ITERATIVELY SOLVING A KIND OF SIGNORINI TRANSMISSION PROBLEM IN A UNBOUNDED DOMAIN

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Abstract. In this paper, we are concerned with a kind of Signorini transmission problem in a unbounded domain. A variational inequality is derived when discretizing this problem by coupled FEM-BEM. To solve such variational inequality, an iterative method, which can be viewed as a variant of the D-N alternative method, will be introduced. In the iterative method, the finite element part and the boundary element part can be solved independently. It will be shown that the convergence speed of this iteration is independent of the mesh size. Besides, a combination between this method and the steepest descent method is also discussed.

1. Introduction

In this paper we analyze the following transmission problem in $\mathbf{R}^2$. Let Ω be a bounded domain with Lipschitz boundary Γ. To describe mixed boundary conditions, let $\Gamma = \overline{\Gamma_t \cup \Gamma_s}$ where Γ_t and Γ_s are nonempty, disjoint, and open in Γ. Let n denote the unit normal on Γ defined almost everywhere pointing from Ω into $\Omega_c := \mathbf{R}^2 \backslash \bar{\Omega}$. Assume that $f \in L^2(\Omega)$, $u_0 \in H^{\frac{1}{2}}(\Gamma)$ and $t_0 \in L^2(\Gamma)$.

As the interior part, we consider the nonlinear partial differential equation

$$\operatorname{div}(p(|\nabla u|) \cdot \nabla u) + f = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $p : [0, \infty) \to [0, \infty)$ is a continuous function with $t \cdot p(t)$ being monotonously increasing with t. In the exterior part, we consider the Laplace equation

$$\triangle u = 0 \quad \text{in } \Omega_c, \tag{1.2}$$

with the radiation condition (note Lem. 3.6 of [2])

$$u(x) = a + o(1) \quad (|x| \to \infty), \tag{1.3}$$

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where a is a real constant. Writing $u_1 := u|_\Omega$ and $u_2 := u|_{\Omega_c}$, the traction on Γ are given by the traces of $p(|\nabla u|)\frac{\partial u_1}{\partial n}$ and $-\frac{\partial u_2}{\partial n}$. We consider transmission conditions on Γ_t,

$$u_1|_{\Gamma_t} = u_2|_{\Gamma_t} + u_0|_{\Gamma_t} \quad and \quad p(|\nabla u|)\frac{\partial u_1}{\partial n}|_{\Gamma_t} = \frac{\partial u_2}{\partial n}|_{\Gamma_t} + t_0|_{\Gamma_t}, \tag{1.4}$$

and Signorini conditions on Γ_s,

$$u_1|_{\Gamma_s} \leq u_2|_{\Gamma_s} + u_0|_{\Gamma_s}, \tag{1.5}$$

$$p(|\nabla u|)\frac{\partial u_1}{\partial n}|_{\Gamma_s} = \frac{\partial u_2}{\partial n}|_{\Gamma_s} + t_0|_{\Gamma_s} \leq 0 \tag{1.6}$$

$$0 = p(|\nabla u|)\frac{\partial u_1}{\partial n}|_{\Gamma_s} \cdot (u_2 + u_0 - u_1)|_{\Gamma_s}. \tag{1.7}$$

This problem can be considered as a scalar model of the two-body contact problem between a linear elastic unbounded medium and a deformable body allowing some nonlinear monotone stress strain relationship (refer to [4,12]). Some similar problems have been considered in [5,8,9,14]. The paper [2] introduced a coupled FEM-BEM variational inequality of the problem (1)–(5), and considered the corresponding approximation problem. Moreover, an asymptotic error estimate has been derived in that paper.

However, there is no literature to study the numerical method for solving this kind of coupled FEM-BEM variational inequality. In the present paper we introduce an iterative method for solving (1)–(5). This method can be viewed as a variant of the D-N alternative method, which has been applied to solving the Poisson equations in bounded domain and unbounded domain (see [7,19,20]). It will be shown that the solution sequence generated by this iteration converges to the solution of the coupled system given in [2], and the convergence speed is independent of the mesh size. The new method has obvious advantages: (1) the finite element problem and boundary element problem are solved respectively; (2) the memory requirement was decreased.

This paper is organized as follows. In Section 2, we describe the coupled variational problem in suitable way. In Section 3, a basic iteration method is described, and its convergence results are given. In Section 4, we analyze the convergence rate. In Section 5, we discuss a combination between this iterative method and the steepest descent method.

2. A COUPLED SYSTEM DERIVED BY FEM AND BEM

In this section, we describe the variational problems considered by [2]. For convenience, we would like to use the equivalent forms to (24) and (28) in [2].

Let $H^1(\Omega), H^{\frac{1}{2}}(\Gamma), H_0^1(\Omega)$ and $H^{-\frac{1}{2}}(\Gamma)$ denote the usual Sobolev spaces (refer to [10]). We define the (nonlinear) functional $A : H^1(\Omega) \times H^1(\Omega) \to \mathbf{R}$ and the linear functional $F : H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma) \to \mathbf{R}$ by

$$A(u, v) = \int_\Omega p(|\nabla u|)(\nabla u)^t \cdot \nabla v \mathrm{d}x, \quad u, \ v \in H^1(\Omega)$$

and

$$F(u, \varphi) = \int_\Omega f \cdot u \mathrm{d}x + \int_\Gamma t_0 \cdot \varphi \mathrm{d}s, \quad u \in H^1(\Omega), \ \varphi \in H^{\frac{1}{2}}(\Gamma).$$

To define a symmetric and positive definite boundary operator, let $\gamma(x, y)$ be the fundamental solution for the Laplacian, i.e.

