ESAIM: M2AN
Vol. 40, N° 4, 2006, pp. 765–784
DOI: 10.1051/m2an:2006029

ESAIM: Mathematical Modelling and Numerical Analysis
www.edpsciences.org/m2an

NEW UNILATERAL PROBLEMS IN STRATIGRAPHY

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Abstract. This work deals with the study of some stratigraphic models for the formation of geological basins under a maximal erosion rate constrain. It leads to introduce differential inclusions of degenerated hyperbolic-parabolic type $0 \in \partial_t u - div\{H(\partial_t u + E)\nabla u\}$, where H is the maximal monotonous graph of the Heaviside function and E is a given non-negative function. Firstly, we present the new and realistic models and an original mathematical formulation, taking into account the *weather-limited* rate constraint in the conservation law, with a unilateral constraint on the outflow boundary. Then, we give a study of the $1 - D$ case with numerical illustrations.

Mathematics Subject Classification. 35K65, 35L80, 35Q35.

Received: October 24, 2005. Revised: March 20, 2006.

1. Introduction

In this paper we are interested in the mathematical analysis of general models arising in geological basin formation. The initial model has been developed by the Institut Français du Pétrole (IFP) and it takes into account sedimentation, transport and accumulation, erosion phenomena at large scales in time and space. The main feature of these models is characterized by a constraint on the time-derivative of the solution. This constraint leads us to consider an original class of conservation laws, *a priori* parabolic, revealing some hyperbolic behaviour because of a diffusive coefficient that depends on the time-derivative of the solution.

A more precise description of these models have been exposed, on the one hand, for the multilithological case, by Eymard *et al.* [9] and Gervais *et al.* [15]; on the second hand, by Antontsev *et al.* [3,4], Gagneux *et al.* [10,12], Etienne [7] and Vallet [18] for the mathematical aspect of the monolithological case and Eymard *et al.* [8] for a theoretical and numerical approach of an inverse problem.

In Section 2, we consider a variational inequality, involving a unilateral constraint on the outflow boundary. After giving a brief presentation of the model and of the mathematical framework, we would be interested in the existence of a solution. The main result proves the existence of an approximating sequence *via* an implicit time-discretization scheme. Then some *a priori* estimates, independent of the time-discretized parameter, are presented. Passing to the limits in the approximate variational inequality is still an open problem, mainly because of a lack of compactness results.

Keywords and phrases. Stratigraphic models, weather limited, degenerated parabolic-hyperbolic conservation laws.

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Then, we present a concrete analysis of examples in the $1-D$ case, where, with some particular hypothesis, one is able to simplify the boundary conditions (and the problem) and overcome the difficulties following Gagneux *et al.* [10] idea. This section ends with numerical simulations and the presentation of open problems.

2. An original hyperbolic-parabolic model

2.1. Presentation of the model

Let us consider a sedimentary basin whose base, denoted by Ω, is a fixed domain of $\mathbb{R}^d$ ($d = 1, 2$ in this framework) with a Lipschitzian boundary Γ. For any positive T, let us denote by $Q =]0, T[\times\Omega$.

The sediment height, denoted by u, naturally satisfies the mass balance equation:

$$\partial_t u + \operatorname{div}\{\vec{q}\} = 0 \quad \text{in } Q,$$

where $\vec{q}$, following a Darcy's law, is proportional to ∇u, the gradient of the topography.

Moreover, a second phenomenon happens in a sedimentary basin formation process: sediments must first be produced *in situ* by weathering processes prior to be transported by surfacing erosion. Thus, Eymard *et al.* [14] introduce a maximum erosion rate E such that:

$$-\partial_t u \leq E \quad \text{in } Q,$$

where E takes into account the composition, the structure and the age of the sediments.

The coupling of these two constraints is clearly an essential issue since both diffusive sedimentation or erosion and weather limited erosion can occur at the same time in a basin.

Then the authors introduce a new unknown λ satisfying $0 \leq \lambda \leq 1$ and playing the role of a flux limiter, according to the Darcy's law:

$$\vec{q} = -\lambda\nabla u \quad \text{in } Q.$$

In order to give a mathematical modelling of λ, Gallouët proposes in [14] the following formulation:

$$1-\lambda \geq 0,\ \partial_t u + E \geq 0,\ (1-\lambda)(\partial_t u + E) = 0 \quad \textit{a.e.} \text{ in } Q.$$

In other words, if one has to correct the flux, *i.e.* $\lambda < 1$, then the constraint has to be observed, *i.e.* $\partial_t u + E = 0$, otherwise $\lambda = 1$.

Remark 2.1. If one assumes that $\lambda = 1$ *a.e.* in Q, the mass balance law and the Darcy's law give $\partial_t u = \Delta u$ and one gets that $E + \Delta u \geq 0$. Then, if one considers a smooth function E, taking the limit when t goes towards 0^+ in the sense of distributions leads to the fact that $\Delta u_0 + E(0,.)$ is necessarily a non-negative Radon measure. Since this condition is not reasonable, λ has to take values less than 1. In that way, the modified flux approach is really required.

Boundary conditions. One considers an initial height given by: $u(0,x) = u_0(x)$, $x \in \Omega$.
One assumes that Γ is separated in two non-trivial distinct parts Γ_s and Γ_e in order to specify the output and the input fluxes.
The input flux is described by a matter transfer through Γ_e, proportional to the gradient of the topography:

$$\lambda\nabla u.\vec{n} + f = 0 \quad \text{on } \Gamma_e, \quad \text{with } f_{|\Gamma_e} \leq 0.$$

On the boundary Γ_s, one imposes the unilateral conditions (the out-flux has to be limited to satisfy the maximum erosion constraint):

$$\lambda\nabla u.\overrightarrow{n} + f \geq 0\ ;\ \partial_t u + E \geq 0\ ;\ (\lambda\nabla u.\overrightarrow{n} + f)(\partial_t u + E) = 0,\ t > 0, \text{ where } f_{|\Gamma_s} \geq 0.$$

Then, one has to look for a couple (λ, u) that satisfies:

$$\partial_t u - \operatorname{div}\{\lambda \nabla u\} = 0 \quad a.e. \text{ in } Q, \tag{1}$$

$$u(0,.) = u_0, \quad a.e. \text{ in } \Omega, \tag{2}$$

$$\lambda \nabla u.\vec{n} + f = 0 \quad a.e. \text{ on }]0,T[\times \Gamma_e, \tag{3}$$

$$\lambda \nabla u.\overrightarrow{n} + f \geq 0,\ \partial_t u + E \geq 0 \text{ and } (\lambda \nabla u.\overrightarrow{n} + f)(\partial_t u + E) = 0,\ a.e. \text{ on }]0,T[\times \Gamma_s, \tag{4}$$

and the global unilateral constraint:

$$1 - \lambda \geq 0,\ \partial_t u + E \geq 0,\ (1-\lambda)(\partial_t u + E) = 0 \quad a.e. \text{ in } Q, \tag{5}$$

where one assumes:

$$u_0 \in H^1(\Omega), \quad E \in L^\infty(0,T;H^1(\Omega)) \text{ satisfying } E \geq 0,$$

$$f \in L^2(]0,T[\times \Gamma) \text{ where: } \begin{cases} f \geq 0 & \text{on }]0,T[\times \Gamma_s, \\ f \leq 0 & \text{on }]0,T[\times \Gamma_e. \end{cases}$$

Mathematical studies of such models have already been done through several papers. Let us cite for example:

(i) Antontsev *et al.* [3,4] and Gagneux *et al.* [12] where λ is assumed to be a fixed parameter. In this paper, the authors prove that problem (1) to (4) is well-posed and propose a condition on λ to obtain (5).

(ii) In the same papers, the authors present explicit solutions (λ, u) to problem (1) to (5) in the $1-D$ dimensional space, by the way of travelling-waves techniques.

(iii) Eymard *et al.* [9] where u is assumed to be given. The authors give an analytical and numerical analysis of the hyperbolic problem (1) to (5) of unknown λ.

(iv) Antontsev *et al.* [4], Gagneux *et al.* [10] and Vallet [18] concerning the problem (1), (2) and (5) with homogeneous Dirichlet boundary conditions. An original conservation law is proposed with partial results of existence. Then, in the $1-D$ sedimentary case (*i.e.* $E=0$), explicit solutions and numerical simulations are proposed. The original mathematical formulation is: if H denotes the maximal monotone graph of the Heaviside function (*i.e.* $H(x)=0$ if $x<0$, $H(x)=1$ if $x>0$ and $H(0)=[0,1]$), one looks for (λ, u) such that:

$$\partial_t u - \operatorname{div}\{\lambda \nabla u\} = 0 \quad a.e. \text{ in } Q \quad \text{with} \quad \lambda \in H(\partial_t u + E),$$

under the same boundary-initial conditions. The unilateral global constraint is then implicitly contained in the formulation (see Vallet [18] and Gagneux *et al.* [10] for information about this).

In the sequel, our goal is to extend the previous results to realistic boundary conditions and non-negative E.

2.2. Definition of a solution

Let us introduce the non-empty closed convex cone $\mathbb{K} = \{v \in H^1(\Omega) \,/\, v \geq 0 \text{ } a.e. \text{ on } \Gamma_s\}$.

At this moment, the variational formulation of the conservation law we are interested in, is:

Definition 2.2. A solution to the problem (1) to (5) is any couple (λ, u) such that

$$u \in H^1(Q) \text{ with } \partial_t u + E \in \mathbb{K} \text{ } a.e. \text{ } t \text{ in }]0,T[,\ \partial_t u + E \geq 0 \text{ } a.e. \text{ in } Q \quad \text{and} \quad \lambda \in L^\infty(Q) \cap H(\partial_t u + E),$$

satisfying $u(0,.) = u_0$ *a.e.* in Ω and, for any $v \in \mathbb{K}$ and *a.e.* t in $]0,T[$,

$$0 \leq \int_\Omega \partial_t u(v - \partial_t u - E)\,\mathrm{d}x + \int_\Omega \lambda \nabla u.\nabla(v - \partial_t u - E)\,\mathrm{d}x + \int_\Gamma f(v - \partial_t u - E)\,\mathrm{d}\sigma.$$

In other words, the resulting model appears to be a non-standard free boundary problem of the form:

$$0 \in \partial_t u - \mathrm{div}\{H(\partial_t u + E)\nabla u\}.$$

Remark 2.3.

