ESAIM: M2AN 52 (2018) 2307–2325
https://doi.org/10.1051/m2an/2018040
**ESAIM: Mathematical Modelling and Numerical Analysis**
www.esaim-m2an.org

# OPTIMAL ERROR ESTIMATES FOR FULLY DISCRETE GALERKIN APPROXIMATIONS OF SEMILINEAR PARABOLIC EQUATIONS

Dominik Meidner[1,*] and Boris Vexler[1]

**Abstract.** We consider a semilinear parabolic equation with a large class of nonlinearities without any growth conditions. We discretize the problem with a discontinuous Galerkin scheme dG(0) in time (which is a variant of the implicit Euler scheme) and with conforming finite elements in space. The main contribution of this paper is the proof of the uniform boundedness of the discrete solution. This allows us to obtain optimal error estimates with respect to various norms.

**Mathematics Subject Classification.** 35K58, 65M15, 65M60.

Received July 25, 2017. Accepted June 22, 2018.

## 1. Introduction

In this paper, we consider the following semilinear parabolic equation.

$$\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) + d(t,x,u(t,x)) &= f(t,x), \quad &(t,x) \in I \times \Omega, \\ u(t,x) &= 0, \quad &(t,x) \in I \times \partial\Omega, \\ u(0,x) &= u_0(x), \quad &x \in \Omega. \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$ is a convex polygonal/polyhedral domain, $I = (0,T)$ is a time interval and $f$ is the right-hand side fulfilling a certain regularity requirement to be specified later.

For the nonlinearity $d(t,x,u)$, we essentially assume that the partial derivative $\partial_u d(t,x,u)$ is bounded from below for all $(t,x) \in I \times \Omega$ and all $u \in \mathbb{R}$, see (2.2b). But we do not require any growth conditions for $d$, see the next section for details. The class of possible nonlinearities includes monotone nonlinearities like $d(u) = u^5$, $d(u) = e^u$ or $d(u) = u^3|u|$ as well as FitzHugh–Nagumo or Allen–Cahn type nonlinearities like $d(u) = u^3 - \alpha u$ with some positive $\alpha \in \mathbb{R}$.

For this class of problems (under a suitable assumption on the right-hand side $f$ and the initial data $u_0$), it is possible to show the existence of a unique bounded solution $u$. The goal of the paper is to prove the uniform boundedness of the discrete approximation $u_{kh}$ to $u$. To this end, we discretize the equation with the discontinuous Galerkin dG(0) method in time and with conforming finite elements in space. The dG(0) time discretization is known to be a variant of the implicit Euler scheme, see Section 3 for details. For this type of

*Keywords and phrases.* Parabolic semilinear equations, finite elements, Galerkin time discretization, error estimates.

[1] Technical University of Munich, Department of Mathematics, Chair of Optimal Control, Garching, Germany.
[*] Corresponding author: meidner@ma.tum.de

discretization we prove that $u_{kh}$ is uniformly bounded, *i.e.*,

$$\|u_{kh}\|_{L^\infty(I\times\Omega)} \le C$$

with a constant $C$ independent of the discretization parameters $k$ and $h$, see Theorem 5.4. Based on this result we are able to prove best-approximation-type error estimates with respect to various norms. We provide such results in particular for the $L^2(I\times\Omega)$, $L^\infty(I;L^2(\Omega))$, and $L^\infty(I\times\Omega)$ norms, *cf.* the Theorems 6.1, 6.3, and 6.5, respectively.

Let us review the related results in the literature. In [10, 25, 26], error estimates for discretization of the semilinear parabolic equation are derived under the assumption that $d$ and $\partial_u d$ are uniformly bounded. In [7,13] growth conditions on $d$ (resp. $\partial_u d$) are assumed for derivation of semi-discrete error estimates. For further results in a different setting we refer to [1]. The most related result is provided in [21], where the uniform boundedness of $u_{kh}$ is shown under a slightly stronger condition $\partial_u d \ge 0$ (*cf.* (2.2b)) in the two-dimensional setting. The technique from [21] does not extend to the three-dimensional situation, due to the inverse inequality used there. Our method here strongly relies on recent discrete maximal parabolic regularity estimates [17], *cf.* also [12] for related results, and extends best approximation estimates from [15] to the semilinear equation.

Our error estimates being of independent interest are important for treatment of optimal control problems. Some recent papers in this context (see, *e.g.*, [4,6]) are restricted to two-dimensional domains only due to the lack of corresponding results in the three-dimensional setting. Thus, our estimates allow to extend the results of these papers to convex polyhedral domains $\Omega \subset \mathbb{R}^3$.

The outline of the paper is as follows: In Section 2, we state the precise functional analytic setting of the problem under consideration and formulate assumptions on the nonlinearity $d$ and the remaining problem data. Under these assumptions, we prove Hölder continuity of the solution $u$ to (1.1). The discrete analog of (1.1) is formulated in Section 3. To this end, we introduce a time discretization by the discontinuous Galerkin dG(0) scheme, whereas the discretization in space is done by means of classical Lagrange finite elements. In this setting, we prove the unique solvability of the discrete nonlinear problem. In the following Section 4, we consider a linear auxiliary equation and its discrete analog. For the solution to this linear discrete problem, we provide maximal parabolic estimates in various norms, which will be the basis for analysis in the remaining two sections. In Section 5, we derive the main result of this paper, namely the boundedness of the solution $u_{kh}$ to the discrete analog of (1.1). Based on this, we provide in the final Section 6 optimal error estimates for the error between $u$ and $u_{kh}$ with respect to the $L^2(I\times\Omega)$, $L^\infty(I;L^2(\Omega))$, and $L^\infty(I\times\Omega)$ norms.

## 2. Continuous problem

To state the precise setting for the problem under consideration, we introduce the following notation: for $r\in[1,\infty]$ and $l\in\{-1,0\}$, we denote the domain in $W^{l,r}(\Omega)$ of the negative Laplacian with homogeneous Dirichlet boundary conditions by

$$\mathrm{Dom}_{l,r}(-\Delta) = \left\{\, u\in W^{l,r}(\Omega) \;\middle|\; -\Delta u \in W^{l,r}(\Omega) \,\right\}.$$

Further, for $p\in[1,\infty]$, we define the space for the initial data by real interpolation as

$$U_{p,r}(\Omega) = (L^r(\Omega), \mathrm{Dom}_{0,r}(-\Delta))_{1-\frac{1}{p},p}. \tag{2.1}$$

The following set of assumptions holds throughout the article.

**Assumption 2.1.**

- *Let* $f\in L^p(I;L^r(\Omega))$ *for some* $p\in(1,\infty)$ *and* $r\in\left(\frac{N}{2},\infty\right)$ *satisfying* $\frac{1}{p}+\frac{N}{2r}<1$.
- *Let* $u_0\in U_{p_0,r_0}(\Omega)$ *for some* $p_0\in(1,\infty)$ *and* $r_0\in\left(\frac{N}{2},\infty\right)$ *satisfying* $\frac{1}{p_0}+\frac{N}{2r_0}<1$.

*Further, for the nonlinearity $d = d(t,x,u)\colon I \times \Omega \times \mathbb{R} \to \mathbb{R}$, we assume the following properties:*

- *$d$ is measurable with respect to $(t,x) \in I \times \Omega$ for all $u \in \mathbb{R}$ and continuously differentiable with respect to $u$ for almost all $(t,x) \in I \times \Omega$.*
- *It holds $d(\cdot,\cdot,0) = 0$.*
- *$\partial_u d$ is locally bounded, i.e., for each $M > 0$ there is $C_M > 0$ such that*

$$|\partial_u d(t,x,u)| \le C_M \tag{2.2a}$$

*for almost all $(t,x) \in I \times \Omega$ and all $u \in [-M, M]$.*

- *There is $\gamma \ge 0$ such that $d$ fulfills the relaxed monotonicity condition*

$$\partial_u d(t,x,u) \ge -\gamma \tag{2.2b}$$

*for almost all $(t,x) \in I \times \Omega$ and all $u \in \mathbb{R}$.*

**Remark 2.2.** A typical setting fulfilling the assumption on $u_0$ would be $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, $u_0 \in U_{p_0,r_0}(\Omega)$ and the relation $\frac{1}{p_0} + \frac{N}{2r_0} < 1$ is valid for $r_0 = 2$ and any $p_0 > \frac{4}{4-N}$.

**Remark 2.3.** Each of the assumptions on $f$ and $u_0$ can be replaced independently by the following assumptions, see the corresponding Remarks 2.5 and 2.9 below.

- Let $f \in L^q(I; W^{-1,s}(\Omega))$ for $q \in (1,\infty)$ and $s \in \big(N,\infty\big)$ satisfying $\frac{1}{q} + \frac{N}{2s} < \frac{1}{2}$.
- Let $u_0 \in \widetilde{U}_{q_0,s_0}(\Omega) = (W^{-1,s_0}(\Omega), \mathrm{Dom}_{-1,s_0}(-\Delta))_{1-\frac{1}{q_0},q_0}$ for $q_0 \in (1,\infty)$ and $s_0 \in \big(N,\infty\big)$ satisfying $\frac{1}{q_0} + \frac{N}{2s_0} < \frac{1}{2}$.

A typical setting fulfilling this assumption on $u_0$ would be $u_0 \in W_0^{1,s_0}(\Omega)$ with some $s_0 > N$. Then, $u_0 \in \widetilde{U}_{q_0,s_0}(\Omega)$ and the relation $\frac{1}{q_0} + \frac{N}{2s_0} < \frac{1}{2}$ is valid for any $q_0 > \frac{2s_0}{s_0-N}$.

To state the existence and boundedness of the solution to (1.1), we need the following lemma.