$$\gamma(x, y) = -\frac{1}{2\pi} \log |x - y|, \quad x, y \in \mathbf{R}^2.$$

Set

$$V\phi(z) = 2\int_{\Gamma} \phi(x) \cdot \gamma(z,x) \mathrm{d}s_x \quad (V : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)),$$

$$K\phi(z) = 2\int_{\Gamma} \phi(x) \cdot \frac{\partial}{\partial n_x}\gamma(z,x) \mathrm{d}s_x \quad (K : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)),$$

$$K'\phi(z) = 2\int_{\Gamma} \phi(x) \cdot \frac{\partial}{\partial n_z}\gamma(z,x) \mathrm{d}s_x \quad (K' : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)),$$

$$W\phi(z) = \frac{\partial}{\partial n_z}K\phi(z) \quad (W : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)).$$

It is well known that $V : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ is symmetric and positive definite, and $W : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ is symmetric and positive semi-definite under suitable assumption (see [2,15]). Thus, the operator $S : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ defined by

$$S = \frac{1}{2}(W + (K' - I)V^{-1}(K - I))$$

is also symmetric and positive definite. Note that the operator S is just the $Dirichlet - Neumann$ map or $Steklov - Poincare$ operator (refer to [2,18]).

Let $\langle \cdot, \cdot \rangle$ be the L^2 inner product on Γ. Set

$$L(u,\varphi) = F(u,\varphi) + \langle Sa, \varphi \rangle, \quad u \in H^1(\Omega), \varphi \in H^{\frac{1}{2}}(\Gamma).$$

For the function $p(t)$, we make the natural assumption: $p_1 \leq p(t) \leq p_2$ and $\alpha \leq p(t) + tp'(t) \leq \beta$ for constants $p_1, p_2, \alpha, \beta > 0$ (refer to [2,11]). Below we fix the constant a in some way to force uniqueness (see [2]).

Let $E = H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$. Set

$$D = \{(u,\varphi) \in E : u = \varphi + u_0 \text{ on } \Gamma_t \text{ and } u \leq \varphi + u_0 \text{ a.e. on } \Gamma_s\}.$$

The variational form of the problem (1)–(5) is: to find $(u,\varphi) \in D$ such that

$$A(u, v - u) + \langle S\varphi, \varphi - \psi \rangle \geq L(v - u, \varphi - \psi),$$

$$\forall (v, \psi) \in D. \tag{2.1}$$

If (u,φ) is the solution of (2.1), then we can obtain the solution of (1)–(5) by $u_1 = u$ and

$$u_2(z) = \frac{1}{2}(K\varphi + VS(\varphi - a))(z) + a, \quad z \in \Omega_c.$$

Without loss of generality, we assume that the domain Ω is a polygon. Let Ω be divided into some regular quasi-uniform triangles with diameter h. Then let H_h^1 denote the finite-dimensional space of continuous and piecewise linear function relating to this partition. The mesh in Ω leads to a mesh of the boundary, so that we set

$$H_h^{\frac{1}{2}} = \{u|_{\Gamma} : u \in H_h^1\}.$$

Moreover, we may consider $H_h^{-\frac{1}{2}}$ as the piecewise constant trial functions. For simplicity of exposition, we assume that $u_0 \in H_h^{\frac{1}{2}}$. Let $E_h = H_h^1 \times H_h^{\frac{1}{2}}$, and set

$$D_h = \{(u_h,\varphi_h) \in E_h : u_h = \varphi_h + u_0 \text{ on } \Gamma_t \text{ and } u_h \leq \varphi_h + u_0 \text{ a.e. on } \Gamma_s\}.$$

Let

$$i_h : H_h^{\frac{1}{2}} \hookrightarrow H^{\frac{1}{2}}(\Gamma)$$

and

$$j_h : H_h^{-\frac{1}{2}} \hookrightarrow H^{-\frac{1}{2}}(\Gamma)$$

denote the canonical imbedding with its dual i_h^* and j_h^*

$$i_h^* : H^{-\frac{1}{2}}(\Gamma) \hookrightarrow (H_h^{\frac{1}{2}})^*$$

and

$$j_h^* : H^{\frac{1}{2}}(\Gamma) \hookrightarrow (H_h^{-\frac{1}{2}})^*.$$

Set

$$S_h = \frac{1}{2}(i_h^* W i_h + i_h^*(K' - I)j_h(j_h^* V j_h)^{-1} j_h^*(K - I)i_h).$$

The discrete problem associated with (2.1) is: to find $(u_h, \varphi_h) \in D_h$ such that

$$A(u_h, v_h - u_h) + \langle S_h \varphi_h, \psi_h - \varphi_h \rangle \geq L_h(v_h - u_h, \psi_h - \varphi_h),$$

$$\forall (v_h, \psi_h) \in D_h \tag{2.2}$$

where

$$L_h(v, \psi) = F(v, \psi) + \langle S_h a, \psi \rangle.$$

It has been shown in [2] that the discrete problem (2.2) (and (2.1)) has a unique solution (by transforming it to an equivalent form). Moreover, an asymptotic estimate of the errors $u_h - u$ and $\varphi_h - \varphi$ was also derived. If (u_h, φ_h) is the solution of (2.2), then we can obtain the approximate solution of (1)–(5) by $u_{1h} = u_h$ and

$$u_{2h}(z) = \frac{1}{2}(K\varphi_h + VS(\varphi_h - a))(z) + a, \quad z \in \Omega_c.$$

Remark 2.1. Since the stiffness matrix of the boundary integral operator S_h is dense, it is very expensive to solve the variational inequality (2.2) by the existing methods (refer to [6,19]). For this reason, we will introduce an iterative method to solve (2.2) in the next section.

