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A SECOND ORDER IN TIME INCREMENTAL PRESSURE CORRECTION FINITE ELEMENT METHOD FOR THE NAVIER-STOKES/DARCY PROBLEM *

Yunxia Wang[1], Shishun Li[2] and Zhiyong Si[2,*]

Abstract. In this paper, we give a second order in time incremental pressure correction finite element method for the Navier-Stokes/Darcy problem. In this method, the Navier-Stokes/Darcy problem is solved in three steps: a convection-diffusion step, a projection correction (incremental pressure correction) step and a Darcy step. In this way, the Navier-Stokes/Darcy equation is solved in a fractional step way, which is a decoupled method. In order to decouple the equation, we use the numerical solutions at the last time level to give the interface conditions. The stability analysis shows that the second order in time incremental pressure correction finite element method is unconditionally stable. The optimal error estimate is also given. Finally, we present some numerical results to show the efficiency of the method.

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1. Introduction

The free flow coupling with the porous media flow has received a great deal of attention in the area of scientific computing due to its many important applications in modeling groundwater flow, filtration processes, petroleum reservoirs. Different numerical methods have been provided, such as unified stabilized finite element formulations for the Stokes/Darcy's flow [2]; super–convergence analysis of finite element method for the Stokes/Darcy system [1]; a posteriori error estimate for the Stokes/Darcy [12]; Robin–Robin domain decomposition methods for the steady–state Stokes–Darcy system [6, 10, 14] and so on. In 2012, Layton *et al.* [32] gave four non–iterative, sub–physics, uncoupling methods by splitting the Stokes/Darcy problems into the Stokes and Darcy problems. In [17], fully–mixed finite element method was given by Gatica *et al.* In [34], a decoupled finite element method for the Stokes/Darcy flow was given. In 2012, Shan, Zheng and Layton [37] presented a decoupled method using different time steps in different domains. In [45], a local discontinuous Galerkin (LDG) method for the Stokes/Darcy flow was given by Vassilev and Yotov. For the Navier–Stokes/Darcy coupling problem,

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[1] School of Materials Science and Engineering, Henan Polytechnic University, 454003, Jiaozuo, P.R. China.

[2] School of Mathematics and Information Science, Henan Polytechnic University, 454003, Jiaozuo, P.R. China.

* Corresponding author: zhiyongsi@gmail.com; sizhiyong@hpu.edu.cn

several iterative methods were presented by Badea, Discacciati and Quarteroni [3, 13]. The two–level method for the Navier–Stokes/Darcy problem was given by Cai *et al.* [5] in 2009. Girault and Riviere [18] presented a discontinuous Galerkin approximation of coupled Navier–Stokes/Darcy equations with BJS interface condition. In [42], we gave the decoupled modified characteristics finite element method for the time–dependent Navier–Stokes/Darcy problem. In this paper, we focus on the model of the Navier–Stokes equation in the surface region coupling the Darcy's law in the subsurface region [13], which is described by a mixed Navier–Stokes/Darcy's model. The first important thing is the interface conditions between the Navier–Stokes flow and the Darcy's flow. This model was firstly studied by Beavers and Joseph [4], they gave the interface condition (named Beavers–Joseph (BJ) condition) on the interface between the surface flow and the subsurface flow. Then, this condition was simplified by Saffman [36], who got the Beavers–Joseph–Saffman (BJS) condition. A coupled finite element method for Navier–Stokes and Darcy problem with the Beavers–Joseph interface condition was present by Zuo and Hou [47].

Projection methods are widely used to approximate the incompressible time–dependent Navier–Stokes equations. Its original version was introduced by Chorin [11] and Temam [44] in the late 1960s. There are three families of projection methods: pressure–correction [15, 29, 38, 43], velocity–correction [27] and consistent splitting scheme [28,35,41], which is also called gauge method [16]. On the other hand, the projection method can be seen as the fractional a splitting step method, but as the pressure is not a dynamic variable, the usual methodology for fractional step method can not be applied directly [21]. The incremental pressure correction method for solving the Navier–Stokes equation was given by [20, 22]. There are many works on this method [23–26, 39, 40]. In this method, an advection–diffusion step was considered and a projection correction (incremental pressure correction) step was considered in the second step.

In this paper, we give a second order in time incremental pressure correction method for solving the free flow coupled with the porous media flow, which is governed by the Navier–Stokes/Darcy equation. This method is implemented in three steps. Firstly, a convection–diffusion equation is solved. Secondly, a projection correction (incremental pressure correction) is done. Finally, the Darcy flow is solved in the porous media domain by the second order in time method. To decouple the Navier–Stokes flows and the Darcy flow, the numerical solutions in the last time level are used on the interface. The stability analysis shows that our method is unconditionally stable. The error estimate yields the optimal convergence order. In order to show the effect of our method, some numerical results are presented. The numerical results show that our method is effective, which confirms our theoretical analysis.

The rest of this paper is organized as follows. In Section 2, the functional settings of the Navier–Stokes/Darcy equation are presented. In Section 3, the second order in time incremental pressure correction finite element method is proposed. In the following Section 4, we derive stability analysis for the new algorithm. Section 5 presents the error analysis. Finally, some numerical examples are presented in Section 6.

2. Functional settings

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a bounded domain, which is decomposed into two non–intersecting sub–domains Ω_f and Ω_p separated by an interface Γ, namely $\Omega = \Omega_f \cup \Omega_p, \Omega_f \cap \Omega_p = \emptyset$ and $\bar{\Omega}_f \cap \bar{\Omega}_p = \Gamma$. We suppose the boundaries $\partial\Omega_f$ and $\partial\Omega_p$ have the Lipschitz conditions. From the physical point of view, Γ is a surface separating the domain Ω_f filled by a fluid from and a domain Ω_p formed by a porous medium.

Let $T > 0$ be a finite constant, the fluid flow is governed by the Navier–Stokes equation as follows

$$\begin{cases} \mathbf{u}_t - \nabla \cdot (2\nu \mathbb{D}(\mathbf{u}) - p\mathbf{I}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = f(x,t), & x \in \Omega_f \times (0, T], \\ \nabla \cdot \mathbf{u} = 0, & x \in \Omega_f \times (0, T], \\ \mathbf{u}(x, 0) = \mathbf{u}_0, & x \in \Omega_f, \\ \mathbf{u} = 0, & x \in \partial\Omega_f \backslash \Gamma \times (0, T], \end{cases} \tag{2.1}$$

where $\mathbf{u}(x,t)$ represents the velocity of the fluid flow in Ω_f, $p(x,t)$ is the pressure, $f(x,t)$ represents the external body force, and ν is the kinematic viscosity, $\mathbb{D}(\mathbf{u})$ is the deformation tensor defined by $\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$.

The porous media flow is governed by the following equation in Ω_p

$$\begin{cases} S_0\phi_t + \nabla\cdot\mathbf{u}_p = f_p(x,t), & x\in\Omega_p\times(0,T], \\ \mathbf{u}_p = -\mathbf{K}\nabla\phi, & x\in\Omega_p\times(0,T], \\ \phi(x,0)=\phi_0, & x\in\Omega_p, \\ \phi = 0, & x\in\partial\Omega_p\backslash\Gamma\times(0,T], \end{cases} \tag{2.2}$$

where ϕ is the piezometric head, $\mathbf{u}_p$ is the fluid velocity in the porous media Ω_p. $\mathbf{K}$ represents the hydraulic conductivity tensor, for simplicity, we assume that $\mathbf{K} = \mathrm{diag}(K_1,\dots,K_d)$ with $0 < K_i \in L^\infty(\Omega_p)$, $i = 1,\dots,d$ which means the porous media is homogeneous. We assume that $\mathbf{K}$ is uniformly bounded and positive definite in Ω_p, *i.e.*, there exist constants $K_{\min}, K_{\max} > 0$ such that

$$K_{\min}|x|^2 \le \mathbf{K}x\cdot x \le K_{\max}|x|^2, \;\; a.e. \;\; x\in\Omega_p.$$

Using Darcy's law, (2.2) can be rewritten in the parabolic form

$$S_0\phi_t - \nabla\cdot(\mathbf{K}\nabla\phi) = g_p(x,t), \;\; x\in\Omega_p\times(0,T].$$

For the Navier–Stokes/Darcy model, the interface conditions of the conservation of mass, balance of forces and the Beavers–Joseph–Saffman condition are imposed by

$$\mathbf{u}\cdot n_f + \mathbf{u}_p\cdot n_p = 0, \quad \text{on} \quad \Gamma\times(0,T], \tag{2.3}$$

$$p - \nu n_f\frac{\partial\mathbf{u}}{\partial n_f} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} = g\phi, \quad \text{on} \quad \Gamma\times(0,T], \tag{2.4}$$

$$-\nu\tau_i\frac{\partial\mathbf{u}}{\partial n_f} = \frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}}\mathbf{u}\cdot\tau_i, i=1,2,\dots,d-1, \quad \text{on} \quad \Gamma\times(0,T]. \tag{2.5}$$

Here, g is the gravitational acceleration, α is a positive parameter depending on the properties of the porous medium and must be determined experimentally. The condition (2.3) can be rewritten as

$$\mathbf{u}\cdot n_f = \mathbf{K}\frac{\partial\phi}{\partial n_p}, \quad \text{on} \quad \Gamma\times(0,T]. \tag{2.6}$$

Remark 2.1. The term $\frac{1}{2}\mathbf{u}\cdot\mathbf{u}$ arises naturally from the momentum equation written in divergence form [7,8,18,31].

Define $W = H_f\times H_p$ and $Q = L^2(\Omega_f)$, where

$$H_f = \left\{\mathbf{v}\in(H^1(\Omega_f))^d : \mathbf{v}=0 \quad \text{on} \quad \partial\Omega_f\backslash\Gamma\right\},$$

$$H_p = \left\{\phi\in H^1(\Omega_p) : \phi = 0 \quad \text{on} \quad \partial\Omega_p\backslash\Gamma\right\},$$

$$\mathbf{V} = \left\{\mathbf{v}\in H_f : \nabla\cdot\mathbf{v} = 0\right\}.$$

The spaces H_f and H_p are equipped with the following norms

$$\|\mathbf{u}\|_{H_f} = \|\mathbb{D}(\mathbf{u})\|_{\Omega_f}, \forall\mathbf{u}\in H_f,$$

$$\|\phi\|_{H_p} = \|\nabla\phi\|_{\Omega_p}, \forall\phi\in H_p,$$

where $\|\cdot\|_D := \|\cdot\|_{L^2(D)}$ means the L^2–norm on the domain D. We equip the space W with the following norms

$$\|w\|_0 = \sqrt{(\mathbf{u},\mathbf{u})_{\Omega_f} + gS_0(\phi,\phi)_{\Omega_p}}, \forall w = (\mathbf{u},\phi) \in W,$$

$$\|w\|_W = \sqrt{\nu(\nabla\mathbf{u},\nabla\mathbf{u})_{\Omega_f} + g(\mathbf{K}\nabla\phi,\nabla\phi)_{\Omega_p}}, \forall w = (\mathbf{u},\phi) \in W,$$

where $(\cdot,\cdot)_D$ refers the inner product in the corresponding domain D for $D = \Omega_f$ or Ω_p. The weak form of the time–dependent Navier–Stokes/Darcy model reads as follows: find $w = (\mathbf{u},\phi) \in W$ and $p \in Q$ such that

$$\begin{aligned} [w_t, z] + a(w,z) + B(\mathbf{u},\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) &= (F,z), \forall z = (\mathbf{v},\psi) \in W,\\ b(\mathbf{u},q) &= 0, \forall q \in Q,\\ w(0) &= w_0, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
[w_t, z] &= (\mathbf{u}_t,\mathbf{v})_{\Omega_f} + gS_0(\phi_t,\psi)_{\Omega_p},\\
a(w,z) &= a_f(\mathbf{u},\mathbf{v}) + a_p(\phi,\psi) + a_\Gamma(w,z),\\
a_f(\mathbf{u},\mathbf{v}) &= 2\nu(\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))_{\Omega_f} + \sum_{i=1}^{d-1}\int_\Gamma \frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}}(\mathbf{u}\cdot\tau_i)(\mathbf{v}\cdot\tau_i)\mathrm{d}s,\\
a_p(\phi,\psi) &= g(\mathbf{K}\nabla\phi,\nabla\psi)_{\Omega_p},\\
a_\Gamma(w,z) &= a_\Gamma(\mathbf{u},\phi;\mathbf{v},\psi) = c_I(\phi,\mathbf{v}) - c_I(\psi,\mathbf{u}),\\
c_I(\psi,\mathbf{u}) &= g\int_\Gamma \psi\mathbf{u}\cdot n_f\mathrm{d}s,\\
B(\mathbf{u},\mathbf{v},\mathbf{w}) &= (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})_{\Omega_f} + \frac{1}{2}((\nabla\cdot\mathbf{u})\mathbf{v},\mathbf{w})_{\Omega_f} - \frac{1}{2}(\mathbf{u}\cdot\mathbf{v},\mathbf{w}\cdot\mathbf{n}_s)_{\Omega_f},\\
b(v,p) &= (p,\nabla\cdot\mathbf{v})_{\Omega_p},\\
(F,z) &= (f,\mathbf{v})_{\Omega_f} + g(f_p,\psi)_{\Omega_p}.
\end{aligned}$$

Here, we define the Stokes operator $\mathcal{A}$ (see [46] and the references therein) by

$$(\mathcal{A}\mathbf{u},\mathbf{v})_{\Omega_f} = (\mathcal{A}^{1/2}\mathbf{u},\mathcal{A}^{1/2}\mathbf{v})_{\Omega_f} = (\nabla\mathbf{u},\nabla\mathbf{v})_{\Omega_f}, \forall\mathbf{u},\mathbf{v} \in \mathbf{V}.$$

