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ADAPTIVE APPROXIMATION OF THE MONGE–KANTOROVICH PROBLEM *VIA* PRIMAL-DUAL GAP ESTIMATES

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Abstract. The Monge–Kantorovich problem arises as a special case for linear cost functionals in optimal transportation problems. It leads to a convex minimization problem with limited regularity properties. The convergent finite element discretization and iterative solution of the problem and its dual are addressed. Based on these approximations a computable upper bound for the primal-dual gap is derived which is suitable for efficient local mesh refinement. Numerical experiments reveal a significant improvement of related adaptive methods.

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1. Introduction

1.1. Optimal mass transfer

Optimal mass transfer is a classical mathematical problem that models the optimal transmission of a Radon measure into another one. It defines the Wasserstein distance which is an important tool in geometry, stochastics, and partial differential equations, with applications in economics, image processing, and data analysis, *cf.* [7, 12–14] and references therein. The general mathematical problem is a continuous linear program. Since this general form provides little information about qualitative properties of the mass transfer and since discretizations are high-dimensional, reduced models for special cost functions have been identified in the literature. Important cases are linear and quadratic cost functions which postulate that transport costs are proportional to distance and squared distance, respectively. The quadratic case leads to the Monge–Ampère equation which is a nonlinear elliptic partial differential equation. The linear case results in the Monge–Kantorovich problem which is a constrained nonsmooth, convex minimization problem. For this problem we address the convergent discretization, the iterative solution for the problem and its dual, and adaptive mesh refinement strategies based on an *a posteriori* error estimate for the primal-dual gap.

1.2. Cost functional and relaxation

The optimal mass transfer problem due to Monge models the available and required amounts of mass by nonnegative Radon measures μ^+ and μ^- on metric spaces X and Y, respectively. An admissible transport map

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is a bijective mapping $s : X \to Y$ which pushes μ^+ into μ^- in the sense that

$$\int_X \psi \circ s \,\mathrm{d}\mu^+ = \int_Y \psi \,\mathrm{d}\mu^-$$

for all $\psi \in C(Y)$ denoted $s_\# \mu^+ = \mu^-$. The total cost associated with a transport map is defined via a cost function $c : X \times Y \to \mathbb{R}$ as

$$I(s) = \int_X c(x, s(x)) \,\mathrm{d}\mu^+(x).$$

Establishing the existence of an optimal admissible transport map is difficult due to the nonlinear character of the constraint. Kantorovich proposed to use transport plans which are nonnegative Radon measures $\mu \in \mathcal{M}(X \times Y)$ on the product space $X \times Y$ and to relax I by considering

$$\widetilde{I}(\mu) = \iint_{X \times Y} c(x, y) \,\mathrm{d}\mu(x, y)$$

subject to the constraint that the projections of μ onto X and Y coincide with μ^+ and μ^- in the sense that

$$\iint_{X \times Y} \phi(x) \,\mathrm{d}\mu(x, y) = \int_X \phi(x) \,\mathrm{d}\mu^+(x),$$
$$\iint_{X \times Y} \psi(y) \,\mathrm{d}\mu(x, y) = \int_Y \psi(y) \,\mathrm{d}\mu^-(x),$$

for all $\phi \in C(X)$ and $\psi \in C(Y)$, respectively. This formulation admits solutions and is consistent with the original formulation. By imposing the constraints *via* Lagrange multipliers, *i.e.*, *via* a maximization of the residuals over ϕ and ψ, in the minimization of $\widetilde{I}$, and carrying out standard duality arguments, one obtains the dual formulation that consists in the maximization of

$$K(\phi, \psi) = \int_X \phi(x) \,\mathrm{d}\mu^+(x) + \int_Y \psi(y) \,\mathrm{d}\mu^-(y)$$

subject to the constraint

$$\phi(x) + \psi(y) \leq c(x, y).$$

The functions ϕ and ψ have the interpretation of shipping costs per unit mass for the producer and the recipient of goods. Since $\mu^\pm$ are nonnegative we may, for given x and y, formally increase the objective by modifying ψ maximally so that we have equality in the constraint. In particular, if $X = Y = \overline{\Omega}$ and c satisfies the triangle inequality we find that for $x \in \overline{\Omega}$ we have

$$\psi(x) = -\phi(x).$$

Assuming further that the Radon measures $\mu^\pm$ are absolutely continuous with respect to Lebesgue measure with nonnegative densities $f^\pm$ and setting $f = f^+ - f^-$, we obtain the reduced functional

$$K(\phi) = \int_\Omega f \phi \,\mathrm{d}x$$

with the constraint

$$\phi(x) - \phi(y) \leq c(x, y)$$

for all $x, y \in \overline{\Omega}$. In the case of the euclidean distance as cost function this means that ϕ is Lipschitz continuous with constant 1. We refer to this case as the Monge–Kantorovich problem. For details of the derivation we refer the reader to [7, 14].

1.3. Monge–Kantorovich problem

Given $f \in L^1(\Omega)$ with vanishing mean we seek a function $\phi \in W^{1,\infty}(\Omega)$ which is maximal for

$$K(\phi) = \int_\Omega f\phi \,\mathrm{d}x - I_{K_1(0)}(\nabla\phi),$$

where $I_{K_1(0)}$ is the indicator functional of the closed unit ball in $K_1(0) \subset L^\infty(\Omega;\mathbb{R}^d)$. Standard duality arguments lead to the dual formulation which determines a minimizing vector field $p \in L^1(\Omega;\mathbb{R}^d)$, whose distributional divergence belongs to $L^1(\Omega)$ and whose normal trace on the boundary vanishes in distributional sense denoted $W^1_N(\mathrm{div};\Omega)$, for

$$D(p) = I_{\{-f\}}(\operatorname{div} p) + \int_\Omega |p| \,\mathrm{d}x$$

with the indicator functional $I_{\{-f\}}$ of the subset $\{-f\} \subset L^1(\Omega)$. We remark that establishing the existence of a maximizer for K is straightforward while it is not trivial to show that a minimizer for D exists. Nevertheless, we have the strong duality relation, *cf.* [7, 10, 14],

$$\max_{\phi\in W^{1,\infty}(\Omega)} K(\phi) = \inf_{p\in W^1_N(\mathrm{div};\Omega)} D(p). \tag{1.1}$$

Since neither K nor D admits uniform convexity properties we will instead of error estimates for ϕ and p consider the approximation of the optimal cost value and in particular the primal-dual gap as a measure for the accuracy of approximations. The convergence of discretizations of the dual formulation has been investigated in [6].

1.4. Error estimation and convergence

The Monge–Kantorovich problem has the interpretation of an infinity Laplace equation with limited regularity of solutions, *cf.* [1, 11]. Approximation schemes can thus greatly benefit from local mesh-refinement. We follow here a well-known concept and use the primal-dual gap to control the approximation of the primal cost, *i.e.*, for a solution $\phi \in W^{1,\infty}(\Omega)$, an approximation $\phi_h \in W^{1,\infty}(\Omega)$, and an arbitrary vector $p_h \in W^1_N(\mathrm{div};\Omega)$ we have

$$0 \le K(\phi) - K(\phi_h) \le D(p_h) - K(\phi_h).$$

Following the arguments from [2, 3] and assuming that ϕ_h and p_h are such that D and K are finite, *i.e.*, $-\operatorname{div} p_h = f$ and $\|\nabla\phi_h\|_{L^\infty(\Omega)} \le 1$, we deduce with an integration by parts that

$$0 \le K(\phi) - K(\phi_h) \le \int_\Omega |p_h| - p_h \cdot \nabla\phi_h \,\mathrm{d}x.$$

The integrand on the right-hand side is nonnegative and serves as a useful indicator for local mesh-refinement. To choose discrete spaces for the approximation of the primal and dual problem with suitable approximation properties, we carry out corresponding *a priori* error analyses. These reveal that for low order conforming $P1$ finite elements we obtain a quadratic consistency error in the primal cost functional while for the choice of lowest order $W^1_N(\mathrm{div};\Omega)$ conforming spaces in discretizing the dual problem we only obtain linear consistency. Hence, a second-order consistent subspace of $W^1_N(\mathrm{div};\Omega)$ has to be chosen in order to benefit from adaptive mesh refinement. We will rigorously analyze fully practical discretizations of K and D which involves incorporating stabilizing terms that are necessary for terminating iterative numerical schemes. A different approach to local mesh-refinement has been used in [6], where refinement indicators are defined via variations of p_h.

1.5. Iterative solution

The iterative solution of the primal and the dual formulation of the Monge–Kantorovich problem is difficult since a constraint on the gradient of the unknown and a nondifferentiable term, respectively, have to be treated appropriately. Since the constraint and the nondifferentiability pose no difficulties when considered pointwise we introduce auxiliary variables that result in augmented Lagrange functionals. Corresponding saddle points will be computed with splitting methods that are unconditionally convergent. Two major difficulties that arise in their practical realization are that degrees of freedom in subspaces of $W^1_N(\mathrm{div};\Omega)$ are typically not nodal values and that nonuniqueness of solutions causes problems in formulating efficient stopping criteria. We will therefore introduce appropriate quadrature or mass lumping and consider optional regularizing terms that do not modify the essential features of the problem. The use of augmented Lagrange functionals is motivated by the results in [5] which reports good performance of the splitting method for the primal problem.

1.6. Outline

The outline of this article is as follows. In Section 2 we introduce some notation and required finite element spaces. Section 3 provides abstract a priori and *a posteriori* error estimates for the primal and dual problem along with their application to specific finite element methods. In Section 4 we address the reliable iterative solution of the maximization and minimization problems via regularization and iterative splitting. The influence of regularizing terms and convergence of adaptively generated approximations are studied in Section 5. Section 6 reports various numerical experiments which show that adaptivity based on our primal-dual gap estimator leads quasi-optimal experimental convergence rates. In Appendix A we include a convergence proof for the employed splitting method.

2. Preliminaries

2.1. Lebesgue and Sobolev spaces

Throughout this article we let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, be a polyhedral Lipschitz domain with finite diameter $d_\Omega > 0$ and denote the inner product and corresponding norm in $L^2(\Omega;\mathbb{R}^\ell)$, $\ell \in \mathbb{N}$, by

$$(v,w) = \int_\Omega v \cdot w \,\mathrm{d}x, \quad \|v\| = (v,v)^{1/2}.$$

We let $W^{s,p}(\Omega;\mathbb{R}^\ell)$ denote the standard Sobolev space with norm and seminorm denoted by $\|v\|_{W^{s,p}(\Omega)}$ and $|v|_{W^{s,p}(\Omega)}$, respectively. If $s = 0$ then we write $L^p(\Omega;\mathbb{R}^\ell)$ instead of $W^{s,p}(\Omega;\mathbb{R}^\ell)$. The set of functions $f \in L^r(\Omega)$ with vanishing mean, *i.e.*,

$$\int_\Omega f \,\mathrm{d}x = 0$$

is denoted by $L^r_0(\Omega)$.