3. A BASIC ITERATIVE METHOD

At first, we describe the algorithm.

For a given $\lambda \in H^{\frac{1}{2}}(\Gamma)$, we define

$$\hat{H}_\lambda^1 = \{v \in H_h^1 : v = \lambda + u_0 \text{ on } \Gamma_t \text{ and } v \leq \lambda + u_0 \text{ a.e. on } \Gamma_s\}.$$

For a given $w \in H_h^{\frac{1}{2}}$, let $\overline{w} \in H_h^1$ denote the zero extension of w (which equals w on Γ, and equals zero at the internal nodes of Ω).

Algorithm 3.1 Let $\varphi_h^0 \in H_h^{\frac{1}{2}}$ be given. When $\varphi_h^n \in H_h^{\frac{1}{2}}$ have been gotten, $\varphi_h^{n+1} \in H_h^{\frac{1}{2}}$ will be obtained by

step $1°$: to find $u_h^{n+1} \in \hat{H}_{\varphi_h^n}^1$ such that

$$A(u_h^{n+1}, v_h - u_h^{n+1}) \geq (f, v_h - u_h^{n+1}), \forall v_h \in \hat{H}_{\varphi_h^n}^1; \tag{3.1}$$

step $2°$: to find $p_h^{n+1} \in H_h^{\frac{1}{2}}$ such that

$$\langle S_h p_h^{n+1}, \psi_h \rangle = (f, \overline{\psi_h}) - A(u_h^{n+1}, \overline{\psi_h}) + \langle t_0 + Sa, \psi_h \rangle, \ \forall \psi_h \in H_h^{\frac{1}{2}}; \tag{3.2}$$

step $3°$: set

$$\varphi_h^{n+1} = \theta p_h^{n+1} + (1 - \theta)\varphi_h^n, \quad 0 < \theta < 1. \tag{3.3}$$

Remark 3.1. In essence, the step 1° and the step 2° can be regarded respectively as solving a Dirichlet (inequality) problem on Ω (by FEM) and as solving a Neumann problem on Ω_c (by BEM). In the implementation of the step 2°, it is unnecessary to consider the operator i_h^* or j_h^*. Instead, only the Galerkin method is used (refer to [1]).

Remark 3.2. The main merits of this algorithm are: (1) the finite element problem and the boundary element problem are solved independently, the variational inequality (3.1) and the boundary integral equation (3.2) can be solved by the existing methods (see [3,6]); (2) it has smaller memory requirement than the algorithms to solve globally the variational inequality (2.2).

Now, we give the convergence result of this iteration.

Since $S_h : H_h^{\frac{1}{2}} \rightarrow (H_h^{\frac{1}{2}})^*$ is a symmetric and positive definite operator, we can define the norm $\|\cdot\|_{S_h} = (\langle S_h \cdot, \cdot \rangle)^{\frac{1}{2}}$. Let $\varphi_h \in H^{\frac{1}{2}}$ be determined by the variational inequality (2.2), and set $e_h^n = \varphi_h - \varphi_h^n$.

Theorem 3.1. *There exists a constant c_0 independent of the mesh size h, such that when $0 < \theta < \frac{1}{c_0+1}$, the D-N alternative algorithm is convergent. Moreover, we have*

$$\|e_h^{n+1}\|_{S_h}^2 \leq (1 - 2\theta + 2(c_0 + 1)\theta^2)\|e_h^n\|_{S_h}^2. \tag{3.4}$$

In particular, when $\theta = \frac{1}{2(c_0+1)}$, the upper bound of (3.4) is minimal and there holds

$$\|e_h^{n+1}\|_{S_h}^2 \leq (1 - \frac{1}{2(c_0 + 1)})\|e_h^n\|_{S_h}^2. \tag{3.5}$$

Remark 3.3. Set $u_{1h}^n = u_h^n$ and

$$u_{2h}^n(z) = \frac{1}{2}(K\varphi_h^n + VS(\varphi_h^n - a))(z) + a, \quad z \in \Omega_c.$$

Theorem 3.1 proves that u_{1h}^n and u_{2h}^n converge to u_{1h} and u_{2h} respectively. In fact, we have

$$\|u_{1h}^n - u_{1h}\|_{1,\Omega} \leq C \left(1 - \frac{1}{2(c_0 + 1)}\right)^{\frac{n}{2}} \|e_h^0\|_{S_h},$$

with C being a constant independent of h and n. The error $u_{2h}^n - u_{2h}$ has a similar estimate, but it involves a particular norm on the unbounded domain Ω_c.

4. ANALYSIS OF THE CONVERGENCE RATE

Since the nonlinear variational inequality is involved, Theorem 3.1 can not be derived by the standard eigenvalue method, which has been used to analyze the convergence rate of the usual iterative method (compare [7,19]). The proof of Theorem 3.1 is based on four lemmas.

Lemma 4.1. *The variational inequality (2.2) can be decomposed into the following sub-problems*

$$A(u_h, v_h - u_h) \geq (f, v_h - u_h), \quad \forall v_h \in \hat{H}_{\varphi_h}^1 \tag{4.1}$$

and

$$\langle S_h\varphi_h, \psi_h \rangle = (f, \overline{\psi}_h) - A(u_h, \overline{\psi}_h) + \langle t_0 + Sa, \psi_h \rangle, \ \forall \psi_h \in H_h^{\frac{1}{2}}. \tag{4.2}$$

Proof. The variational inequality (4.1) follows by setting $\psi_h = \varphi_h$ in (2.2).

