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NUMERICAL ANALYSIS OF THE OSEEN-TYPE PETERLIN VISCOELASTIC MODEL BY THE STABILIZED LAGRANGE–GALERKIN METHOD
PART II: A LINEAR SCHEME

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Abstract. This is the second part of our error analysis of the stabilized Lagrange–Galerkin scheme applied to the Oseen-type Peterlin viscoelastic model. Our scheme is a combination of the method of characteristics and Brezzi–Pitkäranta's stabilization method for the conforming linear elements, which leads to an efficient computation with a small number of degrees of freedom especially in three space dimensions. In this paper, Part II, we apply a semi-implicit time discretization which yields the linear scheme. We concentrate on the diffusive viscoelastic model, *i.e.* in the constitutive equation for time evolution of the conformation tensor a diffusive effect is included. Under mild stability conditions we obtain error estimates with the optimal convergence order for the velocity, pressure and conformation tensor in two and three space dimensions. The theoretical convergence order is confirmed by numerical experiments.

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1. Introduction

The present paper is a continuation of numerical error analysis of the stabilized Lagrange–Galerkin method applied to the Oseen-type Peterlin viscoelastic model. In our previous paper [29], Part I, we dealt with the fully nonlinear implicit scheme, whereas here, in Part II, we investigate a linear semi-implicit scheme.

The development of stable and convergent numerical methods for viscoelastic models, such as the Oldroyd-B type models, is an active research area. In particular, the question of stability when elastic effects are dominant (the so-called high Weissenberg number problem) remains an open problem. We refer the reader to works of Fattal and Kupferman [20, 21], where an interesting approach using the log-conformation representation has been introduced. Furthermore, in Boyaval *et al.* [10] free energy dissipative Lagrange–Galerkin schemes with or without the log-conformation representation has been studied and in Lee and Xu [27] and Lee *et al.* [28]

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finite element schemes using the idea of the generalized Lie derivative have been proposed. Further related numerical schemes and computations can be found, *e.g.*, in [1, 2, 7, 8, 15, 26, 32, 34, 36, 41, 42, 53, 54], see also references therein. To the best of our knowledge there are no results on error estimates of numerical schemes for the Oldroyd-B model, see Picasso and Rappaz [44] and Bonito *et al.* [6] for error analysis of simplified models without convective terms.

In [43] Peterlin proposed a mean-field closure model according to which the average of the elastic force over thermal fluctuations is replaced by the value of the force at the mean-squared polymer extension. This means that a nonlinear spring force law $F(\mathbf{R}) = \gamma(|\mathbf{R}|^2)\mathbf{R}$ that acts in a dumbbell is replaced by the function $F(\mathbf{R}) = \gamma(\operatorname{tr}\mathbf{C})\mathbf{R}$. Here, γ is the spring constant, $\mathbf{C}$ is the so-called conformation tensor and $\mathbf{R}$ is the vector connecting the beads of a dumbbell. Based on this approach Renardy has recently derived a new class of general macroscopic constitutive models, that is motivated by Peterlin dumbbell theories with a nonlinear spring law for an infinitely extensible spring, see Renardy [48, 49] and recent papers by Lukáčová−Medvid'ová *et al.* [30, 31], where the global existence of weak solutions has been obtained.

In this paper, Part II, as well as in our previous paper, Part I, we consider the so-called Oseen-type Peterlin viscoelastic model that is a system of the flow equations and an equation for the conformation tensor, *cf.* [47–49]. We concentrate on the diffusive viscoelastic model, which means that in the constitutive equations for the conformation tensor a diffusive effect is included.

Let us point out that in standard derivations of bead-spring models the diffusive term in the equation for the elastic stress tensor is routinely omitted. In [17] a careful justification of the presence of the diffusive term in the Fokker−Planck equations through the asymptotic analysis is presented. The diffusion coefficient ε is proportional to $(\ell/L)^2/We$, where L and ℓ are characteristic macroscopic and microscopic length scales, respectively, and We is the so-called Weissenberg number. It is a reference number characterizing viscoelastic property of the material. Estimates for $(\ell/L)^2$ presented in [5] show that $(\ell/L)^2$ is in the range of about 10^{-9} and 10^{-7}. As emphasized in [4] the model reduction by neglecting this small diffusive effect is mathematically counterproductive leading to a degenerate parabolic-hyperbolic system (2.1) with $\varepsilon = 0$. On the other hand, when the diffusive term is taken into account, the resulting system (2.1) remains parabolic. We would like to point out that in the analysis presented below we only require $\varepsilon > 0$ and there is no assumption on the size of ε. For the details of the derivation of the diffusive Peterlin model we refer to [30, 35, 48, 49]. Let us mention that, even when the velocity field is given, the equation for the conformation tensor in the Peterlin model is still nonlinear, while the Oldroyd-B model is linear with respect to the extra stress tensor. Hence, we can say that the nonlinearity of the Peterlin model is stronger than that of the Oldroyd-B model. As a starting point of the numerical analysis of the Peterlin model, we consider the Oseen-type model, where the velocity of the material derivative is replaced by a known one, in order to concentrate on the treatment of nonlinear terms arising from the elastic stress.

In the present paper a stabilized Lagrange–Galerkin method for the Peterlin viscoelastic model is studied. It consists of the method of characteristics and Brezzi–Pitkäranta's stabilization method [13] for the conforming linear elements. The method of characteristics derives the robustness in convection-dominated flow problems, and the stabilization method reduces the number of degrees of freedom in computation especially in three space dimensions. In our recent works by Notsu and Tabata [37–39] the stabilized Lagrange–Galerkin method has been applied successfully for the Oseen, Navier–Stokes and natural convection problems and optimal error estimates have been proved. We extend the numerical analysis of the stabilized Lagrange–Galerkin method to the Oseen-type Peterlin model. As already mentioned above, the aim of the present paper is to give a rigorous error analysis of the linear stabilized Lagrange–Galerkin scheme for the diffusive Peterlin model in both two and three space dimensions. We show that under mild stability conditions the obtained error estimates have the optimal convergence rate.

As mentioned in Boyaval *et al.* [10], the positive definiteness of the conformation tensor is important in the analysis of numerical schemes for the viscoelastic models. For the exact strong solution to the Peterlin viscoelastic model this property has been shown in [35]. We remark that our error estimates have been obtained successfully without studying positive definiteness of the conformation tensor. Let us additionally note that this paper includes the error estimate for the pressure in the standard L^2 norm (Thm. 2), which has, as far as we know, never been shown for time-dependent viscoelastic flow problems, *e.g.*, the Oldroyd-B model.

This paper is organized as follows. In Section 2 the mathematical model for the Peterlin viscoelastic fluid is described. In Section 3 a linear stabilized Lagrange–Galerkin scheme is presented. The main results on the convergence with optimal error estimates are stated in Section 4, and proved in Section 5. In Section 6 some numerical experiments confirming the theoretical convergence order are provided.

2. The Oseen-type Peterlin viscoelastic model

The function spaces and the notation to be used throughout the paper are as follows. Let Ω be a bounded domain in $\mathbb{R}^d$ for $d = 2$ or 3, $\Gamma = \partial\Omega$ the boundary of Ω, and T a positive constant. For $m \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty]$ we use the Sobolev spaces $W^{m,p}(\Omega)$, $W_0^{1,\infty}(\Omega)$, $H^m(\Omega)\,(= W^{m,2}(\Omega))$, $H_0^1(\Omega)$ and $L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q \ \mathrm{d}x = 0\}$. Furthermore, we employ function spaces $H^m_{sym}(\Omega) = \{\mathbf{D} \in H^m(\Omega)^{d\times d};\ \mathbf{D} = \mathbf{D}^T\}$ and $C^m_{sym}(\bar{\Omega}) = C^m(\bar{\Omega})^{d\times d} \cap H^m_{sym}(\Omega)$, where the superscript T stands for the transposition. For any normed space S with norm $\|\cdot\|_S$, we define function spaces $H^m(0,T;S)$ and $C([0,T];S)$ consisting of S-valued functions in $H^m(0,T)$ and $C([0,T])$, respectively. We use the same notation $(\cdot,\cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions. The dual pairing between S and the dual space S' is denoted by $\langle\cdot,\cdot\rangle$. The norms on $W^{m,p}(\Omega)$ and $H^m(\Omega)$ and their seminorms are simply denoted by $\|\cdot\|_{m,p}$ and $\|\cdot\|_m\,(= \|\cdot\|_{m,2})$ and by $|\cdot|_{m,p}$ and $|\cdot|_m\,(= |\cdot|_{m,2})$, respectively. The notations $\|\cdot\|_{m,p}$, $|\cdot|_{m,p}$, $\|\cdot\|_m$ and $|\cdot|_m$ are employed not only for scalar-valued functions but also for vector- and matrix-valued ones. We also denote the norm on $H^{-1}(\Omega)^2$ by $\|\cdot\|_{-1}$. For t_0 and $t_1 \in \mathbb{R}$ we introduce the function space,

$$Z^m(t_0,t_1) = \left\{\psi \in H^j(t_0,t_1;H^{m-j}(\Omega));\ j = 0,\dots,m,\ \|\psi\|_{Z^m(t_0,t_1)} < \infty\right\}$$

with the norm

$$\|\psi\|_{Z^m(t_0,t_1)} = \left\{\sum_{j=0}^{m} \|\psi\|^2_{H^j(t_0,t_1;H^{m-j}(\Omega))}\right\}^{1/2},$$

and set $Z^m = Z^m(0,T)$. We often omit $[0,T]$, Ω, and the superscripts d and $d\times d$ for the vector and the matrix if there is no confusion, *e.g.*, we shall write $C(L^\infty)$ in place of $C([0,T];L^\infty(\Omega)^{d\times d})$. For square matrices $\mathbf{A}$ and $\mathbf{B} \in \mathbb{R}^{d\times d}$ we use the notation $\mathbf{A} : \mathbf{B} = \mathrm{tr}\,(\mathbf{A}\mathbf{B}^T) = \sum_{i,j} A_{ij}B_{ij}$.

We consider the system of equations describing the unsteady motion of an incompressible viscoelastic fluid,

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \mathrm{div}\,(2\nu \mathrm{D}(\mathbf{u})) + \nabla p = \mathrm{div}\,[(\mathrm{tr}\,\mathbf{C})\mathbf{C}] + \mathbf{f} \qquad \text{in } \Omega\times(0,T), \tag{2.1a}$$

$$\mathrm{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega\times(0,T), \tag{2.1b}$$

$$\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t} - \varepsilon\Delta\mathbf{C} = (\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T - (\mathrm{tr}\,\mathbf{C})^2\,\mathbf{C} + (\mathrm{tr}\,\mathbf{C})\mathbf{I} + \mathbf{F} \qquad \text{in } \Omega\times(0,T), \tag{2.1c}$$

$$\mathbf{u} = \mathbf{0}, \quad \frac{\partial\mathbf{C}}{\partial\mathbf{n}} = \mathbf{0}, \qquad \text{on } \Gamma\times(0,T), \tag{2.1d}$$

$$\mathbf{u} = \mathbf{u}^0, \quad \mathbf{C} = \mathbf{C}^0, \qquad \text{in } \Omega,\ \text{at } t = 0, \tag{2.1e}$$

where $(\mathbf{u},p,\mathbf{C}) : \Omega\times(0,T) \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d\times d}_{sym}$ are the unknown velocity, pressure and conformation tensor, $\nu \in (0,1]$ is a fluid viscosity, $\varepsilon \in (0,1]$ is an elastic stress viscosity, $(\mathbf{f},\mathbf{F}) : \Omega\times(0,T) \to \mathbb{R}^d \times \mathbb{R}^{d\times d}_{sym}$ is a pair of given external forces, $\mathrm{D}(\mathbf{u}) = (1/2)[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$ is the symmetric part of the velocity gradient, $\mathbf{I}$ is the identity matrix, $\mathbf{n} : \Gamma \to \mathbb{R}^d$ is the outward unit normal, $(\mathbf{u}^0,\mathbf{C}^0) : \Omega \to \mathbb{R}^d \times \mathbb{R}^{d\times d}_{sym}$ is a pair of given initial functions, and $\mathrm{D}/\mathrm{D}t$ is the material derivative defined by

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{w}\cdot\nabla,$$

where $\mathbf{w} : \Omega\times(0,T) \to \mathbb{R}^d$ is a given velocity.