2.2. Standard finite element spaces

For a sequence of regular triangulations $(\mathcal{T}_h)_{h>0}$ of Ω consisting of regular simplices we let

$$\mathcal{L}^k(\mathcal{T}_h) = \{v_h \in L^1(\Omega) : v_h|_T \in P_k(T) \quad \text{for all} \quad T \in \mathcal{T}_h\},$$
$$\mathcal{S}^k(\mathcal{T}_h) = \{v_h \in C(\overline{\Omega}) : v_h|_T \in P_k(T) \quad \text{for all} \quad T \in \mathcal{T}_h\},$$

where $P_k(A)$ denotes the set of polynomials of total degree at most k on the set $A \subset \mathbb{R}^d$. We let

$$\Pi^k_h : L^1(\Omega;\mathbb{R}^\ell) \to \mathcal{L}^k(\mathcal{T}_h)^\ell$$

denote the L^2 projection onto $\mathcal{L}^k(\mathcal{T}_h)^\ell$. The space $C(\mathcal{T}_h)$ consists of all functions in $L^\infty(\Omega)$ that are continuous on every element $T \in \mathcal{T}_h$. The elementwise application of the nodal interpolation operator to an elementwise continuous function is denoted by the operator

$$\widehat{\mathcal{I}}_h : C(\mathcal{T}_h) \to \mathcal{L}^1(\mathcal{T}_h),$$

which can be identified with the standard nodal interpolation operator $\mathcal{I}_h : C(\overline{\Omega}) \to \mathcal{S}^1(\mathcal{T}_h)$ on continuous functions. With the8 set of nodes $\mathcal{N}_h$ and the associated nodal basis $(\varphi_z : z \in \mathcal{N}_h)$ of $\mathcal{S}^1(\mathcal{T}_h)$ we have for all $T \in \mathcal{T}_h$ that

$$\widehat{\mathcal{I}}_h v|_T = \sum_{z \in \mathcal{N}_h \cap T} v|_T(z)\varphi_z|_T.$$

We define a discrete inner product on $C(\mathcal{T}_h)$ by

$$(v, w)_h = \int_\Omega \widehat{\mathcal{I}}_h(vw)\,\mathrm{d}x = \sum_{T \in \mathcal{T}_h} \sum_{z \in \mathcal{N}_h \cap T} \beta_z^T v|_T(z) w|_T(z), \quad \beta_z^T = \int_T \varphi_z\,\mathrm{d}x.$$

Note that the induced norm $\|\cdot\|_h$ is equivalent to the L^2 norm on $\mathcal{L}^1(\mathcal{T}_h)$. We define the mesh-size function $h_\mathcal{T} \in L^\infty(\Omega)$ by

$$h_\mathcal{T}|_T = h_T = \mathrm{diam}(T),$$

for all $T \in \mathcal{T}_h$, and set

$$h_{\min} = \min h_\mathcal{T}, \quad h_{\max} = \max h_\mathcal{T}.$$

Since we consider locally refined meshes we also work with the average mesh-size $\overline{h}$ defined with the number of nodes in $\mathcal{N}_h$ via

$$\overline{h} = (\#\mathcal{N}_h)^{-1/d}.$$

We stress that for a sequence of triangulations $(\mathcal{T}_h)_{h>0}$ we only assume that the average mesh-size $\overline{h}$ tends to zero as $h \to 0$ but not necessarily the maximal mesh size $h_{\max}$ unless stated otherwise.

2.3. Vector fields and weak divergence

We say that the vector field $p \in L^r(\Omega;\mathbb{R}^d)$ has a weak divergence if there exists $f \in L^r(\Omega)$ such that

$$\int_\Omega p \cdot \nabla\phi\,\mathrm{d}x = \int_\Omega f\phi\,\mathrm{d}x$$

for all continuously differentiable, compactly supported functions $\phi \in C_c^1(\Omega)$. In this case we denote $-\mathrm{div}\,p = f$. We say that p has vanishing normal component on $\partial\Omega$ if the identity holds for all $\phi \in C^1(\overline{\Omega})$. For $r \geq 1$ we let

$$W_N^r(\mathrm{div};\Omega) = \left\{p \in L^r(\Omega;\mathbb{R}^d) : \mathrm{div}\,p \in L^r(\Omega),\ p \cdot n = 0 \quad \text{on} \quad \partial\Omega\right\}$$

denote the space of vector fields with weak divergence in $L^r(\Omega)$ and vanishing normal component on $\partial\Omega$ indicated by the subscript N. The space is equipped with the norm

$$\|p\|_{W_N^r(\mathrm{div};\Omega)} = \|p\|_{L^r(\Omega)} + \|\,\mathrm{div}\,p\|_{L^r(\Omega)}.$$

The divergence operator $\mathrm{div} : W_N^r(\mathrm{div};\Omega) \to L_0^r(\Omega)$ is surjective with a bounded left inverse, *i.e.*, for all $f \in L_0^r(\Omega)$ there exists $p \in W_N^r(\mathrm{div};\Omega)$ such that $-\mathrm{div}\,p = f$ and

$$\|p\|_{W_N^r(\mathrm{div};\Omega)} \leq c\|f\|_{L^r(\Omega)},$$

with an r-independent constant $c > 0$, *cf.* [6] for details. For the construction of discrete subspaces of divergence spaces we note that an elementwise polynomial vector field belongs to $W^r_N(\mathrm{div};\Omega)$ if and only if its normal component is continuous across neighboring element boundaries and vanishes on $\partial\Omega$. A suitable discrete subspace of $W^r_N(\mathrm{div};\Omega)$ are the Raviart–Thomas finite element spaces

$$\begin{aligned}\mathcal{R}T^k_N(\mathcal{T}_h) = \big\{q_h \in W^1_N(\mathrm{div};\Omega) : \text{ for all } T \in \mathcal{T}_h \\ q_h|_T(x) = q_T(x) + (x - x_T)p_T(x),\, q_T \in P_k(T)^d,\, p_T \in P_k(T)\big\},\end{aligned}$$

where x_T denotes the barycenter of $T \in \mathcal{T}_h$. This space is compatible with the space $Q_h = \mathcal{L}^k(\mathcal{T}_h) \cap L^1_0(\Omega)$ in the sense that if $V_h = \mathcal{R}T^k_N(\mathcal{T}_h)$ then there exists a bounded left inverse for the divergence operator $\mathrm{div} : V_h \to Q_h$, *i.e.*, for all $f_h \in Q_h$ there exists $p_h \in V_h$ such that for $1 < r < 4/3$ we have

$$-\operatorname{div} p_h = f_h, \quad \|p_h\|_{W^r_N(\mathrm{div};\Omega)} \le c_r \|f_h\|_{L^r(\Omega)},$$

with a constant $c_r > 0$ that depends on $r > 1$ but not on h, *cf.* [6]. This property implies the inf-sup condition. Moreover, there exists a generalized interpolation operator $I_F : W^r_N(\mathrm{div};\Omega) \to V_h$ such that

$$\operatorname{div} I_F \xi = \Pi^k_h \operatorname{div} \xi.$$

For $\xi \in W^r_N(\mathrm{div};\Omega) \cap W^{\beta,r}(\Omega;\mathbb{R}^d)$ with $r > 1$ we have that

$$\|\xi - I_F\xi\|_{L^r(\Omega)} \le c_r h^\beta \|\xi\|_{W^{\beta,r}(\Omega)}, \tag{2.1}$$

for $1 \le \beta \le k+1$, provided that $\mathcal{L}^k(\mathcal{T}_h)^d \cap W^1_N(\mathrm{div};\Omega) \subset V_h$. The operator I_F is stable on $W^{1,r}(\Omega;\mathbb{R}^d)$, *i.e.*,

$$\|I_F\xi\|_{W^{1,r}(\Omega)} \le c_r \|\xi\|_{W^{1,r}(\Omega)}.$$

For details we refer the reader to [4,6,8]. If $f \in L^{r_0}_0(\Omega)$ for some $r_0 > 1$ the operator I_F allows us to construct a sequence $(p_h)_{h>0}$ of discrete vector fields $p_h \in V_h$ with $-\operatorname{div} p_h = f_h = \Pi^1_h f$ and

$$\|p_h\|_{W^1_N(\mathrm{div};\Omega)} \le c\|p_h\|_{W^{1+\varepsilon}_N(\mathrm{div};\Omega)} \le c\|p_h\|_{W^{r_0}_N(\mathrm{div};\Omega)}$$

as $h, \varepsilon \to 0$ by choosing $\xi \in W^{r_0}_N(\mathrm{div};\Omega)$ such that $-\operatorname{div}\xi = f$ and setting $p_h = I_F\xi$. Note that the constant involved in the interpolation estimate for I_F depends on r but the operator itself does not. We finally remark that every $p_h \in \mathcal{R}T^k_N(\mathcal{T}_h)$ admits on every $T \in \mathcal{T}_h$ the local representation

$$p_h|_T = v^T_h + (x - x_T)q^T_h$$

with $v^T_h \in P_k(T)^d$ and $q^T_h \in P_k(T)$ which is homogeneous of degree k, *i.e.*, we have $(x - x_T) \cdot \nabla q^T_h(x) = k q^T_h$, *cf.* [4]. This implies that

$$\operatorname{div} p_h|_T = \operatorname{div} v^T_h + (d+k) q^T_h,$$

and hence

$$p_h|_T = v^T_h + (d+k)^{-1}(x - x_T)(\operatorname{div} p^T_h - \operatorname{div} v^T_h). \tag{2.2}$$

If $\operatorname{div} p_h = 0$ then it follows that $p_h|_T \in P_k(T)^d$ for every $T \in \mathcal{T}_h$.

3. Abstract error estimates

3.1. *A priori* estimate for primal cost

For a finite-dimensional subspace $X_h \subset W^{1,\infty}(\Omega) \cap L^1_0(\Omega)$ and an approximation $f_h \in Q_h \subset L^1_0(\Omega)$ of f we let $\phi_h \in X_h$ be a maximizing function for

$$K_h(\phi_h) = \int_\Omega f_h \phi_h \,\mathrm{d}x - I_{K_1(0)}(\nabla\phi_h).$$

A possibly nonunique solution $\phi_h \in X_h$ exists due to the direct method in the calculus of variations.

Proposition 3.1 (*A priori* estimate I)**.** *Let $\phi_h \in X_h$ be maximal for K_h and $\phi \in W^{1,\infty}(\Omega)$ be maximal for K with vanishing means. For every $1 \le r \le \infty$ we have that*

$$K(\phi) - K(\phi_h) \le \inf_{\substack{\psi_h \in X_h \\ |\nabla \psi_h| \le 1}} \|f\|_{L^1(\Omega)} \|\phi - \psi_h\|_{L^\infty(\Omega)} + c_r \|f - f_h\|_{L^r(\Omega)},$$

where $c_r = 2 d_\Omega |\Omega|^{(r-1)/r}$, $d_\Omega = \operatorname{diam}(\Omega)$, provided that $f \in L^r(\Omega)$.

Proof. For every $\psi_h \in X_h$ with $|\nabla \psi_h| \le 1$ in Ω we have that

$$\begin{aligned} K(\phi) - K(\phi_h) &= K(\phi) - K_h(\phi_h) + \int_\Omega (f_h - f)\phi_h \,\mathrm{d}x \\ &\le K(\phi) - K_h(\psi_h) + \int_\Omega (f_h - f)\phi_h \,\mathrm{d}x \\ &= K(\phi) - K(\psi_h) + \int_\Omega (f_h - f)(\phi_h - \psi_h) \,\mathrm{d}x \\ &= \int_\Omega f(\phi - \psi_h)\,\mathrm{d}x + \int_\Omega (f_h - f)(\phi_h - \psi_h)\,\mathrm{d}x. \end{aligned}$$

In view of the vanishing mean we have the Poincaré inequality $\|\psi\|_{L^\infty(\Omega)} \le d_\Omega \|\nabla \psi\|_{L^\infty(\Omega)}$ and this implies the estimate. □

Remark 3.2. If $f \in L^2(\Omega)$ and $\mathcal{L}^0(\mathcal{T}_h) \cap L^1_0(\Omega) \subset Q_h$ we may choose $r = 2$ and $f_h \in Q_h$ as the L^2 projection $\Pi_h^0 f$ and use the estimate

$$\begin{aligned} \int_\Omega (f_h - f)(\phi_h - \psi_h)\,\mathrm{d}x &\le \|h_\mathcal{T}(f - f_h)\| \|h_\mathcal{T}^{-1}(\phi_h - \psi_h - \Pi_h^0(\phi_h - \psi_h))\| \\ &\le \|h_\mathcal{T}(f - f_h)\| \|\nabla(\phi_h - \psi_h)\|, \end{aligned}$$

to obtain an improved estimate with $\|f - f_h\|_{L^r(\Omega)}$ replaced by $\|h_\mathcal{T}(f - f_h)\|$.

For triangulations $\mathcal{T}_h$ which are right-angled in the sense that every element $T \in \mathcal{T}_h$ has d orthogonal edge vectors we have for $\phi \in W^{1,\infty}(\Omega)$ and its $P1$ interpolant $\mathcal{I}_h \phi \in \mathcal{S}^1(\mathcal{T}_h)$ the stability estimate

$$\|\nabla \mathcal{I}_h \phi\|_{L^\infty(\Omega)} \le \|\nabla \phi\|_{L^\infty(\Omega)}.$$

This implies that $\mathcal{I}_h \phi$ is admissible for K and allows us to deduce the following error estimate.

