p}})| \\
 & \leq L_g k + 2M_f (\bar{\nu}_\varepsilon + 1) k^{2/3} + \frac{2T(q_\varepsilon)}{k^{2/3}} 3M_f k \\
 & \leq L_g k + 2M_f (\bar{\nu}_\varepsilon + 1) k^{2/3} + 6T(q_\varepsilon)M_f k^{1/3},
 \end{aligned} \tag{30}$$

which implies

$$w_k(x_0) \leq U(x) + \varepsilon + L_g k + 2M_f (\bar{\nu} + 1) k^{2/3} + 6T(q_\varepsilon)M_f k^{1/3}. \tag{31}$$

By computing the limit as k goes to zero and recalling that ε was arbitrarily chosen, we obtain

$$\lim_{k \rightarrow 0} w_k(x_0) \leq U(x).$$

Now, let x be any state in Ω and x_0 the beginning of a path as in (26). It is clear that if $q(\cdot)$ is a piecewise constant strategy and $x_0, x_1, \dots, x_{\hat{p}}$ are the switching points of the trajectory

$$J(x, q(\cdot)) \leq F(x_0, x_1, \dots, x_{\hat{p}}) + M_f k.$$

Then, $U(x) \leq F(x_0, x_1, \dots, x_{\hat{p}}) + M_f k$ and,

$$U(x) \leq w_k(x_0) + M_f k.$$

From (27), we have

$$U(x) \leq \lim_{k \rightarrow 0} w_k(x_0). \quad \square$$

Remark 5.11. The term $\frac{2T(q_\varepsilon)}{k^{2/3}} 3M_f k$ in (30) shows that if the number of nodes considered is Ck^{-1} ($h = k$), this scheme does not converge. Moreover, the term $2M_f (\bar{\nu}_\varepsilon + 1) k^{2/3} + 6T(q_\varepsilon)M_f k^{1/3}$ shows that when $h = k^\gamma$ with $\gamma < 1$, the convergence order is $\beta = \min(\gamma, 1 - \gamma)$. Clearly, the optimal convergence order is obtained when $\beta = 1/2$. We choose to work with $\gamma = 2/3$ in order to reduce the number of neighbours involved in each iteration.

6. EXAMPLES

We show two examples to illustrate the performance of the algorithm which computes w_k , see (28).

According to our proposal if $n + 1$ is the number of nodes considering in each direction, the spatial step (size of the mesh) is $k = 1/n$, the time step $h = (1/n)^{2/3}$, and the number of neighbours involved in each direction is of order $n^{1/3}$.

Example 1. First we consider an example where f assumes the shape shown in Figure 1. With $n = 72$, $k = 0.0139$, $h = 0.0578$, the optimal cost results like in Figure 2. Figures 3 and 4 present the level curves of w_k for $n = 50$ and $n = 100$ respectively. Note that the trajectories avoid the highest zones when the number of nodes increases. In Figure 5 the number of nodes is not large enough and then the trajectory does not behave well. Nevertheless, in Figure 6, where a bigger number of nodes is considered, the trajectory follows the expected spiral shape.

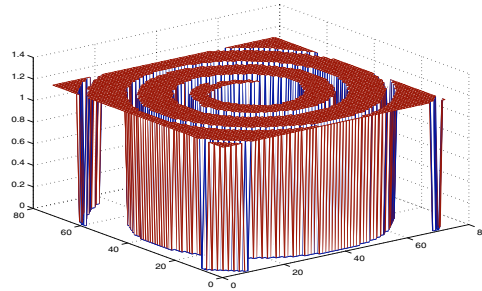


FIGURE 1. Profile of f .

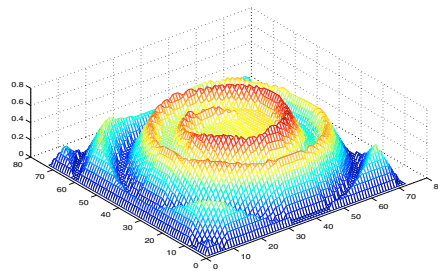


FIGURE 2. Profile of $w_{0.0139}$.