**Lemma 2.4.** *Under the assumptions on $p_0$ and $r_0$ from Assumption 2.1, there is $\alpha > 0$ such that*

$$U_{p_0,r_0}(\Omega) \hookrightarrow C^\alpha(\Omega) \hookrightarrow L^\infty(\Omega).$$

*Proof.* By Assumption 2.1, there are $\varepsilon, \alpha > 0$ such that $1 - \frac{1}{p_0} - \varepsilon > \frac{N}{2r_0} + \frac{\alpha}{2}$. Using Theorems 1.3.3 and 1.15.2 from [27] as well as Theorem 2.10 from [8], we get

$$(L^{r_0}(\Omega), \mathrm{Dom}_{0,r_0}(-\Delta))_{1-\frac{1}{p_0},p_0} \hookrightarrow (L^{r_0}(\Omega), \mathrm{Dom}_{0,r_0}(-\Delta))_{1-\frac{1}{p_0}-\varepsilon,1} \hookrightarrow \mathrm{Dom}_{0,r_0}((-\Delta)^{1-\frac{1}{p_0}-\varepsilon}) \hookrightarrow C^\alpha(\Omega).$$

By the definition of $U_{p_0,r_0}(\Omega)$ from (2.1), this states the assertion. □

**Remark 2.5.** Using Lemma 4.8 from [8], a corresponding result also holds for $\widetilde{U}_{q_0,s_0}(\Omega)$ with $\frac{1}{q_0} + \frac{N}{2s_0} < \frac{1}{2}$.

**Proposition 2.6.** *Under Assumption 2.1, problem (1.1) admits a unique solution $u \in L^\infty(I \times \Omega)$ with a priori estimate*

$$\|u\|_{L^\infty(I\times\Omega)} \le C\big\{\|f\|_{L^p(I;L^r(\Omega))} + \|u_0\|_{L^\infty(\Omega)}\big\}.$$

*Proof.* The property (2.2b) in Assumption 2.1 implies that

$$d(\cdot,\cdot,u)u = (d(\cdot,\cdot,u) - d(\cdot,\cdot,0))u \ge -\gamma u^2 \quad \text{for all } u \in \mathbb{R}.$$

Moreover from Lemma 2.4 we have $u_0 \in L^\infty(\Omega)$. These two facts and the rest of Assumption 2.1 ensure that the nonlinearity $d$ fulfills the assumptions made in (2.6) and (2.7) from [3]. Hence, we can apply Theorem 5.1 from [3], which states the existence and uniqueness of a solution $u \in L^\infty(I \times \Omega)$ to (1.1) as well as the stated *a priori* estimate.

A similar result under the assumption that $f \in L^{\hat{p}}(I \times \Omega)$ for $\hat{p} > \frac{N}{2} + 1$ and $u_0 \in L^\infty(\Omega)$ can be found in Lemma A.1 from [22]. □

The goal of the remaining part of this section is to prove the Hölder continuity of the solution of (1.1). Before doing so, we need to establish some results for the following linear homogeneous and inhomogeneous problems

$$\begin{aligned} \partial_t v(t,x) - \Delta v(t,x) &= g(t,x), && (t,x) \in I \times \Omega, \\ v(t,x) &= 0, && (t,x) \in I \times \partial\Omega, \\ v(0,x) &= 0, && x \in \Omega \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \partial_t w(t,x) - \Delta w(t,x) &= 0, && (t,x) \in I \times \Omega, \\ w(t,x) &= 0, && (t,x) \in I \times \partial\Omega, \\ w(0,x) &= u_0(x), && x \in \Omega. \end{aligned} \tag{2.4}$$

**Proposition 2.7.** *Let $g \in L^p(I; L^r(\Omega))$ with $\frac{1}{p} + \frac{N}{2r} < 1$. Then, there are $\beta, \kappa \in (0,1)$ depending on $p$ and $r$ such that the solution $v$ of (2.3) fulfills $v \in C^\beta(I; C^\kappa(\Omega))$ with*

$$\|v\|_{C^\beta(I;C^\kappa(\Omega))} \le C\|g\|_{L^p(I;L^r(\Omega))}.$$

*Additionally, provided that $g \in L^{\hat p}(I; L^2(\Omega))$ for some $1 < \hat p < \infty$, it holds that $v \in W^{1,\hat p}(I; L^2(\Omega)) \cap L^{\hat p}(I; H^2(\Omega))$ with the estimate*

$$\|\partial_t v\|_{L^{\hat p}(I;L^2(\Omega))} + \|\nabla^2 v\|_{L^{\hat p}(I;L^2(\Omega))} \le C_{\hat p}\|g\|_{L^{\hat p}(I;L^2(\Omega))}$$

*where $C_{\hat p} \le C\frac{\hat p^2}{\hat p - 1}$.*

*Proof.* The first result is proven, *e.g.*, in Theorem 3.1 from [8] setting $u_0 = 0$ there. The second result can be found in Lemma 2.1 from [16], which itself mainly relies on [2, 9]. □

**Proposition 2.8.** *Let $u_0 \in U_{p_0,r_0}(\Omega)$ with $\frac{1}{p_0} + \frac{N}{2r_0} < 1$. Then, there are $\beta, \kappa \in (0,1)$ depending on $p_0$ and $r_0$ such that the solution $w$ of (2.4) fulfills $w \in C^\beta(I; C^\kappa(\Omega))$ with*

$$\|w\|_{C^\beta(I;C^\kappa(\Omega))} \le C\|u_0\|_{U_{p_0,r_0}(\Omega)}.$$

*Additionally, provided that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, it holds that $w \in W^{1,\infty}(I; L^2(\Omega)) \cap L^\infty(I; H^2(\Omega))$ with the estimate*

$$\|\partial_t w\|_{L^\infty(I;L^2(\Omega))} + \|\nabla^2 w\|_{L^\infty(I;L^2(\Omega))} \le C\|\nabla^2 u_0\|_{L^2(\Omega)}.$$

*Proof.* The first result is proven, *e.g.*, in Theorem 3.1 from [8] setting $f = 0$ there. The second result follows from standard estimates for $z = \Delta w$ solving

$$\begin{aligned} \partial_t z - \Delta z &= 0, && \text{in } I \times \Omega, \\ z(0) &= \Delta u_0, && \text{on } \Omega \end{aligned}$$

and elliptic regularity. □

**Remark 2.9.** Using Theorem 4.5 from [8], the results of the Propositions 2.7 and 2.8 can also be proven under the assumptions $f \in L^q(I; W^{-1,s}(\Omega))$ with $\frac{1}{q} + \frac{N}{2s} < \frac{1}{2}$ and $u_0 \in \tilde U_{q_0,s_0}(\Omega)$ with $\frac{1}{q_0} + \frac{N}{2s_0} < \frac{1}{2}$.

Based on these lemmas, we can derive the main result of this section, namely the Hölder continuity of the solution of (1.1).

**Theorem 2.10.** *Let Assumption 2.1 be fulfilled. Then, there are $\beta, \kappa \in (0,1)$ such that the solution $u$ of (1.1) fulfills $u \in C^\beta(I; C^\kappa(\Omega))$ with a priori estimate*

$$\|u\|_{C^\beta(I;C^\kappa(\Omega))} \le C\big\{\|f\|_{L^p(I;L^r(\Omega))} + \|u_0\|_{U_{p_0,r_0}(\Omega)}\big\}.$$

*Proof.* We write the solution $u$ of (1.1) as $u=v+w$ where $v$ solves (2.3) with right-hand side $g=f-d(\cdot,\cdot,u)$ and $w$ solves (2.4). Using Assumption 2.1 and the boundedness of $u$ given by Proposition 2.6, we get by (2.2a)

$$\|d(\cdot,\cdot,u)\|_{L^p(I;L^r(\Omega))}=\|d(\cdot,\cdot,u)-d(\cdot,\cdot,0)\|_{L^p(I;L^r(\Omega))}\le C\|u\|_{L^\infty(I\times\Omega)}\le C\big\{\|f\|_{L^p(I;L^r(\Omega))}+\|u_0\|_{L^\infty(\Omega)}\big\}.$$

Hence, $g$ lies in $L^p(I;L^r(\Omega))$ and Proposition 2.7 implies the existence of $\beta_1,\kappa_1\in(0,1)$ such that

$$\|v\|_{C^{\beta_1}(I;C^{\kappa_1}(\Omega))}\le C\|g\|_{L^p(I;L^r(\Omega))}\le C\big\{\|f\|_{L^p(I;L^r(\Omega))}+\|u_0\|_{L^\infty(\Omega)}\big\}.$$

Further, by Proposition 2.8, there are $\beta_2,\kappa_2\in(0,1)$ such that

$$\|w\|_{C^{\beta_2}(I;C^{\kappa_2}(\Omega))}\le C\|u_0\|_{U_{p_0,r_0}(\Omega)}.$$

Then, setting $\beta=\min\{\beta_1,\beta_2\}$ and $\kappa=\min\{\kappa_1,\kappa_2\}$ and using Lemma 2.4 yields the assertion for $u=v+w$. ☐

## 3. Discrete problem

To introduce the time discontinuous Galerkin discretization for the problem, we partition the interval $(0,T]$ into subintervals $I_m=(t_{m-1},t_m]$ of length $k_m=t_m-t_{m-1}$, where $0=t_0<t_1<\cdots<t_{M-1}<t_M=T$. The maximal and minimal time steps are denoted by $k=\max_m k_m$ and $k_{\min}=\min_m k_m$, respectively.

**Assumption 3.1.** *We impose the following conditions on the temporal mesh (as, e.g., in [17] or [19]):*

- *There are constants $c_1,c_2>0$ independent of $k$ such that $k_{\min}\ge c_1k^{c_2}$.*
- *There is a constant $c>0$ independent of $k$ such that for all $m=1,2,\dots,M-1$ it holds $c^{-1}\le\frac{k_m}{k_{m+1}}\le c$.*
- *It holds $k\le\frac{1}{4}T$.*

*Further, let $\gamma\ge0$ be such that (2.2b) holds. If $\gamma>0$, we make the following assumption on the smallness of $k$:*

- *There is $0<\rho<1$ such that $k$ fulfills $k\le\frac{\rho}{\gamma}$.*

*If $\gamma=0$, no further assumption on $k$ has to be made.*

For the discretization in space with discretization parameter $h>0$, let $\mathcal{T}$ denote a quasi-uniform triangulation of $\Omega$ with mesh size $h$, *i.e.*, $\mathcal{T}=\{\tau\}$ is a partition of $\Omega$ into cells (triangles or tetrahedrons) $\tau$ of diameter $h_\tau$ such that for $h=\max_\tau h_\tau$,

$$\operatorname{diam}(\tau)\le h\le C|\tau|^{\frac{1}{N}},\quad\forall\tau\in\mathcal{T}.$$

Let $V_h$ be the set of all functions in $H^1_0(\Omega)$ that are Lagrange polynomials of order $\nu\ge1$ on each $\tau\in\mathcal{T}$. We consider the space-time finite element space

$$X^{0,\nu}_{k,h}=\left\{\,v_{kh}\in L^2(I;V_h)\;\middle|\;v_{kh,m}:=v_{kh}|_{I_m}\in\mathcal{P}_0(I_m;V_h),\ m=1,2,\dots,M\,\right\},$$

where $\mathcal{P}_0(I;V)$ is the space of constant polynomial functions in time with values in a Banach space $V$.