Corollary 3.3 (Convergence rate I)**.** *Assume that $\mathcal{T}_h$ is right-angled. If $\phi \in W^{1+\alpha,\infty}(\Omega)$ and $f_h \in \mathcal{L}^1(\mathcal{T}_h)$ is the elementwise affine interpolant of the elementwise smooth function $f \in L^2(\Omega)$ then we have for every maximizer $\phi_h \in X_h$ of K_h that*

$$K(\phi) - K(\phi_h) \le c_\mathcal{I} h^{1+\alpha} |\phi|_{W^{1+\alpha,\infty}(\Omega)} + c_2 c_\mathcal{I} h^2 \|D^2_\mathcal{T} f\|_{L^2(\Omega)},$$

where $D^2_\mathcal{T} f$ denotes the elementwise computed Hessian of f.

Proof. The estimate is a consequence of Proposition 3.1 and the interpolation estimates $\|\phi - \mathcal{I}_h \phi\|_{L^\infty(\Omega)} \le c h^{1+\alpha} \|\phi\|_{W^{1+\alpha,\infty}(\Omega)}$ and $\|f - \widehat{\mathcal{I}}_h f\| \le c h^2 \|D^2_\mathcal{T} f\|$. □

Remark 3.4.

(i) The assumption of a right-angled triangulation is restrictive in particular for $d = 3$.

(ii) The maximization of K is a variant of the infinity Laplace problem for which generic solutions $\phi \in W^{1,\infty}(\Omega)$ have at most the regularity property $\phi \in W^{4/3,\infty}(\Omega)$, cf. [1,11], *i.e.*, we cannnot expect a higher convergence rate than $\mathcal{O}(h^{4/3})$ for $P1$ finite element functions which is significantly worse than the formal optimal convergence rate $\mathcal{O}(h^2)$ for $\phi \in W^{2,\infty}(\Omega)$.

3.2. *A posteriori* estimate for primal cost

To obtain a mechanism that allows for local adaptive mesh refinement and thereby improved convergence rates, we derive an *a posteriori* error estimate that is obtained from the strong duality relation with admissible functions.

Proposition 3.5 (*A posteriori* estimate)**.** *Given arbitrary $\phi, \phi_h \in W^{1,\infty}(\Omega) \cap L^1_0(\Omega)$ with*

$$\|\nabla\phi\|_{L^\infty(\Omega)},\ \|\nabla\phi_h\|_{L^\infty(\Omega)} \le 1$$

and an arbitrary vector field $p_h \in W^1_N(\operatorname{div};\Omega)$ with

$$-\operatorname{div} p_h = f_h \ \text{ in } \Omega,$$

we have that

$$K(\phi) - K(\phi_h) \le \eta_h(\phi_h, p_h) = \sum_{T\in\mathcal{T}_h} \eta_T(\phi_h, p_h) + 2d_\Omega \sum_{T\in\mathcal{T}_h} \operatorname{osc}_T(f, f_h),$$

where for every $T \in \mathcal{T}_h$ we have

$$\eta_T(\phi_h, p_h) = \int_T |p_h| - p_h \cdot \nabla\phi_h \,\mathrm{d}x, \quad \operatorname{osc}_T(f, f_h) = \int_T |f - f_h| \,\mathrm{d}x.$$

Proof. Noting the duality relation (1.1) for K_h and D_h which result from replacing f by f_h in K and D, respectively, and integrating by parts we find that

$$\begin{aligned}
K(\phi) - K(\phi_h) &= K_h(\phi) - K_h(\phi_h) + \int_\Omega (f - f_h)(\phi - \phi_h)\,\mathrm{d}x \\
&\le D_h(p_h) - K_h(\phi_h) + \int_\Omega (f - f_h)(\phi - \phi_h)\,\mathrm{d}x \\
&= \int_\Omega |p_h| - f_h\phi_h \,\mathrm{d}x + \int_\Omega (f - f_h)(\phi - \phi_h)\,\mathrm{d}x \\
&= \int_\Omega |p_h| + \operatorname{div} p_h\, \phi_h \,\mathrm{d}x + \int_\Omega (f - f_h)(\phi - \phi_h)\,\mathrm{d}x \\
&\le \int_\Omega |p_h| - p_h \cdot \nabla\phi_h \,\mathrm{d}x + \|\phi - \phi_h\|_{L^\infty(\Omega)} \int_\Omega |f - f_h|\,\mathrm{d}x.
\end{aligned}$$

Using that the function $\phi - \phi_h$ has a root in $\overline{\Omega}$ and that its gradient is uniformly bounded by 2, we deduce $\|\phi - \phi_h\|_{L^\infty(\Omega)} \le 2d_\Omega$, which implies the estimate. □

The *a posteriori* error estimate is based on the primal-dual gap and requires a good approximation of a solution for the dual problem.

3.3. *A priori* estimate for dual cost

For subspaces $V_h \subset W^1_N(\operatorname{div};\Omega)$ and $Q_h \subset L^1_0(\Omega)$ which are compatible in the sense that the divergence operator $\operatorname{div} : V_h \to Q_h$ is a surjection with bounded left inverse, we consider for given $f_h \in Q_h$ the minimization of the discretized dual functional

$$D_h(p_h) = \int_\Omega |p_h|\,\mathrm{d}x + I_{\{-f_h\}}(\operatorname{div} p_h).$$

The following estimate enables us to determine necessary approximation properties of V_h to obtain the same formal consistency error as in the discretization of K.

Proposition 3.6 (A *priori* estimate II)**.** *Let $\delta > 0$ and $p_\delta \in W^1_N(\mathrm{div};\Omega)$ be a δ-minimizer for D, i.e.,*

$$D(p_\delta) \le \inf_{p\in W^1_N(\mathrm{div};\Omega)} D(p) + \delta.$$

Then, for every $q_h \in V_h$ with $-\operatorname{div} q_h = f_h$ we have that

$$|D_h(q_h) - D(p_\delta)| \le \|q_h - p_\delta\|_{L^1(\Omega)} + 2c_{S,r}|\Omega|^{1/2}\|f - f_h\|_{L^r(\Omega)} + \delta,$$

provided that $f \in L^r(\Omega)$ with $r > 1$ if $d \le 2$ and $r \ge 6/5$ if $d = 3$.

Proof. We first define a sequence of corrections $q^{(h)} \in W^2_N(\mathrm{div};\Omega)$ so that for $\widetilde{q}_h = q_h + q^{(h)}$ we have

$$-\operatorname{div}\widetilde{q}_h = f,$$

and hence $D(\widetilde{q}_h)$ is finite. This is achieved by letting $\alpha^{(h)} \in W^{1,2}(\Omega)$ be the unique weak solution with vanishing mean of

$$-\Delta\alpha^{(h)} = f - f_h \text{ in } \Omega, \quad \nabla\alpha^{(h)}\cdot n = 0 \text{ on } \partial\Omega$$

and setting $q^{(h)} = \nabla\alpha^{(h)}$. With the Sobolev inequality $\|\alpha^{(h)}\|_{L^{r'}(\Omega)} \le c_{S,r}\|\nabla\alpha^{(h)}\|$ for $r' \le 6$ if $d = 3$ or $r' < \infty$ if $d \le 2$, we have

$$\|q^{(h)}\|_{L^1(\Omega)} \le |\Omega|^{1/2}\|\nabla\alpha^{(h)}\| \le c_{S,r}|\Omega|^{1/2}\|f - f_h\|_{L^r(\Omega)},$$

for $r \ge 6/5$ if $d = 3$ and $r > 1$ if $d \le 2$. We thus deduce that

$$|D(\widetilde{q}_h) - D_h(q_h)| \le \|\widetilde{q}_h - q_h\|_{L^1(\Omega)} = \|q^{(h)}\|_{L^1(\Omega)} \le c_{S,r}\|f - f_h\|_{L^r(\Omega)}.$$

This leads to

$$\begin{aligned} 0 &\le D(\widetilde{q}_h) - \inf_{q\in W^1_N(\mathrm{div};\Omega)} D(q) \\ &\le D(\widetilde{q}_h) - D(p_\delta) + \delta \\ &\le \|q_h - p_\delta\|_{L^1(\Omega)} + c_{S,r}|\Omega|^{1/2}\|f - f_h\|_{L^r(\Omega)} + \delta, \end{aligned}$$

which implies the estimate. □

Assuming the existence of a bounded sequence of almost-minimizing vector fields $p_\delta \in W^{\beta,2}(\Omega;\mathbb{R}^d)$, $1 \le \beta \le 2$, for the dual functional, we obtain the following formal error estimate.

Corollary 3.7 (Convergence rate II)**.** *Assume that $\beta \in [1, 2]$ and every $\delta = h^\beta_{\max}$ there exists $p_\delta \in W^{\beta,2}(\Omega;\mathbb{R}^d)$ which is a δ-minimizer for D, that $f \in L^2(\Omega)$ is elementwise smooth and $f_h = \Pi^1_h f$, and that for $k \le 1$ we have*

$$\mathcal{L}^k(\mathcal{T}_h)^d \cap W^1_N(\mathrm{div};\Omega) \subset V_h.$$

If $1 \le \beta \le k + 1$ we have for every minimizer $p_h \in V_h$ for D_h that

$$|D_h(p_h) - D(p_\delta)| \le ch^\beta(1 + |p_\delta|_{W^{\beta,2}(\Omega)}) + ch^2\|D^2_{\mathcal{T}} f\|.$$

Proof. The estimate follows from interpolation estimates for the Fortin-like operator I_F, cf. (2.1), with $q_h = I_F p_\delta$ noting $-\operatorname{div} q_h = \Pi^1_h f$, the relation $D_h(p_h) \le D_h(q_h)$, and Proposition 3.6. □

Remark 3.8.

(i) Note that we need $k \ge 1$ to match the formal quasioptimal convergence rate $\mathcal{O}(h^2)$ for the approximation of the optimal cost of the primal problem, *i.e.*, the lowest order Raviart–Thomas finite element space $\mathcal{RT}^0_N(\mathcal{T}_h)$ is not sufficient to obtain an optimally convergent error estimator $\eta_h(\phi_h, p_h)$.

(ii) If f_h is defined via $f_h = \widehat{\mathcal{I}}_h f$ then the difference $\widehat{\mathcal{I}}_h f - \Pi^1_h f$ can be controlled similarly to the difference $f - f_h$ in the proof of Proposition 3.6.

4. Iterative solution

We discuss in this section the iterative solution of the primal and dual problem. For a fixed discretization we do not compute the exact discrete solutions. This does not affect the validity of the a priori error estimates provided that the tolerances and regularization terms remain within the formal consistency errors. The *a posteriori* error estimate only requires feasibility which is guaranteed for the computed approximations. In the following we apply the *alternating direction of multipliers method* (ADMM) to appropriate modifications of the discretized primal and dual problems.

4.1. Primal problem

To enforce a unique solution $\phi_h \in \mathcal{S}^1(\mathcal{T}_h)$ we introduce an optional regularizing term weighted by a factor $\varepsilon \geq 0$ and consider the maximization of the functional

$$K_{h,\varepsilon}(\phi_h) = \int_\Omega f_h \phi_h \,\mathrm{d}x - \frac{\varepsilon}{2} \int_\Omega |\nabla \phi_h|^2 \,\mathrm{d}x - I_{K_1(0)}(\nabla \phi_h).$$

For every $\varepsilon > 0$ we have that $-K_{h,\varepsilon}$ is uniformly convex on the space of functions in $\mathcal{S}^1(\mathcal{T}_h)$ with vanishing mean. The iterative solution introduces the auxiliary variable $s_h = \nabla\phi_h$ and uses the augmented Lagrange functional

$$L_{h,\varepsilon,\tau} : \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h)^d \times \mathcal{L}^0(\mathcal{T}_h)^d \to \mathbb{R} \cup \{-\infty\},$$

which is for a stabilization parameter $\tau > 0$ defined by

$$\begin{aligned} L_{h,\varepsilon,\tau}(\phi_h, s_h; \mu_h) = \int_\Omega f_h \phi_h \,\mathrm{d}x - \frac{\varepsilon}{2}\|s_h\|^2 - I_{K_1(0)}(s_h) \\ - (\mu_h, \nabla\phi_h - s_h) - \frac{\tau}{2}\|\nabla\phi_h - s_h\|^2. \end{aligned}$$

Note that the term weighted by the factor τ is a consistent stabilizing term. The maximization of $K_{h,\varepsilon}$ on $\mathcal{S}^1(\mathcal{T}_h)$ is equivalent to computing a saddle point for $L_{h,\varepsilon,\tau}$, *i.e.*, we have

$$\max_{\phi_h \in \mathcal{S}^1(\mathcal{T}_h)} K_{h,\varepsilon}(\phi_h) = \max_{(\phi_h, s_h) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h)^d} \min_{\mu_h \in \mathcal{L}^0(\mathcal{T}_h)^d} L_{h,\varepsilon,\tau}(\phi_h, s_h; \mu_h).$$

The minimization with respect to μ_h enforces the relation $s_h = \nabla\phi_h$. The splitting of the variable has the advantage that the maximization of $L_{h,\varepsilon,\tau}$ with respect to s_h is pointwise and can be done explicitly. Unconditional convergence of the following iterative scheme follows from general assertions, see [9] and Appendix A. The stabilizing term weighted by ε is included to guarantee uniqueness and convergence of the iterates $(s_h^k)_{k=0,1,\ldots}$. For $\varepsilon = 0$ we only have convergence of subsequences.