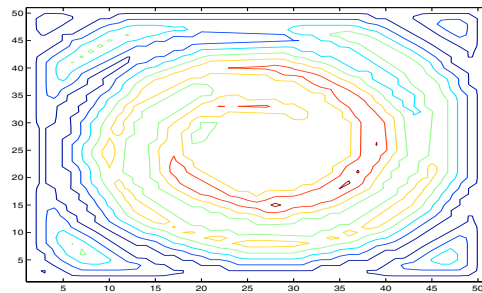


FIGURE 3. Level curves of w_k , $n = 50$.

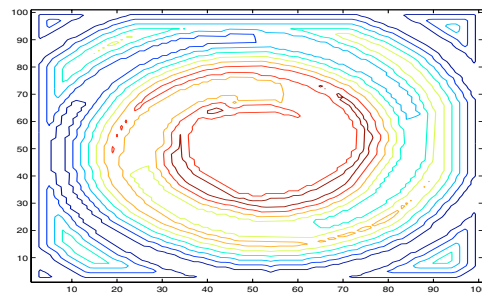


FIGURE 4. Level curves of w_k , $n = 100$.

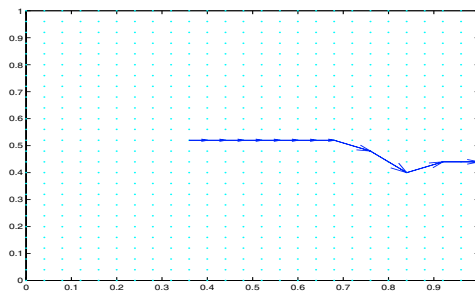


FIGURE 5. A trajectory for $n = 25$.

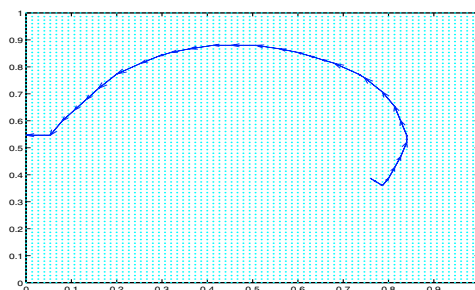


FIGURE 6. A trajectory for $n = 75$.

Example 2. We consider now the same example presented in [3]. The function $f(x, y) = \max\{|x_1 - 1/4| - 1/4, 0\} + |x_2|$ and $\Omega = [-1/4, 1] \times [-1, 1]$ (Fig. 7). The approximate solution level curves are shown in Figure 8. An optimal trajectory is shown in Figure 9.

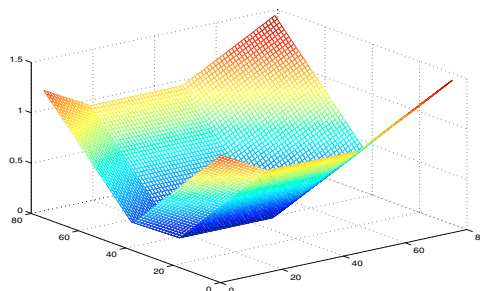


FIGURE 7. Function f .

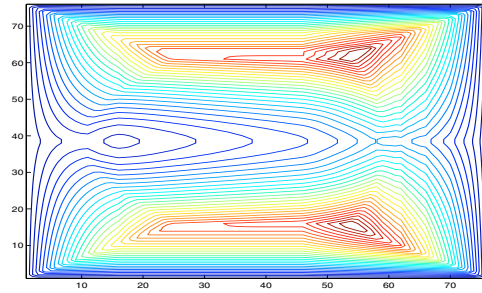


FIGURE 8. Level curves of w_k .

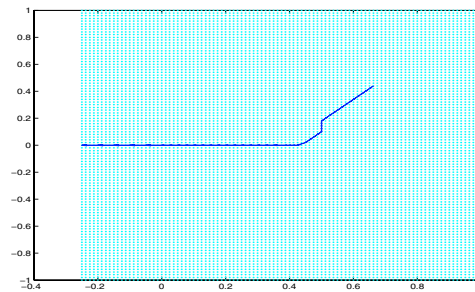


FIGURE 9. A trajectory for $n = 100$.

Now we show the approximate solution for different mesh size (see Figs. 10–12). Since our h also represents the separation from zero (penalization order) it is possible to compare our resulting profiles with those presented in [3].