Throughout, we denote by $P_h\colon L^2(\Omega)\to V_h$ the spatial orthogonal $L^2$ projection and by $R_h\colon H^1_0(\Omega)\to V_h$ the spatial Ritz projection. Moreover, we introduce the discrete Laplace operator $\Delta_h\colon V_h\to V_h$ defined by

$$(-\Delta_hv_h,\varphi_h)_\Omega=(\nabla v_h,\nabla\varphi_h)_\Omega,\quad\forall\varphi_h\in V_h.$$

Further, we denote by $P_k$ the temporal $L^2$ projection given for a function $v\in L^1(I)$ by

$$(P_kv)\big|_{I_m}=\frac{1}{k_m}\int_{I_m}v(t)\,\mathrm{d}t,\qquad m=1,2,\dots,M.$$

Finally, the projection $\Pi_k$ is given for $v \in C(\bar{I})$ by

$$(\Pi_k v)\big|_{I_m} = v(t_m), \qquad m = 1, 2, \ldots, M.$$

The extension of these operators to space- and time-dependent functions is obvious.

We will employ the following notation for time-dependent functions $v$:

$$v_m^+ = \lim_{\varepsilon \to 0^+} v(t_m + \varepsilon), \quad v_m^- = \lim_{\varepsilon \to 0^+} v(t_m - \varepsilon), \quad [v]_m = v_m^+ - v_m^-.$$

Note, that by definition, for $v_{kh} \in X_{k,h}^{0,\nu}$, it holds

$$v_{kh,m}^+ = v_{kh,m+1}, \quad v_{kh,m}^- = v_{kh,m}, \quad [v_{kh}]_m = v_{kh,m+1} - v_{kh,m}.$$

Based on these preparations, we define the bilinear form $B$ by

$$B(u, \varphi) = \sum_{m=1}^{M} \langle \partial_t u, \varphi \rangle_{I_m \times \Omega} + (\nabla u, \nabla \varphi)_{I \times \Omega} + \sum_{m=2}^{M} ([u]_{m-1}, \varphi_{m-1}^+)_\Omega + (u_0^+, \varphi_0^+)_\Omega, \tag{3.1}$$

where $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{I_m \times \Omega}$ are the usual $L^2$ space and space-time inner products, $\langle \cdot,\cdot \rangle_{I_m \times \Omega}$ is the duality pairing between $L^2(I_m; H^{-1}(\Omega))$ and $L^2(I_m; H_0^1(\Omega))$. Rearranging the terms in (3.1), we obtain an equivalent (dual) expression for $B$:

$$B(u, \varphi) = -\sum_{m=1}^{M} \langle u, \partial_t \varphi \rangle_{I_m \times \Omega} + (\nabla u, \nabla \varphi)_{I \times \Omega} - \sum_{m=1}^{M-1} (u_m^-, [\varphi]_m)_\Omega + (u_M^-, \varphi_M^-)_\Omega. \tag{3.2}$$

We note, that the first sum in (3.1) vanishes for $u = u_{kh} \in X_{k,h}^{0,\nu}$ and the first sum in (3.2) for $\varphi = \varphi_{kh} \in X_{k,h}^{0,\nu}$, respectively. Hence, on $X_{k,h}^{0,\nu} \times X_{k,h}^{0,\nu}$, the semilinear form $B$ can be reduced to

$$B(u_{kh}, \varphi_{kh}) = (\nabla u_{kh}, \nabla \varphi_{kh})_{I \times \Omega} + \sum_{m=2}^{M} ([u_{kh}]_{m-1}, \varphi_{kh,m})_\Omega + (u_{kh,1}, \varphi_{kh,1})_\Omega \tag{3.3}$$

and

$$B(u_{kh}, \varphi_{kh}) = (\nabla u_{kh}, \nabla \varphi_{kh})_{I \times \Omega} - \sum_{m=1}^{M-1} (u_{kh,m}, [\varphi_{kh}]_m)_\Omega + (u_{kh,M}, \varphi_{kh,M})_\Omega. \tag{3.4}$$

Then, we define the fully discrete cG($\nu$)dG(0) approximation $u_{kh} \in X_{k,h}^{0,\nu}$ of (1.1) by

$$B(u_{kh}, \varphi_{kh}) + (d(\cdot,\cdot,u_{kh}), \varphi_{kh})_{I \times \Omega} = (f, \varphi_{kh})_{I \times \Omega} + (u_0, \varphi_{kh,1})_\Omega, \quad \forall \varphi_{kh} \in X_{k,h}^{0,\nu}. \tag{3.5}$$

**Theorem 3.2.** *Under the Assumptions 2.1 and 3.1, there is a unique solution $u_{kh} \in X_{k,h}^{0,\nu}$ of (3.5).*

*Proof.* Using (3.3), problem (3.5) can be written as time stepping scheme for $u_{kh,m} = u_{kh}\big|_{I_m}$ for $m = 1, 2, \ldots, M$ as follows:

$$k_m(\nabla u_{kh,m}, \nabla \varphi_h)_\Omega + (u_{kh,m} + k_m \bar{d}_m(\cdot, u_{kh,m}), \varphi_h)_\Omega = (u_{kh,m-1} + k_m \bar{f}_m, \varphi_h)_\Omega, \quad \forall \varphi_h \in V_h,$$

where $u_{kh,0} = P_h u_0$ and the mean values $\bar{d}_m$ and $\bar{f}_m$ are given on $I \times \Omega$ by

$$\bar{d}_m(x, u) = \frac{1}{k_m} \int_{I_m} d(t, x, u)\, dt \text{ for } u \in \mathbb{R} \quad \text{and} \quad \bar{f}_m(x) = \frac{1}{k_m} \int_{I_m} f(t, x)\, dt.$$

Hence, in each time step, the following discrete semilinear elliptic equation for $u_{kh,m}$ with given $u_{kh,m-1}$ has to be solved:

$$k_m(\nabla u_{kh,m}, \nabla\varphi_h)_\Omega + (\tilde{d}_m(\cdot, u_{kh,m}), \varphi_h)_\Omega = (u_{kh,m-1} + k_m \bar{f}_m, \varphi_h)_\Omega, \quad \forall \varphi_h \in V_h. \tag{3.6}$$

The nonlinearity $\tilde{d}_m$ is given for $u \in \mathbb{R}$ as $\tilde{d}_m(\cdot, u) = u + k_m \bar{d}_m(\cdot, u)$. Hence, Assumption 3.1 and (2.2b) imply $\partial_u \tilde{d}(\cdot, u) \geq 1 - k_m\gamma \geq 1 - \rho > 0$ for $\gamma > 0$ and $\partial_u \tilde{d}(\cdot, u) \geq 1$ independent of $k_m$ for $\gamma = 0$. The remaining assumptions on $d$ carry over to $\tilde{d}$ and ensures the unique solvability of (3.6) for $m = 1, 2, \ldots, M$ by application of Brouwer's fixed-point theorem, see, *e.g.*, [5]. □

## 4. Discrete maximal parabolic estimates for a linear auxiliary equation

For given $g \in L^1(I \times \Omega)$, we consider the discrete linear auxiliary equation for $v_{kh} \in X^{0,\nu}_{k,h}$

$$B(v_{kh}, \varphi_{kh}) + (bv_{kh}, \varphi_{kh})_{I\times\Omega} = (g, \varphi_{kh})_{I\times\Omega}, \quad \forall \varphi_{kh} \in X^{0,\nu}_{k,h} \tag{4.1}$$

with a coefficient $b \in L^\infty(I\times\Omega)$ fulfilling $b(t,x) \geq -\gamma$ for $\gamma \geq 0$ from Assumption 2.1 and almost all $(t,x) \in I\times\Omega$.

For the solution $v_{kh}$ of (4.1), discrete maximal parabolic estimates in various norms are available in the literature in the case $b = 0$, see [17]. In this section, we extend these results to the case $b \neq 0$. The extended results will be used later in the Sections 5 and 6 to prove the results for the semilinear problem.

Before doing so, we start with an existence result for (4.1).

**Lemma 4.1.** *Under Assumption 3.1, there is a unique solution $v_{kh} \in X^{0,\nu}_{k,h}$ of* (4.1).

*Proof.* By setting $d(\cdot, \cdot, v_{kh}) = bv_{kh}$, the assertion follows directly from Theorem 3.2. □

**Lemma 4.2.** *Let Assumption 3.1 be fulfilled and $g \in L^1(I; L^2(\Omega))$. Then, for the solution $v_{kh} \in X^{0,\nu}_{k,h}$ of* (4.1) *there holds*

$$\|v_{kh}\|_{L^\infty(I;L^2(\Omega))} \leq C\|g\|_{L^1(I;L^2(\Omega))}$$

*with a constant $C$ independent of $h$, $k$, $g$, and $b$.*

*Proof.* We consider the dual problem for $z_{kh} \in X^{0,\nu}_{k,h}$ given by

$$B(\varphi_{kh}, z_{kh}) + (b\varphi_{kh}, z_{kh})_{I\times\Omega} = (v_{kh,M}, \varphi_{kh,M})_\Omega, \quad \forall \varphi_{kh} \in X^{0,\nu}_{k,h}.$$

Using (3.4), $z_{kh,m}$ satisfies for $m = M-1, M-2, \ldots, 1$ the scheme

$$k_m(\nabla\varphi_h, \nabla z_{kh,m})_\Omega + (z_{kh,m} + k_m \bar{b}_m z_{kh,m}, \varphi_h)_\Omega = (z_{kh,m+1}, \varphi_h)_\Omega, \quad \forall \varphi_h \in V_h, \tag{4.2}$$

where $z_{kh,M} = v_{kh,M}$ and $\bar{b}_m$ is given as before by

$$\bar{b}_m(x) = \frac{1}{k_m}\int_{I_m} b(t,x)\,\mathrm{d}t.$$

To proceed, we will first prove the boundedness of $z_{kh}$ in $L^\infty(I; L^2(\Omega))$. To this end, we employ the discrete transformation argument from [18]. For $\mu > 0$ a sufficient large number to be chosen later let $y_{kh,m}$ be defined as

$$y_{kh,m} = z_{kh,m}\prod_{l=m}^{M}\frac{1}{1+\mu k_l}, \qquad m = 1, 2, \ldots, M.$$