(i) In the one space dimensional case ($d = 1$), the fact that the condition $\partial_t u + E \geq 0$ *a.e.* in Q is implicit to the formulation can be proved, else a conjecture has to be stated (see Gagneux *et al.* [10]).

(ii) As mentioned in Vallet [18] and in Proposition 2.20, the solution to such a problem is not *a priori* unique. Therefore, one is interested in the maximal solution (λ, u) with respect to the multiplier λ, *i.e.* if (μ, v) is any other solution, then $\mu \leq \lambda$. On the other hand, the existence of such a solution is an open problem.

2.3. Mathematical study of the conservation law

This section is devoted to the analysis of an implicit time-discretization scheme. An existence result of such a sequence of approximation and some *a priori* estimates are presented. The section ends with the presentation of the difficulties in passing to the limits in the variational inequality.

Let us start with a remark concerning the regularity of the solution we are looking for.

Remark 2.4. Let us consider the general problem: find $u \in H^1(Q)$ such that $u\,(0,.) = u_0$, *a.e.* in Ω, and, for any $v \in H^1(\Omega)$,

$$\int_\Omega \{\partial_t u v + a(\partial_t u)\nabla u.\nabla v\}\,\mathrm{d}x = 0, \quad t \text{ a.e. in }]0,T[,$$

where a is a lipschitzian-continuous function with $0 \leq a(.) \leq M < \infty$.
Let us assume informally that $v_\varepsilon = \int_0^{\partial_t u} \frac{\mathrm{d}s}{a(s)+\varepsilon}$ ($\varepsilon > 0$) is an admissible test-function in the previous equation.
Then, one obtains: $\frac{1}{M}\left(||\partial_t u||^2_{L^2(Q)} + ||\mathrm{div}(a(\partial_t u)\nabla u)||^2_{L^2(Q)}\right) + ||u||^2_{L^\infty(0,T\,;\,H^1(\Omega))} \leq ||u_0||^2_{H^1(\Omega)}$.
Note that in a time-discretization method, the discrete version of v_ϵ is an admissible test-function.

Usually, the existence of a solution to such problem is proved by the way of an implicit time-discretization scheme. The heart of the following proofs is to state: for any positive integer N, if $h = \frac{T}{N}$, there exists a sequence $\{(\lambda_k, u^k)\}_{k\geq 0}$ such that $u^0 = u_0$,

$$u^k \in H^1(\Omega), \quad \frac{u^k - u^{k-1}}{h} + E_k \in \mathbb{K}, \quad \lambda_k \in L^\infty(\Omega), \quad \lambda_k \in H\left(\frac{u^k - u^{k-1}}{h} + E_k\right),$$

and, by denoting $E_k(x) = \frac{1}{h}\int_{(k-1)h}^{kh} E(t,x)\,\mathrm{d}t$ and $f_k(x) = \frac{1}{h}\int_{(k-1)h}^{kh} f(t,x)\,\mathrm{d}t$, for any $v \in \mathbb{K}$, one gets

$$\begin{aligned} 0 \quad \leq \quad & \int_\Omega \frac{u^k - u^{k-1}}{h}\left(v - \frac{u^k - u^{k-1}}{h} - E_k\right)\mathrm{d}x \\ + \quad & \int_\Omega \lambda_k \nabla u^k.\nabla\left(v - \frac{u^k - u^{k-1}}{h} - E_k\right)\mathrm{d}x + \int_\Gamma f_k.\left(v - \frac{u^k - u^{k-1}}{h} - E_k\right)\mathrm{d}\sigma. \end{aligned}$$

In order to prove the existence of such a sequence, a technique of artificial viscosity and a method of penalization of the constraint on the boundary Γ_s is proposed. Note that, one only needs to establish the result for the first iteration. The sequence will be obtained by induction.

Let us consider the following assumptions on the data for the first iteration:

$$(H): \quad \begin{cases} E_1 \in H^1(\Omega),\ E_1 \geq 0,\ u_0 \in H^1(\Omega), \\ f_1 \in L^2(\Gamma) \text{ such that } f_1 \geq 0 \quad \text{on }]0,T[\times\Gamma_s \quad \text{and} \quad f_1 \leq 0 \quad \text{on }]0,T[\times\Gamma_e. \end{cases}$$

$$\forall\varepsilon \in]0,1[, a_\varepsilon(x) = \max\left(\epsilon, \min\left(1, \frac{1-\varepsilon}{\varepsilon}x + 1\right)\right). \tag{6}$$

Lemma 2.5. *Considering* (H) *and any* ε, δ, h *in* $]0,1[$, *there exists a unique element* $u_{\varepsilon\delta}$ *in* $H^1(\Omega)$ *such that,*

$$\forall v \in H^1(\Omega),\ 0 = \int_\Omega \left\{ \frac{u_{\varepsilon\delta}-u_0}{h} v + a_\varepsilon\left(\frac{u_{\varepsilon\delta}-u_0}{h}+E_1\right)\nabla u_{\varepsilon\delta}.\nabla v \right\} \mathrm{d}x + \int_\Gamma f_1 v\,\mathrm{d}\sigma - \frac{1}{\delta}\int_{\Gamma_s}\left(\frac{u_{\varepsilon\delta}-u_0}{h}+E_1\right)^- v\,\mathrm{d}\sigma.$$

Proof. For any S in $H^1(\Omega)$, the theory of monotone operators (*cf.* Lions [17]) leads to the existence and the uniqueness of the solution to the following problem: find $u_{S,\varepsilon,\delta}$ in $H^1(\Omega)$ such that, for any $v \in H^1(\Omega)$,

$$0 = \int_\Omega \left\{ \frac{u_{S,\varepsilon,\delta}-u_0}{h} v + a_\varepsilon\left(\frac{S-u_0}{h}+E_1\right)\nabla u_{S,\varepsilon,\delta}.\nabla v \right\} \mathrm{d}x + \int_\Gamma f_1 v\,\mathrm{d}\sigma - \frac{1}{\delta}\int_{\Gamma_s}\left(\frac{u_{S,\varepsilon,\delta}-u_0}{h}+E_1\right)^- v\,\mathrm{d}\sigma.$$

Considering the application $\Psi : H^1(\Omega) \to H^1(\Omega),\ S \mapsto u_{S,\varepsilon,\delta}$, the fixed-point theorem of Schauder-Tikhonov in the separable hilbertian framework (see Gagneux *et al.* [11] pp. 29–30) leads to the existence of a solution. The uniqueness is based on a L^1- method by using a suitable approximation of the sign function. □

Lemma 2.6. *Considering* (H) *and any* ε, h *in* $]0,1[$, *there exists a unique element* u_ε *in* $H^1(\Omega)$ *such that:* $u_\varepsilon \geq u_0 - hE_1$ *a.e. on* Γ_s *and, for any* $v \in \mathbb{K}$:

$$\begin{aligned} 0 \leq\ & \int_\Omega \frac{u_\varepsilon - u_0}{h}\left(v - \frac{u_\varepsilon-u_0}{h} - E_1\right) v\,\mathrm{d}x \\ & + \int_\Omega a_\varepsilon\left(\frac{u_\varepsilon-u_0}{h}+E_1\right)\nabla u_\varepsilon.\nabla\left(v - \frac{u_\varepsilon-u_0}{h}-E_1\right)\mathrm{d}x + \int_\Gamma f_1\left(v-\frac{u_\varepsilon-u_0}{h}-E_1\right)\mathrm{d}\sigma. \end{aligned} \tag{7}$$

Proof. Passing to the limits with δ is classical (see Lions [17] p. 372 concerning penalization problems) and provides the existence of an element u_ε.

For the uniqueness of the solution, if one denotes by $w_\varepsilon = \frac{u_\varepsilon - u_0}{h} + E_1$, w_ε is a solution to the problem

$$0 \leq \int_\Omega w_\varepsilon(v-w_\varepsilon)v\,\mathrm{d}x + \int_\Omega a_\varepsilon(w_\varepsilon)\nabla u_\varepsilon.\nabla(v-w_\varepsilon)\,\mathrm{d}x + \int_\Gamma f_1(v-w_\varepsilon)\,\mathrm{d}\sigma + \int_\Omega a_\varepsilon(w_\varepsilon)\nabla(u_0-hE_1).\nabla v\,\mathrm{d}x.$$

Since w_ε is unique by using a L^1- method for variational inequalities thanks to a suitable approximation of the sign function, u_ε is unique too. The reader interested by the technical details of the demonstrations would find some information in Antontsev *et al.* [1]. □

In order to pass to the limits with respect to ε, the following *a priori* estimates are needed.

Lemma 2.7. *Considering* (H) *and any* ε, h *in* $]0,1[$, *there exists a positive constant* C *such that* *(i)* $||w_\varepsilon||_{L^2(\Omega)} \leq C$, *(ii)* $\int_\Omega a_\varepsilon(w_\varepsilon)|\nabla w_\varepsilon|^2\,\mathrm{d}x \leq C$, *and* *(iii)* $||w_\varepsilon^+||_{H^1(\Omega)} \leq C$, *where* $w_\varepsilon = \frac{u_\varepsilon-u_0}{h} + E_1$.