Remark 2.1. The model (2.1) is the Oseen approximation to the fully nonlinear problem, where the material derivative terms,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}, \quad \frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{C}$$

exist in place of $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$ and $\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t}$ in equations (2.1a) and (2.1c). The existence of weak solutions and the uniqueness of regular solutions to the fully nonlinear model have been proved in Lukáčová–Medvid'ová *et al.* ([30], Thms. 1 and 3). The corresponding results are obtained under regularity condition on $\mathbf{w}$ to the model (2.1), which is simpler than the fully nonlinear model. Numerical analysis of the fully nonlinear problem is a future work.

We set an assumption for the given velocity $\mathbf{w}$.

Hypothesis 2.2. *The function* $\mathbf{w}$ *satisfies* $\mathbf{w} \in C([0,T]; W_0^{1,\infty}(\Omega)^d)$.

Let $V = H_0^1(\Omega)^d$, $Q = L_0^2(\Omega)$ and $W = H_{sym}^1(\Omega)$. We define the bilinear forms a_u on $V\times V$, b on $V\times Q$, $\mathcal{A}$ on $(V\times Q)\times(V\times Q)$ and a_c on $W\times W$ by

$$\begin{aligned} a_u(\mathbf{u},\mathbf{v}) &= 2\big(\mathrm{D}(\mathbf{u}),\mathrm{D}(\mathbf{v})\big), \quad b(\mathbf{u},q) = -(\operatorname{div}\mathbf{u},q), \quad \mathcal{A}\big((\mathbf{u},p),(\mathbf{v},q)\big) = \nu a_u(\mathbf{u},\mathbf{v}) + b(\mathbf{u},q) + b(\mathbf{v},p),\\ a_c(\mathbf{C},\mathbf{D}) &= (\nabla\mathbf{C},\nabla\mathbf{D}),\end{aligned}$$

respectively. We present the weak formulation of the problem (2.1); find $(\mathbf{u},p,\mathbf{C}) : (0,T)\to V\times Q\times W$ such that for $t\in(0,T)$

$$\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(t),\mathbf{v}\right) + \mathcal{A}\big((\mathbf{u},p)(t),(\mathbf{v},q)\big) = -\left(\operatorname{tr}\mathbf{C}(t)\,\mathbf{C}(t),\nabla\mathbf{v}\right) + (\mathbf{f}(t),\mathbf{v}), \tag{2.2a}$$

$$\left(\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t}(t),\mathbf{D}\right) + \varepsilon a_c\big(\mathbf{C}(t),\mathbf{D}\big) = 2\big((\nabla\mathbf{u}(t))\mathbf{C}(t),\mathbf{D}\big) - \big((\operatorname{tr}\mathbf{C}(t))^2\mathbf{C}(t),\mathbf{D}\big) + (\operatorname{tr}\mathbf{C}(t)\mathbf{I},\mathbf{D}) + (\mathbf{F}(t),\mathbf{D}), \tag{2.2b}$$

$$\forall(\mathbf{v},q,\mathbf{D})\in V\times Q\times W,$$

with $(\mathbf{u}(0),\mathbf{C}(0)) = (\mathbf{u}^0,\mathbf{C}^0)$.

3. A linear stabilized Lagrange–Galerkin scheme

The aim of this section is to present a linear stabilized Lagrange–Galerkin scheme for the model (2.1).

Let Δt be a time increment, $N_T = \lfloor T/\Delta t\rfloor$ the total number of time steps and $t^n = n\Delta t$ for $n=0,\ldots,N_T$. Let $\mathbf{g}$ be a function defined in $\Omega\times(0,T)$ and $\mathbf{g}^n = \mathbf{g}(\cdot,t^n)$. For the approximation of the material derivative we employ the first-order characteristics method,

$$\frac{\mathrm{D}\mathbf{g}}{\mathrm{D}t}(x,t^n) = \frac{\mathbf{g}^n(x) - \left(\mathbf{g}^{n-1}\circ X_1^n\right)(x)}{\Delta t} + O(\Delta t), \tag{3.1}$$

where $X_1^n : \Omega\to\mathbb{R}^d$ is a mapping defined by

$$X_1^n(x) = x - \mathbf{w}^n(x)\Delta t,$$

and the symbol $\circ$ means the composition of functions,

$$(\mathbf{g}^{n-1}\circ X_1^n)(x) = \mathbf{g}^{n-1}(X_1^n(x)).$$

For the details on deriving the approximation (3.1) of $\mathrm{D}\mathbf{g}/\mathrm{D}t$, see, *e.g.*, [38]. The point $X_1^n(x)$ is called the upwind point of x with respect to $\mathbf{w}^n$. The next proposition, which is a direct consequence of [50] and [52], presents sufficient conditions to ensure that all upwind points defined by X_1^n are in Ω and that its Jacobian $J^n = \det(\partial X_1^n/\partial x)$ is around 1.

Proposition 3.1. *Suppose Hypothesis 2.2 holds. Then, we have the following for* $n \in \{0, \ldots, N_T\}$.

(i) *Under the condition*

$$\Delta t|\mathbf{w}|_{C(W^{1,\infty})} < 1, \tag{3.2}$$

$X_1^n : \Omega \to \Omega$ *is bijective.*

(ii) *Furthermore, under the condition*

$$\Delta t|\mathbf{w}|_{C(W^{1,\infty})} \leq 1/4, \tag{3.3}$$

the estimate $1/2 \leq J^n \leq 3/2$ *holds.*

For the sake of simplicity we suppose that Ω is a polygonal ($d = 2$) or polyhedral ($d = 3$) domain. Let $\mathcal{T}_h = \{K\}$ be a triangulation of $\bar{\Omega}$ $(= \bigcup_{K\in\mathcal{T}_h} K)$, h_K the diameter of $K \in \mathcal{T}_h$ and $h = \max_{K\in\mathcal{T}_h} h_K$ the maximum element size. We consider a regular family of subdivisions $\{\mathcal{T}_h\}_{h\downarrow 0}$ satisfying the inverse assumption [14], *i.e.*, there exists a positive constant α_0 independent of h such that

$$\frac{h}{h_K} \leq \alpha_0, \quad \forall K \in \mathcal{T}_h,\ \forall h.$$

We define the discrete function spaces X_h, M_h, W_h, V_h and Q_h by

$$X_h = \left\{\mathbf{v}_h \in C(\bar{\Omega})^d;\ \mathbf{v}_{h|K} \in P_1(K)^d, \forall K \in \mathcal{T}_h\right\}, \qquad M_h = \left\{q_h \in C(\bar{\Omega});\ q_{h|K} \in P_1(K), \forall K \in \mathcal{T}_h\right\},$$
$$W_h = \left\{\mathbf{D}_h \in C_{sym}(\bar{\Omega});\ \mathbf{D}_{h|K} \in P_1(K)^{d\times d}, \forall K \in \mathcal{T}_h\right\}, \qquad V_h = X_h \cap V, \qquad Q_h = M_h \cap Q,$$

respectively, where $P_1(K)$ is the polynomial space of linear functions on $K \in \mathcal{T}_h$.

Let δ_0 be a small positive constant fixed arbitrarily and $(\cdot,\cdot)_K$ the $L^2(K)^d$ inner product. We define the bilinear forms $\mathcal{A}_h$ on $(V \times H^1(\Omega)) \times (V \times H^1(\Omega))$ and $\mathcal{S}_h$ on $H^1(\Omega) \times H^1(\Omega)$ by

$$\mathcal{A}_h\left((\mathbf{u}, p), (\mathbf{v}, q)\right) = \nu a_u\left(\mathbf{u}, \mathbf{v}\right) + b(\mathbf{u}, q) + b(\mathbf{v}, p) - \mathcal{S}_h(p, q), \qquad \mathcal{S}_h(p, q) = \delta_0 \sum_{K\in\mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K.$$

Let $(\mathbf{f}_h, \mathbf{F}_h) = (\{\mathbf{f}_h^n\}_{n=1}^{N_T}, \{\mathbf{F}_h^n\}_{n=1}^{N_T}) \subset L^2(\Omega)^d \times L^2(\Omega)^{d\times d}$ and $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ be given. A linear stabilized Lagrange–Galerkin scheme for (2.1) is to find $(\mathbf{u}_h, p_h, \mathbf{C}_h) = \{(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h \times W_h$ such that, for $n = 1, \ldots, N_T$,

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\big((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)\big) = -\big((\mathrm{tr}\, \mathbf{C}_h^n)\mathbf{C}_h^{n-1}, \nabla \mathbf{v}_h\big) + (\mathbf{f}_h^n, \mathbf{v}_h), \tag{3.4a}$$

$$\begin{aligned}\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon a_c(\mathbf{C}_h^n, \mathbf{D}_h) = 2\big((\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1}, \mathbf{D}_h\big) - \big((\mathrm{tr}\, \mathbf{C}_h^{n-1})^2\mathbf{C}_h^n, \mathbf{D}_h\big)&\\ + \big((\mathrm{tr}\, \mathbf{C}_h^{n-1})\mathbf{I}, \mathbf{D}_h\big) + (\mathbf{F}_h^n, \mathbf{D}_h),&\\ \forall(\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h.&\end{aligned} \tag{3.4b}$$

4. The main result

In this section we state the main result on error estimates with the optimal convergence order of scheme (3.4), which is proved in the next section.

We use c, c_w, c_s, $c_{w,s}$, c_ν, c_ε and $c_{\nu,\varepsilon}$ to represent generic positive constants independent of the discretization parameters h and Δt, the subscripts imply the dependence of the constants, and the subscripts " w " and " s "

in c_w, c_s and $c_{w,s}$ mean the given velocity $\mathbf{w}$ and the solution $(\mathbf{u}, p, \mathbf{C})$ of (2.2), respectively. For instance, the constant $c_{w,s}$ is dependent on $\mathbf{w}$ and $(\mathbf{u}, p, \mathbf{C})$ and independent of ν and ε, and the constant c has no dependence on $\mathbf{w}$, $(\mathbf{u}, p, \mathbf{C})$, ν nor ε. The symbol "$\prime$ (prime)" is sometimes used in order to distinguish two constants, *e.g.*, c_s and c_s', from each other.

We use the following notation for the norms and seminorms, $\|\cdot\|_V = \|\cdot\|_{V_h} = \|\cdot\|_1$, $\|\cdot\|_Q = \|\cdot\|_{Q_h} = \|\cdot\|_0$,

$$\|(\mathbf{u}, \mathbf{C})\|_{Z^2(t_0,t_1)} = \left\{\|\mathbf{u}\|^2_{Z^2(t_0,t_1)} + \|\mathbf{C}\|^2_{Z^2(t_0,t_1)}\right\}^{1/2},$$

$$\|\mathbf{u}\|_{\ell^\infty(X)} = \max_{n=0,\ldots,N_T} \|\mathbf{u}^n\|_X, \qquad \|\mathbf{u}\|_{\ell^2_m(X)} = \left\{\Delta t \sum_{n=1}^{m} \|\mathbf{u}^n\|^2_X\right\}^{1/2}, \qquad \|\mathbf{u}\|_{\ell^2(X)} = \|\mathbf{u}\|_{\ell^2_{N_T}(X)},$$

$$|p|_h = \left\{\sum_{K\in\mathcal{T}_h} h_K^2 (\nabla p, \nabla p)_K\right\}^{1/2}, \qquad |p|_{\ell^2_m(|.|_h)} = \left\{\Delta t \sum_{n=1}^{m} |p^n|^2_h\right\}^{1/2}, \qquad |p|_{\ell^2(|.|_h)} = |p|_{\ell^2_{N_T}(|.|_h)},$$

for $m \in \{1, \cdots, N_T\}$ and $X = L^\infty(\Omega)$, $L^2(\Omega)$ and $H^1(\Omega)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by $\overline{D}_{\Delta t}\mathbf{u}^n = (\mathbf{u}^n - \mathbf{u}^{n-1})/\Delta t$.

The existence and uniqueness of the solution of scheme (3.4) are ensured by the following proposition, which is also proved in the next section.

Proposition 4.1 (existence and uniqueness)**.** *Suppose Hypothesis* 2.2 *holds. Then, for any h and Δt satisfying* (3.2) *there exists a unique solution $(\mathbf{u}_h, p_h, \mathbf{C}_h) \subset V_h \times Q_h \times W_h$ of scheme* (3.4)*.*

We state the main results after preparing a projection and a hypothesis.