Then, by (4.2), we get

$$k_m \prod_{l=m}^{M} (1+\mu k_l)(\nabla\varphi_h, \nabla y_{kh,m})_\Omega + \prod_{l=m}^{M}(1+\mu k_l)(y_{kh,m} + k_m \bar{b}_m y_{kh,m}, \varphi_h)_\Omega$$
$$= \prod_{l=m+1}^{M}(1+\mu k_l)(y_{kh,m+1}, \varphi_h)_\Omega, \quad \forall \varphi_h \in V_h.$$

Dividing both sides by $\prod_{l=m+1}^{M}(1+\mu k_l)$ yields

$$k_m(1+\mu k_m)(\nabla\varphi_h, \nabla y_{kh,m})_\Omega + (1+\mu k_m)(y_{kh,m} + k_m \bar{b}_m y_{kh,m}, \varphi_h)_\Omega = (y_{kh,m+1}, \varphi_h)_\Omega, \quad \forall \varphi_h \in V_h,$$

which can be rewritten as

$$k_m(1+\mu k_m)(\nabla\varphi_h, \nabla y_{kh,m})_\Omega + (y_{kh,m} + k_m \tilde{b}_m y_{kh,m}, \varphi_h)_\Omega = (y_{kh,m+1}, \varphi_h)_\Omega, \quad \forall \varphi_h \in V_h \tag{4.3}$$

with $\tilde{b}_m = \bar{b}_m + \mu(1 + k_m \bar{b}_m)$. Using Assumption 3.1 and choosing $\mu \ge \frac{\gamma}{1-\rho}$ yields

$$\tilde{b}_m \ge -\gamma + \mu(1 - k_m\gamma) \ge -\gamma + \mu(1-\rho) \ge 0.$$

Then, by testing (4.3) with $\varphi_h = y_{hk,m}$, we get $\|y_{kh,m}\|^2_{L^2(\Omega)} \le (y_{kh,m+1}, y_{kh,m})_\Omega$, which implies $\|y_{kh,m}\|_{L^2(\Omega)} \le \|y_{kh,m+1}\|_{L^2(\Omega)}$. Using this recursively for $m = 1, 2, \ldots, M-1$, we get

$$\|y_{kh,1}\|_{L^2(\Omega)} \le \|y_{kh,M}\|_{L^2(\Omega)} = \|v_{kh,M}\|_{L^2(\Omega)}.$$

Transforming back to $z_{kh,m}$ and using $1 + \mu k_l \le e^{\mu k_l}$ yields

$$\|z_{kh,1}\|_{L^2(\Omega)} = \|y_{kh,1}\|_{L^2(\Omega)} \prod_{l=1}^{M}(1+\mu k_l) \le e^{\mu T}\|v_{kh,M}\|_{L^2(\Omega)}$$

and hence

$$\|z_{kh}\|_{L^\infty(I;L^2(\Omega))} \le e^{\mu T}\|v_{kh,M}\|_{L^2(\Omega)}.$$

Using this and (4.1), we obtain

$$\begin{aligned}\|v_{kh,M}\|^2_{L^2(\Omega)} &= B(v_{kh}, z_{kh}) + (bv_{kh}, z_{kh})_{I\times\Omega} = (g, z_{kh})_{I\times\Omega} \\ &\le \|g\|_{L^1(I;L^2(\Omega))}\|z_{kh}\|_{L^\infty(I;L^2(\Omega))} \le e^{\mu T}\|g\|_{L^1(I;L^2(\Omega))}\|v_{kh,M}\|_{L^2(\Omega)},\end{aligned}$$

which completes the proof. □

The next lemma provides a discrete maximal parabolic estimate for $v_{kh}$ with respect to the $L^\infty(I;L^2(\Omega))$ norm.

**Lemma 4.3.** *Let Assumption 3.1 be fulfilled and $g \in L^\infty(I;L^2(\Omega))$. Then, for the solution $v_{kh} \in X^{0,\nu}_{k,h}$ of (4.1) there holds*

$$\|\Delta_h v_{kh}\|_{L^\infty(I;L^2(\Omega))} + \max_{1\le m\le M}\left\|\frac{[v_{kh}]_{m-1}}{k_m}\right\|_{L^2(\Omega)} \le C \ln\frac{T}{k}\{1 + \|b\|_{L^\infty(I\times\Omega)}\}\|g\|_{L^\infty(I;L^2(\Omega))}$$

*with a constant $C$ independent of $h$, $k$, $g$, and $b$.*

*Proof.* The solution $v_{kh} \in X_{k,h}^{0,\nu}$ of (4.1) fulfills

$$B(v_{kh}, \varphi_{kh}) = (\tilde{g}, \varphi_{kh})_{I\times\Omega}, \quad \forall \varphi_{kh} \in X_{k,h}^{0,\nu}$$

with $\tilde{g} = g - bv_{kh}$. Using Lemma 4.2, we can estimate

$$\begin{aligned}\|\tilde{g}\|_{L^\infty(I;L^2(\Omega))} &\le \|g\|_{L^\infty(I;L^2(\Omega))} + \|b\|_{L^\infty(I\times\Omega)}\|v_{kh}\|_{L^\infty(I;L^2(\Omega))}\\ &\le \|g\|_{L^\infty(I;L^2(\Omega))} + \|b\|_{L^\infty(I\times\Omega)}\|g\|_{L^1(I;L^2(\Omega))}\\ &\le C\big\{1 + \|b\|_{L^\infty(I\times\Omega)}\big\}\|g\|_{L^\infty(I;L^2(\Omega))}.\end{aligned}$$

Applying the discrete maximal parabolic regularity result of Theorem 2 and Corollary 2 from [17], we obtain the desired estimate for $v_{kh}$. □

Before continuing with estimates for the solution of (4.1), we recall for completeness two well-known results for finite element functions.

**Lemma 4.4.** *For any $w_h \in V_h$, it holds*

$$\|w_h\|_{L^\infty(\Omega)} \le C\|\Delta_h w_h\|_{L^2(\Omega)} \qquad \textit{and} \qquad \|w_h\|_{L^2(\Omega)} \le C\|\Delta_h w_h\|_{L^1(\Omega)}.$$

*Proof.* Let $w \in H_0^1(\Omega)$ given as the solution of

$$(\nabla w, \nabla\varphi)_\Omega = (-\Delta_h w_h, \varphi)_\Omega, \quad \forall \varphi \in H_0^1(\Omega).$$

Note, that by construction, it holds $R_h w = w_h$ for the Ritz projection $R_h$. Elliptic regularity yields $w \in H^2(\Omega)$ with $\|w\|_{H^2(\Omega)} \le C\|\Delta_h w_h\|_{L^2(\Omega)}$. Further, it holds $\|w\|_{L^2(\Omega)} \le C\|\Delta_h w_h\|_{L^1(\Omega)}$ For the first assertion, let $i_h\colon C(\bar{\Omega}) \to V_h$ be the nodal interpolant. Since $\nu \ge 1$, by standard estimates for $w_h - w$ and for the interpolation error $w - i_h w$ as well as an inverse estimate, we get

$$\begin{aligned}\|w_h\|_{L^\infty(\Omega)} &\le \|w_h - i_h w\|_{L^\infty(\Omega)} + \|i_h w - w\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}\\ &\le Ch^{-\frac{N}{2}}\big\{\|w_h - w\|_{L^2(\Omega)} + \|w - i_h w\|_{L^2(\Omega)}\big\} + \|i_h w - w\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}\\ &\le C(h^{2-\frac{N}{2}} + 1)\|\nabla^2 w\|_{L^2(\Omega)} \le C\|\Delta_h w_h\|_{L^2(\Omega)}.\end{aligned}$$

Similarly, we get for the second assertion that

$$\begin{aligned}\|w_h\|_{L^2(\Omega)} \le \|w_h - w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} &\le C\big\{h^2\|\Delta_h w_h\|_{L^2(\Omega)} + \|\Delta_h w_h\|_{L^1}\big\}\\ &\le C(h^{2-\frac{N}{2}} + 1)\|\Delta_h w_h\|_{L^1(\Omega)} \le C\|\Delta_h w_h\|_{L^1(\Omega)}.\end{aligned}$$

This completes the proof. □

The next lemma provides a discrete maximal parabolic estimate for $v_{kh}$ with respect to the $L^1(I\times\Omega)$ norm.

**Lemma 4.5.** *Let Assumption 3.1 be fulfilled and $g \in L^1(I\times\Omega)$. Then, for the solution $v_{kh} \in X_{k,h}^{0,\nu}$ of (4.1) there holds*

$$\|\Delta_h v_{kh}\|_{L^1(I\times\Omega)} + \sum_{m=1}^{M} \|[v_{kh}]_{m-1}\|_{L^1(\Omega)} \le C\left(\ln\frac{T}{k}\right)^2 \big\{1 + \|b\|^2_{L^\infty(I\times\Omega)}\big\}\|g\|_{L^1(I\times\Omega)}$$

*with a constant $C$ independent of $h$, $k$, $g$, and $b$.*

*Proof.* We consider the dual problem for $z_{kh} \in X^{0,\nu}_{k,h}$ given by

$$B(\varphi_{kh}, z_{kh}) + (b\varphi_{kh}, z_{kh})_{I\times\Omega} = (\varphi_{kh}, \operatorname{sgn} v_{kh})_{I\times\Omega}, \quad \forall \varphi_{kh} \in X^{0,\nu}_{k,h},$$

where sgn: $L^2(I \times \Omega) \to L^\infty(I \times \Omega)$ denotes the sign function. Then, it holds

$$\|v_{kh}\|_{L^1(I\times\Omega)} = B(v_{kh}, z_{kh}) + (bv_{kh}, z_{kh})_{I\times\Omega} = (g, z_{kh})_{I\times\Omega} \le \|g\|_{L^1(I\times\Omega)}\|z_{kh}\|_{L^\infty(I\times\Omega)}.$$

By Lemma 4.3 applied to the dual solution $z_{kh}$ and Lemma 4.4 applied separately to $w_h = z_{kh,m}$ for $m = 1, 2, \dots, M$, we get

$$\begin{aligned}\|z_{kh}\|_{L^\infty(I\times\Omega)} &\le C\|\Delta_h z_{kh}\|_{L^\infty(I;L^2(\Omega))} \le C\ \ln\frac{T}{k}\big\{1 + \|b\|_{L^\infty(I\times\Omega)}\big\}\|\operatorname{sgn} v_k\|_{L^\infty(I;L^2(\Omega))}\\ &\le C\ln\frac{T}{k}\big\{1 + \|b\|_{L^\infty(I\times\Omega)}\big\}\end{aligned}$$

and consequently

$$\|v_{kh}\|_{L^1(I\times\Omega)} \le C\ln\frac{T}{k}\big\{1 + \|b\|_{L^\infty(I\times\Omega)}\big\}\|g\|_{L^1(I\times\Omega)}. \tag{4.4}$$

As before, this implies for $\tilde{g} = g - bv_{kh}$ that

$$\|\tilde{g}\|_{L^1(I\times\Omega)} \le C\ln\frac{T}{k}\big\{1 + \|b\|^2_{L^\infty(I\times\Omega)}\big\}\|g\|_{L^1(I\times\Omega)},$$

which yields the assertion again by means of Theorem 2 and Corollary 2 from [17]. □

## 5. Boundedness of the discrete solution

In this section, we derive the boundedness of the solution $u_{kh}$ to (3.5) in $L^\infty(I \times \Omega)$. In the case $N = 2$, this was already proven in [21] using a different approach than used here. The technique employed there does not extend to the three-dimensional situation, due to the used inverse inequality.