Proof. Since $\mathbb{K}$ is a convex cone containing 0, (7) leads to

$$\int_\Omega \{w_\varepsilon^2 + h a_\varepsilon(w_\varepsilon)|\nabla w_\varepsilon|^2\}\,\mathrm{d}x \leq -(I_1+I_2+I_3),$$

$$\text{where}\quad I_1 = \int_\Omega E_1 w_\varepsilon\,\mathrm{d}x,\quad I_2 = \int_\Omega a_\varepsilon(w_\varepsilon)\nabla(u_0-hE_1).\nabla w_\varepsilon\,\mathrm{d}x \quad \textit{and} \quad I_3 = \int_\Gamma f_1 w_\varepsilon\,\mathrm{d}\sigma.$$

Moreover, since $f_1 \geq 0$, $w_\varepsilon \geq 0$ on Γ_s and $f_1 \leq 0$ on Γ_e, embedding theorems and Cauchy's inequality lead to:

$$\begin{aligned} -I_3 &= -\left|\int_{\Gamma_s} f_1 w_\varepsilon\,\mathrm{d}\sigma\right| - \left|\int_{\Gamma_e} f_1 w_\varepsilon^-\,\mathrm{d}\sigma\right| + \int_{\Gamma_e}|f_1|w_\varepsilon^+\,\mathrm{d}\sigma \leq \frac{\mu h}{2}\int_\Omega (w_\varepsilon^{+2} + |\nabla w_\varepsilon^+|^2)\,\mathrm{d}x + \frac{1}{2\mu h}\int_{\Gamma_e} f_1^2\,\mathrm{d}x \\ &\leq \frac{\mu}{2}\int_\Omega (w_\varepsilon^2 + h a_\varepsilon(w_\varepsilon)|\nabla w_\varepsilon|^2)\,\mathrm{d}x + \frac{1}{2\mu}\int_{\Gamma_e} f_1^2\,\mathrm{d}x, \end{aligned}$$

since $\int_\Omega |\nabla w_\varepsilon^+|^2 \,\mathrm{d}x \le \int_\Omega a_\varepsilon(w_\varepsilon)|\nabla w_\varepsilon|^2 \,\mathrm{d}x$ according to (6). Next, using Cauchy's inequality, one gets that:

$$\begin{aligned} |I_1| &\le \frac{\mu}{2}\int_\Omega w_\varepsilon^2 \,\mathrm{d}x + \frac{1}{2\mu}\int_\Omega E_1^2 \,\mathrm{d}x, \\ |I_2| &\le \mu\int_\Omega h a_\varepsilon(w_\varepsilon)|\nabla w_\varepsilon|^2 \,\mathrm{d}x + \frac{1}{2\mu}\int_\Omega h a_\varepsilon(w_\varepsilon)E_1^2 \,\mathrm{d}x + \frac{1}{2\mu}\int_\Omega a_\varepsilon(w_\varepsilon)\frac{|\nabla u_0|^2}{h}\,\mathrm{d}x. \end{aligned}$$

Joining last estimates with suitable choice of μ in $]0,1[$, we come to

$$\int_\Omega \{w_\varepsilon^2 + h a_\varepsilon(w_\varepsilon)|\nabla w_\varepsilon|^2\}\,\mathrm{d}x \le C(\mu)\left(\int_\Omega \left((h a_\varepsilon(w_\varepsilon)+1)E_1^2 + a_\varepsilon(w_\varepsilon)\frac{|\nabla u_0|^2}{h}\right)\mathrm{d}x + \int_{\Gamma_e} f_1^2 \,\mathrm{d}x\right).$$

Under assumption $0 < h < 1$, the last estimate can be rewritten in the form

$$\int_\Omega \{w_\varepsilon^2 + h a_\varepsilon(w_\varepsilon)|\nabla w_\varepsilon|^2\}\,\mathrm{d}x \le C(\frac{1}{h})\left(||E_1||^2_{L^2(\Omega)} + ||u_0||^2_{H^1(\Omega)} + ||f_1||^2_{L^2(\Gamma_e)}\right).$$

Then, one gets points i) and ii), and $(w_\varepsilon^+)_\varepsilon$ is bounded in $H^1(\Omega)$, since $a_\varepsilon = 1$ in $\mathbb{R}^+$. □

Lemma 2.8. *Considering* (H) *and any* ε, h *in* $]0,1[$, *there exists a subsequence, still denoted by* $(w_\varepsilon)_\varepsilon$, *such that, when* ϵ *goes to* 0^+,
(i) $(w_\varepsilon+\varepsilon)^- := \max(0, -w_\varepsilon-\varepsilon)$ *weakly converges towards* 0 *in* $H^1(\Omega)$; *(ii)* w_ε^- *converges towards* 0 *in* $L^2(\Omega)$; *(iii)* w_ε^- *converges towards* 0 *in* $L^2(\Gamma)$.

Proof. Let us set $v = w_\varepsilon + (w_\varepsilon+\varepsilon)^-$ in (7). Then, one has:

$$\begin{aligned} 0 \ \ge\ & -\int_\Omega \{(w_\varepsilon+\varepsilon)(-w_\varepsilon-\varepsilon)^+ - (E+\varepsilon)(-w_\varepsilon-\varepsilon)^+\}\,\mathrm{d}x - \int_\Gamma f_1(-w_\varepsilon-\varepsilon)^+ \,\mathrm{d}\sigma \\ & -\int_\Omega h a_\varepsilon(w_\varepsilon)\nabla w_\varepsilon.\nabla(-w_\varepsilon-\varepsilon)^+ \,\mathrm{d}x - \int_\Omega a_\varepsilon(w_\varepsilon)\nabla(u_0-hE_1).\nabla(-w_\varepsilon-\varepsilon)^+ \,\mathrm{d}x. \end{aligned}$$

Remark that $-\int_{\Omega\cap\{w_\varepsilon\le-\varepsilon\}} a_\varepsilon(w_\varepsilon)\nabla w_\varepsilon.\nabla(-w_\varepsilon-\varepsilon)^+ \,\mathrm{d}x = \varepsilon\int_\Omega |\nabla(w_\varepsilon+\varepsilon)^-|^2 \,\mathrm{d}x$.
Since $E+\varepsilon \ge 0$, a constant $C = C(||\nabla(u_0-hE_1)||_{L^2(\Omega)})$ exists such that

$$\int_\Omega |(w_\varepsilon+\varepsilon)^-|^2 \,\mathrm{d}x + \frac{\varepsilon h}{2}\int_\Omega |\nabla(w_\varepsilon+\varepsilon)^-|^2 \,\mathrm{d}x - \int_\Gamma f_1(-w_\varepsilon-\varepsilon)^+ \,\mathrm{d}\sigma \le C\varepsilon.$$

Since $w_\varepsilon \ge 0$ *a.e.* in Γ_s and $f_1 \le 0$ on Γ_e, it follows that

$$-\int_\Gamma f_1(-w_\varepsilon-\varepsilon)^+ \,\mathrm{d}\sigma = -\int_{\Gamma_e} f_1(-w_\varepsilon-\varepsilon)^+ \,\mathrm{d}\sigma \ge 0.$$

So, one gets that

$$\int_\Omega |(w_\varepsilon+\varepsilon)^-|^2 \,\mathrm{d}x + \varepsilon h\int_\Omega |\nabla(w_\varepsilon+\varepsilon)^-|^2 \,\mathrm{d}x \le C\varepsilon.$$

Thus, $(w_\varepsilon+\varepsilon)^-$ is bounded in $H^1(\Omega)$, and when ε goes to 0^+, $(w_\varepsilon+\varepsilon)^-$ converges towards 0 in $L^2(\Omega)$.
So $(w_\varepsilon)^-$ converges towards 0 in $L^2(\Omega)$ and $(w_\varepsilon+\varepsilon)^-$ converges weakly towards 0 in $H^1(\Omega)$. Therefore, $(w_\varepsilon+\varepsilon)^-$ converges strongly towards 0 in $H^s(\Omega)$ for any given $\frac{1}{2} < s < 1$, and $(w_\varepsilon+\varepsilon)^-$ converges towards 0 in $L^2(\Gamma)$ as well as $(w_\varepsilon)^-$. □

Lemma 2.9. *Considering* (H) *and* h *in* $]0,1[$, *there exists* u *in* $H^1(\Omega)$ *and a subsequence, still denoted by* $(w_\varepsilon)_\varepsilon$, *such that when* ε *goes towards* 0^+,

$$A_\varepsilon\left(\frac{u_\varepsilon-u_0}{h}+E_1\right) \quad \text{converges weakly towards} \quad \left(\frac{u-u_0}{h}+E_1\right)^+ \text{ in } H^1(\Omega) \text{ and}$$
$$G_\varepsilon\left(\frac{u_\varepsilon-u_0}{h}+E_1\right) \quad \text{converges weakly towards} \quad \left(\frac{u-u_0}{h}+E_1\right)^+ \text{ in } H^1(\Omega),$$

where $A_\varepsilon(x)=\int_0^x a_\varepsilon(s)\mathrm{d}s$ *and* $G_\varepsilon(x)=\int_0^x\sqrt{a_\varepsilon(s)}\mathrm{d}s$.

Proof. Thanks to lemma 2.7, up to a subsequence still indexed by ε, w_ε^+ converges, weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ towards w with $w\geq 0$ *a.e.* in Ω. Moreover, w_ε^- converges strongly towards 0 in $L^2(\Omega)$, so w_ε converges strongly towards w in $L^2(\Omega)$. Let us denote $u=u_0+h(w-E_1)$. Then, $u\in H^1(\Omega)$ and the lemma is obtained by computation and thanks to the previous *a priori* estimates. □

Lemma 2.10. *Considering* (H), h *in* $]0,1[$ *and* u *given by the previous lemma, there exists* λ_1 *in* $L^\infty(\Omega)$ *and a subsequence still denoted by* $(u_\varepsilon)_\varepsilon$, *such that* $a_\varepsilon(\frac{u_\varepsilon-u_0}{h}+E_1)$ *converges weakly towards* λ_1 *in* $L^\infty(Q)$ *weak*$-*$ *when* ε *goes to* 0^+, *with* $\lambda_1\in H(\frac{u-u_0}{h}+E_1)$.