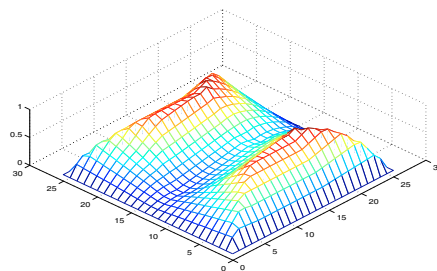
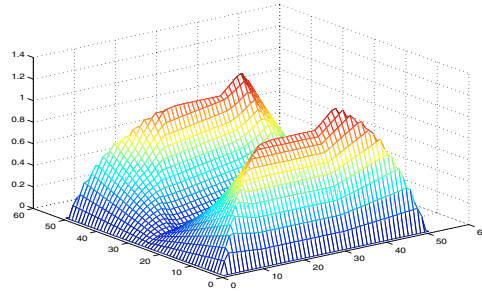
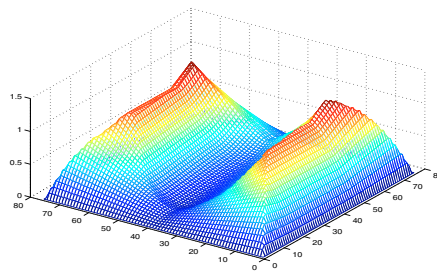


FIGURE 10. Approximate solution $n = 25$, $h = 0.12$.

FIGURE 11. Approximate solution $n = 50$, $h = 0.07$.FIGURE 12. Approximate solution $n = 75$, $h = 0.05$.

CONCLUSIONS

In this work we have presented the different discretizations that must be used in order to get a complete discrete procedure for approximating the optimal cost of a singular optimal control problem. The convergence may be lost once the discretization of the dynamic of the system is introduced. In order to recover the convergence property we used a penalization of the instantaneous cost. In this paper we have used a penalization of order h , being h the time-step employed. Finally, we have obtained a complete discrete procedure of approximation showing that the convergence of this method holds. The main idea is to eliminate some admissible directions in order to accelerate the convergence of the algorithm. This reduction provokes that fewer nodes are taken into account in each iteration.

APPENDIX: A PATHOLOGICAL C^1 FUNCTION

In this appendix we construct a function f which is zero only on a connected set with infinite length. Because of this, this set cannot be an admissible trajectory. The positiveness of the integral $\int_0^T f(\xi(t)) |q(t)| dt$ implies, for every $\xi(\cdot)$ starting in x , that $U(x) > 0$. However, if we chose a control $q(\cdot) \in Q_{x,h}$ such that the resulting trajectory has all their switching points $-\xi(\nu h)$ with $\nu \in \mathbb{N}_0-$ in the connected set where f is zero, the sum $\sum h f(\xi(\nu h)) |q(\nu h)|$ is clearly zero. Then, by considering $g \equiv 0$, it follows that $J_h(x, q) = 0$. This means that (8) is not a good approximation for (1).

In 1935, Whitney [12] had already presented a C^n function defined in a hypercube of \mathbb{R}^{n+1} . For this function there exist two different points, P_1 and P_2 in the hypercube where the function takes different values. At the same time all the derivatives up to order n are zero on a connected set containing P_1 and P_2 . This means that it is, in general, impossible to express the values of a function f along a curve which is not rectifiable, through an integral of a function of partial derivatives of f of order smaller than or equal to n along the curve. For

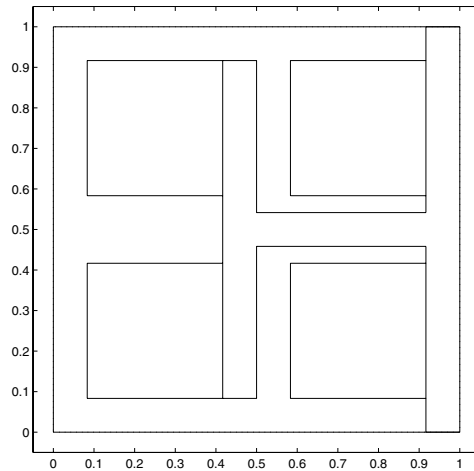


FIGURE 13. The set $S \cup L$.

the sake of completeness, we present here a function with similar properties to by Whitney [12]. For simplicity, we restrict to \mathbb{R}^2 . The connected set where the derivatives are zero is recursively constructed in \mathbb{R}^2 . Since the extension to \mathbb{R}^{n+1} is obvious, we omitted it.