Lemma 2.2. [33] *The bilinear forms $a_f(\cdot,\cdot)$, $a_p(\cdot,\cdot)$ satisfy*

$$a_f(\mathbf{u},\mathbf{v}) \le \max\left\{\nu+1, \frac{C\alpha}{2\sqrt{K_{\min}}}\right\}\|\mathbf{u}\|_{H_f}\|\mathbf{v}\|_{H_f}, \tag{2.8}$$

$$a_f(\mathbf{u},\mathbf{u}) \ge \nu\|\mathbf{u}\|_{H_f}^2 + \frac{\alpha}{\sqrt{K_{\max}}}\sum_{i=1}^{d}\int_\Gamma(\mathbf{u}\cdot\tau_i)^2\mathrm{d}s =: \nu\|\mathbf{u}\|_{H_f}^2 + \frac{\alpha}{\sqrt{K_{\max}}}\|\mathbf{u}\cdot\tau\|_\Gamma^2, \tag{2.9}$$

$$a_p(\phi,\psi) \le K_{\max}\|\phi\|_{H_p}\|\psi\|_{H_p}, \tag{2.10}$$

$$a_p(\phi,\phi) \ge K_{\min}\|\phi\|_{H_p}^2. \tag{2.11}$$

The trilinear form $B(\cdot,\cdot,\cdot)$ satisfies the following properties ([18, 46])

$$B(\mathbf{u},\mathbf{v},\mathbf{w}) = -B(\mathbf{u},\mathbf{w},\mathbf{v}), \forall \mathbf{u} \in H_f(\Omega) \cap V, \mathbf{v},\mathbf{w} \in H_f, \tag{2.12}$$

$$|B(\mathbf{u},\mathbf{v},\mathbf{w})| \le C\|\mathcal{A}\mathbf{u}\|_{\Omega_f}\|\nabla\mathbf{v}\|_{\Omega_f}\|\mathbf{v}\|_{\Omega_f}, \forall \mathbf{u} \in D(\mathcal{A}), \mathbf{v},\mathbf{w} \in X, \tag{2.13}$$

$$|B(\mathbf{u},\mathbf{v},\mathbf{w})| \le C\|\mathcal{A}\mathbf{u}\|_{\Omega_f}^{1/2}\|\mathbf{u}\|_{\Omega_f}^{1/2}\|\nabla\mathbf{v}\|_{\Omega_f}\|\mathbf{w}\|_{\Omega_f}, \forall \mathbf{u} \in D(\mathcal{A}), \mathbf{v},\mathbf{w} \in X. \tag{2.14}$$

3. The second order in time incremental pressure correction finite element method

For each positive integer N, let $\{\mathcal{J}_n : 1 \leqslant n \leqslant N\}$ be a partition of $[0,T]$ into subintervals $\mathcal{J}_n = (t_{n-1}, t_n]$, with $t_n = n\tau,\ \tau = T/N$. With the previous notations, we get the time discrete incremental pressure correction method for the Navier–Stokes/Darcy problem (2.1).

Algorithm 3.1 (Time discrete incremental pressure correction method)**.**

Step 1. Find $\tilde{\mathbf{u}}^{n+1} \in H_f$, such that

$$\frac{3\tilde{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\triangle t} - \nabla\cdot(2\nu\mathbb{D}(\tilde{\mathbf{u}}^{n+1}) - p^n\mathbf{I}) + ((2\mathbf{u}^n - \mathbf{u}^{n-1})\cdot\nabla)\tilde{\mathbf{u}}^{n+1} = f(t_{n+1}). \tag{3.1}$$

Step 2. Find $(\mathbf{u}^{n+1}, p^{n+1}) \in (H_f, Q)$, such that

$$\frac{3\mathbf{u}^{n+1} - 3\tilde{\mathbf{u}}^{n+1}}{2\triangle t} + \nabla(p^{n+1} - p^n) = 0, \tag{3.2}$$

$$\nabla\cdot\mathbf{u}^{n+1} = 0, \tag{3.3}$$

Step 3. Find $\phi_h^{n+1} \in H_p$ such that

$$S_0\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\triangle t} - \nabla\cdot(\mathbf{K}\nabla\phi^n) = f_p(t_{n+1}).$$

The weak form of the method is easily given as follows.

Algorithm 3.2 (Weak form of the time discrete incremental pressure correction method)**.**

Step 1. Find $\tilde{\mathbf{u}}^{n+1} \in H_f$, such that

$$\left(\frac{3\tilde{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\triangle t}, \mathbf{v}\right)_{\Omega_f} + a_f(\tilde{\mathbf{u}}^{n+1}, \mathbf{v}) + B(2\mathbf{u}^n - \mathbf{u}^{n-1}, \tilde{\mathbf{u}}^{n+1}, \mathbf{v}) + c_I(2\phi^n - \phi^{n-1}, \mathbf{v}) - b(\mathbf{v}, p^n) = (f(t_{n+1}), \mathbf{v})_{\Omega_f}. \tag{3.4}$$

Step 2. Find $(\mathbf{u}^{n+1}, p^{n+1}) \in (H_f, Q)$, such that

$$\left(\frac{3\mathbf{u}^{n+1} - 3\tilde{\mathbf{u}}^{n+1}}{2\triangle t}, \mathbf{v}\right)_{\Omega_f} - b(\mathbf{v}, p^{n+1} - p^n) = 0, \tag{3.5}$$

$$b(\mathbf{u}^{n+1}, q) = 0. \tag{3.6}$$

Step 3. Find $\phi^{n+1} \in H_p$, *via*

$$gS_0\left(\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\triangle t}, \psi\right)_{\Omega_p} + g(\mathbf{K}\nabla\phi^{n+1}, \nabla\psi)_{\Omega_p} - c_I(\psi, 2\mathbf{u}^n - \mathbf{u}^{n-1}) = g(f_p(t_{n+1}), \psi)_{\Omega_p}.$$

Let $\Im_h$ be a quasi–uniform partition of $\bar{\Omega}_f \cup \bar{\Omega}_p$ into non–overlapping triangles if d = 2 or tetrahedra if d = 3, indexed by a parameter $h = \max_{K\in\Im_h}\{h_K : h_K = \mathrm{diam}(K)\}$. We introduce the finite element subspaces $W_h = H_{fh} \times H_{ph} \subset W$ and $Q_h \subset Q$,

$$H_{fh} = \{\mathbf{v}_h \in H_f \cap C^0(\bar{\Omega})^d; \mathbf{v}_h|_K \in P_\ell(K)^d, \forall K \in \Im_h\},$$

$$Q_h = \{q_h \in Q \cap H^1(\bar{\Omega}); q_h|_K \in P_k(K), \forall K \in \Im_h\},$$

$$H_{ph} = \{\phi_h \in H_p \cap C^0(\bar{\Omega}); \phi_h|_K \in P_l(K), \forall K \in \Im_h\},$$

where $P_\ell(K)$ is the space of piecewise polynomials of degree ℓ on K, $\ell \geqslant 1$, $k \geqslant 1$ and $l \geqslant 1$ are three integers. Here, we assume that (H_{fh}, Q_h) satisfy the discrete LBB condition, *i.e.*, there exists a constant $\beta > 0$ such that

$$\sup_{\mathbf{v}_h\in H_{fh}} \frac{b(\mathbf{v}_h, \varphi_h)}{\|\nabla \mathbf{v}_h\|_0} \geqslant \beta\|\varphi_h\|_0, \quad \forall \varphi_h \in Q_h.$$

Remark 3.3. Here, we chose $Q_h \subset H^1(\bar{\Omega})$ as in references [19, 22], which is different for the classical finite element space $Q_h \subset L^2(\bar{\Omega})$.

Under the above notations, we give the second order incremental pressure correction finite element method for the Navier–Stokes/Darcy problem as follows.

Algorithm 3.4 (Second order incremental pressure correction finite element method)**.**

Step 1. Find $\tilde{\mathbf{u}}_h^{n+1} \in H_{fh}$, such that

$$\left(\frac{3\tilde{\mathbf{u}}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}}{2\triangle t}, \mathbf{v}_h\right)_{\Omega_f} + a_f(\tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) + B(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) + c_I(2\phi_h^n - \phi_h^{n-1}, \mathbf{v}_h) - b(\mathbf{v}_h, p_h^n) = (f(t_{n+1}), \mathbf{v}_h)_{\Omega_f}. \tag{3.7}$$

Step 2. Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in (H_{fh}, Q_h)$, such that

$$\left(\frac{3\mathbf{u}_h^{n+1} - 3\tilde{\mathbf{u}}_h^{n+1}}{2\triangle t}, \mathbf{v}_h\right)_{\Omega_f} - b(\mathbf{v}_h, p_h^{n+1} - p_h^n) = 0, \tag{3.8}$$

$$b(\mathbf{u}_h^{n+1}, q_h) = 0. \tag{3.9}$$

Step 3. Find $\phi_h^{n+1} \in H_{ph}$, *via*

$$gS_0\left(\frac{3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1}}{2\triangle t}, \psi_h\right)_{\Omega_p} + g(\mathbf{K}\nabla\phi_h^{n+1}, \nabla\psi_h)_{\Omega_p} - c_I(\psi_h, 2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) = g(f_p(x, t_{n+1}), \psi_h)_{\Omega_p}. \tag{3.10}$$

4. Stability analysis

In this section, we turn our attention to the stability analysis of the projection method. We give the following lemma firstly.

Lemma 4.1. [32, 34] *There exists a constant* $C > 0$*, such that*

$$|c_I(\phi, \mathbf{u})| \leq Cg\|\phi\|_{H_p}\|\mathbf{u}\|_{H_f}. \tag{4.1}$$

Then, we can get the stability as follows.

Theorem 4.2 (Stability)**.** *The second order incremental pressure correction finite element method is unconditionally stable, in the sense that*

$$\begin{aligned}
&\|\mathbf{u}_h^{n+1}\|_{\Omega_f}^2 + \|2\mathbf{u}_h^{n+1} - \mathbf{u}^n\|_{\Omega_f}^2 + \sum_{i=1}^{n} \|\mathbf{u}_h^{i+1} - 2\mathbf{u}_h^i + \mathbf{u}_h^{i-1}\|_{\Omega_f}^2 \\
&+\ 4\triangle t\nu \sum_{i=1}^{n} \|\tilde{\mathbf{u}}_h^{i+1}\|_{H_f}^2 + \triangle t \sum_{i=1}^{n} \int_{\Gamma} \frac{\alpha\sqrt{\nu g}}{\sqrt{trace(\mathbf{K})}} (\tilde{\mathbf{u}}_h^{i+1} \cdot \tau)^2 \mathrm{ds} \\
&+\ \frac{4\triangle t^2}{3} \|\nabla p_h^{n+1}\|_{\Omega_f}^2 + 3g\triangle t \sum_{i=1}^{n} \|\mathbf{K}^{1/2} \nabla \phi_h^{i+1}\|_{\Omega_p}^2 \\
&+\ gS_0 \left(\|\phi_h^{n+1}\|_{\Omega_p}^2 + \|2\phi_h^{n+1} - \phi_h^n\|_{\Omega_p}^2 + \sum_{i=1}^{n} \|\phi_h^{i+1} - 2\phi_h^i + \phi_h^{i-1}\|_{\Omega_p}^2 \right) \\
&\le \frac{C\triangle t}{\nu} \sum_{i=1}^{n} \|f(t_{i+1})\|_{H^{-1}(\Omega_f)}^2 + \frac{4\triangle tg}{k_{\min}} \sum_{i=1}^{n} \|g_p(t_{i+1})\|_{\Omega_p}^2.
\end{aligned}$$

Proof. For convenience, we define $\delta_t a^n = a^n - a^{n-1}$, and $\delta_{tt} a^n = \delta_t(\delta_t a^{n-1})$, for all the function a. Letting $\mathbf{v}_h = 4\triangle t\tilde{\mathbf{u}}_h^{n+1}$ in (3.7) and using $2(3a^{n+1} - 4a^n + a^{n-1}, a^{n+1}) = |a^{n+1}|^2 + |2a^{n+1} - a^n|^2 + |\delta_{tt} a^{n+1}|^2 - |a^n|^2 - |2a^n - a^{n-1}|^2$, we have

$$\begin{aligned}
&2\left(3\tilde{\mathbf{u}}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}\right)_{\Omega_f} \\
&= 2\left(3\mathbf{u}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}\right)_{\Omega_f} + 6(\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^{n+1}, \tilde{\mathbf{u}}_h^{n+1}) \\
&= 2\left(3\mathbf{u}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^{n+1}\right)_{\Omega_f} + 2\left(3\mathbf{u}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}\right)_{\Omega_f} + 6(\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^{n+1}, \tilde{\mathbf{u}}_h^{n+1}).
\end{aligned}$$

From (3.8), we have $\left(3\mathbf{u}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^{n+1}\right)_{\Omega_f} = 0$, then

$$\begin{aligned}
&2\left(3\tilde{\mathbf{u}}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}\right)_{\Omega_f} \\
&= \|\mathbf{u}_h^{n+1}\|_{\Omega_f}^2 - \|\mathbf{u}_h^n\|_{\Omega_f}^2 + \|2\mathbf{u}_h^{k+1} - \mathbf{u}^k\|_{\Omega_f}^2 - \|2\mathbf{u}_h^k - \mathbf{u}^{k-1}\|_{\Omega_f}^2 + \|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\|_{\Omega_f}^2 \\
&+ 3\|\tilde{\mathbf{u}}_h^{n+1}\|_{\Omega_f}^2 + 3\|\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^{n+1}\|_{\Omega_f}^2 - 3\|\mathbf{u}_h^{n+1}\|_{\Omega_f}^2. \qquad (4.2)
\end{aligned}$$

Via (2.12), we deduce

$$\begin{aligned}
&3\|\tilde{\mathbf{u}}_h^{n+1}\|_{\Omega_f}^2 - 2\|\mathbf{u}_h^{n+1}\|_{\Omega_f}^2 + \|2\mathbf{u}_h^{k+1} - \mathbf{u}^k\|_{\Omega_f}^2 - \|2\mathbf{u}_h^k - \mathbf{u}^{k-1}\|_{\Omega_f}^2 + \|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\|_{\Omega_f}^2 \\
&-\ \|\mathbf{u}_h^n\|_{\Omega_f}^2 + 3\|\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^{n+1}\|_{\Omega_f}^2 + 4\triangle t\nu\|\tilde{\mathbf{u}}_h^{n+1}\|_{H_f}^2 + \triangle t \int_{\Gamma} \frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}} (\tilde{\mathbf{u}}_h^{n+1} \cdot \tau)^2 \mathrm{d}s \\
&+\ 4\triangle t c_I(2\phi_h^n - \phi_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}) - 4\triangle t(\nabla \cdot \tilde{\mathbf{u}}_h^{n+1}, p_h^n)_{\Omega_f} = 4\triangle t(f(t_{n+1}), \tilde{\mathbf{u}}_h^{n+1})_{\Omega_f}. \qquad (4.3)
\end{aligned}$$