Proof. Since $\frac{u_\varepsilon-u_0}{h}+E_1$ converges towards $\frac{u-u_0}{h}+E_1$ in $L^2(\Omega)$ with ε towards 0^+, it converges *a.e.* in Ω for a subsequence. Moreover $0\leq a_\varepsilon\leq 1$, so $a_\varepsilon(\frac{u_\varepsilon-u_0}{h}+E_1)$ converges weakly towards λ_1 in $L^\infty(\Omega)$ weak$-*$ with $0\leq\lambda_1\leq 1$ *a.e.* in Ω.
Then, by construction, denoting by

$$A=\left\{x\in\Omega\ /\ \left(\frac{u_\varepsilon-u_0}{h}+E_1\right)(x)\to\left(\frac{u-u_0}{h}+E_1\right)(x)\text{ and }\left(\frac{u-u_0}{h}+E_1\right)(x)\neq 0\right\},$$
$$\text{and}\qquad A^+=\left\{x\in A\ /\ \frac{u-u_0}{h}+E_1>0\right\},\quad A^-=\left\{x\in A\ /\ \frac{u-u_0}{h}+E_1<0\right\},$$

$a_\varepsilon(\frac{u_\varepsilon-u_0}{h}+E_1)$ converges *a.e.* in A towards $sign_0^+(\frac{u-u_0}{h}+E_1)$ and $\lambda_1\in H(\frac{u-u_0}{h}+E_1)$. □

Lemma 2.11. *Considering* (H), h *in* $]0,1[$, u *and* λ_1 *given by the previous lemma and a subsequence still denoted by* $(u_\varepsilon)_\varepsilon$, *one has:*

$$\limsup_{\varepsilon\to 0^+}\int_\Omega a_\varepsilon\left(\frac{u_\varepsilon-u_0}{h}+E_1\right)\nabla u_\varepsilon.\nabla\left(v-\frac{u_\varepsilon-u_0}{h}-E_1\right)\mathrm{d}x\leq\int_\Omega\lambda_1\nabla u.\nabla\left(v-\frac{u-u_0}{h}-E_1\right)\mathrm{d}x.$$

Proof. One writes that:

$$\begin{aligned}
I_\varepsilon &= \int_\Omega a_\varepsilon\left(\frac{u_\varepsilon-u_0}{h}+E_1\right)\nabla u_\varepsilon.\nabla\left(v-\frac{u_\varepsilon-u_0}{h}-E_1\right)\mathrm{d}x\\
&= h\int_\Omega a_\varepsilon(w_\varepsilon)\nabla w_\varepsilon.\nabla(v-w_\varepsilon)\,\mathrm{d}x+\int_\Omega a_\varepsilon(w_\varepsilon)\nabla(u_0-hE_1).\nabla(v-w_\varepsilon)\,\mathrm{d}x\\
&= h\int_\Omega\nabla A_\varepsilon(w_\varepsilon).\nabla v\,\mathrm{d}x-h\int_\Omega|\nabla G_\varepsilon(w_\varepsilon)|^2\,\mathrm{d}x+\int_\Omega a_\varepsilon(w_\varepsilon)\nabla(u_0-hE_1).\nabla(v-w_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

Thus, passing to the limits gives:

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} I_\varepsilon \leq\ & h\int_\Omega\nabla\left(\frac{u-u_0}{h}+E_1\right)^+.\nabla v\,\mathrm{d}x-h\int_\Omega|\nabla\left(\frac{u-u_0}{h}+E_1\right)^+|^2\,\mathrm{d}x\\
&+\int_\Omega\nabla(u_0-hE_1).\nabla\left(\frac{u-u_0}{h}+E_1\right)^+\mathrm{d}x+\int_\Omega\lambda_1\nabla(u_0-hE_1).\nabla v\,\mathrm{d}x.
\end{aligned}$$

and the result holds since, thanks to the lemma of Saks, $\nabla(\frac{u-u_0}{h}+E_1)^+ = \lambda_1\nabla(\frac{u-u_0}{h}+E_1)$ *a.e.* □

All these lemmata lead us to the following proposition:

Proposition 2.12. *Considering* (H) *and* h *in* $]0,1[$, *there exists a couple* (λ_1, u^1) *such that:*

$$u^1 \in H^1(\Omega),\ \frac{u^1-u_0}{h}+E_1 \geq 0 \quad a.e.\ in\ \Omega\ and\ on\ \Gamma_s, \quad \lambda_1 \in L^\infty(\Omega),\ \lambda_1 \in H\left(\frac{u^1-u_0}{h}+E_1\right)$$

which satisfies, for any $v \in \mathbb{K}$*:*

$$0 \leq \int_\Omega \frac{u^1-u_0}{h}\left(v-\frac{u^1-u_0}{h}-E_1\right)\mathrm{d}x + \int_\Omega \lambda_1\nabla u^1.\nabla\left(v-\frac{u^1-u_0}{h}-E_1\right)\mathrm{d}x + \int_\Gamma f_1.\left(v-\frac{u^1-u_0}{h}-E_1\right)\mathrm{d}\sigma.$$

By induction of this proposition, one is then able to conclude:

Theorem 2.13. *Let us consider a positive integer* N *such that* $h=\frac{T}{N}<1$, $\forall k=1,..,N$, $E_k \in H^1(\Omega)$ *with* $E_k \geq 0$, $f_k \in L^2(\Gamma)$ *with* $f_k \leq 0$ *on* Γ_e *and* $f_k \geq 0$ *on* Γ_s *and* $u_0 \in H^1(\Omega)$. *Then, there exists a sequence* $\{(\lambda_k,u^k)\}_k$ *such that:*

$$\lambda_k \in L^\infty(\Omega),\ \lambda_k \in H\left(\frac{u^k-u^{k-1}}{h}+E_k\right),$$
$$u^0=u_0,\ u^k \in H^1(\Omega),\ \frac{u^k-u^{k-1}}{h}+E_k \geq 0\ a.e.\ in\ \Omega\ and\ on\ \Gamma_s$$

$$\text{and, } \forall v \in \mathbb{K}, \quad 0 \leq \int_\Omega \frac{u^k-u^{k-1}}{h}\left(v-\frac{u^k-u^{k-1}}{h}-E_k\right)\mathrm{d}x + \int_\Omega \lambda_k\nabla u^k.\nabla\left(v-\frac{u^k-u^{k-1}}{h}-E_k\right)\mathrm{d}x + \int_\Gamma f_k.\left(v-\frac{u^k-u^{k-1}}{h}-E_k\right)\mathrm{d}\sigma.$$

Let us give now some *a priori* estimates that should allow us to pass to limits with h towards 0^+.

Lemma 2.14. *Considering the previous theorem hypothesis with* $f_e=0$ *and* $h<\frac{1}{2}$, *for any integer* n, *one has:*

$$\frac{1}{h}\sum_{k=1}^{n}||u^k-u^{k-1}||^2_{L^2(\Omega)} + ||u^n||^2_{H^1(\Omega)} + \sum_{k=1}^{n}||u^k-u^{k-1}||^2_{H^1(\Omega)} \leq (1+2e^2)\left[||\nabla u_0||^2_{(L^2(\Omega))^d} + h\sum_{k=1}^{N}||E_k||^2_{H^1(\Omega)}\right].$$

Proof. Let us denote $w_k=\frac{u^k-u^{k-1}}{h}+E_k$. Since $v=2w_k$ and $v=0$ are available test-functions, one gets:

$$0=\int_\Omega\{(w_k-E_k)w_k+\lambda^k\nabla u^k.\nabla w_k\}\,\mathrm{d}x+\int_\Gamma f_kw_k\,\mathrm{d}\sigma.$$

Since $\lambda^k \in H(w_k)$ with $w_k \geq 0$, one has $\lambda^k\nabla u^k.\nabla w_k = \nabla u^k.\nabla {w_k}^+ = \nabla u^k.\nabla w_k$ and one gets

$$\int_\Omega |w_k|^2\,\mathrm{d}x - \int_\Omega E_kw_k\,\mathrm{d}x + \frac{1}{2h}\int_\Omega[|\nabla u^k|+|\nabla(u^k-u^{k-1})|^2-|\nabla u^{k-1}|^2]\,\mathrm{d}x = -\int_\Gamma f_kw_k\,\mathrm{d}\sigma - \int_\Omega \nabla u^k.\nabla E_k\,\mathrm{d}x.$$

Since $\int_\Gamma f_k w_k \, d\sigma = \int_{\Gamma_s} f_k w_k \, d\sigma \geq 0$, this leads, thanks to some Young's inequalities, to

$$\frac{1}{h}\sum_{k=1}^{n} ||u^k - u^{k-1}||^2_{L^2(\Omega)} + ||\nabla u^n||^2_{(L^2(\Omega))^d} + \sum_{k=1}^{n} ||\nabla(u^k - u^{k-1})||^2_{(L^2(\Omega))^d}$$
$$\leq ||\nabla u_0||^2_{(L^2(\Omega))^d} + h\sum_{k=1}^{n} ||\nabla u^k||^2_{(L^2(\Omega))^d} + h\sum_{k=1}^{n} ||E_k||^2_{H^1(\Omega)}.$$

Thanks to the discrete Gronwall lemma (see Lions [16]), one gets ($h < \frac{1}{2}$) finally

$$||\nabla u^n||^2_{(L^2(\Omega))^d} \leq 2e^2[||\nabla u_0||^2_{(L^2(\Omega))^d} + h\sum_{k=1}^{N} ||E_k||^2_{H^1(\Omega)}]. \qquad \square$$

The previous lemma leads us to the following proposition:

Proposition 2.15. *Considering the hypothesis of the previous theorem with $f_e = 0$ and $h < \frac{1}{2}$, if one denotes*

$$E_h(x) = \sum_{k=0}^{k=N-1} E_{k+1}.1_{[kh,(k+1)h[}, \quad f_h(x) = \sum_{k=0}^{k=N-1} f_{k+1}.1_{[kh,(k+1)h[},$$

$$\hat{u}_h(x) = \sum_{k=0}^{k=N-1} \left[\frac{u^{k+1} - u^k}{h}(t - kh) + u^k\right].1_{[kh,(k+1)h[} \quad \textit{and} \quad \lambda_h(x) = \sum_{k=0}^{k=N-1} \lambda^{k+1}.1_{[kh,(k+1)h[},$$

then, one gets, for any v in $L^2(0,T;\mathbb{K})$,

$$0 \leq \int_\Omega \partial_t \hat{u}_h(v - \partial_t \hat{u}_h - E_h) \, dx + \int_\Omega \lambda_h \nabla u_h . \nabla(v - \partial_t \hat{u}_h - E_h) \, dx + \int_\Gamma f_h(v - \partial_t \hat{u}_h - E_h) \, d\sigma.$$

The sequence $(\hat{u}_h)$ is bounded in $H^1(Q) \cap L^\infty(0,T;H^1(\Omega))$. Thus, it is relatively compact in $C([0,T];L^2(\Omega))$ and a mild solution exists in the sense of Bénilan et al. [5].

Remark 2.16. Unfortunately, passing to the limits in the above inequality with respect to h is still an open problem, mainly for two reasons:

(i) the presence of two weak convergences does not allow us to pass to the limits in the term $\int_\Omega \lambda_h \nabla \hat{u}_h . \nabla v \, dx$ when $h \to 0^+$, without some additional properties (see Gagneux *et al.* [10] for information about that),

(ii) one needs a priori estimates for $\partial_t \hat{u}_h$ in $L^2(0,T;H^1(\Omega))$ too, in order to consider the condition on Γ_s.