Construction of the function

Let us construct a non-constant C^1 function defined at $[0, 1]^2$ whose derivatives are zero on a connected set which contains the points $P_1 = (1, 0)$ and $P_2 = (1, 1)$.

Let $\varphi(s) = 3s^2 - 2s^3$ and define ψ_0 as the basic C^1 function that we will use to generate the sequence of functions ψ_ν ,

$$\psi_0(x, y) = \varphi(y), \quad \forall (x, y) \in [0, 1]^2. \tag{32}$$

Clearly $\psi_0(P_1) = 0$ and $\psi_0(P_2) = 1$.

Inside the initial square $[0, 1]^2$ we consider four squares Q_1, Q_2, Q_3, Q_4 , and the set S consisting of these squares. S is connected by straight lines between its parts and to the points P_1 and P_2 as shown in Figure 13. These straight lines are denoting by L .

We define the function ψ_1 as a suitable translation and contraction of ψ_0 in the interior squares and suitable interpolations in the complement of those regions. Even though we do not present the details of the construction, it is important to emphasize that we set ψ_1 constant in each polygonal line of L and that $\psi_1 \in C^1([0, 1]^2)$. Besides,

$$\|\nabla\psi_1\|_{C^0([0,1]^2)} \leq 9/2 \quad \text{and} \quad \|\nabla\psi_1 - \nabla\psi_0\|_{C^0([0,1]^2)} \leq 3. \tag{33}$$

Defining recursively ψ_ν as

$$\psi_{\nu+1}(x, y) = \begin{cases} \psi_1(x, y), & \text{in } S, \\ (1/4)\psi_\nu(3(x - 7/12), 3(y - 1/12)), & \text{in } Q_1, \\ 1/4 + (1/4)\psi_\nu(3(x - 1/12), 3(y - 1/12)), & \text{in } Q_2, \\ 1/2 + (1/4)\psi_\nu(3(x - 1/12), 3(y - 7/12)), & \text{in } Q_3, \\ 3/4 + (1/4)\psi_\nu(3(x - 7/12), 3(y - 7/12)), & \text{in } Q_4, \end{cases}$$

it is clear that at level $\nu + 1$ (implicitly and as a result of the recurrence), new four squares are constructed inside each ν -level square. We get a sequence of functions defined in $[0, 1]^2$ where, in each ν -level square, $\psi_{\nu+1}$

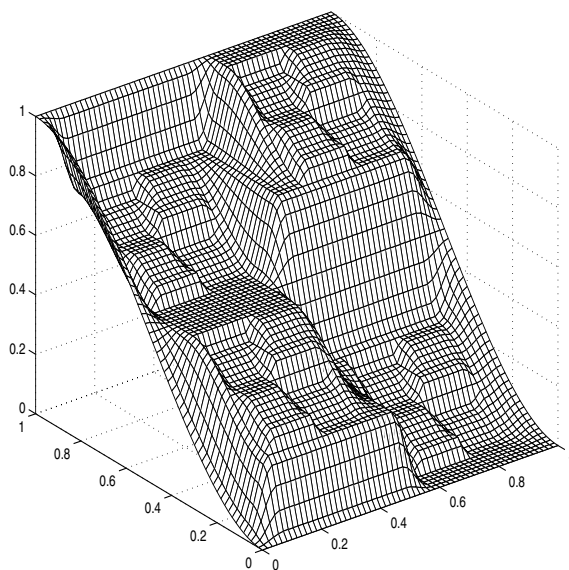


FIGURE 14. Function ψ_ν .

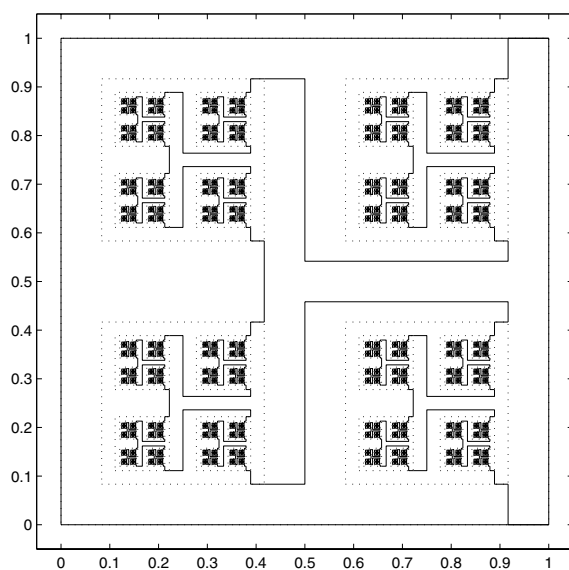


FIGURE 15. Set L_ν .