In order to estimate the term $4\triangle tb(\tilde{u}_h^{n+1}, p_h^n)$, setting $\mathbf{v}_h = 4\triangle t^2 \nabla p_h^n$ in (3.8), we have

$$\begin{aligned}
4\triangle tb(\tilde{u}_h^{n+1}, p_h^n) &= -\frac{8\triangle t^2}{3} (\nabla(p_h^{n+1} - p_h^n), \nabla p_h^n) \\
&= \frac{4\triangle t^2}{3} \|\nabla p_h^n\|_{\Omega_f}^2 - \frac{4\triangle t^2}{3} \|\nabla p_h^{n+1}\|_{\Omega_f}^2 + \frac{4\triangle t^2}{3} \|\nabla(p_h^{n+1} - p_h^n)\|_{\Omega_f}^2. \qquad (4.4)
\end{aligned}$$

Letting $v_h = \triangle t \nabla(p_h^{n+1} - p_h^n)$ in (3.8), we have

$$\triangle t \|p_h^{n+1} - p_h^n\|_{\Omega_f}^2 = -\frac{3}{2}(u_h^{n+1} - \tilde{u}_h^{n+1}, \nabla(p_h^{n+1} - p_h^n))$$
$$\leq \frac{3}{2}\|\nabla(p_h^{n+1} - p_h^n)\|_{\Omega_f}\|u_h^{n+1} - \tilde{u}_h^{n+1}\|_{\Omega_f}.$$

It means that

$$\triangle t\|\nabla(p_h^{n+1} - p_h^n)\|_{\Omega_f} \leq \frac{3}{2}\|u_h^{n+1} - \tilde{u}_h^{n+1}\|_{\Omega_f}. \tag{4.5}$$

Combining (4.4) and (4.5), we have

$$-2\triangle t b(\tilde{u}_h^{n+1}, p_h^n) \geq \frac{4\triangle t^2}{3}\|\nabla p_h^{n+1}\|_{\Omega_f}^2 - \frac{4\triangle t^2}{3}\|\nabla p_h^n\|_{\Omega_f}^2 - 3\triangle t\|u_h^{n+1} - \tilde{u}_h^{n+1}\|_{\Omega_f}^2. \tag{4.6}$$

Using (4.1), we get

$$4\triangle t c_I(2\phi_h^n - \phi_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}) \leq Cg\triangle t(\|\phi_h^n\|_{H_p} + \|\phi_h^{n-1}\|_{H_p})\|\tilde{\mathbf{u}}_h^{n+1}\|_{H_p}$$
$$\leq \frac{\triangle t\nu}{2}\|\tilde{\mathbf{u}}_h^{n+1}\|_{H_f}^2 + \frac{Cg\triangle t}{K_{\min}}(\|K^{1/2}\nabla\phi_h^n\|_{\Omega_p}^2 + \|K^{1/2}\nabla\phi_h^{n-1}\|_{\Omega_p}^2). \tag{4.7}$$

Via Cauchy–Schwarz inequality, we have

$$4\triangle t(f(t_{n+1}), \tilde{\mathbf{u}}_h^{n+1})_{\Omega_f} \leq \frac{\triangle t\nu}{2}\|\tilde{\mathbf{u}}_h^{n+1}\|_{H_f}^2 + \frac{C\triangle t}{\nu}\|f(t_{n+1})\|_{H^{-1}(\Omega_f)}^2. \tag{4.8}$$

Setting $\mathbf{v}_h = 4\triangle t\mathbf{u}_h^{n+1}$ in (3.8) and noticing $b(\mathbf{u}_h^{n+1}, p_h^{n+1} - p_h^n) = 0$, we deduce

$$\|\mathbf{u}_h^{n+1}\|_{\Omega_f}^2 - \|\tilde{\mathbf{u}}_h^{n+1}\|_{\Omega_f}^2 + \|\mathbf{u}_h^{n+1} - \tilde{\mathbf{u}}_h^{n+1}\|_{\Omega_f}^2 = 0. \tag{4.9}$$

Combining (4.3), (4.6), (4.7), (4.8) and (4.9), we arrive at

$$\|\mathbf{u}_h^{n+1}\|_{\Omega_f}^2 - \|\mathbf{u}_h^n\|_{\Omega_f}^2 + \|2\mathbf{u}_h^{n+1} - \mathbf{u}^n\|_{\Omega_f}^2 - \|2\mathbf{u}_h^n - \mathbf{u}^{n-1}\|_{\Omega_f} + \|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\|_{\Omega_f}^2$$
$$+\ 4\triangle t\nu\|\tilde{\mathbf{u}}_h^{n+1}\|_{H_f}^2 + \triangle t\int_\Gamma \frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}}(\tilde{\mathbf{u}}_h^{n+1}\cdot\tau)^2 \mathrm{d}s + \frac{4\triangle t^2}{3}\|\nabla p_h^{n+1}\|_{\Omega_f}^2 - \frac{4\triangle t^2}{3}\|\nabla p_h^n\|_{\Omega_f}^2$$
$$\leq \frac{Cg\triangle t}{K_{\min}}(\|K^{1/2}\nabla\phi_h^n\|_{\Omega_p}^2 + \|K^{1/2}\nabla\phi_h^{n-1}\|_{\Omega_p}^2) + \frac{C\triangle t}{\nu}\|f(t_{n+1})\|_{H^{-1}(\Omega_f)}^2. \tag{4.10}$$

Letting $\psi_h = 4\triangle t\phi_h^{n+1}$ in (3.10), we have

$$gS_0(\|\phi_h^{n+1}\|_{\Omega_p}^2 - \|\phi_h^n\|_{\Omega_p}^2 + \|2\phi_h^{n+1} - \phi_h^n\|_{\Omega_p}^2 - \|2\phi_h^n - \phi_h^{n-1}\|_{\Omega_p}^2 + \|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|_{\Omega_p}^2)$$
$$+\ 4g\triangle t\|\mathbf{K}^{1/2}\nabla\phi_h^{n+1}\|_{\Omega_p}^2 = 4\triangle t c_I(\phi_h^{n+1}, 2\mathbf{u}_h^n - \mathbf{u}_h^n) + 4\triangle t(f_p(t_{n+1}), \phi_h^{n+1})_{\Omega_p}.$$

Using (4.1), we get

$$4\triangle t c_I(\phi_h^{n+1}, 2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) \leq Cg\triangle t\|\phi_h^{n+1}\|_{H_p}(\|\mathbf{u}_h^n\|_{H_f} + \|\mathbf{u}_h^{n-1}\|_{H_f})$$
$$\leq \frac{C\triangle t}{K_{\min}}(\|\mathbf{u}_h^n\|_{H_f}^2 + \|\mathbf{u}_h^{n-1}\|_{H_f}^2) + \frac{g\triangle t}{2}\|\mathbf{K}^{1/2}\nabla\phi_h^{n+1}\|_{\Omega_p}^2.$$

Using the Cauchy-Schwarz inequality, we deduce

$$4g\triangle t(g_p(t_{n+1}),\phi_h^{n+1})_{\Omega_p} \le \frac{C\triangle tg}{k_{\min}}\|g_p(t_{n+1})\|_{\Omega_p}^2 + \frac{g\triangle t}{2}\|\mathbf{K}^{1/2}\nabla\phi_h^{n+1}\|_{\Omega_p}^2.$$

Then, we get

$$\begin{aligned}
&gS_0(\|\phi_h^{n+1}\|_{\Omega_p}^2 - \|\phi_h^n\|_{\Omega_p}^2 + \|2\phi_h^{n+1}-\phi_h^n\|_{\Omega_p}^2 - \|2\phi_h^n-\phi_h^{n-1}\|_{\Omega_p}^2 + \|\phi_h^{n+1}-2\phi_h^n+\phi_h^{n-1}\|_{\Omega_p}^2)\\
&+ 3g\triangle t\|\mathbf{K}^{1/2}\nabla\phi_h^{n+1}\|_{\Omega_p}^2 \le \frac{4\triangle tg}{k_{\min}}\|g_p(t_{n+1})\|_{\Omega_p}^2 + \frac{C\triangle t}{2K_{\min}}(\|\mathbf{u}_h^n\|_{H_f}^2 + \|\mathbf{u}_h^{n-1}\|_{H_f}^2).
\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we arrive at

$$\begin{aligned}
&\|\mathbf{u}_h^{n+1}\|_{\Omega_f}^2 - \|\mathbf{u}_h^n\|_{\Omega_f}^2 + \|2\mathbf{u}_h^{n+1}-\mathbf{u}^n\|_{\Omega_f}^2 - \|2\mathbf{u}_h^n-\mathbf{u}^{n-1}\|_{\Omega_f}^2 + \|\mathbf{u}_h^{n+1}-2\mathbf{u}_h^n+\mathbf{u}_h^{n-1}\|_{\Omega_f}^2\\
&+ 4\triangle t\nu\|\tilde{\mathbf{u}}_h^{n+1}\|_{H_f}^2 + \triangle t\int_\Gamma \frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}}(\tilde{\mathbf{u}}_h^{n+1}\cdot\tau)^2\mathrm{d}s\\
&+ \frac{4\triangle t^2}{3}\|\nabla p_h^{n+1}\|_{\Omega_f}^2 - \frac{4\triangle t^2}{3}\|\nabla p_h^n\|_{\Omega_f}^2 + 3g\triangle t\|\mathbf{K}^{1/2}\nabla\phi_h^{n+1}\|_{\Omega_p}^2\\
&+ gS_0(\|\phi_h^{n+1}\|_{\Omega_p}^2 - \|\phi_h^n\|_{\Omega_p}^2 + \|2\phi_h^{n+1}-\phi_h^n\|_{\Omega_p}^2 - \|2\phi_h^n-\phi_h^{n-1}\|_{\Omega_p}^2 + \|\phi_h^{n+1}-2\phi_h^n+\phi_h^{n-1}\|_{\Omega_p}^2)\\
\le& \frac{Cg\triangle t}{K_{\min}}\left(\|K^{1/2}\nabla\phi_h^n\|_{\Omega_p}^2 + \|K^{1/2}\nabla\phi_h^{n-1}\|_{\Omega_p}^2\right) + \frac{C\triangle t}{\nu}\|f(t_{n+1})\|_{H^{-1}(\Omega_f)}^2 + \frac{4\triangle tg}{k_{\min}}\|g_p(t_{n+1})\|_{\Omega_p}^2\\
&+ \frac{C\triangle t}{2K_{\min}}\left(\|\mathbf{u}_h^n\|_{H_f}^2 + \|\mathbf{u}_h^{n-1}\|_{H_f}^2\right).
\end{aligned}$$

Summing it from 1 to n, we have

$$\begin{aligned}
&\|\mathbf{u}_h^{n+1}\|_{\Omega_f}^2 + \|2\mathbf{u}_h^{n+1}-\mathbf{u}^n\|_{\Omega_f}^2 + \sum_{i=1}^n\|\mathbf{u}_h^{i+1}-2\mathbf{u}_h^i+\mathbf{u}_h^{i-1}\|_{\Omega_f}^2\\
&+ 4\triangle t\nu\sum_{i=1}^n\|\tilde{\mathbf{u}}_h^{i+1}\|_{H_f}^2 + \triangle t\sum_{i=1}^n\int_\Gamma\frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}}(\tilde{\mathbf{u}}_h^{i+1}\cdot\tau)^2\mathrm{d}s\\
&+ \frac{4\triangle t^2}{3}\|\nabla p_h^{n+1}\|_{\Omega_f}^2 + 3g\triangle t\sum_{i=1}^n\|\mathbf{K}^{1/2}\nabla\phi_h^{i+1}\|_{\Omega_p}^2\\
&+ gS_0\left(\|\phi_h^{n+1}\|_{\Omega_p}^2 + \|2\phi_h^{n+1}-\phi_h^n\|_{\Omega_p}^2 + \sum_{i=1}^n\|\phi_h^{i+1}-2\phi_h^i+\phi_h^{i-1}\|_{\Omega_p}^2\right)\\
\le& \frac{Cg\triangle t}{K_{\min}}\sum_{i=1}^n\|K^{1/2}\nabla\phi_h^i\|_{\Omega_p}^2 + \frac{C\triangle t}{\nu}\sum_{i=1}^n\|f(t_{i+1})\|_{H^{-1}(\Omega_f)}^2\\
&+ \frac{4\triangle tg}{k_{\min}}\sum_{i=1}^n\|g_p(t_{i+1})\|_{\Omega_p}^2 + \frac{C\triangle t}{2K_{\min}}\sum_{i=1}^n\|\mathbf{u}_h^i\|_{H_f}^2.
\end{aligned}$$

Using Gronwall's lemma, we deduce

$$
\begin{aligned}
&\|\mathbf{u}_h^{n+1}\|^2_{\Omega_f} + \|2\mathbf{u}_h^{n+1} - \mathbf{u}^n\|^2_{\Omega_f} + \sum_{i=1}^{n} \|\mathbf{u}_h^{i+1} - 2\mathbf{u}_h^i + \mathbf{u}_h^{i-1}\|^2_{\Omega_f} \\
&+ 4\triangle t\nu \sum_{i=1}^{n} \|\tilde{\mathbf{u}}_h^{i+1}\|^2_{H_f} + \triangle t \sum_{i=1}^{n} \int_{\Gamma} \frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}} (\tilde{\mathbf{u}}_h^{i+1}\cdot\tau)^2 \mathrm{d}s \\
&+ \frac{4\triangle t^2}{3} \|\nabla p_h^{n+1}\|^2_{\Omega_f} + 3g\triangle t \sum_{i=1}^{n} \|\mathbf{K}^{1/2}\nabla\phi_h^{i+1}\|^2_{\Omega_p} \\
&+ gS_0 \left(\|\phi_h^{n+1}\|^2_{\Omega_p} + \|2\phi_h^{n+1} - \phi_h^n\|^2_{\Omega_p} + \sum_{i=1}^{n} \|\phi_h^{i+1} - 2\phi_h^i + \phi_h^{i-1}\|^2_{\Omega_p} \right) \\
&\leq \frac{C\triangle t}{\nu} \sum_{i=1}^{n} \|f(t_{i+1})\|^2_{H^{-1}(\Omega_f)} + \frac{4\triangle tg}{k_{\min}} \sum_{i=1}^{n} \|g_p(t_{i+1})\|^2_{\Omega_p}.
\end{aligned}
$$

Therefore, we complete the proof. □

5. Error analysis

In order to get the error estimate of our method, we define the projection operator $P_h : (\mathbf{w}(t), p(t)) \in (W, Q) \rightarrow (P_h\mathbf{w}(t), P_h p(t)) \in (W_h, Q_h), \forall t \in [0, T]$ by

$$
a(P_h\mathbf{w}(t), z_h) + b(z_h, P_h p(t)) = a(\mathbf{w}(t), z_h) + b(z_h, p(t)), \forall z_h \in W_h, \tag{5.1}
$$

$$
b(P_h\mathbf{w}(t), q_h) = 0, \forall q_h \in W_h. \tag{5.2}
$$

We give the following lemma.