2.4. Explicit solution in the $1 - D$ case

By considering particular assumptions, let us overcome in the $1-D$ case the problems of the previous remark. In this section, $\Omega =]0,1[$, $\Gamma_e = \{0\}$, $\Gamma_s = \{1\}$ and f_s, $-f_e$ and E are assumed to be non-negative constants. One is interested in the following problem: find $u \in H^1(0,1)$ and $\lambda \in H(\frac{u-u_0}{h} + E) \cap L^\infty(0,1)$ such that

$$u - u_0 - h(\lambda u')' = 0 \quad \text{in} \quad]0,1[, \tag{8}$$

$$\lambda u'(0) = f_e < 0, \tag{9}$$

$$u(1) \geq u_0(1) - hE, \ \lambda u'(1) + f_s \geq 0 \quad \text{and} \quad (u(1) - u_0(1) + hE)(\lambda u'(1) + f_s) = 0. \tag{10}$$

Note that moreover, $\frac{u-u_0}{h} + E \geq 0$ and then, one gets that:

Proposition 2.17. *If u is any solution to the above problem, then $\varphi : x \mapsto \lambda u'(x) + Ex$ is a non-decreasing continuous function. Moreover, $\varphi(0) = f_e$ and $\varphi(1) \geq E - f_s$.*

In order to know *a priori* if the constraint on the edge Γ_s is carried out by $u(1)-u_0(1)+hE$ or by $\lambda u'(1)+f_s$, let us consider the following remark.

Proposition 2.18. *If $f_e+f_s \geq E$, then condition (10) is equivalent to the Dirichlet one*

$$u(1) = u_0(1) - hE. \tag{11}$$

Proof. Let us consider a solution u to (8-9).
Then, by multiplying (8) by $v = (\frac{u-u_0}{h}+E)^-$ and using the condition (9) one gets

$$|v|^2 \leq -(\lambda u')(1)v(1).$$

On each case, $u(1) \geq u_0(1) - hE$, therefore $v(1) = 0$ and $\frac{u-u_0}{h}+E \geq 0$.
Thus, thanks to equation (8), $\varphi : x \mapsto \lambda u'(x) + Ex$ is a non-decreasing continuous function and one gets

$$0 \leq f_e + f_s - E = \varphi(0) + f_s - E \leq \varphi(1) - E + f_s = \lambda u'(1) + f_s,$$

i.e. $\lambda u'(1) + f_s \geq 0$ is always satisfied.
If u satisfies (8-9-10):
on the one hand, if $u = u_0 - hE$ then (11) holds obviously,
on the other hand, if $u \neq u_0 - hE$, φ has to be increasing at least in an non-empty interval and one gets

$$0 \leq \varphi(0) + f_s - E < \varphi(1) - E + f_s = \lambda u'(1) + f_s.$$

Therefore (10) becomes: $u(1) = u_0(1) - hE$. At last, if u is a solution to (8-9-11), (10) is obvious. □

In the sequel, $f_e + f_s \geq E$ is assumed. This scenario corresponds to a basin that empties too quickly by the boundary Γ_s. The boundary condition is then given by the constraint and the studied time-discretized problem is: find $u \in H^1(0,1)$ and $\lambda \in H(\frac{u-u_0}{h}+E) \cap L^\infty(0,1)$ such that

$$u - u_0 - h(\lambda u')' = 0 \text{ in }]0,1[, \quad \lambda u'(0) = f_e < 0 \quad \text{and} \quad u(1) = u_0(1) - hE. \tag{12}$$

Moreover, one would consider initial conditions that satisfy:

$$\psi : x \mapsto u_0(x) + \frac{E}{2}x^2 \quad \text{is a convex function.} \tag{13}$$

Note that this hypothesis is rather natural since if $\lambda = 1$ in $]a,b[$, the constraint $\frac{u-u_0}{h}+E \geq 0$ implies that $x \mapsto u(x) + \frac{E}{2}x^2$ is a convex function in $[a,b]$. Moreover, $u_0' \in BV(0,1)$ and one denotes by: $u_0'(0) = u_0'(0^+)$, $u_0'(1) = u_0'(1^-)$ and one assumes that for any x, $u_0'(x) = \frac{u_0'(x^+)+u_0'(x^-)}{2}$.

One considers these assumptions mainly to be able to construct easily explicit solutions to problem (8-9-11) and confront them with the numerical results obtained in the same framework and presented in Section 2.5.

Then, numerical results of problem (8-9-10) would be presented in a more general framework.

First of all, let us consider the case when $(\lambda, u_0 - hE)$ is a solution with $\lambda \in [0,1]$.

2.4.1. *If a trivial solution exists*

One assumes in this section that there exists $\bar\lambda$ in $[0,1]$ such that $(\bar\lambda, u_0 - hE)$ is a solution to problem (12).

Lemma 2.19. *If $f_e = 0$ and $E = 0$, the trivial couple $(\bar\lambda = 0, u_0)$ is always a solution.*
If $f_e \neq 0$ or $E \neq 0$, then $\bar\lambda > 0$ and $u_0'(x) = \frac{f_e - Ex}{\bar\lambda}$.

Proof. The result comes from the fact that $\overline{\lambda} u_0'(x) = f_e - Ex$ since $(\bar\lambda u_0')' = -E$ with $\bar\lambda u_0'(0) = f_e$. □

First case $f_e = 0$ and $E = 0$.

Proposition 2.20. *Assume that $f_e = 0$ and $E = 0$, then:*
(i) the trivial couple $(0, u_0)$ is always a solution;
(ii) if $u_0'(0) \geq 0$, then a solution to (12) is given by the couple $(1, w)$ where w is the solution to the problem

$$w - u_0 - hw'' = 0 \quad \textit{in }]0,1[\quad \textit{with} \quad w'(0) = 0 \quad \textit{and} \quad w(1) = u_0(1);$$

(iii) if $u_0'(0) < 0$ and $u_0'(1) \leq 0$, for any solution (λ, u) one has $u = u_0$ and $0 \leq \lambda \leq 1_{\{u_0'=0\}}$;
(iv) if $u_0'(0) < 0$ and $u_0'(1) > 0$, there exists x_0 in $]0,1[$ such that a solution to (12) is given by

$$u = u_0 \quad \textit{with} \quad \lambda = 0 \quad \textit{in }]0, x_0[\quad \textit{and} \quad u = w \quad \textit{with} \quad \lambda = 1 \quad \textit{in }]x_0, 1[$$

where w is the solution to: $w - u_0 - hw'' = 0$ in $]x_0, 1[$, $\quad w'(x_0) = 0, \quad w(x_0) = u_0(x_0) \quad$ and $\quad w(1) = u_0(1)$.

Proof. As (i) is obvious, let us start by:
Claim 1. $u_0'(0) \geq 0$.
Since $E = 0$, u_0 is assumed to be a convex function, thus this condition and the maximum principle lead to $w \geq u_0$. Conclusion: the solution given by $\lambda = 1$ is compatible with the constraint.
Claim 2. $u_0'(0) < 0$.
. If $u_0'(1) \leq 0$. For any non-trivial solution u, one gets: $0 = \varphi(0) < \varphi(1) = \lambda u'(1)$.
Since $u \geq u_0$ and $u(1) = u_0(1)$, one notes that $u'(1) \leq u_0'(1)$ and a contradiction is obtained.
Thus, the trivial solution (*i.e.* $u = u_0$) is the unique solution to the problem.
. If $u_0'(1) > 0$. Let us consider α in $]0,1[$ such that $u_0'(\alpha) \geq 0$ and $u_0'(x) < 0$ if $x < \alpha$.
Then, w is given by $w(x) = u_0(x) - \int_{x_0}^{x} u_0'(y)\,\mathrm{ch}(\frac{y-x}{\sqrt{h}})\,\mathrm{d}y$, where x_0 is the unique point in $]0,\alpha[$, defined by: $\int_{x_0}^{1} u_0'(y)\,\mathrm{ch}(\frac{y-1}{\sqrt{h}})\,\mathrm{d}y = 0$ (see Antontsev *et al.* [1] for some technical details). □

Remark 2.21. *A priori,* w is the solution to an ill-posed elliptic problem since three boundary conditions are considered. This objection disappears since the free boundary x_0 is then characterised by this third condition.

Second case $f_e < 0$ or $E > 0$.

Proposition 2.22. *In that case, only the trivial solution (i.e. $u = u_0 - hE$) exists.*

Proof. Let us consider a non-trivial solution (λ, u)(*i.e.* $u \neq u_0 - hE$). Since these solutions are continuous functions, two real numbers a and b exist verifying $0 \leq a < b \leq 1$ and $\{x \;/\; u > u_0 - hE\} \supset]a,b[$.

Since φ is an increasing function in $]a,b[$, $\lambda = 1$ in $]a,b[$ and one gets that $f_e = \varphi(0) < \varphi(b) = u'(b) + bE$. Moreover, since $u(1) = u_0(1) - hE$, one has $u(b) = u_0(b) - hE$ and $u \geq u_0 - hE$. Thus, $u'(b) \leq u_0'(b)$ and

$$f_e < u_0'(b) + bE = \frac{f_e - bE}{\bar\lambda(b)} + bE, \quad \text{so} \quad f_e(1 - \frac{1}{\bar\lambda(b)}) < E\left(1 - \frac{1}{\bar\lambda(b)}\right),$$

which is not possible since $\bar\lambda(b) \in]0,1]$, $f_e < 0$ and $E > 0$. □

2.4.2. *Non-trivial solutions*

Remark that if $u_0'(1) + E \leq f_e$, thanks to the convex hypothesis (13), for any x in $]0,1[$, $u_0'(x) + Ex \leq f_e < 0$ and $\overline{\lambda}(x) := \frac{f_e - Ex}{u_0'(x)} \in [0,1]$. Therefore, $(\overline{\lambda}, u_0 - Eh)$ is a solution and one finds again the case of the previous section. The reader is invited to look at Figure 1 for an illustration of this remark.

Thus, let us consider in this section that $\exists \overline{b} \in [0,1],\ u_0'(\overline{b}) + \overline{b}E > f_e$, in particular, thanks to the hypothesis on ψ, for any b greater than $\overline{b}$ one has the same inequality.

First case.

Proposition 2.23. *If $u_0'(0) \geq f_e$ then a solution to (12) exists with $\lambda = 1$.*

Proof. In that case, the maximum principle proves that the solution given by $\lambda = 1$ is compatible with the constraint and of course it is the maximal one in the sense of Remark 2.3. $\square$

Remark 2.24. The reader is invited to see Figure 5 for an illustration of this case.