Algorithm 4.1 (Primal splitting)**.** Let $s_h^0, \mu_h^0 \in \mathcal{L}^0(\mathcal{T}_h)^d$, $\tau > 0$, set $k = 1$.

(1) Compute the maximizer $\phi_h^k \in \mathcal{S}^1(\mathcal{T}_h)$ with vanishing mean for

$$\phi_h \mapsto L_{h,\varepsilon,\tau}(\phi_h, s_h^{k-1}; \mu_h^{k-1}).$$

(2) Compute the maximizer $s_h^k \in \mathcal{L}^0(\mathcal{T}_h)^d$ for

$$s_h \mapsto L_{h,\varepsilon,\tau}(\phi_h^k, s_h, \mu_h^{k-1}).$$

(3) Compute the minimizer $\mu_h^k \in \mathcal{L}^0(\mathcal{T}_h)^d$ for

$$\mu_h \mapsto \frac{1}{2\tau}\|\mu_h - \mu_h^{k-1}\|^2 + L_h(\phi_h^k, s_h^k; \mu_h).$$

(4) Stop the iteration if

$$\tau\|s_h^k - s_h^{k-1}\| + \|\mu_h^k - \mu_h^{k-1}\| \le \delta_{\text{stop}}.$$

(5) Increase $k \to k+1$ and continue with (1).

Note that it is essential to involve the variable s_h in the stopping criterion in order to ensure convergence of approximations to a saddle point since the difference $s_h^k - s_h^{k-1}$ occurs as a residual in the optimality equation for ϕ_h.

Remark 4.2.

(i) The solution in step (1) is the uniquely defined function $\phi_h^k \in \mathcal{S}^1(\mathcal{T}_h)$ with vanishing mean that satisfies

$$\tau(\nabla\phi_h^k, \nabla\psi_h) = -(\mu_h^{k-1} - \tau s_h^{k-1}, \nabla\psi_h) + (f_h, \psi_h)$$

for all $\psi_h \in \mathcal{S}^1(\mathcal{T}_h)$.

(ii) The solution in step (2) is the uniquely defined function $s_h^k \in \mathcal{L}^0(\mathcal{T}_h)^d$ that satisfies

$$-(\tau+\varepsilon)s_h^k + \mu_h^{k-1} + \tau\nabla\phi_h^k \in \partial I_{K_1(0)}(s_h^k),$$

which is given by the elementwise shrinkage operation

$$s_h = \frac{r_h}{\max\{1, |r_h|\}}, \quad r_h = \frac{1}{\tau+\varepsilon}\big(\mu_h^{k-1} + \tau\nabla\phi_h^k\big).$$

(iii) Step (3) leads to the explicit updating formula

$$\mu_h^k = \mu_h^{k-1} + \tau(\nabla\phi_h^k - s_h^k).$$

We note that our stopping criterion controls the constraint violation $\nabla\phi_h = s_h$ *via* $\tau(\nabla\phi_h^k - s_h^k) = \mu_h^k - \mu_h^{k-1}$ and this quantity also occurs as a residual in the stationarity equations for the variables s_h and ϕ_h. The residual in the optimality equation for the variable ϕ_h involves also the quantity $s_h^k - s_h^{k-1}$ resulting from the decoupling of the minimizations in s_h and ϕ_h.

4.2. Dual problem

For approximately solving the discretized dual problem we consider a finite element space $V_h \subset W_N^1(\mathrm{div};\Omega)$ and the regularized functional

$$\widetilde{D}_{h,\varepsilon}(p_h) = \frac{1}{1+\varepsilon}\int_\Omega |p_h|^{1+\varepsilon}\,\mathrm{d}x + I_{\{-f_h\}}(\operatorname{div} p_h).$$

The choice of a positive regularization parameter $\varepsilon > 0$ introduces a uniform convexity property and thereby uniqueness of the discrete minimizer. For optimal stability, the iterative solution has to respect the features on the nonregularized problem and should not make explicit use of the regularization. In particular, the devised numerical scheme described below is well defined for $\varepsilon = 0$. To efficiently deal with the practical nondifferentiability we introduce a variable $s_h \approx p_h$ that is an element of a finite element space with nodal degrees of freedom, *i.e.*, in the space $\mathcal{L}^1(\mathcal{T}_h)^d$ of elementwise affine, discontinuous vector fields which are not contained in $W_N^r(\mathrm{div};\Omega)$. Using quadrature, this allows us to express the nondifferentiable functional as a sum of separate functionals applied to nodal values. This leads to the discretization

$$\begin{aligned} D_{h,\varepsilon}(p_h) &= \frac{1}{1+\varepsilon}\int_\Omega \widehat{\mathcal{I}}_h|\Pi_h^1 p_h|^{1+\varepsilon}\,\mathrm{d}x + I_{\{-f_h\}}(\operatorname{div} p_h) \\ &= \frac{1}{1+\varepsilon}\sum_{T\in\mathcal{T}_h}\sum_{z\in\mathcal{N}_h\cap T}\beta_z^T\big|s_h|_T(z)\big|^{1+\varepsilon} + I_{\{-f_h\}}(\operatorname{div} p_h), \end{aligned}$$

where we abbreviate $s_h = \Pi_h^1 p_h$. The discretization is similar to the one considered in [6] for zero-th order Raviart–Thomas elements. The practical minimization is realized *via* the augmented Lagrange functional

$$M_{h,\varepsilon,\tau} : V_h \times \mathcal{L}^1(\mathcal{T}_h)^d \times \mathcal{L}^1(\mathcal{T}_h)^d \to \mathbb{R} \cup \{+\infty\}$$

defined by

$$\begin{aligned} M_{h,\varepsilon,\tau}(p_h, s_h; \lambda_h) = \frac{1}{1+\varepsilon} \int_\Omega \widehat{\mathcal{I}}_h |s_h|^{1+\varepsilon} \,\mathrm{d}x + I_{\{-f_h\}}(\operatorname{div} p_h) \\ + (s_h - \Pi_h^1 p_h, \lambda_h)_{h,w} + \frac{\tau}{2} \|s_h - \Pi_h^1 p_h\|_{h,w}^2. \end{aligned}$$

The weighted scalar product $(\cdot,\cdot)_{h,w}$ is given by

$$(s_h, \lambda_h)_{h,w} = \int_\Omega h_\mathcal{T}^d \, \widehat{\mathcal{I}}_h [s_h \cdot \lambda_h] \,\mathrm{d}x.$$

For the induced norm we have by an elementwise inverse estimate that

$$\|s_h\|_{h,w} \le c \|s_h\|_{L^1(\Omega)}.$$

The use of this weighting is necessary due to the fact that approximations $s_h \approx p_h$ are only bounded in $L^1(\Omega; \mathbb{R}^d)$. The minimization of $D_{h,\varepsilon}$ is equivalent to finding a saddle point for $M_{h,\varepsilon,\tau}$, *i.e.*, we have

$$\min_{p_h \in V_h} D_{h,\varepsilon}(p_h) = \min_{(p_h, s_h) \in V_h \times \mathcal{L}^1(\mathcal{T}_h)^d} \ \max_{\lambda_h \in \mathcal{L}^1(\mathcal{T}_h)^d} M_{h,\varepsilon,\tau}(p_h, s_h; \lambda_h).$$

We use the following splitting algorithm to approximate saddle points for $M_{h,\varepsilon,\tau}$.

Algorithm 4.3 (Splitting method II)**.** Choose $s_h^0, \lambda_h^0 \in \mathcal{L}^1(\mathcal{T}_h)^d$ and $\tau > 0$ and set $k = 1$.

(1) Compute the minimizer $p_h^k \in V_h$ for

$$p_h \mapsto M_{h,\varepsilon,\tau}(p_h, s_h^{k-1}; \lambda_h^{k-1}).$$

(2) Compute the minimizer $s_h^k \in \mathcal{L}^1(\mathcal{T}_h)^d$ for

$$s_h \mapsto M_{h,\varepsilon,\tau}(p_h^k, s_h; \lambda_h^{k-1}).$$

(3) Compute the maximizer $\lambda_h^k \in \mathcal{L}^1(\mathcal{T}_h)^d$ for

$$\lambda_h \mapsto -\frac{1}{2\tau} \|\lambda_h - \lambda_h^{k-1}\|_{h,w}^2 + M_{h,\varepsilon,\tau}(p_h^k, s_h^k; \lambda_h).$$

(4) Stop the iteration if

$$\tau \|s_h^k - s_h^{k-1}\|_{h,w} + \|\lambda_h^k - \lambda_h^{k-1}\|_{h,w} \le \delta_{\mathrm{stop}}.$$

(5) Increase $k \to k+1$ and continue with (1).

The minimization problem in step (1) is equivalent to a linear system of equations, while the iterates of steps (2) and (3) can be computed explicitly.

Remark 4.4.

(i) The minimization in step (1) is a linearly constrained quadratic minimization problem. Its well-posedness in the case of zeroth or first order Raviart–Thomas elements follows from the coercivity of the bilinear form

$$a_{h,\tau}(p_h, q_h) = \tau \int_\Omega h_T^d \widehat{\mathcal{I}}_h \left[\Pi_h^1 p_h \cdot \Pi_h^1 q_h \right] \mathrm{d}x$$

on the subspace of divergence-free vector fields in V_h, *i.e.*, on

$$K_h = \{q_h \in V_h : \operatorname{div} q_h = 0\}.$$

It is shown in [4] and Section 2.3 that for $V_h = \mathcal{RT}_N^\ell(\mathcal{T}_h)$ we have

$$K_h \subset \mathcal{L}^\ell(\mathcal{T}_h) \cap W_N^1(\mathrm{div}; \Omega).$$

Hence, if $\ell = 0, 1$ then for $p_h \in K_h$ we have $\Pi_h^1 p_h = p_h$ and

$$a_{h,\tau}(p_h, p_h) = \tau \int_\Omega h_T^d \widehat{\mathcal{I}}_h |p_h|^2 \,\mathrm{d}x \geq \tau h_{\min}^d \|p_h\|^2_{W_N^1(\mathrm{div};\Omega)},$$

where we used that $\|v_h\|_h \geq \|v_h\|$ for $v_h \in \mathcal{L}^\ell(\mathcal{T}_h)$, $\ell = 0, 1$.

(ii) step (2) is equivalent to nodewise minimization problems, *i.e.*, for every $T \in \mathcal{T}_h$ and every $z \in \mathcal{N}_h \cap T$ the value $x = s_h^k|_T(z)$ is minimal for

$$x \mapsto \frac{1}{1+\varepsilon}|x|^{1+\varepsilon} + x \cdot c + \frac{c'}{2}|x|^2,$$

with $c = h_T^d(\lambda_h^{k-1}|_T(z) - \tau \Pi_h^1 p_h^k|_T(z))$ and $c' = h_T^d \tau$. The optimal vector $x \in \mathbb{R}^d$ is zero if $c = 0$ and a nonnegative multiple of $-c/|c|$ otherwise. This factor $\alpha = |s_h^k|_T(z)|$ is minimal for

$$\alpha \mapsto \frac{1}{1+\varepsilon}\alpha^{1+\varepsilon} - \alpha|c| + \frac{c'}{2}\alpha^2.$$

For $\varepsilon = 0$ the minimizer is given by $\alpha = (1/c')\max\{|c| - 1, 0\}$. We use this value to initialize the Newton scheme for the optimality equation

$$\alpha^\varepsilon - |c| + c'\alpha = 0.$$

To improve the convexity properties and hence the performance of the Newton iteration for this equation, we use the variable transformation $\beta = \alpha^\varepsilon$, which results in the equation

$$g(\beta) = \beta - |c| + c'\beta^{1/\varepsilon} = 0$$

with a uniformly convex function g.