Lemma 5.1. [34, 37] *Note that P_h is a linear operator, and for any $(\mathbf{w}(t), p(t)) \in W \times Q$ and $(\mathbf{v}(t), q(t)) \in W \times Q$,*

$$
P_h(\mathbf{w}(t) - \mathbf{v}(t)) = P_h\mathbf{w}(t) - P_h\mathbf{v}(t),
$$

$$
P_h(p(t) - q(t)) = P_h p(t) - P_h q(t).
$$

Furthermore, under certain smoothness assumption in $(w(t), p(t))$, we have the following error estimates

$$
\|P_h\mathbf{w}(t) - \mathbf{w}(t)\|_W \leq Ch,
$$

$$
\|P_h p(t) - p(t)\|_0 \leq Ch,
$$

$$
\|P_h\mathbf{w}(t) - \mathbf{w}(t)\|_0 \leq Ch^2.
$$

Using this lemma and inverse inequality, we can easily get

$$
\|P_h\mathbf{w}(t)\|_{W^{1,\infty}} \leq C.
$$

Define $\omega_h^n = P_h\mathbf{u}(t_n)$, $\theta_h^n = P_h p(t_n)$ and $\varphi_h^n = P_h\phi(t_n)$, we obtain the following theorem.

Theorem 5.2. *If u, p, u_t, ϕ and ϕ_t are sufficiently smooth, then the second order incremental pressure correction finite element method has the following results*

$$\begin{aligned}&\|\mathbf{e}_h^{n+1}\|^2_{\Omega_f}+\|2\mathbf{e}_h^{n+1}-\mathbf{e}_h^n\|^2_{\Omega_f}+\sum_{i=1}^{n}\|\mathbf{e}_h^{i+1}-2\mathbf{e}_h^i+\mathbf{e}_h^{i-1}\|^2_{\Omega_f}+2\triangle t\sum_{i=1}^{n}\|\tilde{\mathbf{e}}_h^{i+1}\|^2_{H_f}\\&+4\triangle t\sum_{i=1}^{n}\int_\Gamma\frac{\alpha\sqrt{\nu g}}{\sqrt{trace(\mathbf{K})}}(\tilde{\mathbf{e}}_h^{i+1}\cdot\tau)^2\mathrm{d}s+\|\eta_h^{n+1}\|^2_{\Omega_p}+\|2\eta_h^{n+1}-\eta_h^n\|^2_{\Omega_p}+\sum_{i=1}^{n}\|\delta_{tt}\eta_h^{i+1}\|^2_{\Omega_p}\\&+\ 2g\triangle t\sum_{i=1}^{n}\|\mathbf{K}^{1/2}\nabla\eta_h^{i+1}\|^2_{\Omega_p}\le C(\triangle t^2+h^4).\end{aligned}$$

where $\mathbf{e}_h^n=\omega_h^n-\mathbf{u}_h^n,\tilde{\mathbf{e}}_h^n=\omega_h^n-\tilde{\mathbf{u}}_h^n,\eta_h^n=\varphi_h^n-\phi_h^n$ and C is a positive constant independent of $\triangle t$.

Proof. Subtracting (3.7) for (2.7) with $z=(\mathbf{v}_h,0)$ and using (5.1), we get

$$\begin{aligned}&\left(\frac{3\tilde{\mathbf{e}}_h^{n+1}-4\mathbf{e}_h^n+\mathbf{e}_h^{n-1}}{2\triangle t},\mathbf{v}_h\right)_{\Omega_f}+a_f(\tilde{\mathbf{e}}_h^{n+1},\mathbf{v}_h)+c_I(\varphi_h^{n+1}-(2\phi_h^n-\phi_h^{n-1}),\mathbf{v}_h)\\&+\ b(\mathbf{v}_h,\theta_h^{n+1}-p_h^n)+B(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v}_h)-B(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\tilde{\mathbf{u}}_h^{n+1},\mathbf{v}_h)\\&=(R_0^{n+1},\mathbf{v}_h)_{\Omega_f}+\left(R_1^{n+1},\mathbf{v}_h\right)_{\Omega_f},\end{aligned}\tag{5.3}$$

where $R_0^{n+1}=\frac{3\mathbf{u}(t_{n+1})-4\mathbf{u}(t_n)+\mathbf{u}(t_{n-1})}{2\triangle t}-\mathbf{u}_t(t_{n+1})$, $R_1^{n+1}=\frac{3\zeta^{n+1}-4\zeta^n+\zeta^{n-1}}{2\triangle t}$, $\beta_h^n=\theta_h^{n+1}-p_h^n=\delta_t\theta_h^{n+1}+\varepsilon_h^n$ and $\zeta^n=\mathbf{u}(t_n)-\omega_h^n$.

Similar as (4.4), we have

$$\begin{aligned}&2\left(3\tilde{\mathbf{e}}_h^{n+1}-4\mathbf{e}_h^n+\mathbf{e}_h^{n-1},\tilde{\mathbf{e}}_h^{n+1}\right)_{\Omega_f}\\&=\|\mathbf{e}_h^{n+1}\|^2_{\Omega_f}-\|\mathbf{e}_h^n\|^2_{\Omega_f}+\|2\mathbf{e}_h^{n+1}-\mathbf{e}_h^n\|^2_{\Omega_f}-\|2\mathbf{e}_h^n-\mathbf{e}^{n-1}\|_{\Omega_f}+\|\mathbf{e}_h^{n+1}-2\mathbf{e}_h^n+\mathbf{e}_h^{n-1}\|^2_{\Omega_f}\\&+\ 3\|\tilde{\mathbf{e}}_h^{n+1}\|^2_{\Omega_f}+3\|\tilde{\mathbf{e}}_h^{n+1}-\mathbf{e}_h^{n+1}\|^2_{\Omega_f}-3\|\mathbf{e}_h^{n+1}\|^2_{\Omega_f}.\end{aligned}$$

Letting $\mathbf{v}_h=4\triangle t\mathbf{e}_h^{n+1}$ in (5.3), we deduce

$$\begin{aligned}&\|\mathbf{e}_h^{n+1}\|^2_{\Omega_f}-\|\mathbf{e}_h^n\|^2_{\Omega_f}+\|2\mathbf{e}_h^{n+1}-\mathbf{e}_h^n\|^2_{\Omega_f}-\|2\mathbf{e}_h^n-\mathbf{e}^{n-1}\|_{\Omega_f}+\|\mathbf{e}_h^{n+1}-2\mathbf{e}_h^n+\mathbf{e}_h^{n-1}\|^2_{\Omega_f}\\&+\ 3\|\tilde{\mathbf{e}}_h^{n+1}\|^2_{\Omega_f}+3\|\tilde{\mathbf{e}}_h^{n+1}-\mathbf{e}_h^{n+1}\|^2_{\Omega_f}-3\|\mathbf{e}_h^{n+1}\|^2_{\Omega_f}+4\triangle t\nu\|\tilde{\mathbf{e}}_h^{n+1}\|^2_{H_f}\\&+\ 4\triangle t\int_\Gamma\frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}}(\tilde{\mathbf{e}}_h^{n+1}\cdot\tau)^2\mathrm{d}s+\triangle t\theta(\|\tilde{\mathbf{e}}_h^{n+1}\|^2_{H_f}-\|\mathbf{e}_h^n\|^2_{H_f}+\|\tilde{\mathbf{e}}_h^{n+1}-\mathbf{e}_h^n\|^2_{H_f})\\&+\ 4\triangle tc_I(\varphi_h^{n+1}-(2\phi_h^n-\phi_h^{n-1}),\tilde{\mathbf{e}}_h^{n+1})+4\triangle tb(\tilde{\mathbf{e}}_h^{n+1},\beta_h^n)+4\triangle tB(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}_h^{n+1})\\&-4\triangle tB(\mathbf{u}_n^n,\tilde{\mathbf{u}}_h^{n+1},\mathbf{v}_h)\\&=4\triangle t\theta(\nabla(\delta_t\omega_h^{n+1}),\nabla\tilde{\mathbf{e}}_h^{n+1})_{\Omega_f}+4\triangle t(R_0^{n+1},\tilde{\mathbf{e}}_h^{n+1})_{\Omega_f}+4\triangle t(R_1^{n+1},\tilde{\mathbf{e}}_h^{n+1})_{\Omega_f}.\end{aligned}\tag{5.4}$$

Using Lemma 5.1 and Taylor's formula, we obtain

$$\begin{aligned}4\triangle t(R_1^{n+1},\tilde{\mathbf{e}}_h^{n+1})_{\Omega_f}&=2(3(\mathbf{u}(t_{n+1})-\omega_h^{n+1})-4(\mathbf{u}(t_n)-\omega_h^n)+3(\mathbf{u}(t_{n-1})-\omega_h^{n-1}),\tilde{\mathbf{e}}_h^{n+1})_{\Omega_f}\\&\le C\triangle t(\|(\mathbf{u}(t_{n+1})-\omega_h^{n+1})_t\|_{\Omega_f}+\|(\mathbf{u}(t_n)-\omega_h^n)_t\|_{\Omega_f})\|\tilde{\mathbf{e}}_h^{n+1}\|_{\mathbf{H}_f}\\&\le C\triangle th^4+\frac{\triangle t\nu}{16}\|\tilde{\mathbf{e}}_h^{n+1}\|^2_{\mathbf{H}_f}.\end{aligned}$$

Using Taylor's formula and Cauchy-Schwarz inequality, we get

$$\begin{aligned} 2\triangle t(R_0^{n+1}, \tilde{\mathbf{e}}_h^{n+1})_{\Omega_f} &\le 2\triangle t\|R_0^{n+1}\|_{H^{-1}(\Omega_f)}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f} \\ &\le \frac{\nu\triangle t}{8}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2 + C\triangle t^5. \end{aligned} \tag{5.5}$$

Adding some terms, we deduce

$$\begin{aligned} &B(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \tilde{\mathbf{e}}_h^{n+1}) - B(2\mathbf{u}_n^n - \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}, \tilde{\mathbf{e}}_h^{n+1}) \\ &= B(\delta_{tt}\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \tilde{\mathbf{e}}_h^{n+1}) + B(2\zeta^n - \zeta^{n-1}, \mathbf{u}(t_{n+1}), \tilde{\mathbf{e}}_h^{n+1}) + B(2\omega_h^n - \omega_h^{n-1}, \zeta^{n+1}, \tilde{\mathbf{e}}_h^{n+1}) \\ &+ B(2\tilde{\mathbf{e}}_h^n - \tilde{\mathbf{e}}_h^{n-1}, w_h^{n+1}, \tilde{\mathbf{e}}_h^{n+1}) + B(2\omega_h^n - \omega_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1}, \tilde{\mathbf{e}}_h^{n+1}) - B(2\tilde{\mathbf{e}}^n - \tilde{\mathbf{e}}_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1}, \tilde{\mathbf{e}}_h^{n+1}). \end{aligned}$$

We give the estimate for each term on the right-hand side, which is mainly relying on the following inequalities and equalities.

$$\begin{aligned} B(\delta_{tt}\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \tilde{\mathbf{e}}_h^{n+1}) &\le C\|\delta_{tt}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f} \\ &\le C\triangle t^2\|u_{tt}(t_{n+1})\|_{L^\infty(\Omega_f)}\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f} \\ &\le C\triangle t^4 + \frac{\nu}{16}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2. \end{aligned}$$

$$\begin{aligned} B(2\zeta^n - \zeta^{n-1}, \mathbf{u}(t_{n+1}), \tilde{\mathbf{e}}_h^{n+1}) &\le C\|2\zeta^n - \zeta^{n-1}\|_{\Omega_f}\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f} \\ &\le Ch^2(\|u\|_{L^\infty(H^2)} + \|p\|_{L^\infty(H^1)})\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f} \\ &\le Ch^4 + \frac{\nu}{16}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2. \end{aligned}$$

$$\begin{aligned} B(2\omega_h^n - \omega_h^{n-1}, \zeta^{n+1}, \tilde{\mathbf{e}}_h^{n+1}) &\le C\|\mathcal{A}(2\omega_h^n - \omega_h^{n-1})\|_{\Omega_f}\|\zeta^{n+1}\|_{\Omega_f}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f} \\ &\le Ch^4 + \frac{\nu}{16}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2. \end{aligned}$$

$$B(2\mathbf{e}_h^n - \mathbf{e}_h^{n-1}, w_h^{n+1}, \tilde{\mathbf{e}}_h^{n+1}) \le C\|\mathcal{A}\omega_h^{n+1}\|_{\Omega_f}^2(\|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|\mathbf{e}_h^{n-1}\|_{\Omega_f}^2) + \frac{\nu}{16}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2.$$

$$B(2\mathbf{e}^n - \mathbf{e}_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1}, \tilde{\mathbf{e}}_h^{n+1}) = 0.$$

$$B(2\omega_h^n - \omega_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1}, \tilde{\mathbf{e}}_h^{n+1}) = 0.$$

Then, we have

$$\begin{aligned} &2\triangle tB(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \tilde{\mathbf{e}}_h^{n+1}) - 2\triangle tB(2\mathbf{u}_n^n - \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}, \tilde{\mathbf{e}}_h^{n+1}) \\ &\le C\triangle t(\triangle t^4 + h^4) + C\triangle t(\|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|\mathbf{e}_h^{n-1}\|_{\Omega_f}^2) + \frac{\triangle t\nu}{4}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2. \end{aligned} \tag{5.6}$$