Second case. In that case, *i.e.* $u_0'(0) < f_e$, finding a solution (λ, u) with $\lambda = 1$ on $[0,1]$ is irrelevant. Such a remark leads from the following lemma.

Lemma 2.25. *No solution exists such that $\lambda = 1$ in an interval $]0, \varepsilon[$ for any given $\varepsilon > 0$.*

Proof. Let us assume that there exists a solution (λ, u) such that $\lambda = 1$ in $]0, x_0[$ for a given x_0 in $]0,1]$ such that $u(x_0) = u_0(x_0) - hE$. Therefore, for any x in $]0, x_0[$,

$$u(x) = A \operatorname{ch}\left(\frac{x}{\sqrt{h}}\right) + B \operatorname{sh}\left(\frac{x}{\sqrt{h}}\right) - u_0(x) - \int_0^x \operatorname{ch}\left(\frac{y-x}{\sqrt{h}}\right) u_0'(y)\,\mathrm{d}y,$$

$$\text{where:}\qquad A = -f_e\sqrt{h}\operatorname{th}\left(\frac{x_0}{\sqrt{h}}\right) + \left[\operatorname{ch}\left(\frac{x_0}{\sqrt{h}}\right)\right]^{-1} \int_0^{x_0} \operatorname{ch}\left(\frac{y-x_0}{\sqrt{h}}\right) u_0'(y)\,\mathrm{d}y \quad \text{and} \quad B = -f_e\sqrt{h}.$$

Then, it can be proved (see Antontsev *et al.* [1] for technical details), for h small enough, that $\frac{u-u_0}{h} + E < 0$ on an interval $[0, \varepsilon]$ and a contradiction is found. $\square$

Thus, according to this result, let us look for a solution such that $\lambda = 1$ on an interval of the form $[a, 1]$.

Proposition 2.26. *For a suitable a in $]0,1[$, a solution w to the problem*

$$w - hw'' = u_0 \text{ on }]a,1[\quad \text{with} \quad w(a) = u_0(a) - hE, \quad w(1) = u_0(1) - hE \quad \text{and} \quad w'(a) + Ea = f_e,$$

exists with the additional property for any x in $]0, a[$: $\lambda(x) := \frac{f_e - Ex}{u_0'(x)} \in [0,1]$.

In particular, a solution to the problem (12) exists.

Proof. Let us consider $w(x) = A\operatorname{ch}(\frac{x}{\sqrt{h}}) + B\operatorname{sh}(\frac{x}{\sqrt{h}}) - u_0(x) - \int_a^x \operatorname{ch}(\frac{y-x}{\sqrt{h}})u_0'(y)\,\mathrm{d}y$, so that A and B provide the solution to the Dirichlet problem:

$$w - hw'' = u_0 \text{ on }]a,1[\quad \text{with} \quad w(a) = u_0(a) - hE \quad \text{and} \quad w(1) = u_0(1) - hE.$$

Since $u_0'' + E$ is a non-negative measure and $u_0'(1) + E > f_e$, a in $]0,1[$ exists such that $w'(a) + Ea = f_e$ (see Antontsev *et al.* [1]) and the Neumann boundary condition is satisfied.

For the additional property, note that $w'(a) \geq u_0'(a)$ with $w'(a) + Ea = f_e$. So $u_0'(a) + Ea \leq f_e < 0$ and, as $x \mapsto u_0'(x) + Ex$ is an non-decreasing function, for $x \leq a$, one has $u_0'(x) + Ex \leq f_e$, *i.e.* $u_0'(x) \leq f_e - Ex < 0$. And, for the last part of the proposition, note that by construction, this solution is (λ, u) such that

$$\lambda = \frac{f_e - Ex}{u_0'(x)} \quad \text{with} \quad u = u_0 - hE \quad \text{in }]0,a[\quad \textit{and} \quad \lambda = 1 \quad \text{with} \quad \lambda = w \quad \textit{else}. \qquad \square$$

Remark 2.27. The previous result is a way to construct a maximal solution in the sense of Remark 2.3. But, in some particular cases, one may find in Antontsev *et al.* [1] a way to construct several solutions to problem (12).

One has been able to build a non-trivial solution u^1 with $\lambda^1 = 1$ in $]a_1, 1[$ where a_1 is characterized by the condition $u'(a_1) + Ea_1 = f_e$. Note that $u'(a_1) \geq u_0'(a_1)$ and thus $u_0'(a_1) + Ea_1 \leq f_e$.
Therefore, in the construction of (λ^2, u_2) in the same way, a_2 can be chosen less then a_1. By induction, a non-increasing sequence $(a_k)_k$ may be constructed such that:

$$\lambda^k(x) = \mu^k 1_{[0,a_k[} + 1_{]a_k,1]} \quad \text{and} \quad u^k = (u_0 - khE)1_{]0,a_k]} + w^k 1_{]a_k,1]} \quad \text{where} \quad \mu^k = \frac{f_e - Ex}{u'_{k-1}(x)}$$

and w^k is the solution to:

$$w^k - h(w^k)'' = u_0 \text{ in }]a_k, 1] \quad \text{with} \quad w^k(a_k) = u_0(a_k) - khE, \quad w^k(1) = u_0(1) - khE \quad \text{and} \quad (w^k)'(a_k) = f_e.$$

Therefore, considering the notation of Property 2.15, $\lambda_h = 1_{\omega_h} + \lambda_h(1 - 1_{\omega_h})$ where ω_h is a set of uniform bounded perimeter. Thus, passing to the limits is possible in the formulation, for any v in $H^1(0,1)$ with $v(1) = 0$,

$$0 = \int_\Omega \{\partial_t \hat{u}_h v + \lambda_h \nabla u_h.\nabla v\} \, \mathrm{d}x + \int_{\Gamma_e} f_e v \, \mathrm{d}\sigma$$

since u_h is known in the set where λ_h is not equal to 1 (see Gagneux *et al.* [10]).

Remark 2.28. The reader is invited to consult the simulations given in Figures 2–4.

2.5. $1-D$ numerical simulations

Obviously, even in the $1-D$ case, the numerical discretization of the equation defined by (1) to (5) presents lots of difficulties. The most important are the calculus of λ and the unilateral constraint. We present a relatively simple algorithm which allows us to illustrate some theoretical results and to present some interesting simulations.

In order to solve the equation defined by (1) to (5), we discretize the time derivative by using an implicit Euler scheme and the space derivative by using a P_1-conform finite element method. The function λ is approached by constants by piece.

Define $[0,1] = \cup_{i=0}^{n-1} [x_i, x_{i+1}]$, W the space of hat-functions and $u^k(x)$ an approximation of $u(kh, x)$ in W.

The variational formulation of the semi-discretized problem is the following:

$$\left|\begin{array}{l} \text{find } u^{k+1} \text{ in } W \text{ satisfying }, \forall v_j \in W \subset H^1(0,1), \\ \displaystyle\int_\Omega \frac{u^{k+1} - u^k}{h} v_j \mathrm{d}x + \int_\Omega \lambda(u^{k+1}, u^k) u^{k+1'} v_j' \mathrm{d}x + f_e v_j(0) = 0, \\ \lambda u^{k+1\prime}(0) = f_e, \\ \lambda u^{k+1\prime}(1) + f_s \geq 0_e, u^{k+1}(1) - u^k(1) + hE \geq 0 \text{ and } (\lambda u^{k+1\prime}(1) + f_s)(u^{k+1}(1) - u^k(1) + hE) = 0. \end{array}\right.$$

At each time step, a non-linear equation must be solve and a fixed-point algorithm is used. The base of this algorithm is to determine the value of λ and the boundary condition at $x = 1$ from the value of u at the previous iteration.

$$\left|\begin{array}{l} \text{For } l = 1, 2, ..., \ \forall v_j \in W, \\ \displaystyle\int_\Omega \frac{u^{k+1,l+1} - u^k}{h} v_j \mathrm{d}x + \int_\Omega \lambda(u^{k+1,l}, u^k) u^{k+1,l+1'} v_j' \mathrm{d}x + f_e v_j(0) = 0, \\ + \text{ boundary conditions}, \\ \text{with } u^{k+1,0} = u^k. \end{array}\right.$$

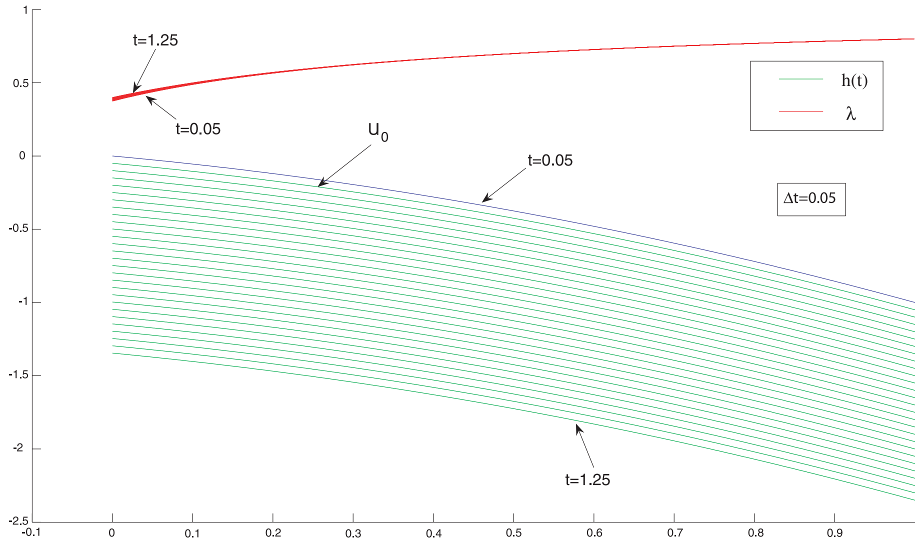

FIGURE 1. $E = 1$, $f_e = -0.2$ and $u_0'(1) = -1.5$.

Supposed given $\lambda(u^{k+1,l}, u^k)$ and the boundary condition at $x = 1$, then $u^{k+1,l+1}$ is solution of a linear system and the process is stopped when the series converges.