(iii) Step (3) is equivalent to the explicit equation

$$h_T^d(\lambda_h^k - \lambda_h^{k-1}) = \tau h_T^d (s_h^k - \Pi_h^1 p_h^k).$$

As in the previous subsection the stopping criterion controls the residuals in the optimality conditions and the constraint violation.

5. Convergence of approximations

5.1. Vanishing gap estimator

We show that if the estimators $\eta_h(\phi_h, p_h)$ converge to zero as $h \to 0$, then the bounded sequences $(\phi_h)_{h>0} \subset W^{1,\infty}(\Omega)$ and $(p_h)_{h>0} \subset W^1_N(\mathrm{div};\Omega)$ accumulate at solutions of the primal and dual problem, respectively. The following proposition guarantees that for vanishing maximal mesh-size and gap estimator numerical approximations for the primal problem accumulate at solutions. For statements on the convergence of the dual variable we refer the reader to discussion below and the results of [6].

Proposition 5.1. *Assume that the sequences $(\phi_h)_{h>0} \subset W^{1,\infty}(\Omega)$ and $(p_h)_{h>0} \subset W^1_N(\mathrm{div};\Omega)$ are such that*

$$\eta_h(\phi_h, p_h) \to 0$$

as $h \to 0$. We then have that

$$K(\phi_h) \to \max_{\phi \in W^{1,\infty}(\Omega)} K(\phi), \quad D_h(p_h) \to \inf_{p \in W^1_N(\mathrm{div};\Omega)} D(p).$$

Moreover, every weak-$\star$ accumulation point $\phi \in W^{1,\infty}(\Omega)$ is a maximizer for K.

Proof. The estimator defines an upper bound for the primal dual gap of the pair (ϕ_h, p_h) and for the difference $\|f - f_h\|_{L^1(\Omega)}$. This implies that we have

$$\lim_{h\to 0} K(\phi_h) \to \max_{\phi \in W^{1,\infty}(\Omega)} K(\phi),$$

i.e., $(\phi_h)_{h>0}$ is an infimizing sequence for $-K$. The weak-* lower semicontinuity of $-K$ implies that every weak-* accumulation point solves the primal problem. □

Remark 5.2. Interpreting the sequence $(p_h)_{h>0}$ as a bounded sequence in a space of vectorial Radon measures, it follows that subsequences converge to generalized solutions of the dual problem, *cf.* [6].

5.2. Consistency of regularizations

The definition of the primal-dual gap estimator is based on the numerical solution of regularized primal and dual formulations. We show that these are consistent regularizations under moderate assumptions on the regularization parameters. The following two results imply that the gap estimator tends to zero as $h \to 0$.

Proposition 5.3 (Primal functional)**.** *Let $(\phi_h)_{h>0}$ be a sequence of maximizers $\phi_h \in X_h \subset W^{1,\infty}(\Omega) \cap L^1_0(\Omega)$ for the functionals*

$$K_{h,\varepsilon}(\phi_h) = \int_\Omega f_h \phi_h \,\mathrm{d}x - \frac{\varepsilon}{2}\|\nabla\phi_h\|^2 - I_{K_1(0)}(\nabla\phi_h).$$

We then have that

$$0 \le K_h(\phi_h) - K_{h,\varepsilon}(\phi_h) \le \frac{\varepsilon}{2}|\Omega|.$$

Proof. The statement is an immediate consequence of the fact that we have $\|\nabla\phi_h\|_{L^\infty(\Omega)} \le 1$ for every $h > 0$. □

To allow for the best possible convergence rate of the discrete costs we choose $\varepsilon = \overline{h}^2$. The analysis for the regularized dual functional is more involved.

Proposition 5.4 (Dual functional)**.** *Assume that* $f_h = \Pi_h^1 f \to f$ *in* $L^1(\Omega)$ *as* $h \to 0$. *For every* $q \in W_N^1(\mathrm{div};\Omega) \cap W^{2,\infty}(\Omega;\mathbb{R}^n)$ *with* $-\operatorname{div} q = f$ *we have*

$$\big|D_{h,\varepsilon}(I_F q) - D(q)\big| \le c\big(h^\sigma + h^2 + \varepsilon\big),$$

with some $\sigma \ge 1$ *or* $\sigma = 2$ *if*

$$\|D_\mathcal{T}^2 |\Pi_h^k I_F q|^{1+\varepsilon}\|_{L^2(\Omega)} \le c,$$

and where $D_{h,\varepsilon}$ *is defined by*

$$D_{h,\varepsilon}(p_h) = \frac{1}{1+\varepsilon}\int_\Omega \widehat{\mathcal{I}}_h |\Pi_h^1 p_h|^{1+\varepsilon}\,\mathrm{d}x + I_{\{-f_h\}}(\operatorname{div} p_h). \qquad \square$$

Proof. The properties of the generalized interpolant I_F imply that $-\operatorname{div} I_F q = f_h$ and that $I_F q \cdot n = 0$ on $\partial\Omega$. To prove the asserted estimate we note that with the abbreviations $q_h = I_F q$ and $s_h = \Pi_h^1 q_h$ we have

$$\begin{aligned}\big|D_{h,\varepsilon}(I_F q) - D(q)\big| &\le \Big|\int_\Omega \widehat{\mathcal{I}}_h |s_h|^{1+\varepsilon} - |s_h|^{1+\varepsilon}\,\mathrm{d}x\Big| \\ &+ \big\||s_h|^{1+\varepsilon} - |s_h|\big\|_{L^1(\Omega)} + \big\|s_h - q\big\|_{L^1(\Omega)} + \varepsilon\|s_h\|_{L^1(\Omega)} = A + B + C + D.\end{aligned}$$

We estimate the terms on the right-hand side using that the local projection operator Π_h^1 is stable in the sense that $\|s_h\|_{W^{\ell,r}(T)} \le c_{\ell,r}\|q_h\|_{W^{\ell,r}(T)}$ for $\ell \in \{0,1\}$, $r \in [1,\infty]$, and every $T \in \mathcal{T}_h$. Basic interpolation estimates imply that, formally, the quadrature term A is bounded by

$$A \le ch^2 \|D_\mathcal{T}^2 |s_h|^{1+\varepsilon}\|_{L^2(\Omega)},$$

Rigorously, we have with $r = 1+\varepsilon$ for $x \in T$ that

$$\big||s_h(x)|^r - \widehat{\mathcal{I}}_h |s_h|^r(x)\big| = \sum_{z \in \mathcal{N}_h \cap T} \varphi_z(x)\big(|s_h(z)|^r - |s_h(x)|^r\big),$$

and the monotonicity estimate

$$|b|^r - |c|^r \le r|b|^{r-2} b \cdot (b-c)$$

yields that

$$A \le r\|s_h\|_{L^\infty(\Omega)}^{r-1} h \|\nabla s_h\|_{L^1(\Omega)}.$$

To bound term B we note that by the mean value theorem we have for $a > 0$ that there exists $0 \le \varepsilon' \le \varepsilon$ such that

$$|a - a^{1+\varepsilon}| \le \varepsilon a^{1+\varepsilon'} |\ln a|.$$

If $a \ge 1$ then we use $a^{1+\varepsilon'} \le a^{1+\varepsilon}$. Otherwise, if $0 < a < 1$ then we set $s = 1+\varepsilon'$ and deduce that

$$|a - a^{1+\varepsilon}| \le \varepsilon a^s |s \ln a| = \varepsilon e^{s\ln a}|s\ln a| \le \varepsilon,$$

where we incorporated that $|x|e^x \le 1$ for $x < 0$. It thus follows that

$$B \le \varepsilon\big(1 + \ln(1 + \|s_h\|_{L^\infty(\Omega)})\big)\int_\Omega |s_h|^{1+\varepsilon}\,\mathrm{d}x.$$

An estimate for term C follows from interpolation estimates, *i.e.*,

$$C \le \|\Pi_h^1 I_F q - I_F q\|_{L^1(\Omega)} + \|I_F q - q\|_{L^1(\Omega)} \le ch^2\|q\|_{H^2(\Omega)}.$$

Incorporating H^2 stability of I_F implies the estimate.

By choosing a regularized infimizing sequence we obtain convergence of the discrete, regularized cost value to the exact one. For completeness we show how convergence of equilibria can be proved. For this we assume that the sequence of approximations of the dual problem accumulates at vector fields in $W^1_N(\mathrm{div};\Omega)$ but this assumption can be avoided by considering generalized minimizers, *cf.* [6]. Minimizers for $D_{h,\varepsilon}$ are characterized by the system

$$\begin{aligned}\left(|\Pi_h^1 p_h|^{\varepsilon-1}\Pi_h^1 p_h, \Pi_h^1 q_h\right)_h - (u_h, \operatorname{div} q_h) &= 0,\\ -(v_h, \operatorname{div} p_h) &= (f_h, v_h).\end{aligned}$$

and we have that the unique solutions (p_h, u_h) obey the uniform bound

$$\|p_h\|_{W^1_N(\mathrm{div};\Omega)} + \|u_h\|_{L^\infty(\Omega)} \le c,$$

which follows from the estimates discussed in Section 2.3, *cf.* [6] for further details. As $(h,\varepsilon) \to 0$ we extract weakly-* convergent subsequences (possibly after embedding the vector fields into a space of Radon measures) with limit $(p,u) \in W^1_N(\mathrm{div};\Omega) \times L^\infty(\Omega)$. To show that this pair is a solution for the continuous dual problem we first note that for every $\xi \in C_0^\infty(\Omega;\mathbb{R}^d)$ we have

$$(u_h, \operatorname{div} I_F\xi) \to (u, \operatorname{div}\xi).$$

This implies that with $r = 1+\varepsilon$ we have

$$\begin{aligned}(u_h, \operatorname{div}[p_h - I_F\xi]) &= (|\Pi_h^1 p_h|^{r-2}\Pi_h^1 p_h, \Pi_h^1[p_h - I_F\xi])_h\\ &\ge \frac{1}{r}(|\Pi_h^1 p_h|^r, 1)_h - \frac{1}{r}(|\Pi_h^1 I_F\xi|^r, 1)_h.\end{aligned}$$

As $(h,r) \to (0,1)$ we find, using Jensen's inequality and weak lower semicontinuity of the L^1 norm, that $-\operatorname{div} p = f$ and

$$(u, \operatorname{div}[p-\xi]) \ge \int_\Omega |q| \,\mathrm{d}x - \int_\Omega |\xi| \,\mathrm{d}x,$$

which characterizes a solution of the continuous dual problem. Note that $u \in W^{1,\infty}(\Omega)$ solves the primal problem, *i.e.*, the multipliers $u_h \in Q_h$ for the constraint $-\operatorname{div} p_h = f_h$ define nonconforming approximations for the primal problem.

6. Numerical experiments

We verify our results and test our adaptive refinement strategy in two experiments. The densities $f^\pm$ are constructed with the help of the Gaussian bell function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$g(x,z) = c_{d,\sigma}\exp\left(-\frac{|x-z|^2}{2\sigma^2}\right),$$

where we always set $\sigma = 0.05$. The first setting uses a convex domain and four Gaussian bells.

Example 6.1 (Convex domain)**.** Let $d = 2$ and $\Omega = (-1/2, 1/2)^2$ and define for $x \in \Omega$

$$f^\pm(x) = \frac{1}{2}g(x, z_\pm) + \frac{1}{2}g(x, -z_\pm),$$

where $z_+ = (1/4, 1/4)$ and $z_- = (-1/4, 1/4)$.

Less regular solutions are expected on nonconvex domains. Considering this case is motivated by the connection of the Monge–Kantorovich problem problem to the infinity Laplace problem.