Via (4.1), we get

$$\begin{aligned} c_I(\varphi_h^{n+1} - (2\phi_h^n - \phi_h^{n-1}), \tilde{\mathbf{e}}_h^{n+1}) &= c_I(\varphi_h^{n+1} - 2\varphi_h^n + \varphi_h^{n-1}) + c_I(2\eta_h^n - \eta_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1}) \\ &\le 2C_2 g\triangle t\|\varphi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|_{H_p}\|\tilde{e}_h^{n+1}\|_{H_f} \\ &+ C_2 g\triangle t(\|\varphi_h^{n+1} - \varphi_h^n\|_{H_p} + \|\eta_h^n\|_{H_p})\|\tilde{e}_h^{n+1}\|_{H_f} \\ &\le Cg\triangle t^4 + C(\|\eta_h^n\|_{H_P}^2 + \|\eta_h^{n-1}\|_{H_P}^2) + \frac{\nu}{4}\|\tilde{e}_h^{n+1}\|_{H_f}^2. \end{aligned} \tag{5.7}$$

Furthermore, by (3.8), we have

$$\left(\frac{3\mathbf{e}_h^{n+1}-3\tilde{\mathbf{e}}_h^{n+1}}{2\triangle t},\mathbf{v}_h\right)_{\Omega_f}-b(\mathbf{v}_h,\varepsilon_h^{n+1}-\beta_h^n)=0. \tag{5.8}$$

In order to give the estimate of $2\triangle tb(\tilde{\mathbf{e}}_h^{n+1},\beta_h^n)$, letting $\mathbf{v}_h=4\triangle t\nabla\beta_h^n$, we deduce

$$\begin{aligned}4\triangle tb(\tilde{\mathbf{e}}_h^{n+1},\beta_h^n)&=-\frac{8}{3}\triangle t^2(\nabla(\varepsilon_h^{n+1}-\beta_h^n),\nabla\beta_h^n)_{\Omega_f}\\&=\frac{4\triangle t^2}{3}\|\nabla\varepsilon_h^{n+1}\|_{\Omega_f}^2-\frac{4\triangle t^2}{3}\|\nabla(\varepsilon_h^n+\delta_t\theta_h^{n+1})\|_{\Omega_f}^2+\frac{4\triangle t^2}{3}\|\nabla(\varepsilon_h^{n+1}-\beta_h^n)\|_{\Omega_f}^2\\&\le\frac{4}{3}\triangle t^2(\|\nabla\varepsilon_h^{n+1}\|_{\Omega_f}^2-\|\nabla\varepsilon_h^n\|_{\Omega_f}^2)+C\triangle t^3+\frac{4\triangle t^2}{3}\|\nabla(\varepsilon_h^{n+1}-\beta_h^n)\|_{\Omega_f}^2.\end{aligned} \tag{5.9}$$

Taking $\mathbf{v}_h=2\triangle t\nabla(\varepsilon_h^{n+1}-\beta_h^n)$ in (5.8), we have

$$\triangle t\|\nabla(\varepsilon_h^{n+1}-\beta_h^n)\|_{\Omega_f}^2\le\frac{3}{2}\|\mathbf{e}_h^{n+1}-\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}\|\nabla(\varepsilon_h^{n+1}-\beta_h^n)\|_{\Omega_f}.$$

It means that,

$$\triangle t\|\nabla(\varepsilon_h^{n+1}-\beta_h^n)\|_{\Omega_f}\le\frac{3}{2}\|\mathbf{e}_h^{n+1}-\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}.$$

Then, we arrive at

$$-4\triangle tb(\tilde{\mathbf{e}}_h^{n+1},\beta_h^n)\ge\frac{4}{3}\triangle t^2(\|\nabla\varepsilon_h^{n+1}\|_{\Omega_f}^2-\|\nabla\varepsilon_h^n\|_{\Omega_f}^2)-C\triangle t^3-3\|\mathbf{e}_h^{n+1}-\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}^2. \tag{5.10}$$

Letting $\mathbf{v}_h=2\triangle t\mathbf{e}_h^{n+1}$ in (5.8), we have

$$\|\mathbf{e}_h^{n+1}\|_{\Omega_f}^2-\|\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}^2+\|\mathbf{e}_h^{n+1}-\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}^2=0. \tag{5.11}$$

Combining (5.4), (5.5), (5.6), (5.7), (5.9) and (5.11), we arrive at

$$\begin{aligned}&\|\mathbf{e}_h^{n+1}\|_{\Omega_f}^2-\|\mathbf{e}_h^n\|_{\Omega_f}^2+\|2\mathbf{e}_h^{n+1}-\mathbf{e}_h^n\|_{\Omega_f}^2-\|2\mathbf{e}_h^n-\mathbf{e}^{n-1}\|_{\Omega_f}+\|\mathbf{e}_h^{n+1}-2\mathbf{e}_h^n+\mathbf{e}_h^{n-1}\|_{\Omega_f}^2\\&+2\triangle t\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2+4\triangle t\int_\Gamma\frac{\alpha\sqrt{\nu g}}{\sqrt{\text{trace}(\mathbf{K})}}(\tilde{\mathbf{e}}_h^{n+1}\cdot\tau)^2\text{d}s\\&\le C\triangle t(\triangle t^2+h^4)+C\triangle t(\|\mathbf{e}_h^n\|_{\Omega_f}^2+\|\mathbf{e}_h^{n-1}\|_{\Omega_f}^2)+C\triangle t\|\eta_h^n\|_{H_p}^2+C\triangle t\|\eta_h^{n-1}\|_{H_p}^2.\end{aligned} \tag{5.12}$$

Subtracting (3.10) from (2.7) with $z=(0,\psi_h)$ and using (5.1), we get

$$\begin{aligned}gS_0\left(\frac{3\eta_h^{n+1}-4\eta_h^n+\eta_h^{n-1}}{2\triangle t},\psi_h\right)_{\Omega_p}&+\;g(\mathbf{K}\nabla\eta_h^{n+1},\nabla\psi_h)_{\Omega_p}-c_I(\psi_h,\omega_h^{n+1}-2\mathbf{u}_h^n+\mathbf{u}_h^{n-1})\\&=(R_2^{n+1},\psi_h)_{\Omega_p}+(R_3^{n+1},\psi_h)_{\Omega_p},\end{aligned} \tag{5.13}$$

where $R_2^{n+1}=\phi_t(t_{n+1})-\frac{3\phi(t_{n+1})-4\phi(t_n)+\phi(t_{n-1})}{2\triangle t}$ and $R_3^{n+1}=\frac{3(\phi(t_{n+1})-\varphi_h^{n+1})-4(\phi(t_n)-\varphi_h^n)+(\phi(t_{n-1})-\varphi_h^n)}{2\triangle t}$. It follows from (5.13) with $\psi_h=4\triangle t\eta_h^{n+1}$

$$\begin{aligned}&gS_0(\|\eta_h^{n+1}\|_{\Omega_p}^2-\|\eta_h^n\|_{\Omega_p}^2+\|2\eta_h^{n+1}-\eta_h^n\|_{\Omega_p}^2+\|\delta_{tt}\eta_h^{n+1}\|_{\Omega_p}^2-\|2\eta_h^n-\eta_h^{n-1}\|_{\Omega_p}^2)+4g\triangle t\|\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{\Omega_p}^2\\&=2\triangle tc_I(\eta_h^{n+1},\omega_h^{n+1}-(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1}))+4\triangle t(R_2^{n+1},\eta_h^{n+1})_{\Omega_p}+4\triangle t(R_3^{n+1},\eta_h^{n+1})_{\Omega_p}.\end{aligned}$$

Using Taylor's formula, we get

$$4\triangle t(R_2^{n+1}, \eta_h^{n+1})_{\Omega_p} \leq C\triangle t^5 + \frac{g\triangle t}{4}\|\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{H_p}^2.$$

Using Lemma 5.1 and Taylor's formula, we have

$$4\triangle t(R_3^{n+1}, \eta_h^{n+1})_{\Omega_p} \leq C\triangle t h^4 + \frac{g\triangle t}{4}\|\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{H_p}^2.$$

Using (4.1) and Lemma 5.1, we deduce

$$\begin{aligned}
&4\triangle t c_I(\eta_h^{n+1}, \omega_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}) \\
&\leq 2\triangle t\|\eta_h^{n+1}\|_{H_p}\|\omega_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\|_{H_f} \\
&\leq g\triangle t\|\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{H_p}^2 + \frac{Cg\triangle t}{K_{\min}}\|\omega_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\|_{H_f}^2 \\
&\leq g\triangle t\|\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{H_p}^2 + \frac{Cg\triangle t}{K_{\min}}(\|\delta_{tt}\omega_h^{n+1}\|_{H_p}^2 + 2\|\mathbf{e}_h^n\|_{H_f}^2 + \|\mathbf{e}_h^{n-1}\|_{H_f}^2) \\
&\leq C\triangle t^5 + \frac{Cg\triangle t}{K_{\min}}(\|\mathbf{e}_h^n\|_{H_f}^2 + \|\mathbf{e}_h^{n-1}\|_{H_f}^2) + \frac{g\triangle t}{4}\|\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{H_p}^2.
\end{aligned}$$

Then, we arrive at

$$\begin{aligned}
&gS_0(\|\eta_h^{n+1}\|_{\Omega_p}^2 - \|\eta_h^n\|_{\Omega_p}^2 + \|2\eta_h^{n+1} - \eta_h^n\|_{\Omega_p}^2 + \|\delta_{tt}\eta_h^{n+1}\|_{\Omega_p}^2 - \|2\eta_h^n - \eta_h^{n-1}\|_{\Omega_p}^2) \\
&+ 2g\triangle t\|\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{\Omega_p}^2 \leq C\triangle t^5 + \frac{Cg\triangle t}{K_{\min}}(\|\mathbf{e}_h^n\|_{H_f}^2 + \|\mathbf{e}_h^{n-1}\|_{H_f}^2).
\end{aligned} \tag{5.14}$$

Combining (5.12) and (5.14), we obtain

$$\begin{aligned}
&\|\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 - \|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|2\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|_{\Omega_f}^2 - \|2\mathbf{e}_h^n - \mathbf{e}^{n-1}\|_{\Omega_f} + \|\delta_{tt}\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 \\
&+ 2\triangle t\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2 + 4\triangle t\int_\Gamma \frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}}(\tilde{\mathbf{e}}_h^{n+1}\cdot\tau)^2\mathrm{d}s + \|\eta_h^{n+1}\|_{\Omega_p}^2 - \|\eta_h^n\|_{\Omega_p}^2 \\
&+ \|2\eta_h^{n+1} - \eta_h^n\|_{\Omega_p}^2 + \|\delta_{tt}\eta_h^{n+1}\|_{\Omega_p}^2 - \|2\eta_h^n - \eta_h^{n-1}\|_{\Omega_p}^2 + 2g\triangle t\|\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{\Omega_p}^2 \\
\leq &C\triangle t(\triangle t^2 + h^4) + C\triangle t(\|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|\mathbf{e}_h^{n-1}\|_{\Omega_f}^2) + C\triangle t\|\eta_h^n\|_{H_P}^2 + C\triangle t\|\eta_h^{n-1}\|_{H_P}^2.
\end{aligned}$$

Summing over the above inequality and using Gronwall's lemma, we have

$$\begin{aligned}
&\|\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 + \|2\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|_{\Omega_f}^2 + \sum_{i=1}^n\|\mathbf{e}_h^{i+1} - 2\mathbf{e}_h^i + \mathbf{e}_h^{i-1}\|_{\Omega_f}^2 + 2\triangle t\sum_{i=1}^n\|\tilde{\mathbf{e}}_h^{i+1}\|_{H_f}^2 \\
&+ 4\triangle t\sum_{i=1}^n\int_\Gamma \frac{\alpha\sqrt{\nu g}}{\sqrt{\mathrm{trace}(\mathbf{K})}}(\tilde{\mathbf{e}}_h^{i+1}\cdot\tau)^2\mathrm{d}s + \|\eta_h^{n+1}\|_{\Omega_p}^2 + \|2\eta_h^{n+1} - \eta_h^n\|_{\Omega_p}^2 + \sum_{i=1}^n\|\delta_{tt}\eta_h^{i+1}\|_{\Omega_p}^2 \\
&+ 2g\triangle t\sum_{i=1}^n\|\mathbf{K}^{1/2}\nabla\eta_h^{i+1}\|_{\Omega_p}^2 \leq C(\triangle t^2 + h^4).
\end{aligned}$$

Therefore, we complete the proof. □

This theorem is not optimal. In order to give the optimal results, we give the following lemma firstly.