Let us denote the discrete operators:

$$D_t u_j = \frac{u_j^{k+1,l} - u_j^k}{h}, \quad D_{x,0} u_j = \frac{u_j^k - u_{j-1}^k}{x_j - x_{j-1}}, \quad D_{x,1} u_j = \frac{u_j^{k+1,l} - u_{j+1}^{k+1,l}}{x_j - x_{j-1}}.$$

To construct $\lambda(u^{k+1,l}, u^k)$, Proposition 2.17 is taken into account (continuity of φ). So we have the following algorithm:

- $\lambda_j = 0$ $(1 \leq j \leq n)$
- if $D_t u_{j-1} + E > 0$ and $D_t u_j + E > 0$ then $\lambda_j = 1$. We denote by $I = \cup_{i=1}^m \left[x_{p_i^1}, x_{p_i^2}\right]$ the set where $\lambda = 0$.
  - for $j \in \left[p_1^1, p_1^2\right], \lambda_j = \dfrac{f_e + Ex_j}{D_{x,0} u_j}$,
  - for $i \in [2, m]$, for $j \in \left[p_i^1, p_i^2\right]$ $\lambda_j = \dfrac{D_{x,1} u_{p_i^1} + Ex_j}{D_{x,0} u_j}$.
- if $\lambda_j > 1$ then $\lambda_j = 1$.

At each step of the process, the unilateral constraint at $x = 1$ can give either a Dirichlet or a Neumann condition. To determine it, we propose the following simple algorithm:

- if the boundary condition is of Dirichlet type at step $k+1, l$ and $\lambda(u^{k+1,l}, u^k) D_{x,0} u_n + f_s < 0$, then at step $k+1, l+1$, we impose $\lambda(u^{k+1,l}, u^k) u^{k+1,l+1\prime}(1) + f_s = 0$.
- if the boundary condition is of Neumann type at step $k+1, l$ and $u^{k+1,l}(1) - u^k(1) + hE < 0$, then at step $k+1, l+1$, we impose $u^{k+1,l+1}(1) - u^k(1) + hE = 0$.

Then, the space of discretization W is modified to take into account the type of boundary condition.

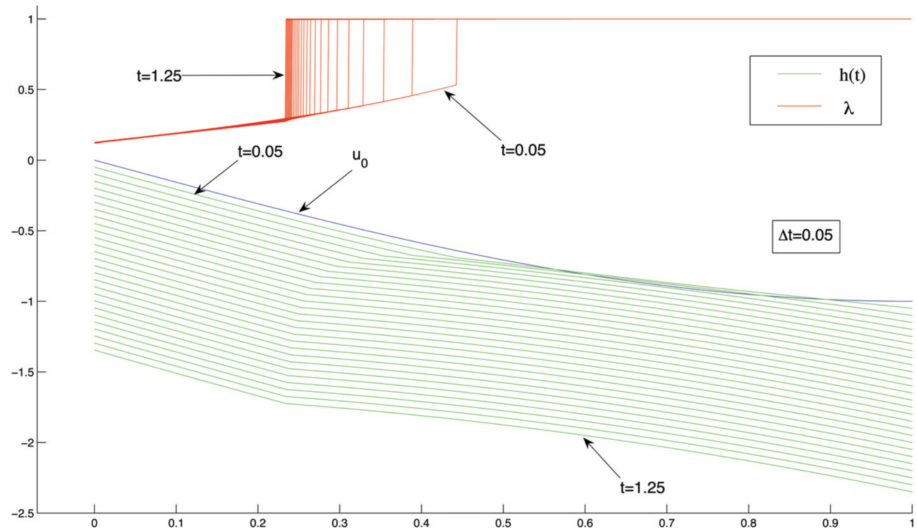

FIGURE 2. $E = 1$, $f_e = -0.2$ and $u_0'(0) = -\frac{\pi}{2}$.

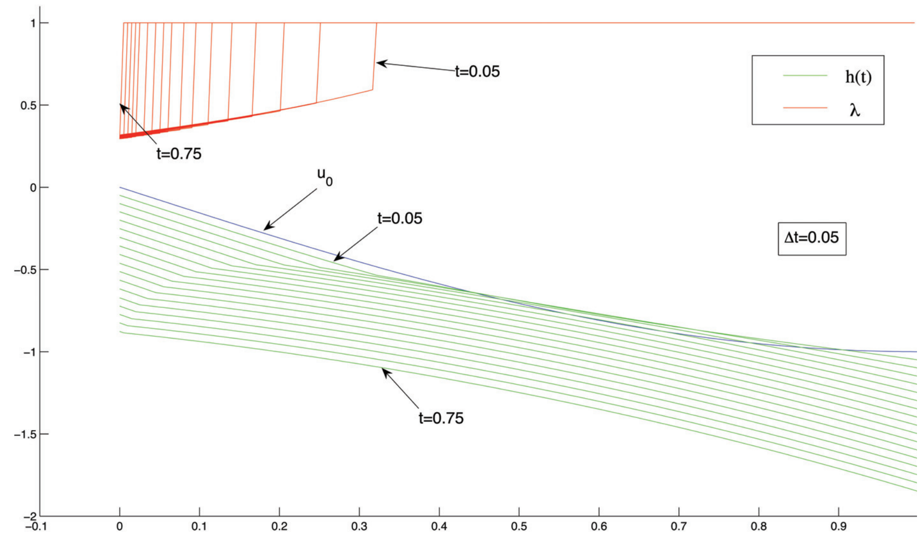

FIGURE 3. $E = 1$, $f_e = -0.5$ and $u_0'(0) = -\frac{\pi}{2}$.

Remark 2.29.

- There are no theoretical results about the convergence of the fixed-point.
- At each step of the fixed-point, the finite element matrix must be assembled again. If an explicit time scheme is used, the fixed-point does not converge.

In Figures 1 to 5, one presents numerical simulations of phenomena described in Section 2.4 when $f_e+f_s \geq E$. As expected, at each iteration, the numerical process chooses the Dirichlet boundary condition at $x = 1$.

Figures 6 to 11 present numerical tests obtained for two initial conditions u_0 and different values of (E, f_e, f_s). With these values, the unilateral boundary condition is not predetermined as previously. Then, the numerical

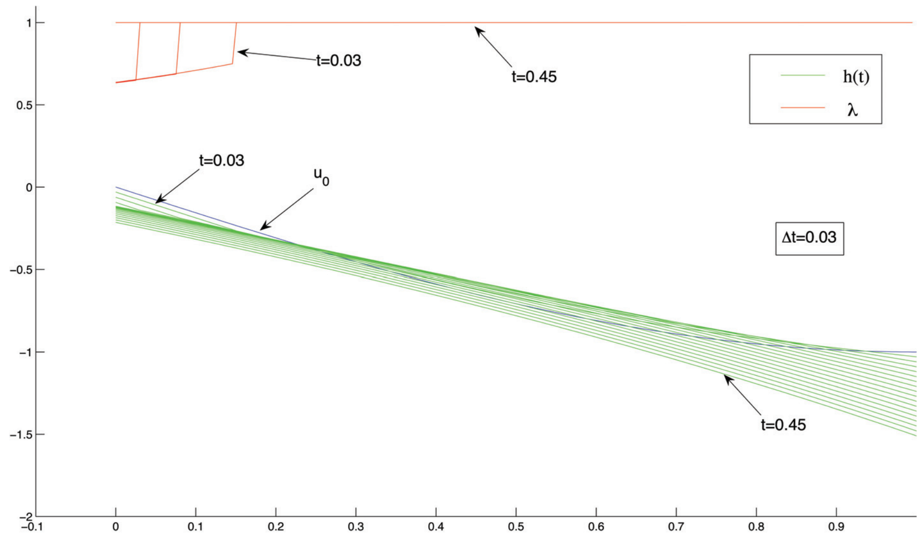

FIGURE 4. $E = 1$, $f_e = -1$ and $u_0'(0) = -\frac{\pi}{2}$.

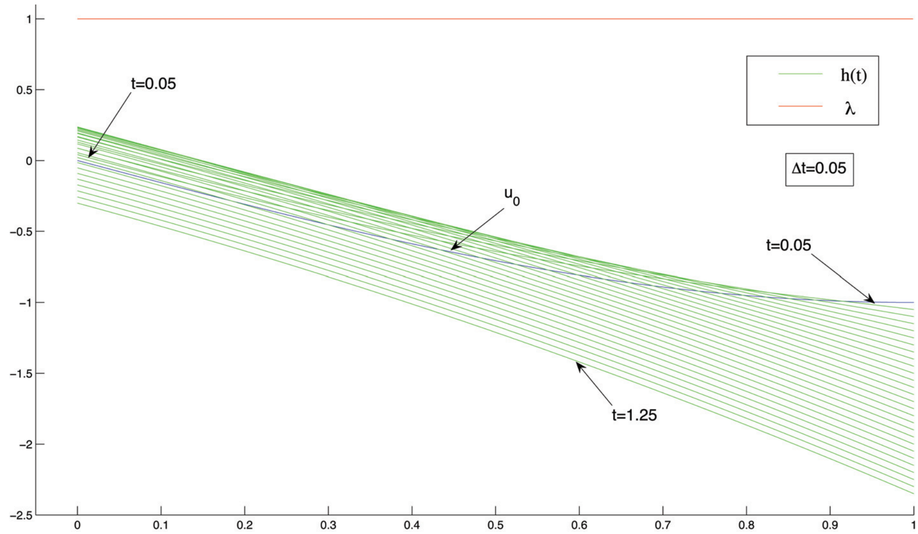

FIGURE 5. $E = 1$, $f_e = -1.6$ and $u_0'(0) = -\frac{\pi}{2}$.

process has to select the right one. In the first example (Figs. 6 to 8), u_0 is decreasing. If $f_e + f_s < E$, the algorithm imposes the Neumann condition (Figs. 6, 7) and the Dirichlet condition else (Fig. 8).

In the second example (Figs. 9 to 11), the initial condition u_0 in not a monotone function and one observes that the boundary type has to change after a time when $f_e + f_s < E$ (Figs. 9, 10)[1]. In the other case, as expected in Section 2.4, only the Dirichlet condition occurs (Fig. 11).

[1]A single line if the Neumann-condition is chosen, a dash line else.