Example 6.2 (Nonconvex domain)**.** Let $d = 2$ and $\Omega = (-1/2, 1/2)^2 \setminus [-1/8, 1/8]^2$, and define for $x \in \Omega$

$$f^{\pm}(x) = g(x, \pm z),$$

where $z = (1/4, 1/4)$.

We use the $P1$ finite element space $X_h = \mathcal{S}^1(\mathcal{T}_h) \cap L^1_0(\mathcal{T}_h)$ for discretizing the primal problem and the first order Raviart–Thomas finite element space $V_h = \mathcal{RT}^1_N(\mathcal{T}_h)$ combined with elementwise affine functions $Q_h = \mathcal{L}^1(\mathcal{T}_h)$ for the dual problem. Corresponding approximations $f_h \in Q_h$ of the functions $f = f^+ - f^-$ on adaptively generated triangulations are displayed in Figure 1. The functions are obtained by nodal interpolation and a correction of their means, *i.e.*, we always use

$$f_h = \frac{1}{(\mathcal{I}_h f^+, 1)} \mathcal{I}_h f^+ - \frac{1}{(\mathcal{I}_h f^-, 1)} \mathcal{I}_h f^-.$$

The adapted triangulations $\mathcal{T}_h$ are obtained from coarse initial triangulations by refining elements $T \in \mathcal{M}_h$ in a minimal set $\mathcal{M}_h \subset \mathcal{T}_h$ for which we have

$$\sum_{T \in \mathcal{M}_h} \widetilde{\eta}_T(p_h, \phi_h) \geq \frac{1}{2} \sum_{T' \in \mathcal{T}_h} \widetilde{\eta}_T(p_h, \phi_h).$$

Further elements are refined to guarantee mesh conformity. The numerical solutions p_h and ϕ_h defining the error indicators $\eta_T(p_h, \phi_h)$ are obtained with Algorithms 4.1 and 4.3, respectively. The auxiliary variables s_h and μ_h needed therein are initialized via prolongations of corresponding functions on coarser triangulations or with the trivial value zero. The involved parameters are defined via the average mesh-size $\overline{h} = (\#\mathcal{N}_h)^{-1/d}$ and

$$\varepsilon = \overline{h}^2, \ \tau = 1/\overline{h}, \ \delta_{\text{stop}} = \overline{h}.$$

All linear systems of equations with a symmetric and positive definite matrix were solved with a conjugate gradient methods and a preconditioning obtained from Cholesky factorizations. Other linear systems, *i.e.*, those that correspond to saddle-point problems were solved with a direct method. Mesh-refinement was always done using bisection of single elements.

Remark 6.3. We implemented computable approximations $\widetilde{\eta}_T$ of the indicators η_T. These are obtained from first replacing p_h by its elementwise projection $\Pi^1_h p_h$ onto elementwise affine vector fields, *i.e.*,

$$\eta_T(\phi_h, p_h) \leq \int_T |\Pi^1_h p_h| - \Pi^1_h p_h \cdot \nabla \phi_h \,\mathrm{d}x + \|\Pi^1_h p_h - p_h\|_{L^1(T)}.$$

Noting that the mapping $x \mapsto |\Pi^1_h p_h(x)|$, $x \in T$, is convex, we deduce the upper bound

$$\eta_T(\phi_h, p_h) \leq \int_T \widehat{\mathcal{I}}_h |\Pi^1_h p_h| - \Pi^1_h p_h \cdot \nabla \phi_h \,\mathrm{d}x + \|\Pi^1_h p_h - p_h\|_{L^1(T)}.$$

To control the L^1 norm on the right-hand side we note that it follows from (2.2) and $\operatorname{div} p_h = f_h$ that on every $T \in \mathcal{T}_h$ we have

$$p_h - \Pi^1_h p_h = \frac{1}{d+k} \big(\big[(x - x_T) f_h\big] - \Pi^1_h \big[(x - x_T) f_h\big] \big).$$

Hence, we have

$$\|p_h - \Pi^1_h p_h\|_{L^1(T)} \leq |T|^{1/2} \|(I - \Pi^1_h)\big[(x - x_T) f_h\big]\|_{L^2(T)} \leq h_T^{1+d/2} \|f_h\|_{L^2(T)}.$$

Omitting this data term motivates defining the approximate estimator

$$\widetilde{\eta}_T(p_h, \phi_h) = \int_T \widehat{\mathcal{I}}_h |\Pi^1_h p_h| - \Pi^1_h p_h \cdot \nabla \phi_h \,\mathrm{d}x.$$

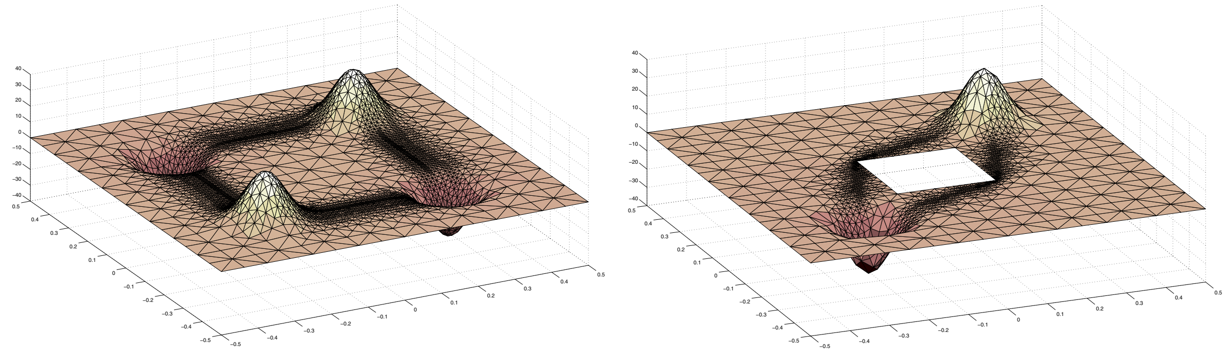

FIGURE 1. Density functions $f = f^+ - f^-$ in Examples 6.1 (*left*) and 6.2 (*right*) on adaptively generated triangulations.

6.1. Experimental convergence rates

To illustrate the benefits of adaptive mesh refinement we plotted in Figure 2 for sequences of uniformly and adaptively refined triangulations the errors in the approximation of the primal cost defined by

$$\delta_h = K_{\text{ref}} - K_h(\phi_h),$$

where the reference value K_{ref} is obtained from an extrapolation of approximations on a sequence of uniformly refined triangulations. In the case of the convex domain from Example 6.1 we observe quadratic experimental rates of convergence for both uniform and adaptive mesh refinement. This is surprising at first glance since the approximated exact potential ϕ is nonsmooth as can be anticipated from the plot of its approximation ϕ_h shown in the left part of Figure 3. The transport flux is however supported in regions in which ϕ_h is smooth, *cf.* Figure 4. In the setting of Example 6.2 the experimental convergence rate for uniform mesh refinement is approximately linear. The adaptive refinement strategy improves this experimental value to the theoretically optimal quadratic rate.

In the experiments reported above we avoided having a good approximation of an exact dual solution by working with a reference value for the primal cost. It turned out that the approximation of the dual variable is more difficult than approximating the primal variable accurately. To illustrate this observation we limit the number of error sources and replace the Gaussian bells in the densities $f^\pm$ of Examples 6.1 and 6.2 by hat functions with respect to the initial triangulations centered at the same points. Due to this choice the approximations of the primal cost are nearly exact. Figure 4 shows the transport flux in the modified versions of Examples 6.1 and 6.2.

Experimental convergence rates for the primal-dual gaps

$$\delta_h = D_{h,\varepsilon}(p_h) - K_{\text{ref}}$$

are illustrated in Figure 5. We observe that for the convex and the nonconvex domain uniform mesh refinement leads to suboptimal experimental convergence rates. The convergence rates relate to the expected $W^{4/3,\infty}(\Omega)$ regularity of solutions for the infinity Laplace operator independently of convexity properties of the domain. The experimental rates are improved to the optimal quadratic rate by adaptive mesh refinement.

6.2. Choice of parameters

We introduced regularizations of the primal and dual problems defined by a parameter $\varepsilon \geq 0$. The splitting algorithms devised for the numerical solution of the primal and dual problems are well defined for $\varepsilon = 0$ but their perfomance is expected to improve for a positive parameter as solutions are then unique. Figure 6 displays

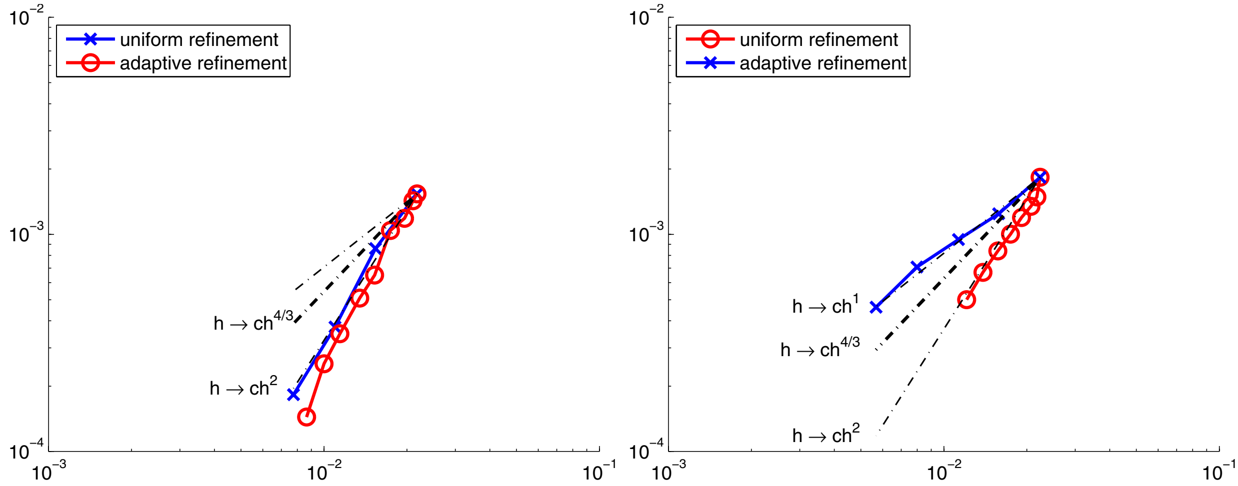

FIGURE 2. Approximated primal cost errors $K_{\mathrm{ref}} - K_h(\phi_h)$ in the convex (*left*) and nonconvex (*right*) settings from Examples 6.1 and 6.2. Adaptive mesh refinement leads to reduced errors and an improved experimental convergence rate in the nonconvex setting.

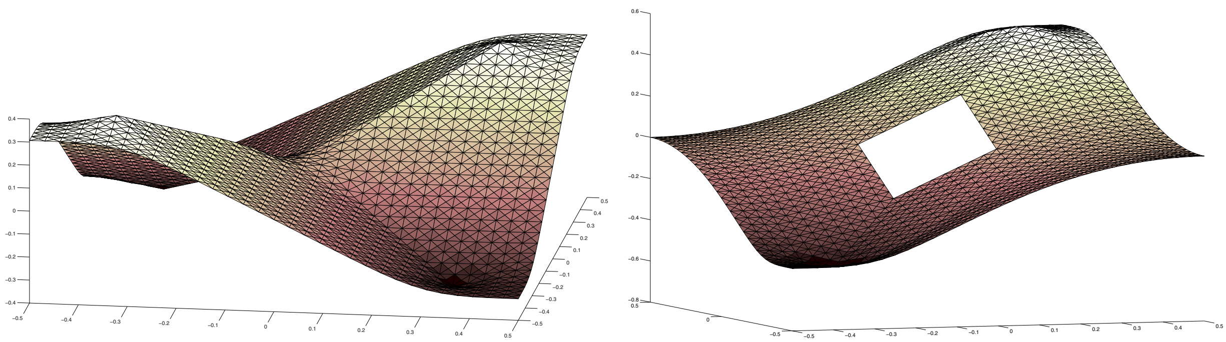

FIGURE 3. Approximate potentials ϕ_h on the initial triangulations of convex (*left*) and nonconvex (*right*) domains in Examples 6.1 and 6.2.

the numbers of iterations needed to satisfy the stopping criteria on triangulations obtained from 8, 9, 10, and 11 uniform refinements of the initial triangulation. The numbers show that regularization does not affect the iteration numbers in case of the primal problem while these are substantially reduced in case of the dual problem.