Definition 4.2 (Stokes–Poisson projection)**.** For $(\mathbf{u}, p, \mathbf{C}) \in V \times Q \times W$ we define the Stokes–Poisson projection $(\hat{\mathbf{u}}_h, \hat{p}_h, \hat{\mathbf{C}}_h) \in V_h \times Q_h \times W_h$ of $(\mathbf{u}, p, \mathbf{C})$ by

$$\mathcal{A}_h\left((\hat{\mathbf{u}}_h, \hat{p}_h), (\mathbf{v}_h, q_h)\right) + a_c(\hat{\mathbf{C}}_h, \mathbf{D}_h) + (\hat{\mathbf{C}}_h, \mathbf{D}_h) = \mathcal{A}\left((\mathbf{u}, p), (\mathbf{v}_h, q_h)\right) + a_c(\mathbf{C}, \mathbf{D}_h) + (\mathbf{C}, \mathbf{D}_h),$$
$$\forall (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h. \tag{4.1}$$

The Stokes–Poisson projection derives an operator $\Pi_h^{\rm SP} : V \times Q \times W \to V_h \times Q_h \times W_h$ defined by $\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C}) = (\hat{\mathbf{u}}_h, \hat{p}_h, \hat{\mathbf{C}}_h)$. We denote the i-th component of $\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})$ by $[\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})]_i$ for $i = 1, 2, 3$ and the pair of the first and third components $(\hat{\mathbf{u}}_h, \hat{\mathbf{C}}_h) = ([\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})]_1, [\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})]_3)$ by $[\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})]_{1,3}$ simply.

Remark 4.3. Identity (4.1) can be decoupled into the Stokes projection and the Poisson projection. For the simplicity of the notation we use (4.1) in the sequel. Since the Neumann boundary condition (2.1d) is imposed on $\mathbf{C}$, we use the Poisson projection corresponding to the operator $-\Delta + I$ for the unique solvability.

Hypothesis 4.4. *The solution $(\mathbf{u}, p, \mathbf{C})$ of* (2.2) *satisfies $\mathbf{u} \in Z^2(0,T)^d \cap H^1(0,T; V \cap H^2(\Omega)^d) \cap C([0,T]; W^{1,\infty}(\Omega)^d)$, $p \in H^1(0,T; Q \cap H^1(\Omega))$ and $\mathbf{C} \in Z^2(0,T)^{d\times d} \cap H^1(0,T; W \cap H^2(\Omega)^{d\times d})$.*

Remark 4.5. Let us note that we assume a higher regularity of the exact solution than that of the weak solution. Such regularity is usually assumed in discussing the convergence rate of numerical solutions of partial differential equations. We remark that our recent theoretical result [30] shows that both velocity and conformation tensor belong to $L^\infty(H^2)$ for the fully nonlinear Peterlin model with $\varepsilon > 0$. The result holds also for the Oseen-type Peterlin model with $\varepsilon > 0$.

We now impose the conditions

$$(\mathbf{u}_h^0, \mathbf{C}_h^0) = [\Pi_h^{\rm SP}(\mathbf{u}^0, 0, \mathbf{C}^0)]_{1,3}, \quad (\mathbf{f}_h, \mathbf{F}_h) = (\mathbf{f}, \mathbf{F}). \tag{4.2}$$

Remark 4.6. For the choice of $(\mathbf{u}_h^0, \mathbf{C}_h^0)$ we employ the Stokes–Poisson projection of $(\mathbf{u}^0, 0, \mathbf{C}^0)$ by (4.1) in (4.2), since the initial condition for the pressure is not given in (2.1). This choice does not influence the convergence order in our results below.

Theorem 4.7 (error estimates I)**.** *Suppose Hypotheses 2.2 and 4.4 hold. Then, there exist positive constants h_0, c_0 and $c_\dagger$ such that, for any pair $(h, \Delta t)$ satisfying*

$$h \in (0, h_0], \quad \Delta t \le \begin{cases} c_0(1+|\log h|)^{-1/2} & (d=2), \\ c_0 h^{1/2} & (d=3), \end{cases} \tag{4.3}$$

the solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (3.4) with (4.2) is estimated as follows.

$$\|\mathbf{C}_h\|_{\ell^\infty(L^\infty)} \le \|\mathbf{C}\|_{C(L^\infty)} + 1, \tag{4.4}$$

$$\|\mathbf{u}_h - \mathbf{u}\|_{\ell^\infty(L^2)}, \ \|\mathbf{u}_h - \mathbf{u}\|_{\ell^2(H^1)}, \ |p_h - p|_{\ell^2(|\cdot|_h)}, \ \|\mathbf{C}_h - \mathbf{C}\|_{\ell^\infty(H^1)}, \ \left\|\overline{D}_{\Delta t}\mathbf{C}_h - \frac{\partial \mathbf{C}}{\partial t}\right\|_{\ell^2(L^2)} \le c_\dagger(\Delta t + h). \tag{4.5}$$

Theorem 4.8 (error estimates II)**.** *Suppose Hypotheses 2.2 and 4.4 hold. Let h_0 and c_0 be the constants stated in Theorem 4.7. Then, there exists a positive constant $c_\ddagger$ such that, for any pair $(h, \Delta t)$ with (4.3) the solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (3.4) with (4.2) satisfies the estimates,*

$$\left\|\overline{D}_{\Delta t}\mathbf{u}_h - \frac{\partial \mathbf{u}}{\partial t}\right\|_{\ell^2(L^2)}, \quad \|p_h - p\|_{\ell^2(L^2)} \le c_\ddagger(\Delta t + h). \tag{4.6}$$

Remark 4.9.

(i) Condition (4.3) is mild in comparison with, *e.g.*, the CFL condition of the form $\|\mathbf{w}\|_{C(L^\infty)}\Delta t \le ch$. We can take $\Delta t = ch^\alpha$ for any $\alpha > 0$ $(d=2)$ or $\alpha \ge 1/2$ $(d=3)$.

(ii) Condition (4.3) is needed to deal with the nonlinearity of the model or, more precisely, to get the boundedness of $\|\mathbf{C}_h^n\|_{0,\infty}$ by using the inverse inequality (5.2), *cf.* the estimate (5.23) with (5.17a). In fact, the stabilized Lagrange–Galerkin scheme for the Oseen equations is stable under only (3.3), *cf.* [38].

5. Proofs

In what follows we prove Proposition 4.1 and Theorems 4.7 and 4.8.

5.1. Preliminaries

Let us list lemmas employed directly in the proofs below. In the lemmas, α_i, $i = 1, \ldots, 4$, are numerical constants independent of h, Δt, ν and ε.

Lemma 5.1 [33]**.** *Let Ω be a bounded domain with a* Lipschitz-*continuous boundary. Then, the following inequalities hold:*

$$\|\mathrm{D}(\mathbf{v})\|_0 \le \|\mathbf{v}\|_1 \le \alpha_1\|\mathrm{D}(\mathbf{v})\|_0, \qquad \forall \mathbf{v} \in H_0^1(\Omega)^d.$$

Let $\Pi_h : C(\bar{\Omega}) \to M_h$ be the Lagrange interpolation operator. The operators defined on $C(\bar{\Omega})^d$ and $C(\bar{\Omega})^{d\times d}$ are also denoted by the same symbol Π_h. We introduce the function

$$D(h) = \begin{cases} (1+|\log h|)^{1/2} & (d=2), \\ h^{-1/2} & (d=3), \end{cases} \tag{5.1}$$

which is used in the sequel.

Lemma 5.2 [11, 14]**.** *The following inequalities hold:*

$$\begin{aligned} \|\Pi_h \mathbf{D}\|_{0,\infty} &\le \|\mathbf{D}\|_{0,\infty}\,, && \forall \mathbf{D} \in C(\bar{\Omega})^{d\times d}, \\ \|\Pi_h \mathbf{D} - \mathbf{D}\|_1 &\le \alpha_{20} h \,\|\mathbf{D}\|_2\,, && \forall \mathbf{D} \in H^2(\Omega)^{d\times d}, \\ \|\mathbf{D}_h\|_{0,\infty} &\le \alpha_{21} D(h)\,\|\mathbf{D}_h\|_1\,, && \forall \mathbf{D}_h \in W_h. \end{aligned} \tag{5.2}$$

The next lemma is obtained by combining the error estimates for the Stokes and the Poisson problems, see, *e.g.*, [12, 14, 23] for the proof.

Lemma 5.3.

(i) *The following inequality holds:*

$$\inf_{(u_h,p_h)\in V_h\times Q_h} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\mathcal{A}_h((u_h,p_h),(v_h,q_h))}{\|(u_h,p_h)\|_{V\times Q}\|(v_h,q_h)\|_{V\times Q}} \ge \nu\alpha_{30}.$$

(ii) *Assume* $(\mathbf{u},p,\mathbf{C}) \in (V\cap H^2(\Omega)^d)\times(Q\cap H^1(\Omega))\times(W\cap H^2(\Omega)^{d\times d})$. *Let* $(\hat{\mathbf{u}}_h,\hat{p}_h,\hat{\mathbf{C}}_h)\in V_h\times Q_h\times W_h$ *be the* Stokes–Poisson *projection of* $(\mathbf{u},p,\mathbf{C})$ *defined by* (4.1). *Then, the following inequalities hold:*

$$\|\hat{\mathbf{u}}_h - \mathbf{u}\|_1\,,\ \|\hat{p}_h - p\|_0\,,\ |\hat{p}_h - p|_h \le \frac{\alpha_{31}}{\nu} h\,\|(\mathbf{u},p)\|_{H^2\times H^1}\,, \qquad \|\hat{\mathbf{C}}_h - \mathbf{C}\|_1 \le \alpha_{32} h\|\mathbf{C}\|_2.$$

Remark 5.4. Let us note that the first part of error estimates in (ii) is based on the generalized inf-sup condition in (i) that is satisfied by the bilinear form $\mathcal{A}_h$ defined above and the pair of the discrete function spaces V_h and Q_h, where the ν-dependence is obtained by a simple modification of the analysis in, *e.g.*, [23] after taking into account the diffusion constant.

Remark 5.5. As pointed out in [8], there are basically three possible approaches to obtain stable and convergent numerical methods for viscoelastic fluid flow problems. Firstly, the usual Galerkin methods using finite element spaces satisfying the inf-sup condition, *e.g.*, [3, 22, 34]. Secondly, the equal-order approximations for the velocity, pressure and stress with stabilization terms added to the usual weak formulation, see for instance [19]. And finally, the elastic viscous split stress (EVSS) method, *e.g.*, [18, 24, 45], in which the stress is split into two parts, the elastic and the viscous part. Scheme (3.4) is classified into the second approach. Theorems 4.7 and 4.8 imply that our method for the Peterlin viscoelastic model is indeed stable and convergent.

Lemma 5.6 [38, 50]**.** *Under Hypothesis 2.2 and condition (3.3) the following inequalities hold for any* $n \in \{0,\ldots,N_T\}$*:*

$$\begin{aligned} \|\mathbf{g}\circ X_1^n\|_0 &\le (1+\alpha_{40}|\mathbf{w}^n|_{1,\infty}\Delta t)\,\|\mathbf{g}\|_0\,, && \forall \mathbf{g} \in L^2(\Omega)^s, \\ \|\mathbf{g} - \mathbf{g}\circ X_1^n\|_0 &\le \alpha_{41}\|\mathbf{w}^n\|_{0,\infty}\Delta t\,|\mathbf{g}|_1, && \forall \mathbf{g} \in H^1(\Omega)^s, \end{aligned}$$

where $s = d$ *or* $d\times d$.

Proof. We prove only the first estimate, and see the proof of ([38], Lem. 6) for the second. Let $n \in \{0,\ldots,N_T\}$ be fixed arbitrarily. By changing the variable from x to $y = X_1^n(x)$, we have

$$\|\mathbf{g}\circ X_1^n\|_0^2 = \int_\Omega \mathbf{g}\left(X_1^n(x)\right)^2\,\mathrm{d}x = \int_\Omega \mathbf{g}(y)^2\frac{1}{J^n}\,\mathrm{d}y \le (1+\alpha_{40}|\mathbf{w}^n|_{1,\infty}\Delta t)^2\,\|\mathbf{g}\|_0^2,$$

where J^n is the Jacobian $\det(\partial y/\partial x)$. Here we have used the estimate,

$$\frac{1}{J^n} \le \frac{1}{1-|1-J^n|} \le 1 + 2|1-J^n| \le 1 + 2\alpha_{40}|\mathbf{w}^n|_{1,\infty}\Delta t \le (1+\alpha_{40}|\mathbf{w}^n|_{1,\infty}\Delta t)^2,$$

which is derived from Proposition 3.1-(ii) and $1/(1-s) \le 1+2s$ $(s\in[0,1/2])$. □

We use the following simplified version of the discrete Gronwall inequality ([25], Lem. 5.1).