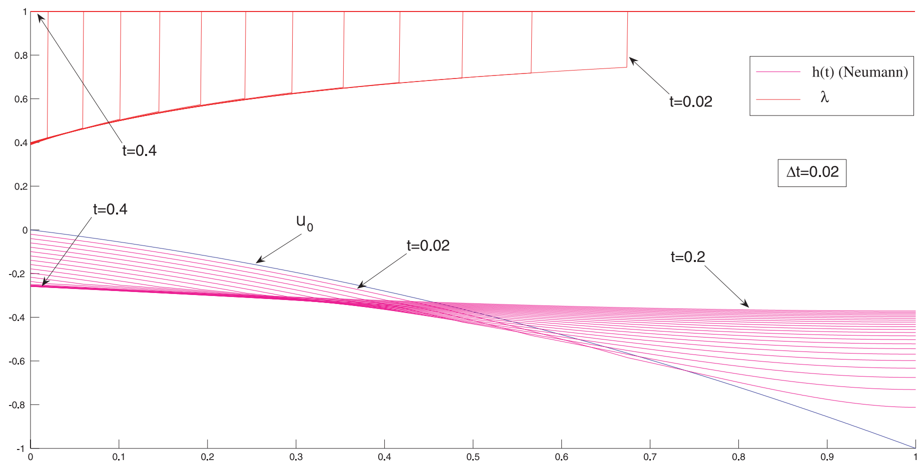

FIGURE 6. $E = 1$, $f_e = -0.2$ and $f_s = 0$.

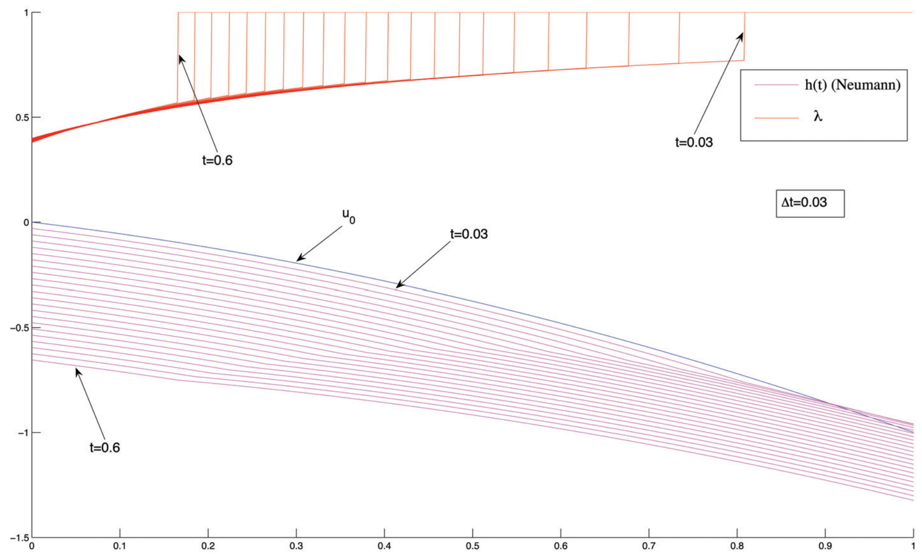

FIGURE 7. $E = 1$, $f_e = -0.2$ and $f_s = 1$.

All figures seem to be in agreement with the phenomena of transport and erosion modelled by the equation and the unilateral constraint.

2.6. **Conclusion and open problem**

One has been able to extend the results already obtained in the case of homogeneous Dirichlet boundary conditions. This study concerns a more acceptable physically problem. The boundary conditions are of unilateral type and lead to a variational inequality, in order to satisfy the maximal erosion constraint.

Of course, the existence of a solution is still an open problem, mainly since one needs more information on the convergence of the term $\int_\Omega \lambda_h \nabla u_h.\nabla v \, dx$. Nevertheless, in the $1-D$ case, as presented in Vallet [18] and

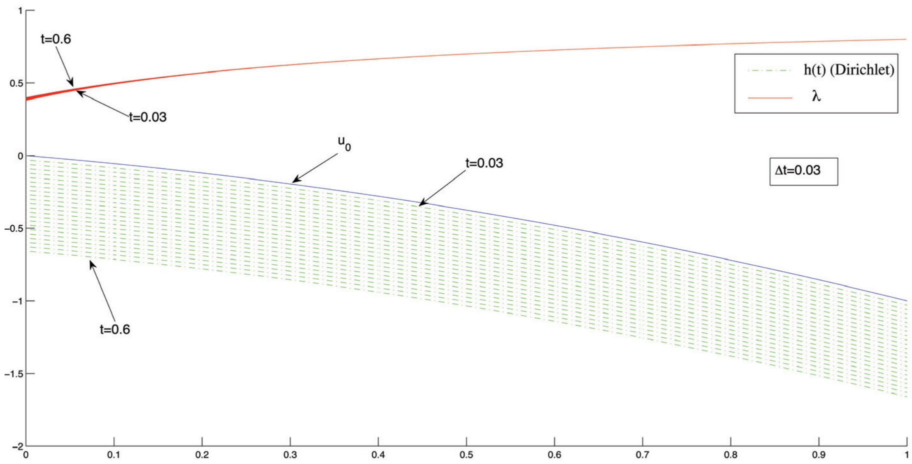

FIGURE 8. $E = 1$, $f_e = -0.2$ and $f_s = 1.8$.

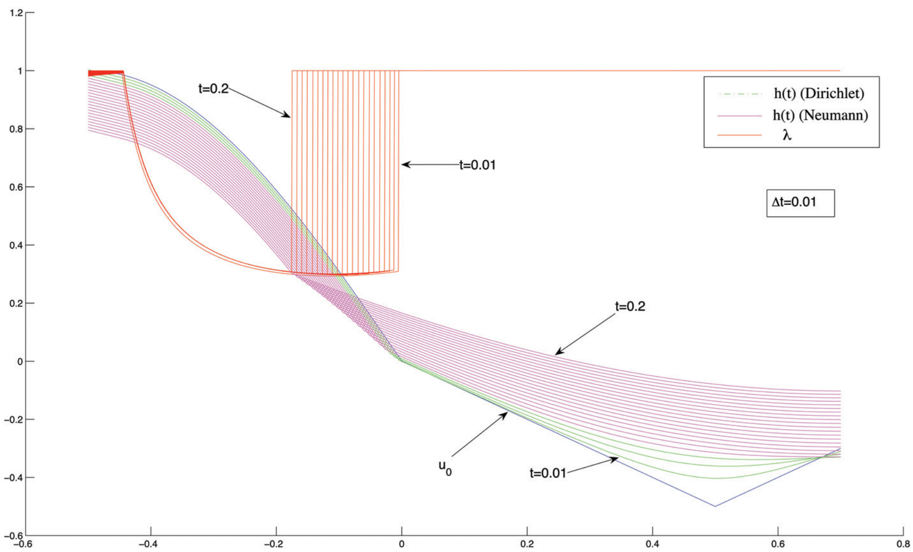

FIGURE 9. $E = 1$, $f_e = -0.5$ and $f_s = 0$.

Gagneux *et al.* [10] for the Dirichlet problem, it has been possible to get rid of the difficulty by an approach based on the properties of bounded variation sets.

An other difficulty appears since in the formulation and in the boundary condition on Γ_s, *a priori* estimates in $L^2(0, T; H^1(\Omega))$ for $\partial_t \hat{u}_h$ is explicitly needed. Therefore, one has to consider a generalization of the Darcy's law in the following sense: $\vec{q} = -\lambda\nabla[u + \tau\partial_t u]$ in Q where $\tau > 0$, according to the Barenblatt model in the non-static situation, see Cuesta *et al.* [6]. One proposes a few words about that in Gagneux *et al.* [13] and Antontsev *et al.* [2].

At last, one has to think about a robust numerical method in order to simulate such non-linear models. The main problem is to consider a fixed-point method with multi-valued applications.

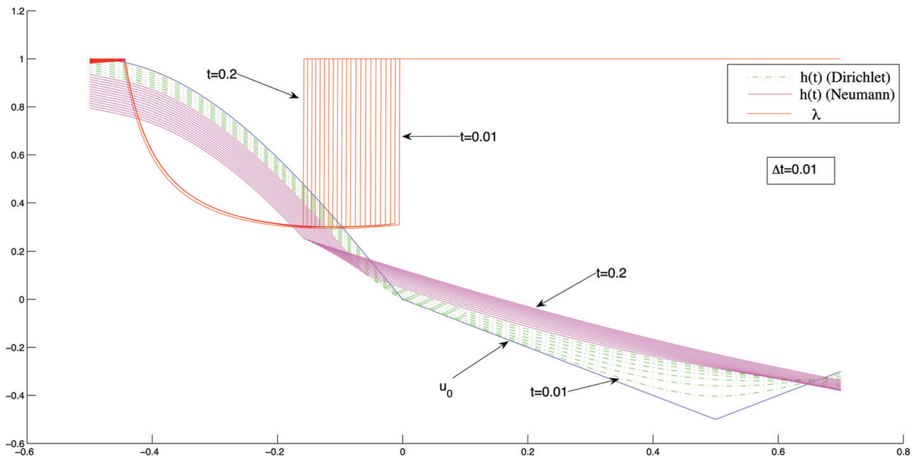

FIGURE 10. $E = 1$, $f_e = -0.5$ and $f_s = 0.5$.

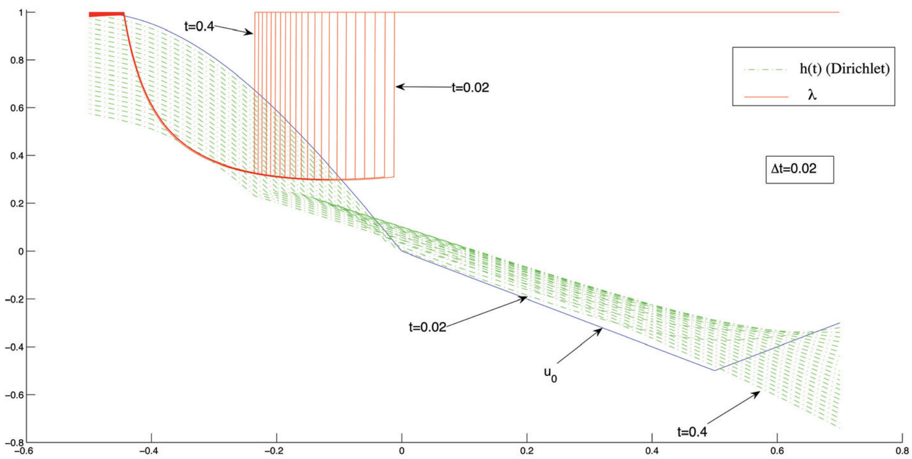

FIGURE 11. $E = 1$, $f_e = -0.5$ and $f_s = 2$.

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