First, we introduce a modified nonlinearity $d_R$ with bounded derivative $\partial_u d_R$, To this end, let for $R > 0$ the nonlinearity $d_R$ be defined by

$$d_R(t, x, u) = \begin{cases} d(t, x, R) + (u - R)\partial_u d(t, x, R), & \text{for } u > R,\\ d(t, x, u), & \text{for } |u| \le R,\\ d(t, x, -R) + (u + R)\partial_u d(t, x, -R), & \text{for } u < -R.\end{cases}$$

Further, let $u^R$ and $u^R_{kh}$ be the solutions of the continuous problem (1.1) and the discrete problem (3.5) with $d_R$ instead of $d$. Assumption (2.2a) on the local boundedness of $\partial_u d$ implies the global boundedness of

$$\partial_u d_R(t, x, u) = \begin{cases} \partial_u d(t, x, R), & \text{for } u > R,\\ \partial_u d(t, x, u), & \text{for } |u| \le R,\\ \partial_u d(t, x, -R), & \text{for } u < -R\end{cases}$$

by a constant $C_R$ depending on $R$:

$$|\partial_u d_R(t, x, u)| \le C_R \quad \text{for almost all } (t, x) \in I \times \Omega \text{ and all } u \in \mathbb{R}. \tag{5.1}$$

Additionally, by (2.2b), it holds

$$\partial_u d_R(t, x, u) \ge -\gamma. \tag{5.2}$$

In the following lemma, we state an quasi best approximation result for the error between $u^R$ and $u^R_{kh}$ with respect to the $L^\infty(I \times \Omega)$ norm:

**Lemma 5.1.** *Let the Assumptions 2.1 and 3.1 be fulfilled, $u^R$ be the solution of (1.1), and $u^R_{kh} \in X^{0,\nu}_{k,h}$ be the solution of (3.5) each with $d_R$ instead of $d$. Then, it holds*

$$\|u^R - u^R_{kh}\|_{L^\infty(I\times\Omega)} \le C_R |\ln h| \left(\ln \frac{T}{k}\right)^2 \|u^R - \chi_{kh}\|_{L^\infty(I\times\Omega)}$$

*for any $\chi_{kh} \in X^{0,\nu}_{k,h}$.*

*Proof.* Let $\chi_{kh}$ be an arbitrary but fixed element of $X^{0,\nu}_{k,h}$. We decompose the error $e = u^R - u^R_{kh}$ as

$$e = (u^R - \chi_{kh}) + (\chi_{kh} - u^R_{kh}) = \eta + \xi_{kh},$$

By Galerkin orthogonality, there holds

$$B(e, \varphi_{kh}) + (d_R(\cdot,\cdot,u^R) - d_R(\cdot,\cdot,u^R_{kh}), \varphi_{kh})_{I\times\Omega} = 0, \qquad \forall \varphi_{kh} \in X^{0,\nu}_{k,h}$$

and therefore

$$B(\xi_{kh}, \varphi_{kh}) + (d_R(\cdot,\cdot,\chi_{kh}) - d_R(\cdot,\cdot,u^R_{kh}), \varphi_{kh})_{I\times\Omega} = -B(\eta, \varphi_{kh}) - (d_R(\cdot,\cdot,u^R) - d_R(\cdot,\cdot,\chi_{kh}), \varphi_{kh})_{I\times\Omega} \quad (5.3)$$

for all $\varphi_{kh} \in X^{0,\nu}_{k,h}$. To formulate an appropriate dual problem, we define the coefficient $b$ by

$$b = \int_0^1 \partial_u d_R(\cdot,\cdot,u^R_{kh} + s(\chi_{kh} - u^R_{kh}))\,\mathrm{d}s.$$

By (5.1), it follows $\|b\|_{L^\infty(I\times\Omega)} \le C_R$ and (5.2) implies $b(t,x) \ge -\gamma$ for almost all $(t,x) \in I \times \Omega$. Further, by construction, it holds

$$b\xi_{kh} = d_R(\cdot,\cdot,\chi_{kh}) - d_R(\cdot,\cdot,u^R_{kh}).$$

We will estimate $\xi_{kh,M}(x_0)$ by using a duality argument. To this end, let $\tilde{\delta}_{x_0}\colon \Omega \to \mathbb{R}$ be a smoothed Dirac function with support contained in a single spatial cell $\bar{\tau} \ni x_0$ fulfilling

$$\int_\tau \tilde{\delta}_{x_0}(x)\chi(x)\,\mathrm{d}x = \chi(x_0), \quad \forall \chi \in \mathcal{P}_\nu(\tau) \quad \text{and} \quad \|\tilde{\delta}_{x_0}\|_{L^1(\Omega)} \le C.$$

The explicit construction of such a function is given for instance in Appendix from [24]. Further, let $\theta_M\colon I \to \mathbb{R}$ be a smooth function with support contained in $I_M$ and fulfilling $\theta_M \ge 0$ as well as

$$\int_{I_M} \theta_M(t)\,\mathrm{d}t = 1.$$

Them, let $z_{kh} \in X^{0,\nu}_{k,h}$ be given as solution of

$$B(\varphi_{kh}, z_{kh}) + (b\varphi_{kh}, z_{kh})_{I\times\Omega} = (\theta_M \tilde{\delta}_{x_0}, \varphi_{kh})_{I\times\Omega}, \quad \forall \varphi_{kh} \in X^{0,\nu}_{k,h}.$$

Using (5.3), we obtain

$$\begin{aligned}
\xi_{kh,M}(x_0) &= (\theta_M \tilde{\delta}_{x_0}, \xi_{kh})_{I\times\Omega} = B(\xi_{kh}, z_{kh}) + (b\xi_{kh}, z_{kh})_{I\times\Omega} \\
&= B(\xi_{kh}, z_{kh}) + (d_R(\cdot,\cdot,\chi_{kh}) - d_R(\cdot,\cdot,u^R_{kh}), z_{kh})_{I\times\Omega} \\
&= -B(\eta, z_{kh}) - (d_R(\cdot,\cdot,u^R) - d_R(\cdot,\cdot,\chi_{kh}), z_{kh})_{I\times\Omega} \\
&= -(\nabla\eta, \nabla z_{kh})_{I\times\Omega} + \sum_{m=1}^M (\eta_m, [z_{kh}]_m)_\Omega - (d_R(\cdot,\cdot,u^R) - d_R(\cdot,\cdot,\chi_{kh}), z_{kh})_{I\times\Omega},
\end{aligned} \quad (5.4)$$

where $\eta_m = u^R(t_m) - \chi_{kh,m}$. For the first term on the right-hand side of (5.4), we get

$$\begin{aligned}|(\nabla\eta,\nabla z_{kh})_{I\times\Omega}| &= |(\nabla R_h\eta,\nabla z_{kh})_{I\times\Omega}| = |(R_h\eta,\Delta_h z_{kh})_{I\times\Omega}| \le \|R_h\eta\|_{L^\infty(I\times\Omega)}\|\Delta_h z_{kh}\|_{L^1(I\times\Omega)}\\ &\le C|\ln h|\|\eta\|_{L^\infty(I\times\Omega)}\|\Delta_h z_{kh}\|_{L^1(I\times\Omega)},\end{aligned}$$

where the stability of $R_h$ in $L^\infty(\Omega)$ from [23] for $N=2$ and from Theorem 12 of [14] for $N=3$ was used. For the second term on the right-hand side of (5.4), it follows

$$\left|\sum_{m=1}^M (\eta_m,[z_{kh}]_m)_\Omega\right| \le \sum_{m=1}^M \|\eta_m\|_{L^\infty(\Omega)}\|[z_{kh}]_m\|_{L^1(\Omega)} \le \|\eta\|_{L^\infty(I\times\Omega)}\sum_{m=1}^M \|[z_{kh}]_m\|_{L^1(\Omega)}.$$

Finally, for the third term on the right-hand side of (5.4), we obtain due to (5.1) that

$$|(d_R(\cdot,\cdot,u^R) - d_R(\cdot,\cdot,\chi_{kh}), z_{kh})_{I\times\Omega}| \le C_R\|\eta\|_{L^\infty(I\times\Omega)}\|z_{kh}\|_{L^1(I\times\Omega)}.$$

Combining the previous estimates and applying Lemma 4.5 to the dual problem considered here as well as Lemma 4.4 for $\|z_{kh}\|_{L^1(I\times\Omega)}$ leads to

$$\begin{aligned}\xi_{kh,M}(x_0) &\le C_R|\ln h|\|\eta\|_{L^\infty(I\times\Omega)}\left\{\|\Delta_h z_{kh}\|_{L^1(I\times\Omega)} + \sum_{m=1}^M \|[z_{kh}]_m\|_{L^1(\Omega)} + \|z_{kh}\|_{L^1(I\times\Omega)}\right\}\\ &\le C_R|\ln h|\left(\ln\frac{T}{k}\right)^2\|\eta\|_{L^\infty(I\times\Omega)}\|\theta_M\tilde{\delta}_{x_0}\|_{L^1(I\times\Omega)}.\end{aligned}$$

Using the bound

$$\|\theta_M\tilde{\delta}_{x_0}\|_{L^1(I\times\Omega)} = \|\theta_M\|_{L^1(I)}\|\tilde{\delta}_{x_0}\|_{L^1(\Omega)} \le C$$

concludes the estimate of $\xi_{kh}$. Then, we get for the error

$$\|e\|_{L^\infty(I\times\Omega)} \le \|\eta\|_{L^\infty(I\times\Omega)} + \|\xi_{kh}\|_{L^\infty(I\times\Omega)} \le C_R|\ln h|\left(\ln\frac{T}{k}\right)^2\|\eta\|_{L^\infty(I;L^\infty(\Omega))},$$

which states the assertion. □

To formulate the boundedness result for $u_{kh}\in X^{0,\nu}_{k,h}$, we require the following mild assumption on $k$ and $h$.