Lemma 5.7. *Let α and β be non-negative numbers, Δt a positive number, and $\{x^n\}_{n\geq 0}$ and $\{y^n\}_{n\geq 1}$ non-negative sequences. Suppose the inequality*

$$x^m + \Delta t \sum_{n=1}^{m} y^n \leq \alpha \Delta t \sum_{n=0}^{m-1} x^n + \beta, \quad \forall m \geq 0,$$

holds. Then, it holds that

$$x^m + \Delta t \sum_{n=1}^{m} y^n \leq (1+\alpha\Delta t)^m \beta, \quad \forall m \geq 0.$$

5.2. Proof of Proposition 4.1

For each time step n scheme (3.4) can be rewritten as

$$\left(\frac{\mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h\right) + \nu a_u(\mathbf{u}_h^n, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^n) + ((\operatorname{tr}\mathbf{C}_h^n)\mathbf{C}_h^{n-1}, \nabla \mathbf{v}_h) = (\mathbf{g}_h^n, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in V_h, \quad (5.3a)$$

$$b(\mathbf{u}_h^n, q_h) - \mathcal{S}_h(p_h^n, q_h) = 0, \qquad \forall q_h \in Q_h, \quad (5.3b)$$

$$\left(\frac{\mathbf{C}_h^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon a_c(\mathbf{C}_h^n, \mathbf{D}_h) - 2\left((\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1}, \mathbf{D}_h\right) + \left((\operatorname{tr}\mathbf{C}_h^{n-1})^2 \mathbf{C}_h^n, \mathbf{D}_h\right) = (\mathbf{G}_h^n, \mathbf{D}_h), \qquad \forall \mathbf{D}_h \in W_h, \quad (5.3c)$$

where $\mathbf{g}_h^n = (1/\Delta t)(\mathbf{u}_h^{n-1} \circ X_1^n) + \mathbf{f}_h^n$ and $\mathbf{G}_h^n = (1/\Delta t)(\mathbf{C}_h^{n-1} \circ X_1^n) + (\operatorname{tr}\mathbf{C}_h^{n-1})\mathbf{I} + \mathbf{F}_h^n$. Selecting specific bases of V_h, Q_h and W_h and expanding $\mathbf{u}_h^n$, p_h^n and $\mathbf{C}_h^n$ in terms of the associated basis functions, we can derive the system of linear equations from (5.3). The existence and uniqueness of the solution is equivalent to the invertibility of the coefficient matrix of the system, which is obtained by proving $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n) = (\mathbf{0}, 0, \mathbf{0})$ below when $(\mathbf{g}_h^n, \mathbf{G}_h^n) = (\mathbf{0}, \mathbf{0})$. Substituting $(\mathbf{u}_h^n, -p_h^n, \frac{1}{2}(\operatorname{tr}\mathbf{C}_h^n)\mathbf{I})$ into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in (5.3) and adding (5.3b) to (5.3a), we have

$$\frac{1}{\Delta t}\|\mathbf{u}_h^n\|_0^2 + 2\nu\|\mathrm{D}(\mathbf{u}_h^n)\|_0^2 + \delta_0|p_h^n|_h^2 + \left((\operatorname{tr}\mathbf{C}_h^n)\mathbf{C}_h^{n-1}, \nabla \mathbf{u}_h^n\right) = 0, \qquad (5.4a)$$

$$\frac{1}{2\Delta t}\|\operatorname{tr}\mathbf{C}_h^n\|_0^2 + \frac{\varepsilon}{2}\|\nabla \operatorname{tr}\mathbf{C}_h^n\|_0^2 - \left(\operatorname{tr}[(\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1}], \operatorname{tr}\mathbf{C}_h^n\right) + \frac{1}{2}\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\operatorname{tr}\mathbf{C}_h^n\right\|_0^2 = 0. \qquad (5.4b)$$

By the identity

$$\left((\operatorname{tr}\mathbf{C}_h^n)\mathbf{C}_h^{n-1}, \nabla \mathbf{u}_h^n\right) - \left(\operatorname{tr}[(\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1}], \operatorname{tr}\mathbf{C}_h^n\right) = 0,$$

the sum of (5.4a) and (5.4b) yields

$$\frac{1}{\Delta t}\|\mathbf{u}_h^n\|_0^2 + 2\nu\|\mathrm{D}(\mathbf{u}_h^n)\|_0^2 + \delta_0|p_h^n|_h^2 + \frac{1}{2\Delta t}\|\operatorname{tr}\mathbf{C}_h^n\|_0^2 + \frac{\varepsilon}{2}\|\nabla \operatorname{tr}\mathbf{C}_h^n\|_0^2 + \frac{1}{2}\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\operatorname{tr}\mathbf{C}_h^n\right\|_0^2 = 0.$$

Hence, we have $(\mathbf{u}_h^n, p_h^n) = (\mathbf{0}, 0)$. Substituting $\mathbf{C}_h^n$ into $\mathbf{D}_h$ in (5.3c) and noting that $\mathbf{u}_h^n = \mathbf{0}$, we obtain

$$\frac{1}{\Delta t}\|\mathbf{C}_h^n\|_0^2 + \varepsilon\|\nabla \mathbf{C}_h^n\|_0^2 + \left\|(\operatorname{tr}\mathbf{C}_h^{n-1})\mathbf{C}_h^n\right\|_0^2 = 0,$$

which implies $\mathbf{C}_h^n = 0$. Thus, we get $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n) = (\mathbf{0}, 0, \mathbf{0})$, which completes the proof.

5.3. An estimate at each time step

In this subsection we present a proposition which is employed in the proof of Theorem 4.7.
Let $(\hat{\mathbf{u}}_h, \hat{p}_h, \hat{\mathbf{C}}_h)(t) = \Pi_h^{\mathrm{SP}}(\mathbf{u}, p, \mathbf{C})(t) \in V_h \times Q_h \times W_h$ for $t \in [0, T]$ and let

$$\mathbf{e}_h^n = \mathbf{u}_h^n - \hat{\mathbf{u}}_h^n, \qquad \epsilon_h^n = p_h^n - \hat{p}_h^n, \qquad \mathbf{E}_h^n = \mathbf{C}_h^n - \hat{\mathbf{C}}_h^n, \qquad \boldsymbol{\eta}(t) = (\mathbf{u} - \hat{\mathbf{u}}_h)(t), \qquad \boldsymbol{\Xi}(t) = (\mathbf{C} - \hat{\mathbf{C}}_h)(t).$$

Then, from (3.4), (4.1) and (2.2), we have for $n \geq 1$

$$\left(\frac{\mathbf{e}_h^n - \mathbf{e}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\big((\mathbf{e}_h^n, \epsilon_h^n), (\mathbf{v}_h, q_h)\big) = \langle \mathbf{r}_h^n, \mathbf{v}_h\rangle, \qquad \forall(\mathbf{v}_h, q_h)\in V_h\times Q_h, \tag{5.5a}$$

$$\left(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \varepsilon a_c(\mathbf{E}_h^n, \mathbf{D}_h) = \langle \mathbf{R}_h^n, \mathbf{D}_h\rangle, \qquad \forall \mathbf{D}_h \in W_h, \tag{5.5b}$$

where

$$\begin{aligned}
\mathbf{r}_h^n &= \sum_{i=1}^{4} \mathbf{r}_{hi}^n \in V_h', \qquad \mathbf{R}_h^n = \sum_{i=1}^{11} \mathbf{R}_{hi}^n \in W_h', \\
\langle \mathbf{r}_{h1}^n, \mathbf{v}_h\rangle &= \left(\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t} - \frac{\mathbf{u}^n - \mathbf{u}^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\right), \\
\langle \mathbf{r}_{h2}^n, \mathbf{v}_h\rangle &= \frac{1}{\Delta t}\left(\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}\circ X_1^n, \mathbf{v}_h\right), \\
\langle \mathbf{r}_{h3}^n, \mathbf{v}_h\rangle &= \big((\mathrm{tr}\,\mathbf{C}^n)(\mathbf{C}^n - \mathbf{C}^{n-1} + \boldsymbol{\Xi}^{n-1} - \mathbf{E}_h^{n-1}), \nabla\mathbf{v}_h\big), \\
\langle \mathbf{r}_{h4}^n, \mathbf{v}_h\rangle &= \left([\mathrm{tr}\,(\boldsymbol{\Xi}^n - \mathbf{E}_h^n)]\mathbf{C}_h^{n-1}, \nabla\mathbf{v}_h\right), \\
\langle \mathbf{R}_{h1}^n, \mathbf{D}_h\rangle &= \left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t} - \frac{\mathbf{C}^n - \mathbf{C}^{n-1}\circ X_1^n}{\Delta t}, \mathbf{D}_h\right), \\
\langle \mathbf{R}_{h2}^n, \mathbf{D}_h\rangle &= \frac{1}{\Delta t}\left(\boldsymbol{\Xi}^n - \boldsymbol{\Xi}^{n-1}\circ X_1^n, \mathbf{D}_h\right), \\
\langle \mathbf{R}_{h3}^n, \mathbf{D}_h\rangle &= -\varepsilon(\boldsymbol{\Xi}^n, \mathbf{D}_h), \\
\langle \mathbf{R}_{h4}^n, \mathbf{D}_h\rangle &= 2\left((\nabla\mathbf{e}_h^n)\mathbf{C}_h^{n-1}, \mathbf{D}_h\right), \\
\langle \mathbf{R}_{h5}^n, \mathbf{D}_h\rangle &= -2\left((\nabla\boldsymbol{\eta}^n)\mathbf{C}_h^{n-1}, \mathbf{D}_h\right), \\
\langle \mathbf{R}_{h6}^n, \mathbf{D}_h\rangle &= -2\left((\nabla\mathbf{u}^n)(\mathbf{C}^n - \mathbf{C}^{n-1} + \boldsymbol{\Xi}^{n-1} - \mathbf{E}_h^{n-1}), \mathbf{D}_h\right), \\
\langle \mathbf{R}_{h7}^n, \mathbf{D}_h\rangle &= \left((\mathrm{tr}\,\mathbf{C}_h^{n-1})^2(\boldsymbol{\Xi}^n - \mathbf{E}_h^n), \mathbf{D}_h\right), \\
\langle \mathbf{R}_{h8}^n, \mathbf{D}_h\rangle &= -\big([\mathrm{tr}\,(\mathbf{C}_h^{n-1} + \hat{\mathbf{C}}_h^{n-1})](\mathrm{tr}\,\mathbf{E}_h^{n-1})\mathbf{C}^n, \mathbf{D}_h\big), \\
\langle \mathbf{R}_{h9}^n, \mathbf{D}_h\rangle &= \big([\mathrm{tr}\,(\mathbf{C}^{n-1} + \hat{\mathbf{C}}_h^{n-1})](\mathrm{tr}\,\boldsymbol{\Xi}^{n-1})\mathbf{C}^n, \mathbf{D}_h\big), \\
\langle \mathbf{R}_{h10}^n, \mathbf{D}_h\rangle &= \left([\mathrm{tr}\,(\mathbf{C}^n + \mathbf{C}^{n-1})][\mathrm{tr}\,(\mathbf{C}^n - \mathbf{C}^{n-1})]\mathbf{C}^n, \mathbf{D}_h\right), \\
\langle \mathbf{R}_{h11}^n, \mathbf{D}_h\rangle &= -\left([\mathrm{tr}\,(\mathbf{C}^n - \mathbf{C}^{n-1} + \boldsymbol{\Xi}^{n-1} - \mathbf{E}_h^{n-1})]\mathbf{I}, \mathbf{D}_h\right).
\end{aligned} \tag{5.6}$$

We note that

$$(\mathbf{e}_h^0, \mathbf{E}_h^0) = (\mathbf{u}_h^0, \mathbf{C}_h^0) - (\hat{\mathbf{u}}_h^0, \hat{\mathbf{C}}_h^0) = [\varPi_h^{\mathrm{SP}}(0, -p^0, 0)]_{1,3}. \tag{5.7}$$

In the following we use the constants α_i defined in Lemma i, $i = 1, \ldots, 4$, and the notation $\mathbb{H}^2 = H^2(\varOmega)^2 \times H^1(\varOmega) \times H^2(\varOmega)^{2\times 2}$.