For any $w_h \in H_h^{\frac{1}{2}}$, we set $v_h = u_h + \overline{w_h}$ (or $u_h - \overline{w_h}$) and $\psi_h = \varphi_h + w_h$ (or $\varphi_h - w_h$) in (2.2). Then, the function $\varphi_h \in H_h^{\frac{1}{2}}$ satisfies

$$A(u_h, \overline{w_h}) + \langle S_h\varphi_h, w_h \rangle \geq (f, \overline{w_h}) + \langle t_0 + Sa, w_h \rangle$$

and

$$A(u_h, -\overline{w_h}) + \langle S_h\varphi_h, -w_h \rangle \geq (f, -\overline{w_h}) + \langle t_0 + Sa, -w_h \rangle.$$

These two inequalities imply the equation (4.2). $\qquad\square$

Set

$$\mu_h^n = p_h^{n+1} - \varphi_h.$$

Lemma 4.2. *The following inequality is valid:*

$$A(u_h, u_h - u_h^{n+1}) - A(u_h^{n+1}, u_h - u_h^{n+1}) \leq \langle S_h\mu_h^n, e_h^n \rangle. \tag{4.3}$$

Proof. It can be verified directly that

$$
\begin{aligned}
\langle S_h\mu_h^n, e_h^n \rangle &= \langle S_h\mu_h^n, u_h - u_h^{n+1} \rangle + \langle S_h\mu_h^n, \varphi_h + u_0 - u_h \rangle + \langle S_h\mu_h^n, u_h^{n+1} - (\varphi_h^n + u_0) \rangle \\
&= \langle S_h\mu_h^n, u_h - u_h^{n+1} \rangle + \langle S_hp_h^{n+1}, \varphi_h + u_0 - u_h \rangle - \langle S_h\varphi_h, \varphi_h + u_0 - u_h \rangle \\
&\quad + \langle S_hp_h^{n+1}, u_h^{n+1} - (\varphi_h^n + u_0) \rangle - \langle S_h\varphi_h, u_h^{n+1} - (\varphi_h^n + u_0) \rangle.
\end{aligned}
\tag{4.4}
$$

Set $w_h = \varphi_h + u_0 - u_h|_\Gamma$. From the definition of p_h^{n+1} (see step 2°), we have

$$
\begin{aligned}
\langle S_hp_h^{n+1}, \varphi_h + u_0 - u_h \rangle = \langle S_hp_h^{n+1}, w_h \rangle &= (f, \overline{w_h}) - A(u_h^{n+1}, \overline{w_h}) + \langle t_0 + Sa, \overline{w_h} \rangle \\
&= \langle t_0 + Sa, w_h \rangle - [(f, -\overline{w_h}) - A(u_h^{n+1}, -\overline{w_h})] \\
&= \langle t_0 + Sa, w_h \rangle - [(f, (u_h^{n+1} - \overline{w_h}) - u_h^{n+1}) \\
&\quad - A(u_h^{n+1}, (u_h^{n+1} - \overline{w_h}) - u_h^{n+1})].
\end{aligned}
\tag{4.5}
$$

Since $u_h \in \hat{H}_{\varphi_h}^1$, we know that

$$w_h|_{\Gamma_t} = 0 \text{ and } w_h|_{\Gamma_s} \geq 0 \text{ (a.e. on } \Gamma_s). \tag{4.6}$$

Note that $u_h^{n+1} \in \hat{H}_{\varphi_h^n}^1$, we have

$$u_h^{n+1}|_{\Gamma_t} = \varphi_h^n + u_0 \text{ and } u_h^{n+1}|_{\Gamma_t} \leq \varphi_h^n + u_0 \text{ (a.e. on } \Gamma_s),$$

by (4.6), this leads to

$$(u_h^{n+1} - \overline{w_h})|_{\Gamma_t} = \varphi_h^n + u_0 \text{ and } (u_h^{n+1} - \overline{w_h})|_{\Gamma_t} \leq \varphi_h^n + u_0 \text{ (a.e. on } \Gamma_s).$$

Namely, $u_h^{n+1} - \overline{w_h} \in \hat{H}_{\varphi_h^n}^1$. Thus, by (4.5) and the definition of u_h^{n+1} (see step 1°) we obtain

$$\langle S_hp_h^{n+1}, \varphi_h + u_0 - u_h \rangle \geq \langle t_0 + Sa, w_h \rangle = \langle t_0 + Sa, \varphi_h + u_0 - u_h \rangle. \tag{4.7}$$

Similarly, we can prove

$$
\begin{aligned}
-\langle S_h\varphi_h, u_h^{n+1} - (\varphi_h^n + u_0) \rangle &= \langle S_h\varphi_h, \varphi_h^n + u_0 - u_h^{n+1} \rangle \\
&\geq \langle t_0 + Sa, \varphi_h^n + u_0 - u_h^{n+1} \rangle.
\end{aligned}
\tag{4.8}
$$

On the other hand, let $\hat{v}_h \in H_h^1$ denote a suitable extension of $\varphi_h + u_0$ such that $\hat{v}_h - u_h$ is just the zero extension of $\varphi_h + u_0 - u_h|_\Gamma$. From (4.2), we have

$$\langle S_h\varphi_h, \varphi_h + u_0 - u_h \rangle = (f, \hat{v}_h - u_h) - A(u_h, \hat{v}_h - u_h) + \langle t_0 + Sa, \varphi_h + u_0 - u_h \rangle). \tag{4.9}$$