**Assumption 5.2.** *There exist $\sigma > 0$ (arbitrary small) and a constant $C > 0$ such that*

$$k \le Ch^\sigma.$$

**Remark 5.3.** This is a very mild condition, since $\sigma$ can be choosen arbitrary small. We do not require any conditions like $k\le Ch$ or even $k\le Ch^2$ in the whole paper.

**Theorem 5.4.** *Let the Assumptions 2.1, 3.1, and 5.2 be fulfilled. Then, there exists $h_0 > 0$ and a constant $C > 0$ independent of $k$ and $h$ such that for all $h < h_0$ the solution $u_{kh}\in X^{0,\nu}_{k,h}$ of (3.5) fulfills*

$$\|u_{kh}\|_{L^\infty(I\times\Omega)} \le \|u\|_{L^\infty(I\times\Omega)} + 1.$$

*Proof.* Let $R = \|u\|_{L^\infty(I\times\Omega)} + 1$. By the boundedness of $u$, see Proposition 2.6, we have $R < \infty$. Due to this choice, it holds $u^R = u$. Using the estimate from Lemma 5.1, setting $\chi_{kh} = P_kP_hu$ and using the stability of the temporal $L^2$ projection $P_k$ in $L^\infty(I\times\Omega)$, we get

$$\begin{aligned}\|u - u^R_{kh}\|_{L^\infty(I\times\Omega)} &\le C_R|\ln h|\left(\ln\frac{T}{k}\right)^2\left\{\|u^R - P_ku^R\|_{L^\infty(I\times\Omega)} + \|P_k(u^R - P_hu^R)\|_{L^\infty(I\times\Omega)}\right\}\\ &\le C_R|\ln h|\left(\ln\frac{T}{k}\right)^2\left\{\|u^R - P_ku^R\|_{L^\infty(I\times\Omega)} + \|u^R - P_hu^R\|_{L^\infty(I\times\Omega)}\right\}.\end{aligned}$$

By standard estimates for $P_h$ and $P_k$ together with the regularity of $u$ from Theorem 2.10, it follows

$$\begin{aligned}\|u-u_{kh}^R\|_{L^\infty(I\times\Omega)} &\le C_R|\ln h|\left(\ln\frac{T}{k}\right)^2\left\{\|u-P_ku\|_{L^\infty(I\times\Omega)}+\|u-P_hu\|_{L^\infty(I\times\Omega)}\right\}\\ &\le C_R|\ln h|\left(\ln\frac{T}{k}\right)^2\left(k^\beta+h^\kappa\right)\|u\|_{C^\beta(I;C^\kappa(\Omega))}\\ &\le C_R|\ln h|\left(\ln\frac{T}{k}\right)^2\left(k^\beta+h^\kappa\right)\left\{\|f\|_{L^p(I;L^r(\Omega))}+\|u_0\|_{U_{p_0,r_0}(\Omega)}\right\}.\end{aligned}$$

Using Assumptions 5.2, it follows with $\delta=\min\{\sigma\beta,\kappa\}>0$

$$\|u-u_{kh}^R\|_{L^\infty(I\times\Omega)}\le C_R|\ln h|^3h^\delta.$$

Consequently, there exists $h_0>0$, such that for all $h<h_0$ we have $\|u-u_{kh}^R\|_{L^\infty(I\times\Omega)}\le 1$. This yields

$$\|u_{kh}^R\|_{L^\infty(I\times\Omega)}\le\|u\|_{L^\infty(I\times\Omega)}+\|u-u_{kh}^R\|_{L^\infty(I\times\Omega)}\le\|u\|_{L^\infty(I\times\Omega)}+1=R,$$

and therefore $u_{kh}=u_{kh}^R$. This gives the boundedness of $u_{kh}$. □

**Remark 5.5.** As already mentioned in the introduction, the result of this theorem is available in the literature for $N=2$, see Theorem 4.1 from [21]. The proof from [21] can not be extended to the three-dimensional case ($N=3$) due to application of an inverse inequality.

## 6. Error estimates

In this section, we provide (quasi) best approximation results and error estimates of the discretization error between the continuous solution $u$ of (1.1) and the discrete solution $u_{kh}$ of (3.5) in various norms. Basis of all given estimates is the boundedness of $u_{kh}$ given by Theorem 5.4.

We start with a best-approximation-type result in the $L^2(I\times\Omega)$ norm.

**Theorem 6.1.** *Let the Assumptions 2.1, 3.1 and 5.2 be fulfilled. Further, let $u$ be the solution of (1.1), and $u_{kh}\in X_{k,h}^{0,\nu}$ be the solution of (3.5). Then, it holds*

$$\|u-u_{kh}\|_{L^2(I\times\Omega)}\le C\left\{\|u-\chi_{kh}\|_{L^2(I\times\Omega)}+\|u-\Pi_ku\|_{L^2(I\times\Omega)}+\|u-R_hu\|_{L^2(I\times\Omega)}\right\}$$

*for any $\chi_{kh}\in X_{k,h}^{0,\nu}$.*

*Proof.* Due to the boundedness of $u$ by Proposition 2.6 and the boundedness of $u_{kh}$ by Theorem 5.4, we have

$$R_u=\|u\|_{L^\infty(I\times\Omega)}<\infty \quad\text{and}\quad R_{u_{kh}}=\sup_{k,h}\|u_{kh}\|_{L^\infty(I\times\Omega)}<\infty.$$

Choosing $R=\max(R_u,R_{u_{kh}})$ in Lemma 5.1, we directly obtain $u=u^R$ and $u_{kh}=u_{kh}^R$. Proceeding as in the proof of Lemma 5.1, we decompose

$$e=u-u_{kh}=(u-\chi_{kh})+(\chi_{kh}-u_{kh})=\eta+\xi_{kh}$$

and introduce the following dual problem for $z_{kh}\in X_{k,h}^{0,\nu}$:

$$B(\varphi_{kh},z_{kh})+(b\varphi_{kh},z_{kh})_{I\times\Omega}=(\xi_{kh},\varphi_{kh})_{I\times\Omega},\quad\forall\varphi_{kh}\in X_{k,h}^{0,\nu}$$

with $b$ as in the proof of Lemma 5.1. Testing with $\varphi_{kh} = \xi_{kh}$ yields

$$\begin{aligned}\|\xi_{kh}\|^2_{L^2(I\times\Omega)} &= B(\xi_{kh}, z_{kh}) + (b\xi_{kh}, z_{kh})_{I\times\Omega} \\ &= -(\nabla\eta, \nabla z_{kh})_{I\times\Omega} + \sum_{m=1}^{M} (\eta_m, [z_{kh}]_m)_\Omega - (d_R(\cdot,\cdot,u) - d_R(\cdot,\cdot,\chi_{kh}), z_{kh})_{I\times\Omega}.\end{aligned} \tag{6.1}$$

For the first term on the right-hand side of (6.1), we get

$$|(\nabla\eta, \nabla z_{kh})_{I\times\Omega}| = |(R_h\eta, \Delta_h z_{kh})_{I\times\Omega}| \le \|R_h\eta\|_{L^2(I\times\Omega)} \|\Delta_h z_{kh}\|_{L^2(I\times\Omega)}.$$

For the second term on the right-hand side of (6.1), it follows from the definition of $\Pi_k$ that

$$\eta_m = u(t_m) - \chi_{kh,m} = u(t_m) - \chi_{kh}(t_m) = (\Pi_k u)(t_m) - \Pi_k(\chi_{kh})(t_m) = (\Pi_k\eta)_m$$

and thus

$$\begin{aligned}\left|\sum_{m=1}^{M} (\eta_m, [z_{kh}]_m)_\Omega\right| &= \left|\sum_{m=1}^{M} ((\Pi_k\eta)_m, [z_{kh}]_m)_\Omega\right| \le \sum_{m=1}^{M} \|(\Pi_k\eta)_m\|_{L^2(\Omega)} \|[z_{kh}]_m\|_{L^2(\Omega)} \\ &\le \left(\sum_{m=1}^{M} k_m \|\Pi_k\eta\|^2_{L^2(\Omega)}\right)^{\frac12} \left(\sum_{m=1}^{M} k_m^{-1} \|[z_{kh}]_m\|^2_{L^2(\Omega)}\right)^{\frac12} \\ &= \|\Pi_k\eta\|_{L^2(I\times\Omega)} \left(\sum_{m=1}^{M} k_m^{-1} \|[z_{kh}]_m\|^2_{L^2(\Omega)}\right)^{\frac12}.\end{aligned}$$

Finally, for the third term on the right-hand side of (6.1), we obtain due to (5.1)

$$|(d_R(\cdot,\cdot,u) - d_R(\cdot,\cdot,\chi_{kh}), z_{kh})_{I\times\Omega}| \le C_R \|\eta\|_{L^2(I\times\Omega)} \|z_{kh}\|_{L^2(I\times\Omega)}.$$

It remains to bound the arising terms involving $z_{kh}$. By Lemma 4.2 applied to the dual problem for $z_{kh}$, we have $\|z_{kh}\|_{L^\infty(I;L^2(\Omega))} \le \|\xi_{kh}\|_{L^1(I;L^2(\Omega))}$ and consequently

$$\|bz_{kh}\|_{L^2(I\times\Omega)} \le \|b\|_{L^\infty(I\times\Omega)} \|z_{kh}\|_{L^2(I\times\Omega)} \le \|b\|_{L^\infty(I\times\Omega)} \|\xi_{kh}\|_{L^2(I\times\Omega)}.$$