Lemma 5.3. *Under the hypotheses of Theorem 4.1, we have*

$$
\begin{aligned}
&3\|\delta_t\tilde{\mathbf{e}}_h^{n+1}\|^2_{\Omega_f}+3\sum_{i=1}^{n}\|\delta_t\tilde{\mathbf{e}}_h^{i+1}-\delta_t\mathbf{e}_h^{i+1}\|^2_{\Omega_f}+\|\delta_t\mathbf{e}_h^{n+1}\|^2_{\Omega_f}+\|2\delta_t\mathbf{e}_h^{n+1}-\delta_t\mathbf{e}_h^{n}\|^2_{\Omega_f}\\
&+\sum_{i=1}^{n}\|\delta_{ttt}\mathbf{e}_h^{n+1}\|^2_{\Omega_f}+\frac{3\triangle t^2}{3}\|\delta_t\nabla\varepsilon_h^{n+1}\|^2_{\Omega_f}\\
&+\ gS_0(\|\delta_t\eta_h^{n+1}\|^2_{\Omega_p}+\|2\delta_t\eta_h^{n+1}-\delta_t\eta_h^{n}\|^2_{\Omega_p}+\sum_{i=1}^{n}\|\delta_{ttt}\eta_h^{n+1}\|^2_{\Omega_p})+3\nu\sum_{i=1}^{n}\|\delta_t\tilde{\mathbf{e}}_h^{i+1}\|^2_{H_f}\\
&\le C(\triangle t^2+h^4)
\end{aligned}
$$

and

$$\|\mathbf{e}_h^{n+1}-\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}\le C(\triangle t^2+h^2).$$

Proof. From (5.3), we get

$$
\begin{aligned}
&\left(\frac{3\delta_t\tilde{\mathbf{e}}_h^{n+1}-4\delta_t\mathbf{e}_h^{n}+\delta_t\mathbf{e}_h^{n-1}}{2\triangle t},\mathbf{v}_h\right)_{\Omega_f}+a_f(\delta_t\tilde{\mathbf{e}}_h^{n+1},\mathbf{v}_h)+c_I(\delta_t\varphi_h^{n+1}-\delta_t(2\phi_h^n-\phi_h^{n-1}),\mathbf{v}_h)\\
&+\ b(v_h,\delta_t\theta_h^{n+1}-\delta_tp_h^n)+B(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v}_h)-B(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\tilde{\mathbf{u}}_h^{n+1},\mathbf{v}_h)\\
&+\ B(\mathbf{u}(t_n),\mathbf{u}(t_n),\mathbf{v}_h)-B(2\mathbf{u}_h^{n-1}-\mathbf{u}_h^{n-2},\tilde{\mathbf{u}}_h^{n+1},\mathbf{v}_h)=(\delta_tR_0^{n+1},\mathbf{v}_h)_{\Omega_f}+(\delta_tR_1^{n+1},\mathbf{v}_h)_{\Omega_f}.
\end{aligned}
$$

Letting $\mathbf{v}_h=4\triangle t\delta_t\tilde{\mathbf{e}}_h^{n+1}$, we deduce

$$
\begin{aligned}
&2\left(3\delta_t\tilde{\mathbf{e}}_h^{n+1}-4\delta_t\mathbf{e}_h^{n}+\delta_t\mathbf{e}_h^{n-1},\delta_t\tilde{\mathbf{e}}_h^{n+1}\right)_{\Omega_f}+4\triangle ta_f(\delta_t\tilde{\mathbf{e}}_h^{n+1},\delta_t\tilde{\mathbf{e}}_h^{n+1})+4\triangle tb(\delta_t\tilde{\mathbf{e}}_h^{n+1},\delta_t\beta_h^n)\\
&+\ 4\triangle tc_I(\delta_t\varphi_h^{n+1}-\delta_t(2\phi_h^n-\phi_h^{n-1}),\delta_t\tilde{\mathbf{e}}_h^{n+1})+4\triangle tB(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\delta_t\tilde{\mathbf{e}}_h^{n+1})\\
&-\ 4\triangle tB(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\tilde{\mathbf{u}}_h^{n+1},\delta_t\tilde{\mathbf{e}}_h^{n+1})+4\triangle tB(\mathbf{u}(t_n),\mathbf{u}(t_n),\delta_t\tilde{\mathbf{e}}_h^{n+1})\\
&-4\triangle tB(2\mathbf{u}_h^{n-1}-\mathbf{u}_h^{n-2},\tilde{\mathbf{u}}_h^{n+1},\delta_t\tilde{\mathbf{e}}_h^{n+1})=(\delta_tR_0^{n+1},\delta_t\tilde{\mathbf{e}}_h^{n+1})_{\Omega_f}+(\delta_tR_1^{n+1},\delta_t\tilde{\mathbf{e}}_h^{n+1})_{\Omega_f},
\end{aligned}
\tag{5.15}
$$

For the first term, we have

$$
\begin{aligned}
&2\left(3\delta_t\tilde{\mathbf{e}}_h^{n+1}-4\delta_t\mathbf{e}_h^{n}+\delta_t\mathbf{e}_h^{n-1},\delta_t\tilde{\mathbf{e}}_h^{n+1}\right)_{\Omega_f}\\
&=6(\delta_t\tilde{\mathbf{e}}_h^{n+1}-\delta_t\mathbf{e}_h^{n+1},\delta_t\tilde{\mathbf{e}}_h^{n+1})_{\Omega_f}+2(3\delta_t\mathbf{e}_h^{n+1}-4\delta_t\mathbf{e}_h^{n}+\delta_t\mathbf{e}_h^{n-1},\delta_t\tilde{\mathbf{e}}_h^{n+1}-\delta_t\mathbf{e}_h^{n+1})_{\Omega_f}\\
&+\ 2(3\delta_t\mathbf{e}_h^{n+1}-4\delta_t\mathbf{e}_h^{n}+\delta_t\mathbf{e}_h^{n-1},\delta_t\mathbf{e}_h^{n+1})_{\Omega_f}\equiv I_1+I_2+I_3.
\end{aligned}
$$

Then, we get

$$I_1=3\|\delta_t\tilde{\mathbf{e}}_h^{n+1}\|^2_{\Omega_f}+3\|\delta_t\tilde{\mathbf{e}}_h^{n+1}-\delta_t\mathbf{e}_h^{n+1}\|^2_{\Omega_f}-3\|\delta_t\tilde{\mathbf{e}}_h^{n+1}\|^2_{\Omega_f}.$$

Using (5.16) and noticing $b(\mathbf{e}_h^n,q_h)=0,\forall q_h\in M_h$, we deduce

$$I_2=2b(3\delta_t\mathbf{e}_h^{n+1}-4\delta_t\mathbf{e}_h^{n}+\delta_t\mathbf{e}_h^{n-1},\delta_t\varepsilon_h^{n+1}-\delta_t\beta_h^n)=0.$$

We can also have

$$I_3 = \|\delta_t \mathbf{e}_h^{n+1}\|_{\Omega_f}^2 + \|2\delta_t \mathbf{e}_h^{n+1} - \delta_t \mathbf{e}_h^n\|_{\Omega_f}^2 + \|\delta_{tt} \mathbf{e}_h^{n+1}\|_{\Omega_f}^2 - \|\delta_t \mathbf{e}_h^n\|_{\Omega_f}^2 - \|2\delta_t \mathbf{e}_h^n - \delta_t \mathbf{e}_h^{n-1}\|_{\Omega_f}^2.$$

From (5.8), we have

$$\left(\frac{3\delta_t \mathbf{e}_h^{n+1} - 3\delta_t \tilde{\mathbf{e}}_h^{n+1}}{2\triangle t}, \mathbf{v}_h\right)_{\Omega_f} - b(\mathbf{v}_h, \delta_t \varepsilon_h^{n+1} - \delta_t \beta_h^n) = 0. \tag{5.16}$$

Letting $\mathbf{v}_h = 8\triangle t^2 \delta_t \nabla \beta_h^n$ in (5.16), we deduce

$$\begin{aligned}
4\triangle t b(\delta_t \tilde{\mathbf{e}}_h^{n+1}, \delta_t \beta_h^{n+1}) &= -\frac{8\triangle t^2}{3}(\delta_t \nabla \beta_h^n, \delta_t \nabla \varepsilon_h^{n+1} - \delta_t \nabla \beta_h^n)_{\Omega_f} \\
&= \frac{4\triangle t^2}{3}(\|\delta_t \nabla \beta_h^n\|_{\Omega_f}^2 - \|\delta_t \nabla \varepsilon_h^{n+1}\|_{\Omega_f}^2) + \frac{4\triangle t^2}{3}\|\delta_t \nabla \varepsilon_h^{n+1} - \delta_t \nabla \beta_h^n\|_{\Omega_f}^2 \\
&\le \frac{4\triangle t^2}{3}(\|\delta_t \nabla \varepsilon_h^n\|_{\Omega_f}^2 - \|\delta_t \nabla \varepsilon_h^{n+1}\|_{\Omega_f}^2) + C\triangle t^3 + \frac{4\triangle t^2}{3}\|\delta_t \nabla \varepsilon_h^{n+1} - \delta_t \nabla \beta_h^n\|_{\Omega_f}^2.
\end{aligned} \tag{5.17}$$

Taking $\mathbf{v}_h = \triangle t \nabla(\delta_t \varepsilon_h^{n+1} - \delta_t \beta_h^n)$ in (5.16), we have

$$\triangle t \|\nabla(\delta_t \varepsilon_h^{n+1} - \delta_t \beta_h^n)\|_0^2 \le \frac{3}{2}\|\delta_t \varepsilon_h^{n+1} - \delta_t \beta_h^n\|_{\Omega_f}\|\delta_t \mathbf{e}_h^{n+1} - \delta_t \tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}.$$

It means

$$\triangle t \|\nabla(\delta_t \varepsilon_h^{n+1} - \delta_t \beta_h^n)\|_0 \le \frac{3}{2}\|\delta_t \mathbf{e}_h^{n+1} - \delta_t \tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}.$$

Then, we get

$$4\triangle t b(\delta_t \tilde{\mathbf{e}}_h^{n+1}, \delta_t \beta_h^{n+1}) \ge \frac{4\triangle t^2}{3}(\|\delta_t \nabla \varepsilon_h^{n+1}\|_{\Omega_f}^2 - \|\delta_t \nabla \varepsilon_h^n\|_{\Omega_f}^2) - C\triangle t^3 - 3\|\delta_t \varepsilon_h^{n+1} - \delta_t \beta_h^n\|_{\Omega_f}^2. \tag{5.18}$$

Using (4.1), we get

$$\begin{aligned}
&c_I(\delta_t \varphi_h^{n+1} - (2\delta_t \phi_h^n - \delta_t \phi_h^{n-1}), \tilde{\mathbf{e}}_h^{n+1}) \\
&= c_I(\delta_t \varphi_h^{n+1} - 2\delta_t \varphi_h^n + \delta_t \varphi_h^{n-1}, \delta_t \tilde{\mathbf{e}}_h^{n+1}) + c_I(2\delta_t \eta_h^n - \delta_t \eta_h^{n-1}, \delta_t \tilde{\mathbf{e}}_h^{n+1}) \\
&\le 2C_2 g\|\delta_t \varphi_h^{n+1} - 2\delta_t \varphi_h^n + \delta_t \varphi_h^{n-1}\|_{H_p}\|\delta_t \tilde{e}_h^{n+1}\|_{H_f} \\
&+ C_2 g(\|\delta_t \eta_h^{n+1}\|_{H_p} + \|\delta_t \eta_h^n\|_{H_p})\|\delta_t \tilde{e}_h^{n+1}\|_{H_f} \\
&\le Cg\triangle t^4 + C(\|\delta_t \eta_h^n\|_{H_P}^2 + \|\delta_t \eta_h^{n-1}\|_{H_P}^2) + \frac{\nu}{4}\|\delta_t \tilde{e}_h^{n+1}\|_{H_f}^2.
\end{aligned} \tag{5.19}$$

Similar as (5.6), we have

$$\begin{aligned}
&4\triangle t B(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \delta_t \tilde{\mathbf{e}}_h^{n+1}) - 4\triangle t B(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}, \delta_t \tilde{\mathbf{e}}_h^{n+1}) \\
&+ 4\triangle t B(\mathbf{u}(t_n), \mathbf{u}(t_n), \delta_t \tilde{\mathbf{e}}_h^{n+1}) - 4\triangle t B(2\mathbf{u}_h^{n-1} - \mathbf{u}_h^{n-2}, \tilde{\mathbf{u}}_h^{n+1}, \delta_t \tilde{\mathbf{e}}_h^{n+1}) \\
&\le C\triangle t(\triangle t^4 + h^4) + C\triangle t(\|\delta_t \mathbf{e}_h^n\|_{\Omega_f}^2 + \|\delta_t \mathbf{e}_h^{n-1}\|_{\Omega_f}^2 + \|\delta_t \mathbf{e}_h^{n-2}\|_{\Omega_f}^2) + \frac{\triangle t \nu}{4}\|\delta_t \tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2.
\end{aligned} \tag{5.20}$$

Then, we arrive at

$$
\begin{aligned}
&3\|\delta_t\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}^2+3\|\delta_t\tilde{\mathbf{e}}_h^{n+1}-\delta_t\mathbf{e}_h^{n+1}\|_{\Omega_f}^2-3\|\delta_t\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}^2+\|\delta_t\mathbf{e}_h^{n+1}\|_{\Omega_f}^2+\|2\delta_t\mathbf{e}_h^{n+1}-\delta_t\mathbf{e}_h^{n}\|_{\Omega_f}^2\\
&+\|\delta_{ttt}\mathbf{e}_h^{n+1}\|_{\Omega_f}^2-\|\delta_t\mathbf{e}_h^{n}\|_{\Omega_f}^2-\|2\delta_t\mathbf{e}_h^{n}-\delta_t\mathbf{e}_h^{n-1}\|_{\Omega_f}^2+\frac{3\triangle t^2}{3}(\|\delta_t\nabla\varepsilon_h^{n+1}\|_{\Omega_f}^2-\|\delta_t\nabla\varepsilon_h^{n}\|_{\Omega_f}^2)\\
&+3\nu\|\delta_t\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2\le C\triangle t(\triangle t^2+h^4)+C\triangle t(\|\delta_t\mathbf{e}_h^{n}\|_{\Omega_f}^2+\|\delta_t\mathbf{e}_h^{n-1}\|_{\Omega_f}^2+\|\delta_t\mathbf{e}_h^{n-2}\|_{\Omega_f}^2)\\
&+C\triangle t(\|\delta_t\eta_h^{n}\|_{H_p}^2+\|\delta_t\eta_h^{n-1}\|_{H_p}^2). \qquad (5.21)
\end{aligned}
$$

From (5.13), we have

$$
\begin{aligned}
&gS_0\left(\frac{3\delta_t\eta_h^{n+1}-4\delta_t\eta_h^{n}+\delta_t\eta_h^{n-1}}{2\triangle t},\psi_h\right)_{\Omega_p}+g(\mathbf{K}\delta_t\nabla\eta_h^{n+1},\nabla\psi_h)_{\Omega_p}\\
&-c_I(\psi_h,\delta_t\omega_h^{n+1}-2\delta_t\mathbf{u}_h^{n}+\delta_t\mathbf{u}_h^{n-1})=(\delta_tR_2^{n+1},\psi_h)_{\Omega_p}+(\delta_tR_3^{n+1},\psi_h)_{\Omega_p}, \qquad (5.22)
\end{aligned}
$$