Proposition 5.8. *Suppose that Hypotheses 2.2 and 4.4 hold and assume (3.3). Let $M_0 \geq 1$ be a positive constant independent of h and Δt. Let $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ be the solution of scheme (3.4) with (4.2). Suppose that for an $n \in \{1, \ldots, N_T\}$*

$$\|\mathbf{C}_h^{n-1}\|_{0,\infty} \leq M_0. \tag{5.8}$$

Then, there exist positive constants c_1 and c_2, dependent on M_0, ν and ε but independent of h and Δt, such that

$$\begin{aligned}
&\overline{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2+\frac{1}{2}\|\mathbf{E}_h^n\|_0^2+\frac{\nu\varepsilon}{64\alpha_1^2d^2M_0^2}|\mathbf{E}_h^n|_1^2\right)+\frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^n\|_1^2+\delta_0|\epsilon_h^n|_h^2+\frac{\nu}{64\alpha_1^2d^2M_0^2}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2\\
&\le c_1\left(\frac{1}{2}\|\mathbf{e}_h^{n-1}\|_0^2+\frac{1}{2}\|\mathbf{E}_h^{n-1}\|_0^2+\frac{\nu\varepsilon}{64\alpha_1^2d^2M_0^2}|\mathbf{E}_h^{n-1}|_1^2+\frac{1}{2}\|\mathbf{E}_h^n\|_0^2\right)\\
&\quad+c_2\left[\Delta t\|(\mathbf{u},\mathbf{C})\|_{Z^2(t^{n-1},t^n)}^2+h^2\left(\frac{1}{\Delta t}\|(\mathbf{u},p,\mathbf{C})\|_{H^1(t^{n-1},t^n;\mathbb{H}^2)}^2+1\right)\right].
\end{aligned}\tag{5.9}$$

For the proof we use the next lemma, which is proved in Appendix A.1.

Lemma 5.9. *Suppose Hypotheses 2.2 and 4.4 hold. Let $n\in\{1,\dots,N_T\}$ be any fixed number. Then, under condition (3.3) it holds that*

$$\|\mathbf{r}_{h1}^n\|_0\le c_w\sqrt{\Delta t}\|\mathbf{u}\|_{Z^2(t^{n-1},t^n)},\tag{5.10a}$$
$$\|\mathbf{r}_{h2}^n\|_0\le \frac{c_wh}{\nu\sqrt{\Delta t}}\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)},\tag{5.10b}$$
$$\|\mathbf{r}_{h3}^n\|_{-1}\le c_s\big(\|\mathbf{E}_h^{n-1}\|_0+\sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)}+h\big),\tag{5.10c}$$
$$\|\mathbf{r}_{h4}^n\|_{-1}\le c_s\|\mathbf{C}_h^{n-1}\|_{0,\infty}\left(\|\mathbf{E}_h^n\|_0+h\right),\tag{5.10d}$$
$$\|\mathbf{R}_{h1}^n\|_0\le c_w\sqrt{\Delta t}\|\mathbf{C}\|_{Z^2(t^{n-1},t^n)},\tag{5.10e}$$
$$\|\mathbf{R}_{h2}^n\|_0\le \frac{c_wh}{\sqrt{\Delta t}}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^2)},\tag{5.10f}$$
$$\|\mathbf{R}_{h3}^n\|_0\le c_sh,\tag{5.10g}$$
$$\|\mathbf{R}_{h4}^n\|_0\le 2d\|\mathbf{C}_h^{n-1}\|_{0,\infty}\|\mathbf{e}_h^n\|_1,\tag{5.10h}$$
$$\|\mathbf{R}_{h5}^n\|_0\le c_s\|\mathbf{C}_h^{n-1}\|_{0,\infty}h,\tag{5.10i}$$
$$\|\mathbf{R}_{h6}^n\|_0\le c_s\big(\|\mathbf{E}_h^{n-1}\|_0+\sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)}+h\big),\tag{5.10j}$$
$$\|\mathbf{R}_{h7}^n\|_0\le c_s\|\mathbf{C}_h^{n-1}\|_{0,\infty}^2(\|\mathbf{E}_h^n\|_0+h),\tag{5.10k}$$
$$\|\mathbf{R}_{h8}^n\|_0\le c_s(\|\mathbf{C}_h^{n-1}\|_{0,\infty}+1)\|\mathbf{E}_h^{n-1}\|_0,\tag{5.10l}$$
$$\|\mathbf{R}_{h9}^n\|_0\le c_sh,\tag{5.10m}$$
$$\|\mathbf{R}_{h10}^n\|_0\le c_s\sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)},\tag{5.10n}$$
$$\|\mathbf{R}_{h11}^n\|_0\le c_s(\|\mathbf{E}_h^{n-1}\|_0+\sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)}+h).\tag{5.10o}$$

Proof of Proposition 5.8. Substituting $(\mathbf{e}_h^n,-\epsilon_h^n)$ into $(\mathbf{v}_h,q_h)$ in (5.5a) and noting that

$$\begin{aligned}
\left(\frac{\mathbf{e}_h^n-\mathbf{e}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{e}_h^n\right)&\ge\frac{1}{2\Delta t}\big(\|\mathbf{e}_h^n\|_0^2-\|\mathbf{e}_h^{n-1}\circ X_1^n\|_0^2\big)\ge\frac{1}{2\Delta t}\Big[\|\mathbf{e}_h^n\|_0^2-(1+\alpha_{40}|\mathbf{w}^n|_{1,\infty}\Delta t)^2\|\mathbf{e}_h^{n-1}\|_0^2\Big]\\
&\ge\overline{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2\right)-c_w\|\mathbf{e}_h^{n-1}\|_0^2,\\
\mathcal{A}_h\big((\mathbf{e}_h^n,\epsilon_h^n),(\mathbf{e}_h^n,-\epsilon_h^n)\big)&\ge\frac{2\nu}{\alpha_1^2}\|\mathbf{e}_h^n\|_1^2+\delta_0|p_h^n|_h^2,\\
\langle\mathbf{r}_h^n,\mathbf{e}_h^n\rangle&\le\|\mathbf{r}_h^n\|_{-1}\|\mathbf{e}_h^n\|_1\le\frac{\alpha_1^2}{4\nu}\|\mathbf{r}_h^n\|_{-1}^2+\frac{\nu}{\alpha_1^2}\|\mathbf{e}_h^n\|_1^2,
\end{aligned}$$

we have

$$\overline{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2\right)+\frac{\nu}{\alpha_1^2}\|\mathbf{e}_h^n\|_1^2+\delta_0|\epsilon_h^n|_h^2\le\frac{\alpha_1^2}{4\nu}\|\mathbf{r}_h^n\|_{-1}^2+c_w\|\mathbf{e}_h^{n-1}\|_0^2. \tag{5.11}$$

Similarly, substituting $\mathbf{E}_h^n$ and $\overline{D}_{\Delta t}\mathbf{E}_h^n$ into $\mathbf{D}_h$ in (5.5b) and noting that

$$\begin{aligned}
\left(\frac{\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{E}_h^n\right)&\ge\overline{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{E}_h^n\|_0^2\right)-c_w\|\mathbf{E}_h^{n-1}\|_0^2,\\
\varepsilon a_c(\mathbf{E}_h^n,\mathbf{E}_h^n)&=\varepsilon|\mathbf{E}_h^n|_1^2\ge 0,\\
\langle\mathbf{R}_h^n,\mathbf{E}_h^n\rangle&\le\|\mathbf{R}_h^n\|_0\|\mathbf{E}_h^n\|_0\le\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\|\mathbf{R}_{hi}^n\|_0\|\mathbf{E}_h^n\|_0+\|\mathbf{R}_{h4}^n\|_0\|\mathbf{E}_h^n\|_0\\
&\le\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\left(\frac{5}{2}\|\mathbf{R}_{hi}^n\|_0^2+\frac{1}{10}\|\mathbf{E}_h^n\|_0^2\right)+2dM_0\|\mathbf{e}_h^n\|_1\|\mathbf{E}_h^n\|_0\quad(\text{by (5.10h),(5.8)})\\
&\le\frac{5}{2}\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\|\mathbf{R}_{hi}^n\|_0^2+\|\mathbf{E}_h^n\|_0^2+\frac{\nu}{4\alpha_1^2}\|\mathbf{e}_h^n\|_1^2+\frac{4\alpha_1^2d^2M_0^2}{\nu}\|\mathbf{E}_h^n\|_0^2\\
&=\frac{5}{2}\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\|\mathbf{R}_{hi}^n\|_0^2+\left(1+\frac{4\alpha_1^2d^2M_0^2}{\nu}\right)\|\mathbf{E}_h^n\|_0^2+\frac{\nu}{4\alpha_1^2}\|\mathbf{e}_h^n\|_1^2,\\
\left(\frac{\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t},\overline{D}_{\Delta t}\mathbf{E}_h^n\right)&=\left(\overline{D}_{\Delta t}\mathbf{E}_h^n+\frac{\mathbf{E}_h^{n-1}-\mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t},\overline{D}_{\Delta t}\mathbf{E}_h^n\right)\\
&\ge\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2-\alpha_{41}\|\mathbf{w}^n\|_{0,\infty}|\mathbf{E}_h^{n-1}|_1\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0,\\
&\ge\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2-c_w|\mathbf{E}_h^{n-1}|_1^2-\frac{1}{4}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2,\\
&=\frac{3}{4}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2-c_w|\mathbf{E}_h^{n-1}|_1^2,\\
\varepsilon a_c\left(\mathbf{E}_h^n,\overline{D}_{\Delta t}\mathbf{E}_h^n\right)&\ge\overline{D}_{\Delta t}\left(\frac{\varepsilon}{2}|\mathbf{E}_h^n|_1^2\right),\\
\langle\mathbf{R}_h^n,\overline{D}_{\Delta t}\mathbf{E}_h^n\rangle&\le\|\mathbf{R}_h^n\|_0\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0\le\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\|\mathbf{R}_{hi}^n\|_0\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0+\|\mathbf{R}_{h4}^n\|_0\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0\\
&\le\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\left(20\|\mathbf{R}_{hi}^n\|_0^2+\frac{1}{80}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2\right)+2dM_0\|\mathbf{e}_h^n\|_1\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0\\
&\qquad\qquad(\text{by (5.10h),(5.8)})\\
&\le 20\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\|\mathbf{R}_{hi}^n\|_0^2+\frac{1}{8}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2+8d^2M_0^2\|\mathbf{e}_h^n\|_1^2+\frac{1}{8}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2\\
&=20\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\|\mathbf{R}_{hi}^n\|_0^2+\frac{1}{4}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2+8d^2M_0^2\|\mathbf{e}_h^n\|_1^2,
\end{aligned}$$

we have the following two inequalities,

$$\overline{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{E}_h^n\|_0^2\right)\le\frac{5}{2}\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\|\mathbf{R}_{hi}^n\|_0^2+\left(1+\frac{4\alpha_1^2d^2M_0^2}{\nu}\right)\|\mathbf{E}_h^n\|_0^2+c_w\|\mathbf{E}_h^{n-1}\|_0^2+\frac{\nu}{4\alpha_1^2}\|\mathbf{e}_h^n\|_1^2, \tag{5.12a}$$

$$\overline{D}_{\Delta t}\left(\frac{\varepsilon}{2}|\mathbf{E}_h^n|_1^2\right)+\frac{1}{2}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2\le 20\sum_{i\in\{1,\dots,11\}\setminus\{4\}}\|\mathbf{R}_{hi}^n\|_0^2+c_w|\mathbf{E}_h^{n-1}|_1^2+8d^2M_0^2\|\mathbf{e}_h^n\|_1^2. \tag{5.12b}$$