Then, Corollary 4.2 from [20] applied to the rewritten dual problem for $z_{kh}$

$$B(\varphi_{kh}, z_{kh}) = (\xi_{kh} - bz_{kh}, \varphi_{kh})_{I\times\Omega}, \quad \forall \varphi_{kh} \in X^{0,\nu}_{k,h}$$

yields

$$\begin{aligned}\|\Delta_h z_{kh}\|_{L^2(I\times\Omega)} + \left(\sum_{m=1}^{M} k_m^{-1} \|[z_{kh}]_m\|^2_{L^2(\Omega)}\right)^{\frac12} &\le \|\xi_{kh} - bz_{kh}\|_{L^2(I\times\Omega)} \\ &\le \left\{1 + \|b\|_{L^\infty(I\times\Omega)}\right\} \|\xi_{kh}\|_{L^2(I\times\Omega)}.\end{aligned}$$

Using Lemma 4.4 to bound $\|z_{kh}\|_{L^2(I\times\Omega)}$ by $\|\Delta_h z_{kh}\|_{L^2(I\times\Omega)}$ and the boundedness of $\|b\|_{L^\infty(I\times\Omega)}$ due to (5.1), we obtain

$$\|\xi_{kh}\|_{L^2(I\times\Omega)} \le C\left\{\|\eta\|_{L^2(I\times\Omega)} + \|\Pi_k\eta\|_{L^2(I\times\Omega)} + \|R_h\eta\|_{L^2(I\times\Omega)}\right\}.$$

Then, the triangle inequality implies the assertion. □

Under slightly strengthened assumptions on $f$ and $u_0$ Theorem 6.1 yields an error estimate in the $L^2(I \times \Omega)$ norm of second order, which is optimal for $\nu = 1$, *i.e.*, for linear ansatz functions in space.

**Corollary 6.2.** *Let the Assumptions 2.1, 3.1 and 5.2 be fulfilled and additionally* $p, r \geq 2$ *and* $u_0 \in H_0^1(\Omega)$. *Then, for the solution* $u$ *of* (1.1), *it holds* $u \in H^1(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega))$ *with*

$$\|\partial_t u\|_{L^2(I\times\Omega)} + \|\nabla^2 u\|_{L^2(I\times\Omega)} \leq C\big\{\|f\|_{L^p(I;L^r(\Omega))} + \|\nabla u_0\|_{L^2(\Omega)} + \|u_0\|_{L^\infty(\Omega)}\big\}.$$

*Further, for the error between* $u$ *and the solution* $u_{kh} \in X_{k,h}^{0,\nu}$ *of* (3.5), *it holds*

$$\|u - u_{kh}\|_{L^2(I\times\Omega)} \leq C(k + h^2)\big\{\|f\|_{L^p(I;L^r(\Omega))} + \|\nabla u_0\|_{L^2(\Omega)} + \|u_0\|_{L^\infty(\Omega)}\big\}.$$

*Proof.* By putting the nonlinearity $d$ to the right-hand side as

$$\partial_t u - \Delta u = f - d(\cdot, \cdot, u),$$

regularity theory for the linear equation (*cf.*, *e.g.*, [11], Chap. 7, Thm. 5) yields as in the proof of Theorem 2.10 by means of Proposition 2.6 that

$$\begin{aligned}\|\partial_t u\|_{L^2(I\times\Omega)} + \|\nabla^2 u\|_{L^2(I\times\Omega)} &\leq C\big\{\|f - d(\cdot,\cdot,u)\|_{L^2(I\times\Omega)} + \|\nabla u_0\|_{L^2(\Omega)}\big\}\\ &\leq C\big\{\|f\|_{L^2(I\times\Omega)} + \|u\|_{L^\infty(I\times\Omega)} + \|\nabla u_0\|_{L^2(\Omega)}\big\}\\ &\leq C\big\{\|f\|_{L^p(I;L^r(\Omega))} + \|\nabla u_0\|_{L^2(\Omega)} + \|u_0\|_{L^\infty(\Omega)}\big\},\end{aligned}$$

since $p, r \geq 2$.

From Theorem 6.1, we have

$$\|u - u_{kh}\|_{L^2(I\times\Omega)} \leq C\big\{\|u - \chi_{kh}\|_{L^2(I\times\Omega)} + \|u - \Pi_k u\|_{L^2(I\times\Omega)} + \|u - R_h u\|_{L^2(I\times\Omega)}\big\}.$$

Choosing $\chi_{kh} = P_k P_h u$ as in the proof of Theorem 5.4, we get by the stability of $P_k$ in $L^2(I \times \Omega)$

$$\|u - \chi_{kh}\|_{L^2(I\times\Omega)} \leq C\big\{\|u - P_k u\|_{L^2(I\times\Omega)} + \|u - P_h u\|_{L^2(I\times\Omega)}\big\}.$$

Then, the standard estimates

$$\begin{aligned}\|u - P_k u\|_{L^2(I\times\Omega)} + \|u - \Pi_k u\|_{L^2(I\times\Omega)} &\leq Ck\|\partial_t u\|_{L^2(I\times\Omega)},\\ \|u - P_h u\|_{L^2(I\times\Omega)} + \|u - R_h u\|_{L^2(I\times\Omega)} &\leq Ch^2\|\nabla^2 u\|_{L^2(I\times\Omega)}\end{aligned}$$

yield the assertion. □

Next, we derive a best-approximation-type result in the $L^\infty(I; L^2(\Omega))$ norm.

**Theorem 6.3.** *Let the Assumptions 2.1, 3.1 and 5.2 be fulfilled. Further, let* $u$ *be the solution of* (1.1), *and* $u_{kh} \in X_{k,h}^{0,\nu}$ *be the solution of* (3.5). *Then, it holds for all* $1 \leq \hat{p} \leq \infty$

$$\|u - u_{kh}\|_{L^\infty(I;L^2(\Omega))} \leq C \ln\frac{T}{k}\big\{\|u - \chi_{kh}\|_{L^\infty(I;L^2(\Omega))} + k^{-\frac{1}{\hat{p}}}\|u - R_h u\|_{L^{\hat{p}}(I;L^2(\Omega))}\big\}$$

*Proof.* Similar to the proof of Theorem 6.1, we obtain by the boundedness of $u$ (see Prop. 2.6) and of $u_{kh}$ (see Thm. 5.4) that

$$R_u = \|u\|_{L^\infty(I\times\Omega)} < \infty \quad \text{and} \quad R_{u_{kh}} = \sup_{k,h}\|u_{kh}\|_{L^\infty(I\times\Omega)} < \infty.$$

Hence, setting $R = \max(R_u, R_{u_{kh}})$ in Lemma 5.1 yields $u^R = u$ and $u_{kh}^R = u_{kh}$. Like in the proof of Lemma 5.1, we decompose

$$e = u - u_{kh} = (u - \chi_{kh}) + (\chi_{kh} - u_{kh}) = \eta + \xi_{kh}.$$

and introduce the following dual problem for $z_{kh} \in X^{0,\nu}_{k,h}$:

$$B(\varphi_{kh}, z_{kh}) + (b\varphi_{kh}, z_{kh})_{I\times\Omega} = (\xi_{kh,M}\theta_M, \varphi_{kh})_{I\times\Omega}, \quad \forall \varphi_{kh} \in X^{0,\nu}_{k,h}.$$

with $b$ and $\theta_M$ as in the proof of Lemma 5.1. Testing with $\varphi_{kh} = \xi_{kh}$ yields

$$\begin{aligned}\|\xi_{kh,M}\|^2_{L^2(\Omega)} &= B(\xi_{kh}, z_{kh}) + (b\xi_{kh}, z_{kh})_{I\times\Omega}\\ &= -(\nabla\eta, \nabla z_{kh})_{I\times\Omega} + \sum_{m=1}^{M}(\eta_m, [z_{kh}]_m)_\Omega - (d_R(\cdot,\cdot,u) - d_R(\cdot,\cdot,\chi_{kh}), z_{kh})_{I\times\Omega}.\end{aligned} \tag{6.2}$$

For the first term on the right-hand side of (6.2), we get by an inverse estimate for $\frac{1}{\hat{p}} + \frac{1}{\hat{p}'} = 1$ that

$$\begin{aligned}|(\nabla\eta, \nabla z_{kh})_{I\times\Omega}| &= |(R_h\eta, \Delta_h z_{kh})_{I\times\Omega}| \le |(u - R_h u, \Delta_h z_{kh})_{I\times\Omega}| + |(\eta, \Delta_h z_{kh})_{I\times\Omega}|\\ &\le \|u - R_h u\|_{L^{\hat{p}}(I;L^2(\Omega))}\|\Delta_h z_{kh}\|_{L^{\hat{p}'}(I;L^2(\Omega))} + \|\eta\|_{L^\infty(I;L^2(\Omega))}\|\Delta_h z_{kh}\|_{L^1(I;L^2(\Omega))}\\ &\le C\big\{k^{-\frac{1}{\hat{p}}}\|u - R_h u\|_{L^{\hat{p}}(I;L^2(\Omega))} + \|\eta\|_{L^\infty(I;L^2(\Omega))}\big\}\|\Delta_h z_{kh}\|_{L^1(I;L^2(\Omega))}.\end{aligned}$$

For the second term on the right-hand side of (6.2), we obtain

$$\left|\sum_{m=1}^{M}(\eta_m, [z_{kh}]_m)_\Omega\right| \le \sum_{m=1}^{M}\|\eta_m\|_{L^2(\Omega)}\|[z_{kh}]_m\|_{L^2(\Omega)} \le \|\eta\|_{L^\infty(I;L^2(\Omega))}\sum_{m=1}^{M}\|[z_{kh}]_m\|_{L^2(\Omega)}.$$

Finally, for the third term on the right-hand side of (6.2), we obtain due to (5.1) that

$$|(d_R(\cdot,\cdot,u) - d_R(\cdot,\cdot,\chi_{kh}), z_{kh})_{I\times\Omega}| \le C_R\|\eta\|_{L^\infty(I;L^2(\Omega))}\|z_{kh}\|_{L^1(I;L^2(\Omega))}.$$