Letting $\psi_h=4\delta_t\eta_h^{n+1}$, we deduce

$$
\begin{aligned}
&gS_0(\|\delta_t\eta_h^{n+1}\|_{\Omega_p}^2-\|\delta_t\eta_h^{n}\|_{\Omega_p}^2+\|2\delta_t\eta_h^{n+1}-\delta_t\eta_h^{n}\|_{\Omega_p}^2+\|\delta_{ttt}\eta_h^{n+1}\|_{\Omega_p}^2-\|2\delta_t\eta_h^{n}-\delta_t\eta_h^{n-1}\|_{\Omega_p}^2)\\
&+4g\triangle t\|\mathbf{K}^{1/2}\delta_t\nabla\eta_h^{n+1}\|_{\Omega_p}^2=2\triangle tc_I(\delta_t\eta_h^{n+1},\delta_t\omega_h^{n+1}-2\delta_t\mathbf{u}_h^{n}+\delta_t\mathbf{u}_h^{n-1})\\
&+(\delta_tR_2^{n+1},\delta_t\eta_h^{n+1})_{\Omega_p}+(\delta_tR_3^{n+1},\delta_t\eta_h^{n+1})_{\Omega_p}.
\end{aligned}
$$

Using Lemma 5.1, we get

$$
\begin{aligned}
&4\triangle tc_I(\delta_t\eta_h^{n+1},\delta_t\omega_h^{n+1}-2\delta_t\mathbf{u}_h^{n}+\delta_t\mathbf{u}_h^{n-1})\\
&\le 2\triangle t\|\delta_t\eta_h^{n+1}\|_{H_p}\|\delta_t\omega_h^{n+1}-2\delta_t\mathbf{u}_h^{n}+\delta_t\mathbf{u}_h^{n-1}\|_{H_f}\\
&\le g\triangle t\|\mathbf{K}^{1/2}\delta_t\nabla\eta_h^{n+1}\|_{H_p}^2+\frac{Cg\triangle t}{K_{\min}}\|\delta_t\omega_h^{n+1}-2\delta_t\mathbf{u}_h^{n}+\delta_t\mathbf{u}_h^{n-1}\|_{H_f}^2\\
&\le g\triangle t\|\mathbf{K}^{1/2}\delta_t\nabla\eta_h^{n+1}\|_{H_p}^2+\frac{Cg\triangle t}{K_{\min}}(\|\delta_{ttt}\omega_h^{n+1}\|_{H_p}^2+2\|\delta_t\mathbf{e}_h^{n}\|_{H_f}^2+\|\delta_t\mathbf{e}_h^{n-1}\|_{H_f}^2)\\
&\le C\triangle t^5+\frac{Cg\triangle t}{K_{\min}}(\|\delta_t\mathbf{e}_h^{n}\|_{H_f}^2+\|\delta_t\mathbf{e}_h^{n-1}\|_{H_f}^2)+g\triangle t\|\delta_t\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{H_p}^2.
\end{aligned}
$$

Then, we arrive at

$$
\begin{aligned}
&gS_0(\|\delta_t\eta_h^{n+1}\|_{\Omega_p}^2-\|\delta_t\eta_h^{n}\|_{\Omega_p}^2+\|2\delta_t\eta_h^{n+1}-\delta_t\eta_h^{n}\|_{\Omega_p}^2+\|\delta_{ttt}\eta_h^{n+1}\|_{\Omega_p}^2-\|2\delta_t\eta_h^{n}-\delta_t\eta_h^{n-1}\|_{\Omega_p}^2)\\
&+2g\triangle t\|\mathbf{K}^{1/2}\delta_t\nabla\eta_h^{n+1}\|_{\Omega_p}^2\le C\triangle t(\triangle t^4+h^4)+\frac{Cg\triangle t}{K_{\min}}(\|\delta_t\mathbf{e}_h^{n}\|_{H_f}^2+\|\delta_t\mathbf{e}_h^{n-1}\|_{H_f}^2). \qquad (5.23)
\end{aligned}
$$

Combining (5.21) and (5.23) and using Gronwall's lemma gives that

$$
\begin{aligned}
&3\|\delta_t\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}^2 + 3\sum_{i=1}^{n}\|\delta_t\tilde{\mathbf{e}}_h^{i+1} - \delta_t\mathbf{e}_h^{i+1}\|_{\Omega_f}^2 + \|\delta_t\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 + \|2\delta_t\mathbf{e}_h^{n+1} - \delta_t\mathbf{e}_h^{n}\|_{\Omega_f}^2 \\
&+ \sum_{i=1}^{n}\|\delta_{ttt}\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 + \frac{3\triangle t^2}{3}\|\delta_t\nabla\varepsilon_h^{n+1}\|_{\Omega_f}^2 \\
&+ gS_0(\|\delta_t\eta_h^{n+1}\|_{\Omega_p}^2 + \|2\delta_t\eta_h^{n+1} - \delta_t\eta_h^{n}\|_{\Omega_p}^2 + \sum_{i=1}^{n}\|\delta_{ttt}\eta_h^{n+1}\|_{\Omega_p}^2) + 3\nu\sum_{i=1}^{n}\|\delta_t\tilde{\mathbf{e}}_h^{i+1}\|_{H_f}^2 \\
&\le C(\triangle t^2 + h^4). \qquad (5.24)
\end{aligned}
$$

From (5.16), we get

$$
\|\delta_t\nabla\varepsilon_h^{n+1}\|_{\Omega_f} \le \frac{3}{2\triangle t}\left(\|\tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f} + \|\delta_t\beta_h^{n+1}\|_0\right) \le C(\triangle t^2 + h^4)/\triangle t.
$$

Letting $\mathbf{v}_h = \mathbf{e}_h^{n+1} - \tilde{\mathbf{e}}_h^{n+1}$ in (5.8), we have

$$
\begin{aligned}
3\|\mathbf{e}_h^{n+1} - \tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}^2 &= 2\triangle t b(\mathbf{e}_h^{n+1} - \tilde{\mathbf{e}}_h^{n+1}, \delta_t\varepsilon_h^{n+1} - \delta_t\theta_h^{n+1}) \\
&\le 2\triangle t\|\mathbf{e}_h^{n+1} - \tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f}\left(\|\nabla\delta_t\varepsilon_h^{n+1}\|_{\Omega_f} + \|\delta_t\theta_h^{n}\|_{\Omega_f}\right).
\end{aligned}
$$

Namely,

$$
\begin{aligned}
\|\mathbf{e}_h^{n+1} - \tilde{\mathbf{e}}_h^{n+1}\|_{\Omega_f} &\le 2\triangle t\left(\|\nabla\delta_t\varepsilon_h^{n+1}\|_{\Omega_f} + \|\delta_t\theta_h^{n}\|_{\Omega_f}\right) \\
&\le C(\triangle t^2 + h^2).
\end{aligned}
$$

Based on Theorem 5.3, we can get the following theorems, which give the optimal error estimate. □

Theorem 5.4. *If u, p, u_t, ϕ and ϕ_t are sufficiently smooth, then the numerical method has the following results*

$$
\begin{aligned}
&\|\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 + \|2\mathbf{e}_h^{n+1} - \mathbf{e}^{n}\|_{\Omega_f}^2 + \sum_{i=1}^{n}\|\mathbf{e}_h^{i+1} - 2\mathbf{e}_h^{i} + \mathbf{e}_h^{i-1}\|_{\Omega_f}^2 \\
&+ 2\triangle t\nu\sum_{i=1}^{n}\|\mathbf{e}_h^{i+1}\|_{H_f}^2 + 2g\triangle t\sum_{i=1}^{n}\|\mathbf{K}^{1/2}\nabla\eta_h^{i+1}\|_{\Omega_p}^2 \\
&+ gS_0\left(\|\eta_h^{n+1}\|_{\Omega_p}^2 + \|2\eta_h^{n+1} - \eta_h^{n}\|_{\Omega_p}^2 + \sum_{i=1}^{n}\|\delta_{tt}\eta_h^{i+1}\|_{\Omega_p}^2\right) \le C(\triangle t^4 + h^4).
\end{aligned}
$$

Proof. Using (5.3) and (5.8), we have

$$
\begin{aligned}
&\left(\frac{3\mathbf{e}_h^{n+1} - 4\mathbf{e}_h^{n} + \mathbf{e}_h^{n-1}}{2\triangle t}, \mathbf{v}_h\right)_{\Omega_f} + a_f(\tilde{\mathbf{e}}_h^{n+1}, \mathbf{v}_h) + c_I(\varphi_h^{n+1} - 2\phi_h^{n} + \phi_h^{n-1}, \mathbf{v}_h) + b(v_h, \varepsilon_h^{n+1}) \\
&+ B(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{v}_h) - B(2\mathbf{u}_h^{n} - \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) = (R_0^{n+1}, \mathbf{v}_h)_{\Omega_f} + (R_1^{n+1}, \mathbf{v}_h)_{\Omega_f}. \qquad (5.25)
\end{aligned}
$$

Letting $v_h = 4\triangle t \mathbf{e}_h^{n+1}$ in (5.25), we deduce

$$\begin{aligned}
&\|\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 - \|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|2\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|_{\Omega_f}^2 - \|2\mathbf{e}_h^n - \mathbf{e}^{n-1}\|_{\Omega_f} + \|\mathbf{e}_h^{n+1} - 2\mathbf{e}_h^n + \mathbf{e}_h^{n-1}\|_{\Omega_f}^2 \\
&+ 4\triangle t a_f(\tilde{\mathbf{e}}_h^{n+1}, \mathbf{e}_h^{n+1}) + 4\triangle t c_I(\varphi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}, \mathbf{e}_h^{n+1}) + 4\triangle t B(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{e}_h^{n+1}) \\
&- 4\triangle t B(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1}) = 4\triangle t(R_0^{n+1}, \mathbf{e}_h^{n+1})_{\Omega_f} + 4\triangle t(R_1^{n+1}, \mathbf{e}_h^{n+1})_{\Omega_f},
\end{aligned} \tag{5.26}$$

By (5.8), we have

$$(\tilde{\mathbf{e}}_h^{n+1}, \mathbf{v}_h)_{\Omega_f} = (\mathbf{e}_h^{n+1}, \mathbf{v}_h)_{\Omega_f}, \forall \mathbf{v}_h \in V_h.$$

So, we get

$$(-\Delta\tilde{\mathbf{e}}_h^{n+1}, \mathbf{v}_h)_{\Omega_f} = -(\Delta\mathbf{e}_h^{n+1}, \mathbf{v}_h)_{\Omega_f}, \forall \mathbf{v}_h \in V_h.$$

Namely,

$$a_f(\tilde{\mathbf{e}}_h^{n+1}, \mathbf{v}_h)_{\Omega_f} = a_f(\mathbf{e}_h^{n+1}, \mathbf{v}_h)_{\Omega_f}, \forall \mathbf{v}_h \in V_h. \tag{5.27}$$

Then, we get

$$a_f(\tilde{\mathbf{e}}_h^{n+1}, \mathbf{e}_h^{n+1})_{\Omega_f} = a_f(\mathbf{e}_h^{n+1}, \mathbf{e}_h^{n+1})_{\Omega_f}.$$

Adding some terms, we deduce

$$\begin{aligned}
&B(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{e}_h^{n+1}) - B(2\mathbf{u}_n^n - \mathbf{u}_h^{n-1}, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1}) \\
&= B(\delta_{tt}\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{e}_h^{n+1}) + B(2\zeta^n - \zeta^{n-1}, \mathbf{u}(t_{n+1}), \mathbf{e}_h^{n+1}) + B(2\omega_h^n - \omega_h^{n-1}, \zeta^{n+1}, \mathbf{e}_h^{n+1}) \\
&+ B(2\mathbf{e}_h^n - \mathbf{e}_h^{n-1}, w_h^{n+1}, \mathbf{e}_h^{n+1}) + B(2\omega_h^n - \omega_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1}, \mathbf{e}_h^{n+1}) - B(2\mathbf{e}^n - \mathbf{e}_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1}, \mathbf{e}_h^{n+1}).
\end{aligned}$$

We give the estimate for each term on the right-hand side.

$$\begin{aligned}
B(\delta_{tt}\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{e}_h^{n+1}) &\leq C\|\delta_{tt}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\mathbf{e}_h^{n+1}\|_{H_f} \\
&\leq C\triangle t^2\|u_{tt}(t_{n+1})\|_{L^\infty(\Omega_f)}\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\mathbf{e}_h^{n+1}\|_{H_f} \\
&\leq C\triangle t^4 + \frac{\nu}{16}\|\mathbf{e}_h^{n+1}\|_{H_f}^2.
\end{aligned}$$

$$\begin{aligned}
B(\mathbf{u}(2\zeta^n - \zeta^{n-1}, \mathbf{u}(t_{n+1}), \mathbf{e}_h^{n+1}) &\leq C\|2\zeta^n - \zeta^{n-1}\|_{\Omega_f}\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\mathbf{e}_h^{n+1}\|_{H_f} \\
&\leq Ch^2(\|u\|_{L^\infty(H^2)} + \|p\|_{L^\infty(H^1)})\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\mathbf{e}_h^{n+1}\|_{H_f} \\
&\leq Ch^4 + \frac{\nu}{16}\|\mathbf{e}_h^{n+1}\|_{H_f}^2.
\end{aligned}$$

$$\begin{aligned}
B(2\omega_h^n - \omega_h^{n-1}, \zeta^{n+1}, \mathbf{e}_h^{n+1}) &\leq C\|\mathcal{A}(2\omega_h^n - \omega_h^{n-1})\|_{\Omega_f}\|\zeta^{n+1}\|_{\Omega_f}\|\mathbf{e}_h^{n+1}\|_{H_f} \\
&\leq Ch^4 + \frac{\nu}{16}\|\mathbf{e}_h^{n+1}\|_{H_f}^2.
\end{aligned}$$

$$B(2\mathbf{e}_h^n - \mathbf{e}_h^{n-1}, w_h^{n+1}, \mathbf{e}_h^{n+1}) \leq C\|\mathcal{A}\omega_h^{n+1}\|_{\Omega_f}^2(\|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|\mathbf{e}_h^{n-1}\|_{\Omega_f}^2) + \frac{\nu}{16}\|\mathbf{e}_h^{n+1}\|_{H_f}^2.$$

$$B(2\mathbf{e}^n - \mathbf{e}_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1}, \mathbf{e}_h^{n+1}) \le C(\|\mathbf{e}_h^n\|_{H_f} + \|\mathbf{e}_h^{n-1}\|_{H_f})\|\tilde{\mathbf{e}}^{n+1}\|_{\mathbf{H}_f}\|\mathbf{e}_h^{n+1}\|_{H_f}$$
$$\le C(\|e_h^n\|_{H_f}^2 + \|\mathbf{e}_h^{n-1}\|_{H_f}^2)\|\tilde{\mathbf{e}}^{n+1}\|_{\mathbf{H}_f}^2 + \frac{\nu}{16}\|\mathbf{e}_h^{n+1}\|_{H_f}^2.$$

$$B(2\omega_h^n - \omega_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1}, \mathbf{e}_h^{n+1}) = B(2\omega_h^n - \omega_h^{n-1}, \tilde{\mathbf{e}}_h^{n+1} - \mathbf{e}_h^{n+1}, \mathbf{e}_h^{n+1})$$
$$\le C(\|\mathcal{A}\omega_h^n\|_{\Omega_f} + \|\mathcal{A}\omega_h^{n-1}\|_{\Omega_f})\|\tilde{\mathbf{e}}_h^{n+1} - \mathbf{e}_h^{n+1}\|_{\Omega_f}\|\mathbf{e}_h^{n+1}\|_{H_f}$$
$$\le C(\triangle t^4 + h^4) + \frac{\nu}{16}\|\mathbf{e}_h^{n+1}\|_{H_f}^2.$$