Lemma 5.9, (5.6) and (5.8) imply that

$$\|\mathbf{r}_h^n\|_{-1}^2 \le c_{w,s}\big(M_0^2\|\mathbf{E}_h^n\|_0^2 + \|\mathbf{E}_h^{n-1}\|_0^2\big) + \frac{c'_{w,s}}{\nu}\left[\Delta t\|(\mathbf{u},\mathbf{C})\|_{Z^2(t^{n-1},t^n)}^2 + h^2\left(\frac{1}{\Delta t}\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}^2 + M_0^2 + 1\right)\right], \tag{5.13a}$$

$$\sum_{i\in\{1,\dots,11\}\setminus\{4\}} \|\mathbf{R}_{hi}^n\|_0^2 \le c_{w,s}\left[M_0^4\|\mathbf{E}_h^n\|_0^2 + (M_0^2+1)\|\mathbf{E}_h^{n-1}\|_0^2\right] + c'_{w,s}\left[\Delta t\|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}^2 + h^2\left(\frac{1}{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^2)}^2 + M_0^4 + M_0^2 + 1\right)\right]. \tag{5.13b}$$

Multiplying (5.12b) by $\nu/(32\alpha_1^2 d^2 M_0^2)$, adding it and (5.12a) to (5.11) and using (5.13), we get

$$\begin{aligned}
&\overline{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{1}{2}\|\mathbf{E}_h^n\|_0^2 + \frac{\nu\varepsilon}{64\alpha_1^2 d^2 M_0^2}|\mathbf{E}_h^n|_1^2\right) + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \delta_0|\epsilon_h^n|_h^2 + \frac{\nu}{64\alpha_1^2 d^2 M_0^2}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2\\
&\le p_1(M_0)\left(\frac{1}{2}\|\mathbf{e}_h^{n-1}\|_0^2 + \frac{1}{2}\|\mathbf{E}_h^{n-1}\|_0^2 + \frac{\nu\varepsilon}{64\alpha_1^2 d^2 M_0^2}|\mathbf{E}_h^{n-1}|_1^2 + \frac{1}{2}\|\mathbf{E}_h^n\|_0^2\right)\\
&\quad + p_2(M_0)\left[\Delta t\|(\mathbf{u},\mathbf{C})\|_{Z^2(t^{n-1},t^n)}^2 + h^2\left(\frac{1}{\Delta t}\|(\mathbf{u},p,\mathbf{C})\|_{H^1(t^{n-1},t^n;\mathbb{H}^2)}^2 + 1\right)\right],
\end{aligned}$$

where $p_1(\xi) = p_1(\xi;\nu,\varepsilon)$ and $p_2(\xi) = p_2(\xi;\nu)$ are polynomials in ξ defined by

$$\begin{aligned}
p_1:\quad & c_{w,s}\left[\frac{1}{\nu}(\xi^2+1) + \left(1+\frac{\nu}{\xi^2}\right)(\xi^4+\xi^2+1) + \left(1+\frac{\xi^2}{\nu}\right) + \frac{1}{\varepsilon}\right] \le \frac{c_{w,s}}{\nu\varepsilon}(\xi^4+4\xi^2+6) = p_1(\xi;\nu,\varepsilon), \\
p_2:\quad & c_{w,s}\left[\frac{1}{\nu^2}(\xi^2+1) + \left(1+\frac{\nu}{\xi^2}\right)(\xi^4+\xi^2+1)\right] \le \frac{c_{w,s}}{\nu^2}(\xi^4+3\xi^2+4) = p_2(\xi;\nu).
\end{aligned} \tag{5.14}$$

In the inequalities above the assumptions $\nu,\varepsilon \in (0,1]$ and $M_0 \ge 1$ have been employed. By taking $c_1 = p_1(M_0;\nu,\varepsilon)$ and $c_2 = p_2(M_0;\nu)$ we finally obtain (5.9). □

5.4. Proof of Theorem 4.7

We prove Theorem 4.7 through three steps, where the function $D(h)$ defined in (5.1) is often used.

Step 1. (Setting c_0 and h_0): From (4.2) and (5.7) we have

$$\|\mathbf{e}_h^0\|_0 \le \|\mathbf{u}_h^0 - \mathbf{u}^0\|_1 + \|\mathbf{u}^0 - \hat{\mathbf{u}}_h^0\|_1 \le 2\frac{\alpha_{31}}{\nu}h\|(u,p)^0\|_{H^2\times H^1} = \sqrt{2}c_I h \tag{5.15}$$

for $c_I = (\sqrt{2}\alpha_{31}/\nu)\|(u,p)^0\|_{H^2\times H^1}$. The constants c_1 and c_2 in Proposition 5.8 depend on M_0. Now, we take $M_0 = \|\mathbf{C}\|_{C(L^\infty)} + 1$. Then, c_1 and c_2 are fixed. Let c_3 and c_* be constants defined by

$$c_3 = \exp\left(\frac{3c_1T}{2}\right)\max\left\{\sqrt{c_2}\|(\mathbf{u},\mathbf{C})\|_{Z^2}, \sqrt{c_2}\big(\|(\mathbf{u},p,\mathbf{C})\|_{H^1(\mathbb{H}^2)} + \sqrt{T}\big) + c_I\right\}. \tag{5.16}$$

and $c_* = c_3\,(8\alpha_1 d M_0/\sqrt{\nu\varepsilon}\,)$. We can choose sufficiently small positive constants c_0 and h_0 such that

$$\alpha_{21}\left[c_*\{c_0 + h_0 D(h_0)\} + (\alpha_{20}+\alpha_{32})h_0 D(h_0)\|\mathbf{C}\|_{C(H^2)}\right] \le 1, \tag{5.17a}$$

$$(\Delta t \le)\quad \frac{c_0}{D(h_0)} \le \frac{1}{2c_1}, \tag{5.17b}$$

$$(\Delta t|\mathbf{w}|_{1,\infty} \le)\quad \frac{c_0|\mathbf{w}|_{1,\infty}}{D(h_0)} \le \frac{1}{4}, \tag{5.17c}$$

since $hD(h)$ and $1/D(h)$ tend to zero as h tends to zero.

Let $(h, \Delta t)$ be any pair satisfying (4.3). Since condition (3.2) is satisfied, Proposition 4.1 ensures the existence and uniqueness of the solution $(\mathbf{u}_h, p_h, \mathbf{C}_h) = \{(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h \times W_h$ of scheme (3.4) with (4.2).

Step 2. (Induction): By induction we show that the following property $\mathrm{P}(n)$ holds for $n \in \{0, \ldots, N_T\}$,

$$\mathrm{P}(n): \begin{cases} \text{(a) } \frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{1}{2}\|\mathbf{E}_h^n\|_0^2 + \frac{\nu\varepsilon}{64\alpha_1^2 d^2 M_0^2}|\mathbf{E}_h^n|_1^2 + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h\|_{\ell_n^2(H^1)}^2 + \delta_0|\epsilon_h|_{\ell_n^2(|\cdot|_h)}^2 + \frac{\nu}{64\alpha_1^2 d^2 M_0^2}\|\overline{D}_{\Delta t}\mathbf{E}_h\|_{\ell_n^2(L^2)}^2 \\ \quad \leq \exp(3c_1 n\Delta t)\Big[\frac{1}{2}\|\mathbf{e}_h^0\|_0^2 + \frac{1}{2}\|\mathbf{E}_h^0\|_0^2 + \frac{\nu\varepsilon}{64\alpha_1^2 d^2 M_0^2}|\mathbf{E}_h^0|_1^2 \\ \qquad + c_2\Big\{\Delta t^2\|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,t^n)}^2 + h^2\big(\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(0,t^n;\mathbb{H}^2)}^2 + n\Delta t\big)\Big\}\Big], \\ \text{(b) } \|\mathbf{C}_h^n\|_{0,\infty} \leq \|\mathbf{C}\|_{C(L^\infty)} + 1, \end{cases}$$

where $\|\mathbf{e}_h\|_{\ell_n^2(H^1)} = |\epsilon_h|_{\ell_n^2(|\cdot|_h)} = \|\overline{D}_{\Delta t}\mathbf{E}_h\|_{\ell_n^2(L^2)} = 0$ for $n = 0$.

$\mathrm{P}(n)$-(a) can be rewritten as

$$x_n + \Delta t\sum_{i=1}^{n} y_i \leq \exp(3c_1 n\Delta t)\left(x_0 + \Delta t\sum_{i=1}^{n} b_i\right), \tag{5.18}$$

where

$$x_n = \frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{1}{2}\|\mathbf{E}_h^n\|_0^2 + \frac{\nu\varepsilon}{64\alpha_1^2 d^2 M_0^2}|\mathbf{E}_h^n|_1^2, \qquad y_i = \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^i\|_1^2 + \delta_0|\epsilon_h^i|_h^2 + \frac{\nu}{64\alpha_1^2 d^2 M_0^2}\|\overline{D}_{\Delta t}\mathbf{E}_h^i\|_0^2,$$
$$b_i = c_2\left\{\Delta t\|(\mathbf{u}, \mathbf{C})\|_{Z^2(t^{i-1},t^i)}^2 + h^2\left(\frac{1}{\Delta t}\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(t^{i-1},t^i;\mathbb{H}^2)}^2 + 1\right)\right\}.$$

We firstly prove the general step in the induction. Supposing that $\mathrm{P}(n-1)$ holds true for an integer $n \in \{1, \ldots, N_T\}$, we prove that $\mathrm{P}(n)$ also holds. We prove $\mathrm{P}(n)$-(a). Since (3.3) and (5.8) with $M_0 = \|\mathbf{C}\|_{C(L^\infty)} + 1$ (≥ 1) are satisfied from (5.17c) and $\mathrm{P}(n-1)$-(b), respectively, we have (5.9) from Proposition 5.8. Inequality (5.9) implies that

$$\overline{D}_{\Delta t} x_n + y_n \leq c_1(x_n + x_{n-1}) + b_n,$$

which leads to

$$x_n + \Delta t y_n \leq \exp(3c_1\Delta t)(x_{n-1} + \Delta t b_n) \tag{5.19}$$

by $(1 + c_1\Delta t)/(1 - c_1\Delta t) \leq (1 + c_1\Delta t)(1 + 2c_1\Delta t) \leq \exp(3c_1\Delta t)$, where $c_1\Delta t \leq 1/2$ from (5.17b). From (5.19) and $\mathrm{P}(n-1)$-(a) we have

$$\begin{aligned} x_n + \Delta t\sum_{i=1}^{n} y_i &\leq \exp(3c_1\Delta t)(x_{n-1} + \Delta t b_n) + \Delta t\sum_{i=1}^{n-1} y_i \leq \exp(3c_1\Delta t)\left(x_{n-1} + \Delta t\sum_{i=1}^{n-1} y_i + \Delta t b_n\right) \\ &\leq \exp(3c_1\Delta t)\left[\exp\{3c_1(n-1)\Delta t\}\left(x_0 + \Delta t\sum_{i=1}^{n-1} b_i\right) + \Delta t b_n\right] \\ &\leq \exp(3c_1 n\Delta t)\left(x_0 + \Delta t\sum_{i=1}^{n} b_i\right). \end{aligned}$$

Thus, we obtain $\mathrm{P}(n)$-(a).

For the proof of P(n)-(b) we prepare the estimate of $\|\mathbf{E}_h^n\|_1$. We have

$$x_0 = \frac{1}{2}\|\mathbf{e}_h^0\|_0^2 + \frac{1}{2}\|\mathbf{E}_h^0\|_0^2 + \frac{\nu\varepsilon}{64\alpha_1^2 d^2 M_0^2}|\mathbf{E}_h^0|_1^2 = \frac{1}{2}\|\mathbf{e}_h^0\|_0^2 \le c_I^2 h^2 \tag{5.20}$$

from (5.15). P(n)-(a) with (5.20) implies that

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{1}{2}\|\mathbf{E}_h^n\|_0^2 + \frac{\nu\varepsilon}{64\alpha_1^2 d^2 M_0^2}|\mathbf{E}_h^n|_1^2 + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h\|_{\ell_n^2(H^1)}^2 + \delta_0|\epsilon_h|_{\ell_n^2(|\cdot|_h)}^2 + \frac{\nu}{64\alpha_1^2 d^2 M_0^2}\|\overline{D}_{\Delta t}\mathbf{E}_h\|_{\ell_n^2(L^2)}^2 \\
&\quad\le \exp(3c_1T)\Big[c_I^2h^2 + c_2\Big\{\Delta t^2\|(\mathbf{u},\mathbf{C})\|_{Z^2}^2 + h^2\big(\|(\mathbf{u},p,\mathbf{C})\|_{H^1(\mathbb{H}^2)}^2 + T\big)\Big\}\Big] \\
&\quad\le \exp(3c_1T)\Big[c_2\Delta t^2\|(\mathbf{u},\mathbf{C})\|_{Z^2}^2 + h^2\Big\{c_2\big(\|(\mathbf{u},p,\mathbf{C})\|_{H^1(\mathbb{H}^2)}^2 + T\big) + c_I^2\Big\}\Big] \\
&\quad\le \big\{c_3(\Delta t + h)\big\}^2,
\end{aligned} \tag{5.21}$$

which yields

$$\|\mathbf{E}_h^n\|_1 \le \frac{8\alpha_1 d M_0}{\sqrt{\nu\varepsilon}}c_3(\Delta t + h) = c_*(\Delta t + h) \tag{5.22}$$

from $\nu\varepsilon/(64\alpha_1^2 d^2 M_0^2) \le 1/(64d^2) < 1/2$.