is a suitable contraction and translation of ψ_ν , while outside $\psi_{\nu+1} = \psi_\nu$ holds. The resulting function is shown in Figure 14.

Properties of the function

Lemma 6.1. *The function $\psi = \lim_{\nu \rightarrow \infty} \psi_\nu$ is non constant, and $\psi \in C^0([0, 1]^2)$.*

Proof. Clearly, $\psi(P_1) = \psi_1(P_1) = 0$ and $\psi(P_2) = \psi_1(P_2) = 1$. Moreover, it is easy to check that $\{\psi_\nu\}$ is a Cauchy sequence in C^0 . \square

Lemma 6.2. $\psi \in C^1([0, 1]^2)$.

Proof. Outside the ν -level squares, $\nabla\psi_{\nu+1}(x, y) - \nabla\psi_\nu(x, y) = 0$ while, inside them, we have

$$\nabla\psi_{\nu+1} - \nabla\psi_\nu = (3/4) \begin{pmatrix} \frac{\partial\psi_\nu}{\partial x} - \frac{\partial\psi_{\nu-1}}{\partial x} \\ \frac{\partial\psi_{\nu+1}}{\partial y} - \frac{\partial\psi_{\nu-1}}{\partial y} \end{pmatrix}.$$

It follows that

$$\|\nabla\psi_{\nu+1} - \nabla\psi_\nu\|_{C^0([0,1]^2)} \leq 3/4 \|\nabla\psi_\nu - \nabla\psi_{\nu-1}\|_{C^0([0,1]^2)}.$$

By induction, and recalling (33), we get $\|\nabla\psi_{\nu+1} - \nabla\psi_\nu\|_{C^0([0,1]^2)} \leq 3(3/4)^\nu$. Iterating we get

$\|\nabla\psi_{\nu+m} - \nabla\psi_\nu\|_{C^0([0,1]^2)} \leq 12(3/4)^\nu$ for any $m \in \mathbb{N}$. In consequence, $\{\nabla\psi_\nu\}$ converges uniformly to $\nabla\psi$ and $\psi \in C^1([0, 1]^2)$. \square

Lemma 6.3. $\nabla\psi$ is zero on a connected set which contains P_1 and P_2 .

Proof. We define $\Gamma_1 = S \cup L$, where S is the set consisting of the four squares Q_i with $i = 1, \dots, 4$, and L , the straight lines appearing in Figure 13. On the connecting lines, $\|\nabla\psi_1\|_{C^0(L)} = 0$. Besides, from the definition, it is clear that $\|\nabla\psi_0\|_{C^0(S)} \leq 3/2$.

In the same way, for the ν -th iteration, let $\Gamma_\nu = S_\nu \cup L_\nu$ be the set consisting of the 4^ν squares and the lines joining them. The set L_ν can be seen in Figure 15. On the joining lines, $\|\nabla\psi_\nu\|_{C^0(L_\nu)} = 0$. Analogously, $\|\nabla\psi_\nu\|_{C^0(S_\nu)} \leq (3/2) ((3/4)^\nu)$ and then $\|\nabla\psi_\nu\|_{C^0(\Gamma_\nu)} \leq (3/2) (3/4)^\nu$. Moreover, by construction, it is clear that $\Gamma_{\nu+1} \subseteq \Gamma_\nu$ and $\Gamma = \bigcap_{\nu \in \mathbb{N}} \Gamma_\nu$ is a non empty compact set. Then, for every $\gamma \in \Gamma$,

$$|\nabla\psi(\gamma)| \leq \lim_{\nu \rightarrow \infty} \|\nabla\psi_\nu\|_{C^0(\Gamma_\nu)} = 0. \quad \square$$

Remark 6.4. The function $\nabla\psi$, null on a infinite length connected set, is the announced example in Section 4.2.

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