It is clear that $\hat{v_h} \in \hat{H}^1_{\varphi_h}$. Thus, by (4.1) and (4.9) we obtain

$$\langle S_h \varphi_h, \varphi_h + u_0 - u_h \rangle \leq \langle t_0 + Sa, \varphi_h + u_0 - u_h \rangle.$$

Namely,

$$-\langle S_h \varphi_h, \varphi_h + u_0 - u_h \rangle \geq -\langle t_0 + Sa, \varphi_h + u_0 - u_h \rangle. \tag{4.10}$$

In an analogous way, we can prove

$$
\begin{aligned}
\langle S_h p_h^{n+1}, u_h^{n+1} - (\varphi_h^n + u_0) \rangle &= -\langle S_h p_h^{n+1}, \varphi_h^n + u_0 - u_h^{n+1} \rangle \\
&\geq -\langle t_0 + Sa, \varphi_h^n + u_0 - u_h^{n+1} \rangle.
\end{aligned} \tag{4.11}
$$

Using (4.4), together with (4.7)–(4.8) and (4.10)–(4.11), yields

$$\langle S_h \mu_h^n, e_h^n \rangle \geq \langle S_h \mu_h^n, u_h - u_h^{n+1} \rangle = \langle S_h p_h^{n+1}, u_h - u_h^{n+1} \rangle - \langle S_h \varphi_h, u_h - u_h^{n+1} \rangle. \tag{4.12}$$

Subtracting (4.2) from (3.2), and choosing $\psi_h = (u_h - u_h^{n+1})|_\Gamma$, we obtain

$$\langle S_h p_h^{n+1}, u_h - u_h^{n+1} \rangle - \langle S_h \varphi_h, u_h - u_h^{n+1} \rangle = A(u_h, \overline{e_{h\Gamma}}) - A(u_h^{n+1}, \overline{e_{h\Gamma}}) \tag{4.13}$$

with $e_{h\Gamma} = (u_h - u_h^{n+1})|_\Gamma$. It follows by (3.1) and (4.1) that

$$A(u_h^{n+1}, w_h) = (f, w_h), \quad \forall w_h \in H_h^1 \cap H_0^1(\Omega)$$

and

$$A(u_h, w_h) = (f, w_h), \quad \forall w_h \in H_h^1 \cap H_0^1(\Omega).$$

Hence,

$$A(u_h^{n+1}, \overline{e_{h\Gamma}} - (u_h - u_h^{n+1})) = (f, \overline{e_{h\Gamma}} - (u_h - u_h^{n+1})) = A(u_h, \overline{e_{h\Gamma}} - (u_h - u_h^{n+1})).$$

Namely,

$$A(u_h, \overline{e_{h\Gamma}}) - A(u_h^{n+1}, \overline{e_{h\Gamma}}) = A(u_h, u_h - u_h^{n+1}) - A(u_h^{n+1}, u_h - u_h^{n+1}). \tag{4.14}$$

This, together with (4.12) and (4.13), gives the desired result. $\qquad\square$

By (4.3) and the monotonicity of the (nonlinear) functional $A(\cdot, \cdot)$ (see [2]), we obtain

Corollary 4.1. *We have the inequality*

$$\langle S_h \mu_h^n, e_h^n \rangle \geq 0. \tag{4.15}$$

For a given $\lambda \in H_h^{\frac{1}{2}}$, let $\widetilde{\lambda} \in H_h^1$ denote the discrete harmonic extension of λ.

Lemma 4.3. *There exists a positive constant c such that*

$$|\widetilde{\mu_h^n}|_{1,\Omega}^2 \leq c \langle S_h \mu_h^n, \mu_h^n \rangle. \tag{4.16}$$

Proof. Let $H(\mu_h^n) \in H^1(\Omega)$ denote the (continuous) harmonic extension. Since $\widetilde{\mu_h^n}$ is just the orthogonal projection of $H(\mu_h^n)$, we have

$$|\widetilde{\mu_h^n}|_{1,\Omega}^2 \leq |H(\mu_h^n)|_{1,\Omega}^2. \tag{4.17}$$

It follows by the well-known property of $H(\mu_h^n)$ that there is a positive constant c_1 independent of h, such that

$$|H(\mu_h^n)|_{1,\Omega}^2 \leq c_1 \|\mu_h^n\|_{\frac{1}{2},\Gamma}^2.$$

Thus, by (4.17) and Lemma 5.1 in [2], we obtain

$$|\widetilde{\mu_h^n}|_{1,\Omega}^2 \leq c_1 \|\mu_h^n\|_{\frac{1}{2},\Gamma}^2 \leq c \langle S_h \mu_h^n, \mu_h^n \rangle. \qquad\square$$

Lemma 4.4. *There exists a positive constant c_0 such that*

$$\langle S_h \mu_h^n, \mu_h^n \rangle \leq c_0 \langle S_h e_h^n, e_h^n \rangle. \tag{4.18}$$

Proof. Subtracting (4.2) from (3.2), and choosing $\psi_h = \mu_h^n$, yields

$$
\begin{aligned}
\langle S_h \mu_h^n, \mu_h^n \rangle &= \langle S_h p_h^{n+1}, \mu_h^n \rangle - \langle S_h \varphi_h, \mu_h^n \rangle \\
&= A(u_h, \overline{\mu_h^n}) - A(u_h^{n+1}, \overline{\mu_h^n}).
\end{aligned} \tag{4.19}
$$

Since $(\overline{\mu_h^n} - \widetilde{\mu_h^n})|_\Gamma = 0$, we have (refer to (4.14))

$$A(u_h, \overline{\mu_h^n}) - A(u_h^{n+1}, \overline{\mu_h^n}) = A(u_h, \widetilde{\mu_h^n}) - A(u_h^{n+1}, \widetilde{\mu_h^n}).$$