We prove P(n)-(b) as follows:

$$\begin{aligned}
\|\mathbf{C}_h^n\|_{0,\infty} &\le \|\mathbf{C}_h^n - \Pi_h\mathbf{C}^n\|_{0,\infty} + \|\Pi_h\mathbf{C}^n\|_{0,\infty} \le \alpha_{21}D(h)\|\mathbf{C}_h^n - \Pi_h\mathbf{C}^n\|_1 + \|\Pi_h\mathbf{C}^n\|_{0,\infty} \\
&\le \alpha_{21}D(h)\big(\|\mathbf{C}_h^n - \hat{\mathbf{C}}_h^n\|_1 + \|\hat{\mathbf{C}}_h^n - \mathbf{C}^n\|_1 + \|\mathbf{C}^n - \Pi_h\mathbf{C}^n\|_1\big) + \|\Pi_h\mathbf{C}^n\|_{0,\infty} \\
&\le \alpha_{21}D(h)\big[c_*(\Delta t + h) + \alpha_{32}h\|\mathbf{C}^n\|_2 + \alpha_{20}h\|\mathbf{C}^n\|_2\big] + \|\mathbf{C}^n\|_{0,\infty} \\
&\le \alpha_{21}\big[c_*\{c_0 + h_0D(h_0)\} + (\alpha_{20} + \alpha_{32})h_0D(h_0)\|\mathbf{C}\|_{C(H^2)}\big] + \|\mathbf{C}\|_{C(L^\infty)} \\
&\le 1 + \|\mathbf{C}\|_{C(L^\infty)},
\end{aligned} \tag{5.23}$$

from (5.22), (4.3) and (5.17a). Therefore, P(n) holds true.

The proof of P(0) is easier than that of the general step. P(0)-(a) obviously holds with equality. P(0)-(b) is obtained as follows:

$$\begin{aligned}
\|\mathbf{C}_h^0\|_{0,\infty} &\le \|\mathbf{C}_h^0 - \Pi_h\mathbf{C}^0\|_{0,\infty} + \|\Pi_h\mathbf{C}^0\|_{0,\infty} \le \alpha_{21}D(h)(\|\mathbf{C}_h^0 - \mathbf{C}^0\|_1 + \|\mathbf{C}^0 - \Pi_h\mathbf{C}^0\|_1) + \|\Pi_h\mathbf{C}^0\|_{0,\infty} \\
&\le \alpha_{21}(\alpha_{20} + \alpha_{32})hD(h)\|\mathbf{C}^0\|_2 + \|\mathbf{C}^0\|_{0,\infty} \\
&\le 1 + \|\mathbf{C}\|_{C(L^\infty)}.
\end{aligned}$$

Thus, the induction is completed.

Step 3. Finally we derive (4.4) and (4.5). Since P(N_T) holds true, we have (4.4) and

$$\|\mathbf{e}_h\|_{\ell^\infty(L^2)\cap\ell^2(H^1)}, \quad |\epsilon_h|_{\ell^2(|\cdot|_h)}, \quad \|\overline{D}_{\Delta t}\mathbf{E}_h\|_{\ell^2(L^2)} \le c_{\nu,\varepsilon}c_{w,s}(\Delta t + h) \tag{5.24}$$

from (5.21). Combining (5.24) and the estimates

$$\|\mathbf{u}_h-\mathbf{u}\|_{\ell^\infty(L^2)}\le\|\mathbf{e}_h\|_{\ell^\infty(L^2)}+\|\boldsymbol{\eta}\|_{\ell^\infty(L^2)}\le\|\mathbf{e}_h\|_{\ell^\infty(L^2)}+\frac{\alpha_{31}}{\nu}h\|(\mathbf{u},p)\|_{C(H^2\times H^1)},$$

$$\begin{aligned}\left\|\overline{D}_{\Delta t}\mathbf{C}_h^n-\frac{\partial\mathbf{C}^n}{\partial t}\right\|_0&\le\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0+\|\overline{D}_{\Delta t}\boldsymbol{\Xi}^n\|_0+\left\|\overline{D}_{\Delta t}\mathbf{C}^n-\frac{\partial\mathbf{C}^n}{\partial t}\right\|_0\\&\le\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0+\frac{\alpha_{32}h}{\sqrt{\Delta t}}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^2)}+\sqrt{\frac{\Delta t}{3}}\left\|\frac{\partial^2\mathbf{C}}{\partial t^2}\right\|_{L^2(t^{n-1},t^n;L^2)},\end{aligned}$$

we can obtain the first and the last inequalities of (4.5) with a positive constant $c_\dagger$ independent of h and Δt. The other inequalities of (4.5) are similarly proved by using (5.22) and (5.24).

Remark 5.10. We note that the error constant behaves like $\mathcal{O}(\exp[c_{w,s}T/(\nu\varepsilon)])$ $(\nu,\varepsilon\downarrow 0)$ with respect to the viscosity ν and the elastic diffusion coefficient ε, since the main contribution is the exponential part of c_3 in (5.16), *i.e.*, $\exp[3c_1T/2]=\exp[3p_1(\|\mathbf{C}\|_{C(L^\infty)}+1;\nu,\varepsilon)T/2]=\mathcal{O}(\exp[c_{w,s}T/(\nu\varepsilon)])$, where (5.14) is used for the last equality. Although the dependence on ν and ε of the coefficient is not good, it seems hard to avoid it. Similar coefficient $\mathcal{O}(\exp[c_sT/\nu])$ appears in the estimate of the Navier–Stokes equations, [9,51]. As for the estimate independent of ν, we refer to [40] for the Stokes equations and to [16] for the Oseen equations.

5.5. A lemma for the proof of Theorem 4.8

In the proof of Theorem 4.8 we use the next lemma.

Lemma 5.11. *Suppose that Hypotheses 2.2 and 4.4 and the inequalities (4.4) and (4.5) hold. Let $m\in\{1,\ldots,N_T\}$ be any fixed number. Then, under condition (3.3) we have the following:*

$$\Delta t\sum_{n=1}^m\langle\mathbf{r}_{h1}^n,\overline{D}_{\Delta t}\mathbf{e}_h^n\rangle,\quad\Delta t\sum_{n=1}^m\langle\mathbf{r}_{h2}^n,\overline{D}_{\Delta t}\mathbf{e}_h^n\rangle\le\frac{\Delta t}{6}\sum_{n=1}^m\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0^2+c_{\nu,\varepsilon}c_{w,s}(\Delta t^2+h^2),\tag{5.25a}$$

$$\Delta t\sum_{n=1}^m\langle\mathbf{r}_{h3}^n,\overline{D}_{\Delta t}\mathbf{e}_h^n\rangle,\quad\Delta t\sum_{n=1}^m\langle\mathbf{r}_{h4}^n,\overline{D}_{\Delta t}\mathbf{e}_h^n\rangle\le\frac{\nu}{4}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2+c_{\nu,\varepsilon}c_{w,s}(\Delta t^2+h^2).\tag{5.25b}$$

Proof. The inequalities (5.25a) are obtained by combining (5.10a) and (5.10b) with

$$\langle\mathbf{r}_{hi}^n,\overline{D}_{\Delta t}\mathbf{e}_h^n\rangle\le\|\mathbf{r}_{hi}^n\|_0\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0\le\frac{3}{2}\|\mathbf{r}_{hi}^n\|_0^2+\frac{1}{6}\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0^2,\quad i=1,2.$$

We prove (5.25b). For $i=3,4$ we have

$$\begin{aligned}\Delta t\sum_{n=1}^m\langle\mathbf{r}_{hi}^n,\overline{D}_{\Delta t}\mathbf{e}_h^n\rangle&=\sum_{n=1}^m(\mathbf{r}_{hi}^n,\nabla\mathbf{e}_h^n-\nabla\mathbf{e}_h^{n-1})=(\mathbf{r}_{hi}^m,\nabla\mathbf{e}_h^m)-\sum_{n=1}^{m-1}(\mathbf{r}_{hi}^{n+1}-\mathbf{r}_{hi}^n,\nabla\mathbf{e}_h^n)-(\mathbf{r}_{hi}^1,\nabla\mathbf{e}_h^0)\\&\le\alpha_1\|\mathbf{r}_{hi}^m\|_{-1}\|\mathrm{D}(\mathbf{e}_h^m)\|_0+\sum_{n=1}^{m-1}\|\mathbf{r}_{hi}^{n+1}-\mathbf{r}_{hi}^n\|_0\|\mathbf{e}_h^n\|_1+\|\mathbf{r}_{hi}^1\|_{-1}\|\mathbf{e}_h^0\|_1\\&\le\frac{\nu}{4}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2+\frac{\alpha_1^2}{\nu}\|\mathbf{r}_{hi}^m\|_{-1}^2+\alpha_1\sum_{n=1}^{m-1}\|\mathbf{r}_{hi}^{n+1}-\mathbf{r}_{hi}^n\|_0\|\mathrm{D}(\mathbf{e}_h^n)\|_0+\frac{1}{2}\|\mathbf{r}_{hi}^1\|_{-1}^2+\frac{1}{2}\|\mathbf{e}_h^0\|_1^2\\&\le\frac{\nu}{4}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2+\alpha_1\sum_{n=1}^{m-1}\|\mathbf{r}_{hi}^{n+1}-\mathbf{r}_{hi}^n\|_0\|\mathrm{D}(\mathbf{e}_h^n)\|_0+c_{\nu,\varepsilon}c_{w,s}\big(\Delta t^2+h^2\big)\quad\text{(by (5.10c), (5.10d), (5.15), Thm. 4.7).}\end{aligned}\tag{5.26}$$