Although the primal and dual splitting algorithms are unconditionally convergent in terms of the step size τ, the numbers of iteration depend sensitively on this quantity. Our choices are motivated by the stability estimate for the iterations which involve the upper bounds

$$\|\lambda_h - \lambda_h^0\|^2 + \tau^2 \|p_h - p_h^0\|^2,$$

i.e., the squared distances of starting values to a solution. Assuming that within a multilevel scheme our starting values (λ_h^0, p_h^0) approximate the pair (λ_h, λ_h^0) comparable to the optimal approximation rates $\mathcal{O}(h^k)$, we choose $\tau = h^{-k}$. The iteration numbers shown in Table 1 reveal that for the primal problem a step size in the range $[1, 1/\overline{h}]$ and for the dual problem in the range $[1/\overline{h}, 1/\overline{h}^2]$ lead to the best results. On the triangulations $\mathcal{T}_j$, $j \geq 1$, good starting values are available via prolongation from coarser triangulations.

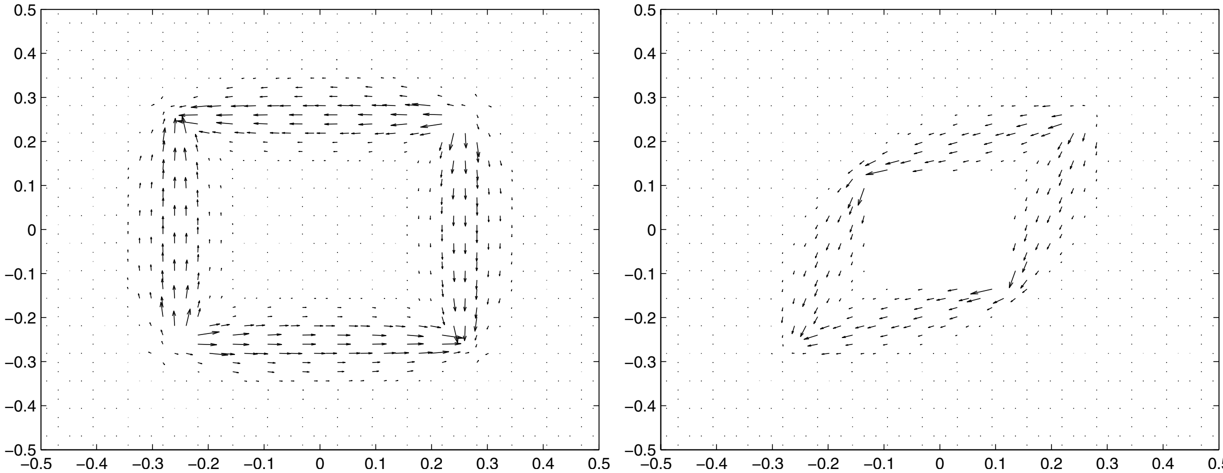

FIGURE 4. Approximated transport flux on a convex (*left*) and a nonconvex (*right*) domain. Nearly no transport occurs through the center of the convex domain.

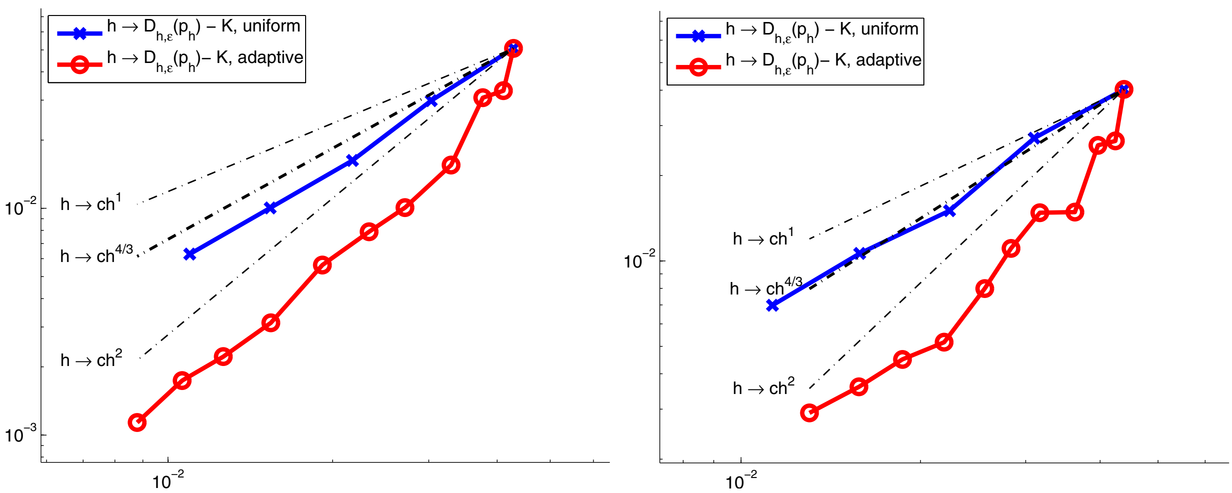

FIGURE 5. Primal-dual gap δ_h with extrapolated reference value K_{ref} in the modified versions of Examples 6.1 and 6.2.

TABLE 1. Iteration numbers for different step sizes in the primal (*left*) and dual (*right*) splitting algorithms on uniform triangulations of a nonconvex domain with different mesh-sizes.

τ	$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$
1	86	72	136	227	356
$1/\overline{h}$	**82**	**51**	**74**	**176**	**246**
$1/\overline{h}^2$	1068	158	342	1458	–

τ	$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$
1	81	157	312	618	1230
$1/\overline{h}$	17	**31**	**41**	**94**	**212**
$1/\overline{h}^2$	**13**	175	346	552	1142

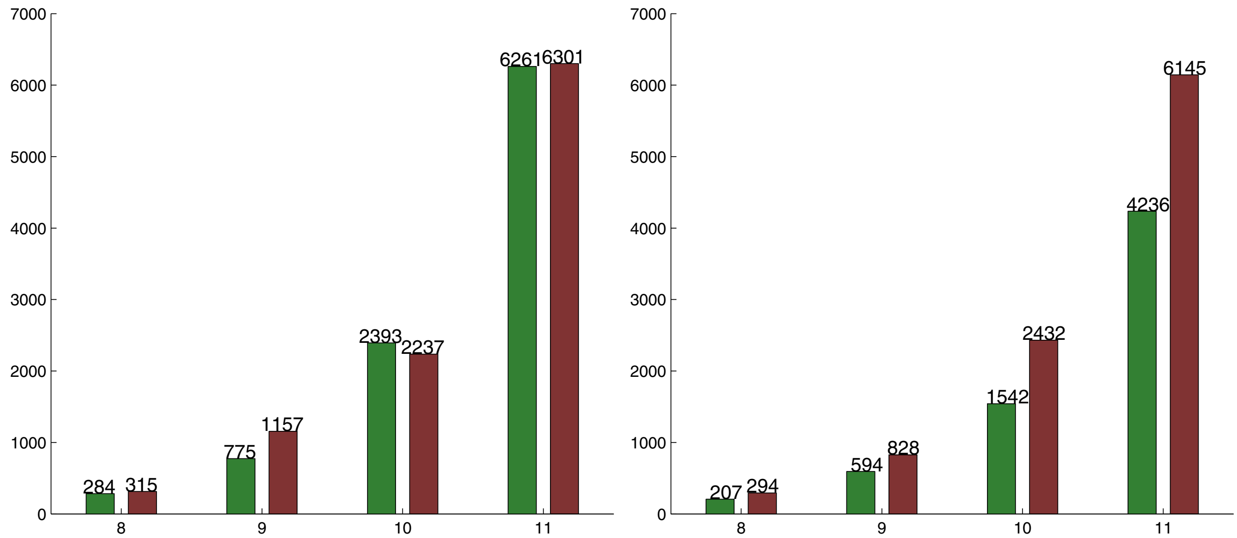

FIGURE 6. Iteration numbers for the primal (*left*) and dual (*right*) splitting algorithms in the case of nonconvex domain and uniformly refined triangulations with mesh-size $\overline{h} \sim 2^{-j}$, $j = 8, 9, 10, 11$. White bars refer to unregularized functionals, i.e., $\varepsilon = 0$, while gray bars correspond to regularized functions with $\varepsilon = \overline{h}^2$.

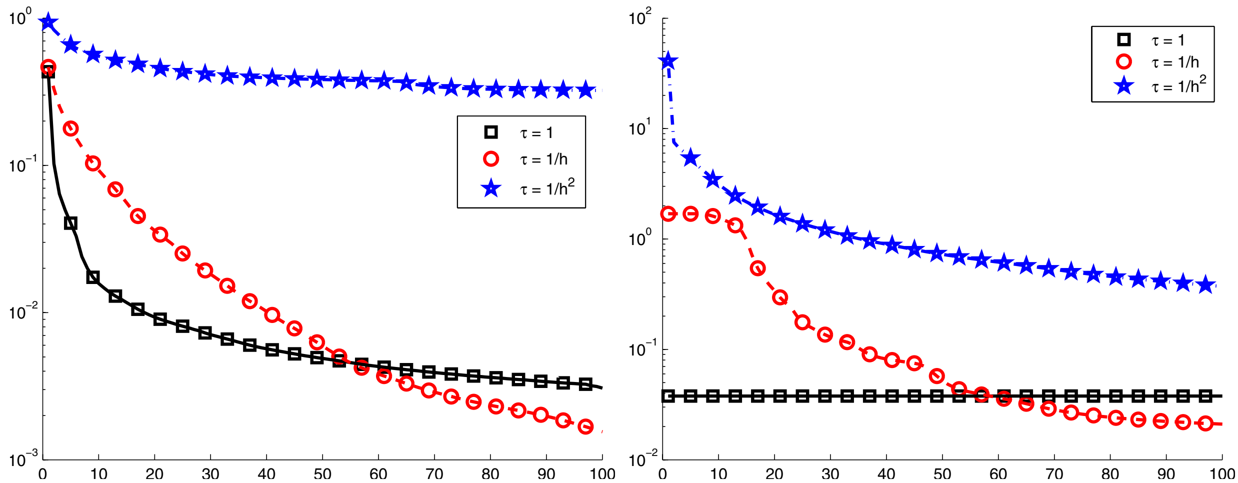

FIGURE 7. Convergence of the discrete residual for the primal (*left*) and dual (*right*) splitting algorithms on a fixed triangulation resulting from 12 uniform refinements and different step sizes.

To further illustrate the importance of a good choice of the step size τ we displayed in Figure 7 the decay of the residuals, *i.e.*, the quantities that serve as stopping criteria in Step (4) of Algorithms 4.1 and 4.3 on a fixed triangulation using different step sizes. We observe that the best convergence behavior is obtained for $\tau = 1/h$. A similar conclusion can be drawn from the plots in Figure 8 where we displayed the primal and dual cost functionals for the iterates of the algorithms. For $\tau = 1/h$ the value $\max I(\phi) = \min D(p) \approx 0.772$ is attained most rapidly.

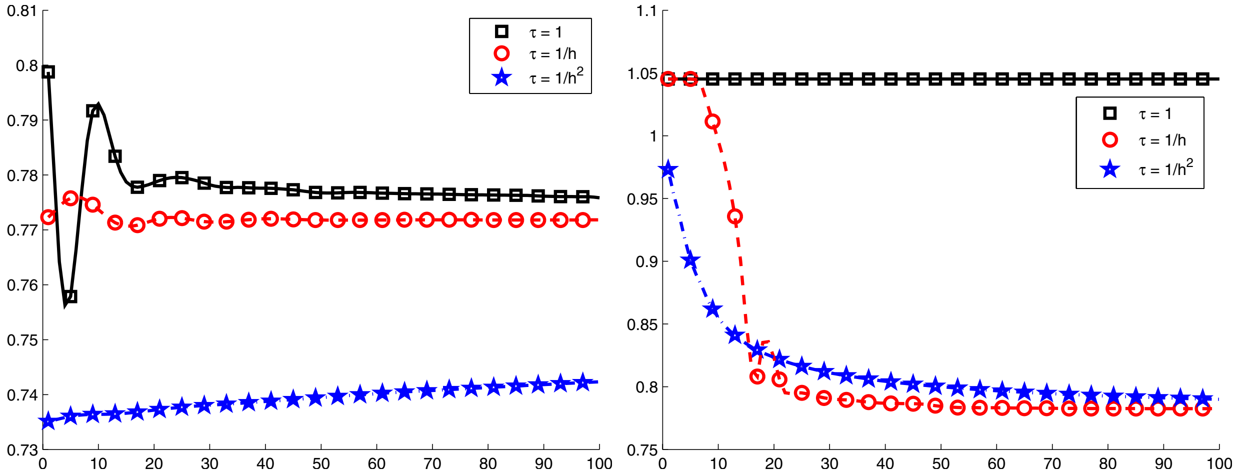

FIGURE 8. Convergence of the primal (*left*) and dual (*right*) cost functionals during the iteration of the primal and dual splitting algorithms.