Hence, combining the above inequalities, we arrive at

$$4\triangle tB(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{e}_h^{n+1}) - 4\triangle tB(\mathbf{u}_n^n, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1})$$
$$\le C\triangle t(\triangle t^4 + h^4) + C\triangle t(\|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|\mathbf{e}_h^{n-1}\|_{\Omega_f}^2) + \frac{\triangle t\nu}{4}\|\tilde{\mathbf{e}}_h^{n+1}\|_{H_f}^2. \tag{5.28}$$

Via (4.1), we get

$$c_I(\varphi_h^{n+1} - (2\phi_h^n - \phi_h^{n-1}), \mathbf{e}_h^{n+1}) = c_I(\varphi_h^{n+1} - 2\varphi_h^n + \varphi_h^{n-1}, \mathbf{e}_h^{n+1}) + c_I(2\eta_h^n - \eta_h^{n-1}, \mathbf{e}_h^{n+1})$$
$$\le 2C_2g\|\varphi_h^{n+1} - 2\varphi_h^n + \varphi_h^{n-1}\|_{H_p}\|\mathbf{e}_h^{n+1}\|_{H_f} + C_2g(\|\eta_h^n\|_{H_p} + \|\eta_h^{n-1}\|_{H_p})\|\mathbf{e}_h^{n+1}\|_{H_f}$$
$$\le Cg(\triangle t^4 + h^4) + C(\|\mathbf{K}^{1/2}\nabla\eta_h^n\|_{\Omega_P}^2 + \|\mathbf{K}^{1/2}\nabla\eta_h^{n-1}\|_{\Omega_P}^2) + \frac{\nu}{4}\|\mathbf{e}_h^{n+1}\|_{H_f}^2. \tag{5.29}$$

Then, we arrive at

$$\|\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 - \|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|2\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|_{\Omega_f}^2 - \|2\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|_{\Omega_f}^2$$
$$+ \|\mathbf{e}_h^{n+1} - 2\mathbf{e}_h^n + \mathbf{e}_h^{n-1}\|_{\Omega_f}^2 + 2\triangle t\nu\|\mathbf{e}_h^{n+1}\|_{H_f}^2$$
$$\le Cg\triangle t(\triangle t^4 + h^4) + C\triangle t(\|\mathbf{K}^{1/2}\nabla\eta_h^n\|_{\Omega_P}^2 + \|\mathbf{K}^{1/2}\nabla\eta_h^{n-1}\|_{\Omega_P}^2)$$
$$+ C\triangle t(\|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|\mathbf{e}_h^{n-1}\|_{\Omega_f}^2). \tag{5.30}$$

Combining it with (5.14), we have

$$\|\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 - \|\mathbf{e}_h^n\|_{\Omega_f}^2 + \|2\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|_{\Omega_f}^2 - \|2\mathbf{e}_h^n - \mathbf{e}^{n-1}\|_{\Omega_f}^2 + \|\mathbf{e}_h^{n+1} - 2\mathbf{e}_h^n + \mathbf{e}_h^{n-1}\|_{\Omega_f}^2$$
$$+ 2\triangle t\nu\|\mathbf{e}_h^{n+1}\|_{H_f}^2 + 2g\triangle t\|\mathbf{K}^{1/2}\nabla\eta_h^{n+1}\|_{\Omega_p}^2$$
$$+ gS_0(\|\eta_h^{n+1}\|_{\Omega_p}^2 - \|\eta_h^n\|_{\Omega_p}^2 + \|2\eta_h^{n+1} - \eta_h^n\|_{\Omega_p}^2 + \|\delta_{tt}\eta_h^{n+1}\|_{\Omega_p}^2 - \|2\eta_h^n - \eta_h^{n-1}\|_{\Omega_p}^2)$$
$$\le C\triangle t(\triangle t^4 + h^4) + C\triangle t(\|\mathbf{K}^{1/2}\nabla\eta_h^n\|_{\Omega_P}^2 + \|\mathbf{K}^{1/2}\nabla\eta_h^{n-1}\|_{\Omega_P}^2) + Cg\triangle t(\|\mathbf{e}_h^n\|_{H_f}^2 + \|\mathbf{e}_h^{n-1}\|_{H_f}^2).$$

Summing it for all n and using Gronwall's lemma, we deduce

$$\|\mathbf{e}_h^{n+1}\|_{\Omega_f}^2 + \|2\mathbf{e}_h^{n+1} - \mathbf{e}^n\|_{\Omega_f}^2 + \sum_{i=1}^n \|\mathbf{e}_h^{i+1} - 2\mathbf{e}_h^i + \mathbf{e}_h^{i-1}\|_{\Omega_f}^2$$
$$+ 2\triangle t\nu \sum_{i=1}^n \|\mathbf{e}_h^{i+1}\|_{H_f}^2 + 2g\triangle t \sum_{i=1}^n \|\mathbf{K}^{1/2}\nabla\eta_h^{i+1}\|_{\Omega_p}^2$$
$$+ gS_0(\|\eta_h^{n+1}\|_{\Omega_p}^2 + \|2\eta_h^{n+1} - \eta_h^n\|_{\Omega_p}^2 + \sum_{i=1}^n \|\delta_{tt}\eta_h^{i+1}\|_{\Omega_p}^2) \le C(\triangle t^4 + h^4).$$

Using triangle inequality, Lemma 4.1 and Theorem 4.2, we have the following theorem. □

Theorem 5.5. *If u, p, u_t, ϕ and ϕ_t are sufficiently smooth, then the numerical method has unconditionally optimal convergence in the sense that*

$$\begin{aligned} \|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\|_{\Omega_f} &\leq C(\triangle t^2 + h^2), \\ \triangle t \sum_{i=1}^{n} \|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\|_{H_f}^2 &\leq C(\triangle t^4 + h^2), \\ \|\phi(t_{n+1}) - \phi_h^{n+1}\|_{\Omega_p} &\leq C(\triangle t^2 + h^2), \\ \triangle t \sum_{i=1}^{n} \|\phi(t_{n+1}) - \phi_h^{n+1}\|_{H_p}^2 &\leq C(\triangle t^4 + h^2). \end{aligned}$$

6. Numerical experiments

In order to show the effect of our method, we give some numerical results in this section. Let $\Omega_f = [0,1]\times[1,2]$ and $\Omega_p = [0,1] \times [0,1]$ with interface $\Gamma = (0,1) \times \{1\}$. The exact solution is

$$\begin{aligned} u_1(x,y,t) &= [x^2(y-1)^2 + y]\cos(t), \\ u_2(x,y,t) &= \left[-\frac{2}{3}x(y-1)^3\right]\cos(t) + [2 - \pi\sin(\pi x)]\cos(t), \\ p(x,y,t) &= [2 - \pi\sin(\pi x)]\sin(0.5\pi y)\cos(t), \\ \phi(x,y,t) &= [2 - \pi\sin(\pi x)][1 - y - \cos(\pi y)]\cos(t). \end{aligned}$$

The initial conditions, boundary conditions and the forcing terms are given by the exact solution. The finite element spaces choosing the MINI elements ($P1b - p1$) for the Navier-Stokes equation in Ω_f, and the linear Lagrangian elements ($P1$) for the Darcy flow in Ω_p. Here, we use the software package FreeFEM++ [30] for our program.

We introduce a more accurate approach [34, 37] to examine the orders of convergence with respect to the time step $\triangle t$ or the mesh size h due to the approximation errors $\mathcal{O}(\triangle t^{r_1}) + \mathcal{O}(h^{r_2})$. Then, we assume that

$$v_h^{\triangle t}(x,t_m) \approx v(x,t_m) + C_1(x,t_m)\triangle t^{r_1} + C_2(x,t_m)h^{r_2}.$$

Thus, we can get

$$\begin{aligned} \rho_{v,h,i} &= \frac{\|v_h^{\triangle t}(x,t_m) - v_{h/2}^{\triangle t}(x.t_m)\|_i}{\|v_{h/2}^{\triangle t}(x,t_m) - v_{h/4}^{\triangle t}(x.t_m)\|_i} \approx \frac{4^{r_2} - 2^{r_2}}{2^{r_2} - 1}, \\ \rho_{v,\triangle t,i} &= \frac{\|v_h^{\triangle t}(x,t_m) - v_h^{\triangle t/2}(x.t_m)\|_i}{\|v_h^{\triangle t/2}(x,t_m) - v_h^{\triangle t/4}(x.t_m)\|_i} \approx \frac{4^{r_1} - 2^{r_1}}{2^{r_1} - 1}. \end{aligned}$$

Here, v can be u, p or ϕ, and i can be 0 or 1. We see that $\rho_{v,h,i}, \rho_{v,\triangle t,i}$ approach 4.0 or 2.0, the convergence order will be 2.0 or 1.0, respectively.

We focus on the convergence orders with respect to the spacing step h, and study the errors by fixing time step $\triangle t = 0.01$ and varying spacing steps $h = 1/2, 1/4, 1/8, 1/16$ and $1/32$, respectively. Tables 1 and 3 present the errors between the numerical solutions and the exact solutions for $\nu = 1.0$ and 0.1, respectively. We see that the errors diminish with the spacing step h becoming small. Tables 2 and 4 show the convergence orders with

TABLE 1. The numerical results at $T = 1$ with $\triangle t = 0.01$ and $\nu = 1.0$ for different h.

$1/h$	$\frac{\|u(T)-u_h^N\|_0}{\|u\|_0}$	$\frac{\|\nabla(u(T)-u_h^N)\|_0}{\|u\|_0}$	$\frac{\|\phi(T)-\phi_h^N\|_0}{\|\phi(T)\|_0}$	$\frac{\|\nabla(\phi(T)-\phi_h^N)\|_0}{\|\nabla\phi(T)\|_0}$
2	0.257995	0.413694	0.599459	0.628726
4	0.0676178	0.208795	0.19486	0.358805
8	0.0266151	0.113212	0.0543885	0.187161
16	0.0254737	0.07329	0.0146734	0.0948262
32	0.0265727	0.0599137	0.00634115	0.0478979

TABLE 2. Convergence order of $\mathcal{O}(h^r)$ $T = 1$ with $\triangle t = 0.01$ and $\nu = 1.0$ for different h.

$1/h$	$\|u_h^N - u_{h/2}^N\|_0$	$\|\nabla(u_h^N - u_{h/2}^N)\|_0$	$\|\phi_h^N - \phi_{h/2}^N\|_0$	$\|\nabla(\phi_h^N - \phi_{h/2}^N)\|_0$
4	0.211291	1.41723	0.158424	1.4032
8	0.05651	0.724435	0.0511751	0.795905
16	0.0148815	0.365974	0.0145124	0.414817
32	0.00409845	0.184518	0.00373865	0.208951
	$\rho_{u,h,0}$	$\rho_{u,h,1}$	$\rho_{\phi,h,0}$	$\rho_{\phi,h,1}$
8	3.73901	1.95632	3.09573	1.76302
16	3.79733	1.97947	3.5263	1.91869
32	3.63101	1.98341	3.88172	1.98524

TABLE 3. The numerical results at $T = 1$ with $\triangle t = 0.01$ and $\nu = 0.1$ for different h.

$1/h$	$\frac{\|u(T)-u_h^N\|_0}{\|u\|_0}$	$\frac{\|\nabla(u(T)-u_h^N)\|_0}{\|u\|_0}$	$\frac{\|\phi(T)-\phi_h^N\|_0}{\|\phi(T)\|_0}$	$\frac{\|\nabla(\phi(T)-\phi_h^N)\|_0}{\|\nabla\phi(T)\|_0}$
2	0.273388	1.79847	0.204781	1.61703
4	0.0729364	0.872048	0.0667661	0.922836
8	0.0614678	0.58033	0.0185896	0.482675
16	0.0710409	0.518779	0.00638197	0.247708
32	0.0750778	0.513098	0.0053065	0.131462

TABLE 4. Convergence order of $\mathcal{O}(h^r)$ $T = 1$ with $\triangle t = 0.01$, and $\nu = 0.1$ for different h.

$1/h$	$\|u_h^N - u_{h/2}^N\|_0$	$\|\nabla(u_h^N - u_{h/2}^N)\|_0$	$\|\phi_h^N - \phi_{h/2}^N\|_0$	$\|\nabla(\phi_h^N - \phi_{h/2}^N)\|_0$
4	0.216266	1.60015	0.158111	1.40296
8	0.0693707	0.832826	0.0516311	0.797819
16	0.0191222	0.416906	0.0148361	0.4166
32	0.00642024	0.213692	0.00383535	0.210017
	$\rho_{u,h,0}$	$\rho_{u,h,1}$	$\rho_{\phi,h,0}$	$\rho_{\phi,h,1}$
8	3.11755	1.92135	3.06232	1.7585
16	3.62775	1.99763	3.48009	1.91507
32	2.97843	1.95097	3.86827	1.98365

respect to the spacing step h for $\nu = 1.0$ and 0.1, respectively. We see that $\rho_{u,h,0}$ and $\rho_{\phi,h,0}$ are nearly 4.0, and $\rho_{u,h,1}$, $\rho_{\phi,h,1}$ approach 2.0. It suggests that the convergence order in space for the L^2-norm of u_h and ϕ_h are $\mathcal{O}(h^2)$, the convergence orders in space for the H^1-norm of u_h and ϕ_h are $\mathcal{O}(h^1)$.

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