Substituting the above equality into (4.19), yields

$$\langle S_h \mu_h^n, \mu_h^n \rangle = A(u_h, \widetilde{\mu_h^n}) - A(u_h^{n+1}, \widetilde{\mu_h^n}). \tag{4.20}$$

On the other hand, by the assumptions to the function $p(t)$ (see Sect. 2), we can prove (refer to [2,11])

$$A(u_h, \widetilde{\mu_h^n}) - A(u_h^{n+1}, \widetilde{\mu_h^n}) \leq \beta |u_h - u_h^{n+1}|_{1,\Omega} \cdot |\widetilde{\mu_h^n}|_{1,\Omega} \tag{4.21}$$

and

$$\alpha |u_h - u_h^{n+1}|_{1,\Omega}^2 \leq A(u_h, u_h - u_h^{n+1}) - A(u_h^{n+1}, u_h - u_h^{n+1}). \tag{4.22}$$

Hence, by (4.20)–(4.22), we obtain

$$\langle S_h \mu_h^n, \mu_h^n \rangle \leq \alpha^{-\frac{1}{2}} \beta [A(u_h, u_h - u_h^{n+1}) - A(u_h^{n+1}, u_h - u_h^{n+1})]^{\frac{1}{2}} \cdot |\widetilde{\mu_h^n}|_{1,\Omega}.$$

Furthermore, it follows by (4.3) that

$$\langle S_h \mu_h^n, \mu_h^n \rangle \leq \alpha^{-\frac{1}{2}} \beta \langle S_h \mu_h^n, e_h^n \rangle^{\frac{1}{2}} \cdot |\widetilde{\mu_h^n}|_{1,\Omega}, \tag{4.23}$$

which, together with (4.16), yields

$$\langle S_h \mu_h^n, \mu_h^n \rangle \leq \alpha^{-1} \beta^2 c \langle S_h \mu_h^n, e_h^n \rangle.$$

Since $S_h : H_h^{\frac{1}{2}} \to (H_h^{\frac{1}{2}})^*$ is a symmetric and positive definite, by the Cauchy-Schwarz inequality, we obtain

$$\langle S_h \mu_h^n, \mu_h^n \rangle \leq \alpha^{-1} \beta^2 c \langle S_h \mu_h^n, \mu_h^n \rangle^{\frac{1}{2}} \cdot \langle S_h e_h^n, e_h^n \rangle^{\frac{1}{2}}.$$

Namely,

$$\langle S_h \mu_h^n, \mu_h^n \rangle \leq \alpha^{-2} \beta^4 c^2 \langle S_h e_h^n, e_h^n \rangle.$$

This means that the inequality (4.18) is valid with $c_0 = \alpha^{-2} \beta^2 c^2$. □

Now, we can prove our main result easily.

Proof of Theorem 3.1. From (3.3), we have

$$e_h^{n+1} = e_h^n - \theta g_h^n. \tag{4.24}$$

where $g_h^n = p_h^{n+1} - \varphi_h^n$. It follows by (4.24) that

$$\|e_h^{n+1}\|_{S_h}^2 = \|e_h^n\|_{S_h}^2 - 2\theta \langle S_h e_h^n, g_h^n \rangle + \theta^2 \|g_h^n\|_{S_h}^2. \tag{4.25}$$

Namely,

$$\|e_h^{n+1}\|_{S_h}^2 = \left(1 - 2\theta \frac{\langle S_h e_h^n, g_h^n \rangle}{\|e_h^n\|_{S_h}^2} + \theta^2 \frac{\|g_h^n\|_{S_h}^2}{\|e_h^n\|_{S_h}^2}\right) \|e_h^n\|_{S_h}^2 \quad (e_h^n \neq 0). \tag{4.26}$$

By Corollary 4.1, yields (note that $g_h^n = e_h^n + \mu_h^n$)

$$\langle S_h e_h^n, g_h^n \rangle = \|e_h^n\|_{S_h}^2 + \langle S_h e_h^n, \mu_h^n \rangle \geq \|e_h^n\|_{S_h}^2. \tag{4.27}$$

Moreover, it follows by Lemma 4.4 that

$$\begin{aligned}
\|g_h^n\|_{S_h}^2 &= \|\mu_h^n + e_h^n\|_{S_h}^2 \\
&\leq 2(\|\mu_h^n\|_{S_h}^2 + \|e_h^n\|_{S_h}^2) \\
&\leq 2(c_0 + 1)\|e_h^n\|_{S_h}^2).
\end{aligned} \tag{4.28}$$

Using (4.26), together with (4.27) and (4.28), we deduce to (3.4).

It is clear that when $0 < \theta < \frac{1}{c_0+1}$, we have

$$\|e_h^{n+1}\|_{S_h}^2 < \|e_h^n\|_{S_h}^2.$$

It can be verified directly that when $\theta = \frac{1}{2(c_0+1)}$, the function

$$g(\theta) = 1 - 2\theta + 2(c_0 + 1)\theta^2$$

reaches the minimum value $1 - \frac{1}{2(c_0+1)}$. $\qquad\qquad\square$

Remark 4.1. Some iterative methods for nonlinear minimization problems have been developed by [16,17]. It is easy to see that the methods as well as the convergence results can be extended to the case of nonlinear equation. On the other hand, the unknown φ_h satisfies a nonlinear equation, and Algorithm 3.1 can be interpreted as a preconditioned Richardson iteration (refer to the next section). Thus, Theorem 3.1 may be derived by using the extended results.