APPENDIX A. CONVERGENCE OF THE SPLITTING METHOD

Given convex, proper, and lower-semicontinuous functionals $F : Y \to \mathbb{R} \cup \{+\infty\}$, $G : X \to \mathbb{R} \cup \{+\infty\}$, and a bounded linear operator $B : X \to Y$ we consider the minimization problem

$$\inf_{u \in X} F(Bu) + G(u).$$

Upon introducing $p = Bu$ and choosing $\tau \geq 0$ we obtain the equivalent, consistently stabilized saddle-point problem defined by

$$\inf_{(u,p)\in X\times Y} \sup_{\lambda \in Y} F(p) + G(u) + (\lambda, Bu - p)_Y + \frac{\tau}{2}\|Bu - p\|_Y^2 = \mathcal{L}_\tau(u, p; \lambda).$$

Here we assume that Y is a Hilbert space and let $(\cdot,\cdot)_Y$ be an inner product on Y with associated norm $\|\cdot\|_Y$. Possible strong convexity of F or G is characterized by nonnegative functionals $\varrho_F : Y \times Y \to \mathbb{R}$ and $\varrho_G : X \times X \to \mathbb{R}$ satisfying

$$\begin{aligned}(r, q - p)_Y + F(p) + \varrho_F(q, p) &\leq F(q),\\ \langle w, v - u\rangle + G(u) + \varrho_G(v, u) &\leq G(v),\end{aligned}$$

for all $p, q, r \in Y$ and $u, v \in X$ and $w \in X'$ such that $r \in \partial F(p)$ and $w \in \partial G(u)$.

Lemma A.1 (Optimality conditions). *A triple (u, p, λ) is a saddle point for $\mathcal{L}_\tau$ if and only if $Bu = p$ and*

$$\begin{aligned}-\big(\lambda, B(v - u)\big)_Y + G(u) + \varrho_G(v, u) &\leq G(v),\\ \big(\lambda, q - p\big)_Y + F(p) + \varrho_F(q, p) &\leq F(q),\end{aligned}$$

for all $(v, q) \in X \times Y$.

Proof. The variational inequalities characterize stationarity with respect to u and p, respectively, *i.e.*, that, *e.g.*, $0 \in \partial_u \mathcal{L}_\tau(u, p; \lambda)$. □

The arbitrary nonnegative parameter $\tau \geq 0$ is assumed to be positive in the following iterative algorithm. Typical choices are $\tau \geq 1$ as the iteration is unconditionally convergent.

Algorithm A.2 (Splitting algorithm)**.** Choose $(p^0, \lambda^0) \in Y \times Y$ and $\tau > 0$, set $k = 1$.

(1) Compute the minimizer $u^k \in X$ for
$$u \mapsto \mathcal{L}_\tau(u, p^{k-1}; \lambda^{k-1}).$$
(2) Compute the minimizer $p^k \in Y$ for
$$p \mapsto \mathcal{L}_\tau(u^k, p; \lambda^{k-1}).$$
(3) Compute the maximizer λ^k for
$$\lambda \mapsto \frac{-1}{2\tau}\|\lambda - \lambda^{k-1}\|_Y^2 + \mathcal{L}_\tau(u^k, p^k; \lambda),$$
i.e., update $\lambda^k = \lambda^{k-1} + \tau(Bu^k - p^k)$.

(4) Stop if
$$\|\lambda^k - \lambda^{k-1}\|_Y + \tau\|p^{k-1} - p^k\|_Y \le \delta_{\text{stop}}.$$
(5) Set $k \to k+1$ and continue with (1).

Note that the order of minimization in u and p can be exchanged. In this case the contribution $\tau\|p^k - p^{k-1}\|_Y$ in the stopping criterion has to be replaced by $\tau\|u^k - u^{k-1}\|_X$ as this difference then occurs as a residual in the optimality equation for the minimization with respect to the p variable. The quantity $\lambda^k - \lambda^{k-1} = \tau(Bu^k - p^k)$ measures the violation of the constraint. To guarantee termination of the iteration, the minimization should first be done in a variable in which no coercivity is available, *cf*. Theorem A.4 below.

Lemma A.3 (Decoupled optimality conditions)**.** *The well defined iterates* $(u^k, p^k, \lambda^k)_{k=0,1,\ldots}$ *satisfy the variational inequalities*
$$\begin{aligned} -\big(\lambda^k + \tau(p^k - p^{k-1}), B(v - u^k)\big)_Y + G(u^k) + \varrho_G(v, u^k) &\le G(v),\\ \big(\lambda^k, q - p^k\big)_Y + F(p^k) + \varrho_F(q, p^k) &\le F(q),\\ \lambda^k - \lambda^{k-1} &= \tau(Bu^k - p^k), \end{aligned}$$
for all $(v, q) \in X \times Y$*, i.e., the triple* (u^k, p^k, λ^k) *is a saddle point for* $\mathcal{L}_\tau$ *if* $\lambda^k - \lambda^{k-1} = 0$ *and* $\tau(p^k - p^{k-1}) = 0$.

Proof. The variational inequalities characterize stationarity of iterates. □

In what follows, we use for a sequence $(a^k)_{k=0,1,\ldots}$ the backward difference quotient
$$d_t a^k = (a^k - a^{k-1})/\tau,$$
and note that we have the discrete product rule
$$2 d_t a^k \cdot a^k = d_t|a^k|^2 + \tau|d_t a^k|^2.$$
With this, the updating step in Algorithm A.2 can be written as $d_t\lambda^k = Bu^k - p^k$. For ease of presentation we introduce the symmetrized coercivity functionals
$$\begin{aligned} \widehat{\varrho}_F(p, q) &= \varrho_F(p, q) + \varrho_F(q, p),\\ \widehat{\varrho}_G(u, v) &= \varrho_G(u, v) + \varrho_G(v, u). \end{aligned}$$
We assume that these are nonnegative functionals.

Theorem A.4 (Convergence). *Let (u,p,λ) be a saddle point for $\mathcal{L}_\tau$. For the iterates $(u^k,p^k,\lambda^k)_{k=0,1,\dots}$ of Algorithm A.2 and corresponding errors $\delta_\lambda^k = \lambda - \lambda^k$, $\delta_p^k = p - p^k$, and $\delta_u^k = u - u^k$, and every $K \geq 0$ we have that*

$$\frac{1}{2}\Big(\|\delta_\lambda^K\|_Y^2 + \tau^2\|\delta_p^K\|_Y^2\Big) + \tau\sum_{k=1}^K \Big\{\widehat{\varrho}_G(u^k,u) + \widehat{\varrho}_F(p^k,p) + \widehat{\varrho}_F(p^k,p^{k-1}) + \frac{\tau}{2}\Big(\|d_t\delta_\lambda^k\|_Y^2 + \tau^2\|d_t\delta_p^k\|^2\Big)\Big\}$$
$$\leq \frac{1}{2}\Big(\|\delta_\lambda^0\|_Y^2 + \tau^2\|\delta_p^{-1}\|_Y^2\Big).$$

In particular, $-d_t\delta_\lambda^k = d_t\lambda^k = Bu^k - p^k \to 0$ and $-d_t\delta p^k = d_t p^k \to 0$ as $k \to \infty$ so that Algorithm A.2 terminates.

Proof. We choose $(v,q) = (u^k,p^k)$ in the optimality conditions for the saddle point and $(v,q) = (u,p)$ in the optimality conditions for the iterates, and add corresponding equations to verify that

$$-\big([\lambda - \lambda^k] - \tau(p^k - p^{k-1}), B[u^k - u]\big)_Y + \widehat{\varrho}_G(u^k,u) \leq 0,$$
$$\big(\lambda - \lambda^k, p^k - p\big)_Y + \widehat{\varrho}_F(p^k,p) \leq 0.$$

Adding and using $Bu^k - p^k = d_t\lambda^k = -d_t\delta_\lambda^k$ and $Bu = p$ we find that

$$\widehat{\varrho}_G(u^k,u) + \widehat{\varrho}_F(p^k,p) \leq -\tau(p^k - p^{k-1}, B[u^k - u])_Y + (\lambda - \lambda^k, Bu^k - p^k)_Y$$
$$= \tau^2(d_t p^k, B\delta_u^k)_Y - (\delta_\lambda^k, d_t\delta_\lambda^k)_Y.$$

This implies that

$$\frac{d_t}{2}\|\delta_\lambda^k\|_Y^2 + \frac{\tau}{2}\|d_t\delta_\lambda^k\|_Y^2 + \widehat{\varrho}_G(u^k,u) + \widehat{\varrho}_F(p^k,p) \leq \tau^2\big(d_t p^k, B\delta_u^k\big)_Y. \tag{A.1}$$

We choose $q = p^{k-1}$ and $q = p^k$ in the variational inequalities that characterize p^k and p^{k-1}, respectively, to verify that

$$\big(\lambda^k, -\tau d_t p^k\big)_Y + F(p^k) - F(p^{k-1}) + \varrho_F(p^k,p^{k-1}) \leq 0,$$
$$\big(\lambda^{k-1}, \tau d_t p^k,\big)_Y + F(p^{k-1}) - F(p^k) + \varrho_F(p^{k-1},p^k) \leq 0.$$

Adding these inequalities and using $d_t\lambda^k = Bu^k - p^k$ leads to

$$-\tau^2\big(Bu^k - p^k, d_t p^k\big)_Y + \widehat{\varrho}_F(p^k,p^{k-1}) \leq 0.$$

Inserting $Bu = p$ and using $d_t p^k = -d_t\delta_p^k$ implies that

$$\tau^2\frac{d_t}{2}\|\delta_p^k\|_Y^2 + \tau^2\frac{\tau}{2}\|d_t\delta_p^k\|_Y^2 + \widehat{\varrho}_F(p^k,p^{k-1}) \leq -\tau^2\big(B\delta_u^k, d_t p^k\big)_Y. \tag{A.2}$$

Adding (A.1) and (A.2) and summing over $k = 1,\dots,K$ proves the theorem. ☐

Convergence of the iterates requires uniqueness of the corresponding limiting objects. Otherwise, only convergence of subsequences can be established.

Remark A.5.

(i) For large step sizes τ the convergence of (u^k) and (p^k) may be slow whereas the consistency error $Bu^k - p^k$ converges rapidly.

(ii) Upper and lower estimates for the cost functional follow from choosing $(v,q) = (u^k, p^k)$ in the optimality conditions and $(v,q) = (u,p)$ in the decoupled optimality conditions. In particular, we find that

$$\delta_I^k = \big[F(p^k) + G(u^k)\big] - \big[F(p) + G(u)\big] \geq -(\lambda, Bu^k - p^k)_Y \geq -\|\lambda\|_Y \frac{\delta_{\text{stop}}}{\tau},$$

and

$$\begin{aligned}\delta_I^k + \widehat{\varrho}_F(p,p^k) &\leq \big(\lambda^k + \tau(p^k - p^{k-1}), B(u-u^k)\big)_Y - (\lambda^k, p - p^k)_Y \\ &\leq \tau\|p^k - p^{k-1}\|_Y\big(\|p-p^k\|_Y + \|p^k - Bu^k\|_Y\big) + \|\lambda^k\|_Y \frac{\delta_{\text{stop}}}{\tau}.\end{aligned}$$

This indicates that $\delta_{\text{stop}}/\tau$ has to be sufficiently small to guarantee convergence of approximating costs.

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