It remains to bound the arising terms involving $z_{kh}$. By Lemma 4.2 applied to the dual problem for $z_{kh}$, we have $\|z_{kh}\|_{L^\infty(I;L^2(\Omega))} \le \|\xi_{kh,M}\theta_M\|_{L^1(I;L^2(\Omega))}$ and consequently

$$\begin{aligned}\|bz_{kh}\|_{L^1(I;L^2(\Omega))} &\le \|b\|_{L^\infty(I\times\Omega)}\|z_{kh}\|_{L^1(I;L^2(\Omega))}\\ &\le \|b\|_{L^\infty(I\times\Omega)}\|\xi_{kh,M}\theta_M\|_{L^1(I;L^2(\Omega)} = \|b\|_{L^\infty(I\times\Omega)}\|\xi_{kh,M}\|_{L^2(\Omega)}\end{aligned}$$

due to the properties of $\theta_M$. By Theorem 11 from [17] applied to the rewritten dual problem for $z_{kh}$

$$B(\varphi_{kh}, z_{kh}) = (\xi_{kh,M}\theta_M - bz_{kh}, \varphi_{kh})_{I\times\Omega}, \quad \forall \varphi_{kh} \in X^{0,\nu}_{k,h}$$

yields

$$\begin{aligned}\|\Delta_h z_{kh}\|_{L^1(I;L^2(\Omega))} + \sum_{m=1}^{M}\|[z_{kh}]_m\|_{L^2(\Omega)} &\le C\ln\frac{T}{k}\|\xi_{kh,M}\theta_M - bz_{kh}\|_{L^1(I;L^2(\Omega))}\\ &\le C\ln\frac{T}{k}\big\{1 + \|b\|_{L^\infty(I\times\Omega)}\big\}\|\xi_{kh,M}\|_{L^2(\Omega)}.\end{aligned}$$

Using Lemma 4.4 for $\|z_{kh}\|_{L^1(I;L^2(\Omega))}$ and the boundedness of $\|b\|_{L^\infty(I\times\Omega)}$ due to (2.2a), we obtain

$$\|\xi_{kh,M}\|_{L^2(\Omega)} \le C\ln\frac{T}{k}\big\{\|\eta\|_{L^\infty(I;L^2(\Omega))} + Ck^{-\frac{1}{\hat{p}}}\|u - R_h u\|_{L^{\hat{p}}(I;L^2(\Omega))}\big\},$$

which yields the assertion. ☐

Under further strengthened assumptions on $f$ and $u_d$, also this quasi best approximation result implies an error estimate of second order, which is optimal (up to logarithmic terms) for $\nu = 1$.

**Corollary 6.4.** *Let the Assumptions 2.1, 3.1 and 5.2 be fulfilled and additionally $r \geq 2$, $f \in L^\infty(I, L^r(\Omega))$, and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, for the solution $u$ of (1.1), it holds $u \in W^{1,\hat{p}}(I; L^2(\Omega)) \cap L^{\hat{p}}(I; H^2(\Omega))$ for all $1 < \hat{p} < \infty$ and there exists a constant $C_{\hat{p}} \leq C\frac{\hat{p}^2}{\hat{p}-1}$ with*

$$\|\partial_t u\|_{L^{\hat{p}}(I;L^2(\Omega))} + \|\nabla^2 u\|_{L^{\hat{p}}(I;L^2(\Omega))} \leq C_{\hat{p}}\big\{\|f\|_{L^\infty(I;L^r(\Omega))} + \|\nabla^2 u_0\|_{L^2(\Omega)}\big\}.$$

*Further, for the error between $u$ and the solution $u_{kh} \in X_{k,h}^{0,\nu}$ of (3.5), it holds*

$$\|u - u_{kh}\|_{L^\infty(I;L^2(\Omega))} \leq C(k + h^2)\left(\ln \frac{T}{k}\right)^2 \big\{\|f\|_{L^\infty(I;L^r(\Omega))} + \|\nabla^2 u_0\|_{L^2(\Omega)}\big\}.$$

*Proof.* We put the nonlinearity $d$ to the right-hand side as

$$\begin{aligned} \partial_t u - \Delta u &= f - d(\cdot,\cdot,u) && \text{in } I \times \Omega,\\ u(0) &= u_0 && \text{on } \Omega, \end{aligned}$$

and split the solution as $u = v + w$ where $v$ solves (2.3) with $g = f - d(\cdot,\cdot,u)$ and $w$ solves (2.4). Then the Propositions 2.7 and 2.8 imply

$$\|\partial_t v\|_{L^{\hat{p}}(I;L^2(\Omega))} + \|\nabla^2 v\|_{L^{\hat{p}}(I;L^2(\Omega))} \leq C_{\hat{p}}\|f - d(\cdot,\cdot,u)\|_{L^{\hat{p}}(I;L^2(\Omega))}.$$

with $C_{\hat{p}} \leq C\frac{\hat{p}^2}{\hat{p}-1}$ and

$$\|\partial_t w\|_{L^\infty(I;L^2(\Omega))} + \|\nabla^2 w\|_{L^\infty(I;L^2(\Omega))} \leq C\|\nabla^2 u_0\|_{L^2(\Omega)}.$$

Combining these estimates and proceeding similarly to the proof of Theorem 2.10 by means of Proposition 2.6 then implies

$$\begin{aligned} \|\partial_t u\|_{L^{\hat{p}}(I;L^2(\Omega))} + \|\nabla^2 u\|_{L^{\hat{p}}(I;L^2(\Omega))} &\leq C_{\hat{p}}\|f - d(\cdot,\cdot,u)\|_{L^{\hat{p}}(I;L^2(\Omega))} + C\|\nabla^2 u_0\|_{L^2(\Omega)}\\ &\leq C_{\hat{p}}\big\{\|f\|_{L^\infty(I;L^2(\Omega))} + \|u\|_{L^\infty(I\times\Omega)}\big\} + C\|\nabla^2 u_0\|_{L^2(\Omega)}\\ &\leq C_{\hat{p}}\big\{\|f\|_{L^\infty(I;L^r(\Omega))} + \|\nabla^2 u_0\|_{L^2(\Omega)}\big\}, \end{aligned}$$

since $r \geq 2$ and $\hat{p} < \infty$.

From Theorem 6.3, we have

$$\|u - u_{kh}\|_{L^\infty(I;L^2(\Omega))} \leq C \ln\frac{T}{k}\big\{\|u - \chi_{kh}\|_{L^\infty(I;L^2(\Omega))} + k^{-\frac{1}{\hat{p}}}\|u - R_h u\|_{L^{\hat{p}}(I;L^2(\Omega))}\big\}.$$

Choosing $\chi_{kh} = P_k P_h u$ as in the proof of Theorem 5.4, we get

$$\begin{aligned} \|u - \chi_{kh}\|_{L^\infty(I;L^2(\Omega))} &\leq \|u - P_k u\|_{L^\infty(I;L^2(\Omega))} + \|P_k(u - P_h u)\|_{L^\infty(I;L^2(\Omega))}\\ &\leq \|u - P_k u\|_{L^\infty(I;L^2(\Omega))} + Ck^{-\frac{1}{\hat{p}}}\|P_k(u - P_h u)\|_{L^{\hat{p}}(I;L^2(\Omega))}\\ &\leq \|u - P_k u\|_{L^\infty(I;L^2(\Omega))} + Ck^{-\frac{1}{\hat{p}}}\|u - P_h u\|_{L^{\hat{p}}(I;L^2(\Omega))}. \end{aligned}$$

From the stability of $P_k$ in $L^\infty(I; L^2(\Omega))$ and standard interpolation estimates, we have

$$\|u - P_k u\|_{L^\infty(I;L^2(\Omega))} \leq Ck^{1-\frac{1}{\hat{p}}}\|\partial_t u\|_{L^{\hat{p}}(I;L^2(\Omega))}.$$

Further, standard estimates for $\|u - P_h u\|_{L^2(\Omega)}$ and $\|u - R_h u\|_{L^2(\Omega)}$ imply

$$\|u - P_h u\|_{L^{\hat{p}}(I;L^2(\Omega))} + \|u - R_h u\|_{L^{\hat{p}}(I;L^2(\Omega))} \leq Ch^2\|\nabla^2 u\|_{L^{\hat{p}}(I;L^2(\Omega))}.$$

Using these estimates, we get

$$\begin{aligned}\|u-u_{kh}\|_{L^\infty(I;L^2(\Omega))} &\le C\ln\frac{T}{k}k^{-\frac{1}{\hat p}}\left\{k\|\partial_t u\|_{L^{\hat p}(I;L^2(\Omega))}+h^2\|\nabla^2 u\|_{L^{\hat p}(I;L^2(\Omega))}\right\}\\ &\le C_{\hat p}k^{-\frac{1}{\hat p}}(k+h^2)\ln\frac{T}{k}\left\{\|f\|_{L^\infty(I;L^r(\Omega))}+\|\nabla^2 u_0\|_{L^2(\Omega)}\right\}.\end{aligned}$$

Then, by setting $\hat p=\ln\frac{T}{k}$ we have $C_{\hat p}k^{-\frac{1}{\hat p}}\le C\ln\frac{T}{k}$, since $\frac{T}{k}\ \ge\ 4$ by assumption. This implies the assertion. □

Finally, in the following Theorem, a best approximation result in $L^\infty(I\times\Omega)$ is stated. This is a direct consequence of Theorem 5.4.

**Theorem 6.5.** *Let the Assumptions 2.1, 3.1 and 5.2 be fulfilled. Further, let $u$ be the solution of (1.1), and $u_{kh}\in X^{0,\nu}_{k,h}$ be the solution of (3.5). Then, it holds*

$$\|u-u_{kh}\|_{L^\infty(I\times\Omega)}\le C|\ln h|\left(\ln\frac{T}{k}\right)^2\|u-\chi_{kh}\|_{L^\infty(I\times\Omega)}$$

*for any $\chi_{kh}\in X^{0,\nu}_{k,h}$.*

*Proof.* Due to the boundedness of $u$ by Proposition 2.6 and the boundedness of $u_{kh}$ by Theorem 5.4, we have

$$R_u=\|u\|_{L^\infty(I\times\Omega)}<\infty\quad\text{and}\quad R_{u_{kh}}=\sup_{k,h}\|u_{kh}\|_{L^\infty(I\times\Omega)}<\infty.$$

Choosing $R=\max(R_u,R_{u_{kh}})$ in Lemma 5.1, we directly obtain

$$\|u-u_{kh}\|_{L^\infty(I\times\Omega)}=\|u^R-u^R_{kh}\|_{L^\infty(I\times\Omega)}\le C|\ln h|\left(\ln\frac{T}{k}\right)^2\|u-\chi_{kh}\|_{L^\infty(I\times\Omega)}.$$

This concludes the short proof. □

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