5. A COMBINATION BETWEEN THE ALGORITHM 3.1 AND THE STEEPEST DESCENT METHOD

In most applications, it is difficult to estimate exactly the constant c in (4.16) (or c_0 in Theorem 3.1). To solve this problem, we consider the steepest descent method, by which we can determine a (variable) relaxation parameter θ_n in the $n - th$ iteration (refer to [6,13]).

For a given $\lambda \in H_h^{\frac{1}{2}}$, let $\Psi(\lambda) \in \hat{H}_\lambda^1$ denote the solution of the variational inequality

$$A(\Psi(\lambda), v - \Psi(\lambda)) \geq (f, v - \Psi(\lambda)), \ \forall v \in \hat{H}_\lambda^1. \tag{5.1}$$

We define the (nonlinear) operator $S_\Omega : H_h^{\frac{1}{2}} \to (H_h^{\frac{1}{2}})^*$ by

$$\langle S_\Omega \lambda, \mu \rangle = A(\Psi(\lambda), \bar{\mu}) - (f, \bar{\mu}), \ \forall \mu \in H_h^{\frac{1}{2}}.$$

Since $\Psi(\varphi_h) = u_h$, it follows by Lemma 4.1 that φ_h satisfies the interface equation

$$\hat{S}\varphi_h = g, \tag{5.2}$$

where

$$\hat{S} = S_\Omega + S_h \ \ and \ \ g = t_0 + S_h a.$$

It is clear that $p_h^{n+1} = S_h^{-1}(g - S_\Omega \varphi_h^n)$. Thus, the algorithm 3.1 given in Section 3 can be regarded as the preconditioned Richardson iteration

$$\varphi_h^{n+1} = \varphi_h^n + \theta S_h^{-1}(g - \hat{S}\varphi_h^n). \tag{5.3}$$

Now, the (variable) parameter θ $(=\theta_n)$ in (5.3) is chosen such that the function

$$G(\theta) = \|\hat{S}\varphi_h - \hat{S}\varphi_h^{n+1}\|_{S_h^{-1}}^2$$

reaches its minimum value when $\theta = \theta_n$.

Some algorithms of determining the parameter θ_n are given in [13]. For the combined method, the main cost still results from the Algorithm 3.1 described in Section 3.

Let $\{\varphi_h^n\}$ denote the solution sequence generated by this method.

Theorem 5.1. *For the combined method, we have the following convergence result:*

$$\|\varphi_h - \varphi_h^n\|_{S_h}^2 \leq \left(1 - \frac{1}{2(c_0+1)}\right)^n \|\hat{S}\varphi_h - \hat{S}\varphi_h^0\|_{S_h^{-1}}^2, \tag{5.4}$$

where c_0 is the constant given in the last section.

To prove this theorem, we need two lemmas.

The following result can be proved like (4.3) (refer to Cor. 4.1).

Lemma 5.1. *The following inequality is valid for any $\lambda, \mu \in H_h^{\frac{1}{2}}$*

$$\langle \lambda - \mu, S_\Omega \lambda - S_\Omega \mu \rangle \geq A(\Psi(\lambda), \Psi(\lambda) - \Psi(\mu)) - A(\Psi(\mu), \Psi(\lambda) - \Psi(\mu)) \geq 0. \tag{5.5}$$

Lemma 5.2. *For any $\lambda, \mu \in H_h^{\frac{1}{2}}$, we have*

$$\|S_\Omega \lambda - S_\Omega \mu\|_{S_h^{-1}}^2 \leq c_0 \|\lambda - \mu\|_{S_h}^2. \tag{5.6}$$

Proof. From the definition of the operator S_Ω, we have

$$\|S_\Omega \lambda - S_\Omega \mu\|_{S_h^{-1}}^2 = A(\Psi(\lambda), \widetilde{w}) - A(\Psi(\mu), \widetilde{w}),$$

where

$$w = S_h^{-1}(S_\Omega \lambda - S_\Omega \mu).$$

Thus, we can prove, in an analogous way with (4.18), that

$$\|S_\Omega \lambda - S_\Omega \mu\|_{S_h^{-1}}^2 \leq \alpha^{-1} \beta^2 c A(\Psi(\lambda), \Psi(\lambda) - \Psi(\mu)) - A(\Psi(\mu), \Psi(\lambda) - \Psi(\mu)). \tag{5.7}$$

On the other hand, it follows by (5.5) that

$$\begin{aligned} A(\Psi(\lambda), \Psi(\lambda) - \Psi(\mu)) - A(\Psi(\mu), \Psi(\lambda) - \Psi(\mu)) &\leq \langle \lambda - \mu, S_\Omega \lambda - S_\Omega \mu \rangle \\ &\leq \|S_h^{\frac{1}{2}}(\lambda - \mu)\| \cdot \|S_h^{-\frac{1}{2}}(S_\Omega \lambda - S_\Omega \mu)\|. \end{aligned} \tag{5.8}$$

Using (5.7) and (5.8), we deduce to (5.6). $\square$

Proof of Theorem 5.1. It follows, by a well-known identity, that

$$\|\hat{S}\varphi_h - \hat{S}\varphi_h^{n+1}\|_{S_h^{-1}}^2 - \|\hat{S}\varphi_h - \hat{S}\varphi_h^n\|_{S_h^{-1}}^2 = 2\langle \hat{S}\varphi_h - \hat{S}\varphi_h^n, \hat{S}\varphi_h^n - \hat{S}\varphi_h^{n+1} \rangle_{S_h^{-1}} + \|\hat{S}\varphi_h^n - \hat{S}\varphi_h^{n+1}\|_{S_h^{-1}}^2. \tag{5.9}$$

Since
$$\varphi_h^n - \varphi_h^{n+1} = -\theta_n S_h^{-1}(\hat{S}\varphi_h - \hat{S}\varphi_h^n),$$
we have
$$\langle \hat{S}\varphi_h - \hat{S}\varphi_h^n, \hat{S}\varphi_h^n - \hat{S}\varphi_h^{n+1}\rangle_{S_h^{-1}} = -\theta^{-1}\langle \varphi_h^n - \varphi_h^{n+1}, \hat{S}\varphi_h^n - \hat{S}\varphi_h^{n+1}\rangle. \tag{5.10}$$
The inequality (5.5) infers that
$$\langle \varphi_h^n - \varphi_h^{n+1}, S_\Omega\varphi_h^n - S_\Omega\varphi_h^{n+1}\rangle \geq 0. \tag{5.11}$$
Thus (note that $\hat{S} = S_\Omega + S_h$)
$$\begin{aligned}\langle \varphi_h^n - \varphi_h^{n+1}, \hat{S}\varphi_h^n - \hat{S}\varphi_h^{n+1}\rangle &\geq \langle \varphi_h^n - \varphi_h^{n+1}, S_h(\varphi_h^n - \varphi_h^{n+1})\rangle \\ &= (\theta_n)^2 \|\hat{S}\varphi_h - \hat{S}\varphi_h^n\|_{S_h^{-1}}^2.\end{aligned}$$

This, together with (5.10), leads to
$$\langle \hat{S}\varphi_h - \hat{S}\varphi_h^n, \hat{S}\varphi_h^n - \hat{S}\varphi_h^{n+1}\rangle_{S_h^{-1}} \leq -\theta\|\hat{S}\varphi_h - \hat{S}\varphi_h^n\|_{S_h^{-1}}^2. \tag{5.12}$$

On the other hand, it follows by (5.6) that (note that $\hat{S} = S_\Omega + S_h$)
$$\begin{aligned}\|\hat{S}\varphi_h^n - \hat{S}\varphi_h^{n+1}\|_{S_h^{-1}}^2 &\leq 2(\|S_\Omega\varphi_h^n - S_\Omega\varphi_h^{n+1}\|_{S_h^{-1}}^2 + \|\varphi_h^n - \varphi_h^{n+1}\|_{S_h}^2) \\ &\leq 2(1+c_0)\|\varphi_h^n - \varphi_h^{n+1}\|_{S_h}^2) \\ &= 2(1+c_0)\theta_n^2\|\hat{S}\varphi_h - \hat{S}\varphi_h^n\|_{S_h^{-1}}^2.\end{aligned} \tag{5.13}$$

Using (5.9), together with (5.12) and (5.13), yields
$$\|\hat{S}\varphi_h - \hat{S}\varphi_h^{n+1}\|_{S_h^{-1}}^2 \leq (1 - 2\theta_n + 2(1+c_0)\theta_n^2)\|\hat{S}\varphi_h - \hat{S}\varphi_h^n\|_{S_h^{-1}}^2. \tag{5.14}$$

Since the parameter θ_n is chosen such that
$$\|\hat{S}\varphi_h - \hat{S}\varphi_h^{n+1}\|_{S_h^{-1}}^2$$

reaches its minimum value, by (5.14) we obtain (set $\theta_n = \frac{1}{2(c_0+1)}$)
$$\|\hat{S}\varphi_h - \hat{S}\varphi_h^{n+1}\|_{S_h^{-1}}^2 \leq \left(1 - \frac{1}{2(c_0+1)}\right)\|\hat{S}\varphi_h - \hat{S}\varphi_h^n\|_{S_h^{-1}}^2, \quad n = 0, 1, \cdots.$$

Hence,
$$\|\hat{S}\varphi_h - \hat{S}\varphi_h^n\|_{S_h^{-1}}^2 \leq (1 - \frac{1}{2(c_0+1)})^n \|\hat{S}\varphi_h - \hat{S}\varphi_h^0\|_{S_h^{-1}}^2.$$
To prove (5.4), we only need to verify
$$\|\hat{S}\varphi_h - \hat{S}\varphi_h^n\|_{S_h^{-1}}^2 \geq \|\varphi_h - \varphi_h^n\|_{S_h}^2. \tag{5.15}$$

In fact, it is clear that
$$\begin{aligned}\|\hat{S}\varphi_h - \hat{S}\varphi_h^n\|_{S_h^{-1}}^2 &= \|(S_\Omega\varphi_h - S_\Omega\varphi_h^n) + S_h(\varphi_h - \varphi_h^n)\|_{S_h^{-1}}^2 \\ &\geq 2\langle S_\Omega\varphi_h - S_\Omega\varphi_h^n, \varphi_h - \varphi_h^n\rangle + \|S_h(\varphi_h - \varphi_h^n)\|_{S_h^{-1}}^2,\end{aligned}$$

which, together with (5.5), gives (5.15). □

Remark 5.1. Since the (nonlinear) operator S_Ω can not be expressed explicitly, it is impossible to calculate directly its Gateaux derivative (or Frechet derivative). Thus, Theorem 5.1 can not be proved in the usual way (compare [13] and [16]).

Remark 5.2. Since the convergence speed of the methods discussed in this paper is independent of the mesh size h, it is not sensitive to the errors of the solutions u_h^n, p_h^n (and the parameter θ_n). The analysis to this dependence is standard (refer to [16]).

Remark 5.3. A similar algorithm with Algorithm 3.1 has been designed for solving linear elliptic problems in unbounded domains in [19], where the efficiency of such algorithm was confirmed by numerical experiments.

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