Applying Hölder's inequality, we have

$$
\begin{aligned}
&\big\|\mathrm{tr}\,\mathbf{C}^{n+1}(\mathbf{C}^{n+1}-\mathbf{C}^{n})-\mathrm{tr}\,\mathbf{C}^{n}(\mathbf{C}^{n}-\mathbf{C}^{n-1})\big\|_0=\left\|\int_{t^n}^{t^{n+1}}\frac{\partial}{\partial t}\big\{\mathrm{tr}\,\mathbf{C}(t)\big[\mathbf{C}(t)-\mathbf{C}(t-\Delta t)\big]\big\}\mathrm{d}t\right\|_0\\
&\le\left\|\int_{t^n}^{t^{n+1}}\mathrm{tr}\,\frac{\partial\mathbf{C}}{\partial t}(t)\big[\mathbf{C}(t)-\mathbf{C}(t-\Delta t)\big]\,\mathrm{d}t\right\|_0+\left\|\int_{t^n}^{t^{n+1}}\mathrm{tr}\,\mathbf{C}(t)\left[\frac{\partial\mathbf{C}}{\partial t}(t)-\frac{\partial\mathbf{C}}{\partial t}(t-\Delta t)\right]\mathrm{d}t\right\|_0\\
&=\left\|\int_{t^n}^{t^{n+1}}\mathrm{tr}\,\frac{\partial\mathbf{C}}{\partial t}(t)\,\mathrm{d}t\int_{t-\Delta t}^{t}\frac{\partial\mathbf{C}}{\partial t}(s)\,\mathrm{d}s\right\|_0+\left\|\int_{t^n}^{t^{n+1}}\mathrm{tr}\,\mathbf{C}(t)\,\mathrm{d}t\int_{t-\Delta t}^{t}\frac{\partial^2\mathbf{C}}{\partial t^2}(s)\,\mathrm{d}s\right\|_0\\
&\le\int_{t^n}^{t^{n+1}}\left\|\mathrm{tr}\,\frac{\partial\mathbf{C}}{\partial t}(t)\right\|_{0,4}\mathrm{d}t\int_{t^{n-1}}^{t^{n+1}}\left\|\frac{\partial\mathbf{C}}{\partial t}(s)\right\|_{0,4}\mathrm{d}s+\int_{t^n}^{t^{n+1}}\|\mathrm{tr}\,\mathbf{C}(t)\|_{0,\infty}\,\mathrm{d}t\int_{t^{n-1}}^{t^{n+1}}\left\|\frac{\partial^2\mathbf{C}}{\partial t^2}(s)\right\|_0\mathrm{d}s\\
&\le\Delta t^{3/4}\left(\int_{t^n}^{t^{n+1}}\left\|\frac{\partial\mathbf{C}}{\partial t}(t)\right\|_{0,4}^4\mathrm{d}t\right)^{1/4}(2\Delta t)^{3/4}\left(\int_{t^{n-1}}^{t^{n+1}}\left\|\frac{\partial\mathbf{C}}{\partial t}(s)\right\|_{0,4}^4\mathrm{d}s\right)^{1/4}\\
&\quad+d\Delta t\|\mathbf{C}\|_{C(L^\infty)}\sqrt{2\Delta t}\|\mathbf{C}\|_{H^2(t^{n-1},t^{n+1};L^2)}\\
&\le c_s\Delta t^{3/2}\left(\left\|\frac{\partial\mathbf{C}}{\partial t}\right\|_{L^4(t^{n-1},t^{n+1};L^4)}^2+\|\mathbf{C}\|_{H^2(t^{n-1},t^{n+1};L^2)}\right),\\
&\|\mathrm{tr}\,\mathbf{C}^{n+1}\boldsymbol{\Xi}^n-\mathrm{tr}\,\mathbf{C}^n\boldsymbol{\Xi}^{n-1}\|_0\le\|(\mathrm{tr}\,\mathbf{C}^{n+1}-\mathrm{tr}\,\mathbf{C}^n)\boldsymbol{\Xi}^n\|_0+\|\mathrm{tr}\,\mathbf{C}^n(\boldsymbol{\Xi}^n-\boldsymbol{\Xi}^{n-1})\|_0\\
&\le\|\mathrm{tr}\,\mathbf{C}^{n+1}-\mathrm{tr}\,\mathbf{C}^n\|_{0,3}\|\boldsymbol{\Xi}^n\|_{0,6}+\|\mathrm{tr}\,\mathbf{C}^n\|_{0,3}\|\boldsymbol{\Xi}^n-\boldsymbol{\Xi}^{n-1}\|_{0,6}\\
&\le\sqrt{\Delta t}\left\|\frac{\partial(\mathrm{tr}\,\mathbf{C})}{\partial t}\right\|_{L^2(t^n,t^{n+1};L^3)}\|\boldsymbol{\Xi}^n\|_{0,6}+c\|\mathbf{C}^n\|_1\sqrt{\Delta t}\left\|\frac{\partial\boldsymbol{\Xi}}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^6)}\\
&\le c'\sqrt{\Delta t}\big(\|\mathbf{C}\|_{H^1(t^n,t^{n+1};H^1)}\|\boldsymbol{\Xi}^n\|_1+\|\mathbf{C}^n\|_1\|\boldsymbol{\Xi}\|_{H^1(t^{n-1},t^n;H^1)}\big)\\
&\le c'\sqrt{\Delta t}(\|\mathbf{C}\|_{H^1(t^n,t^{n+1};H^1)}\alpha_{32}h\|\mathbf{C}^n\|_2+\|\mathbf{C}^n\|_1\alpha_{32}h\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^2)}\big)\\
&\le c_sh\sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^{n+1};H^2)},\\
&\|\mathrm{tr}\,\mathbf{C}^{n+1}\mathbf{E}_h^n-\mathrm{tr}\,\mathbf{C}^n\mathbf{E}_h^{n-1}\|_0\le\|(\mathrm{tr}\,\mathbf{C}^{n+1}-\mathrm{tr}\,\mathbf{C}^n)\mathbf{E}_h^n\|_0+\|\mathrm{tr}\,\mathbf{C}^n(\mathbf{E}_h^n-\mathbf{E}_h^{n-1})\|_0\\
&\le\|\mathrm{tr}\,\mathbf{C}^{n+1}-\mathrm{tr}\,\mathbf{C}^n\|_{0,3}\|\mathbf{E}_h^n\|_{0,6}+\|\mathrm{tr}\,\mathbf{C}^n\|_{0,\infty}\|\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\|_0\\
&\le c\sqrt{\Delta t}\left\|\frac{\partial(\mathrm{tr}\,\mathbf{C})}{\partial t}\right\|_{L^2(t^n,t^{n+1};L^3)}\|\mathbf{E}_h^n\|_1+\|\mathrm{tr}\,\mathbf{C}^n\|_{0,\infty}\Delta t\,\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0\\
&\le c_{\nu,\varepsilon}c_{w,s}\sqrt{\Delta t}\big[(\Delta t+h)\|\mathbf{C}\|_{H^1(t^n,t^{n+1};H^1)}+\sqrt{\Delta t}\,\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0\big]\qquad\text{(by Thm. 4.7)}.
\end{aligned}
$$

Hence, $\|\mathbf{r}_{h3}^{n+1}-\mathbf{r}_{h3}^n\|_0$ is evaluated as follows:

$$
\begin{aligned}
&\|\mathbf{r}_{h3}^{n+1}-\mathbf{r}_{h3}^n\|_0=\|(\mathrm{tr}\,\mathbf{C}^{n+1})(\mathbf{C}^{n+1}-\mathbf{C}^n+\boldsymbol{\Xi}^n-\mathbf{E}_h^n)-(\mathrm{tr}\,\mathbf{C}^n)(\mathbf{C}^n-\mathbf{C}^{n-1}+\boldsymbol{\Xi}^{n-1}-\mathbf{E}_h^{n-1})\|_0\\
&\le\|(\mathrm{tr}\,\mathbf{C}^{n+1})(\mathbf{C}^{n+1}-\mathbf{C}^n)-(\mathrm{tr}\,\mathbf{C}^n)(\mathbf{C}^n-\mathbf{C}^{n-1})\|_0+\|(\mathrm{tr}\,\mathbf{C}^{n+1})\boldsymbol{\Xi}^n-(\mathrm{tr}\,\mathbf{C}^n)\boldsymbol{\Xi}^{n-1}\|_0\\
&\quad+\|(\mathrm{tr}\,\mathbf{C}^{n+1})\mathbf{E}_h^n-(\mathrm{tr}\,\mathbf{C}^n)\mathbf{E}_h^{n-1}\|_0\le c_{\nu,\varepsilon}c_{w,s}\sqrt{\Delta t}\\
&\times\left[\Delta t\left\|\frac{\partial\mathbf{C}}{\partial t}\right\|_{L^4(t^{n-1},t^{n+1};L^4)}^2+\Delta t\|\mathbf{C}\|_{H^2(t^{n-1},t^{n+1};L^2)}+(\Delta t+h)\|\mathbf{C}\|_{H^1(t^{n-1},t^{n+1};H^2)}+\sqrt{\Delta t}\,\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0\right].
\end{aligned}
\tag{5.27}
$$

Combining (5.27) with (5.26) with $i=3$, we get

$$
\begin{aligned}
&\Delta t\sum_{n=1}^{m}\langle \mathbf{r}_{h3}^n,\overline{D}_{\Delta t}\mathbf{e}_h^n\rangle\\
&\le \frac{\nu}{4}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2+c_{\nu,\varepsilon}c_{w,s}\Bigg\{\left(\Delta t^2+h^2\right)+\sum_{n=1}^{m-1}\Bigg[\Delta t\left\|\frac{\partial \mathbf{C}}{\partial t}\right\|_{L^4(t^{n-1},t^{n+1};L^4)}^2+\Delta t\|\mathbf{C}\|_{H^2(t^{n-1},t^{n+1};L^2)}\\
&\quad+(\Delta t+h)\|\mathbf{C}\|_{H^1(t^{n-1},t^{n+1};H^2)}+\sqrt{\Delta t}\,\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0\Bigg]\sqrt{\Delta t}\|\mathrm{D}(\mathbf{e}_h^n)\|_0\Bigg\}\\
&\le \frac{\nu}{4}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2+c_{\nu,\varepsilon}c_{w,s}\Bigg\{(\Delta t^2+h^2)+\|\mathbf{e}_h\|_{\ell^2(H^1)}^2+\|\overline{D}_{\Delta t}\mathbf{E}_h\|_{\ell^2(L^2)}^2+2(\Delta t^2+h^2)\|\mathbf{C}\|_{H^1(H^2)}^2\\
&\quad+\Delta t^2\left(\left\|\frac{\partial \mathbf{C}}{\partial t}\right\|_{L^4(L^4)}^4+\|\mathbf{C}\|_{H^2(L^2)}^2\right)\Bigg\}\\
&\le \frac{\nu}{4}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2+c'_{\nu,\varepsilon}c'_{w,s}(\Delta t^2+h^2),
\end{aligned}
\tag{5.28}
$$

where in the last inequality we have employed Theorem 4.7 and the relation $[L^2(0,T;H^1(\Omega))\cap H^1(0,T;L^2(\Omega))]\hookrightarrow L^4(0,T;L^4(\Omega))$ yielding the inequality $\|\partial\mathbf{C}/\partial t\|_{L^4(L^4)}\le c\|\partial\mathbf{C}/\partial t\|_{L^2(H^1)\cap H^1(L^2)}\le c\|\mathbf{C}\|_{H^1(H^1)\cap H^2(L^2)}\le c_s$. Thus, the first inequality of (5.25b) is proved. We prove the second inequality of (5.25b). For $\|\mathbf{r}_{h4}^{n+1}-\mathbf{r}_{h4}^n\|_0$ we have

$$
\begin{aligned}
&\|\mathbf{r}_{h4}^{n+1}-\mathbf{r}_{h4}^n\|_0=\|[\mathrm{tr}\,(\boldsymbol{\Xi}^{n+1}-\mathbf{E}_h^{n+1})]\mathbf{C}_h^n-[\mathrm{tr}\,(\boldsymbol{\Xi}^n-\mathbf{E}_h^n)]\mathbf{C}_h^n+[\mathrm{tr}\,(\boldsymbol{\Xi}^n-\mathbf{E}_h^n)]\mathbf{C}_h^n-[\mathrm{tr}\,(\boldsymbol{\Xi}^n-\mathbf{E}_h^n)]\mathbf{C}_h^{n-1}\|_0\\
&\quad=\left\|[\mathrm{tr}\,(\boldsymbol{\Xi}^{n+1}-\boldsymbol{\Xi}^n)]\mathbf{C}_h^n-[\mathrm{tr}\,(\mathbf{E}_h^{n+1}-\mathbf{E}_h^n)]\mathbf{C}_h^n+[\mathrm{tr}\,(\boldsymbol{\Xi}^n-\mathbf{E}_h^n)]\Big[\Delta t\overline{D}_{\Delta t}\mathbf{E}_h^n-(\boldsymbol{\Xi}^n-\boldsymbol{\Xi}^{n-1})+(\mathbf{C}^n-\mathbf{C}^{n-1})\Big]\right\|_0\\
&\quad\le c\Big[\|\mathbf{C}_h^n\|_{0,\infty}(\|\boldsymbol{\Xi}^{n+1}-\boldsymbol{\Xi}^n\|_0+\Delta t\|\overline{D}_{\Delta t}\mathbf{E}_h^{n+1}\|_0)+\|\boldsymbol{\Xi}^n-\mathbf{E}_h^n\|_{0,\infty}(\Delta t\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0+\|\boldsymbol{\Xi}^n-\boldsymbol{\Xi}^{n-1}\|_0)\\
&\qquad+(\|\boldsymbol{\Xi}^n\|_0+\|\mathbf{E}_h^n\|_0)\|\mathbf{C}^n-\mathbf{C}^{n-1}\|_{0,\infty}\Big]\\
&\quad\le c\Big[(2\|\mathbf{C}\|_{C(L^\infty)}+1)\Big\{\sqrt{\Delta t}\|\boldsymbol{\Xi}\|_{H^1(t^n,t^{n+1};L^2)}+\Delta t(\|\overline{D}_{\Delta t}\mathbf{E}_h^{n+1}\|_0+\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0)+\sqrt{\Delta t}\|\boldsymbol{\Xi}\|_{H^1(t^{n-1},t^n;L^2)}\Big\}\\
&\qquad+(\|\boldsymbol{\Xi}^n\|_0+\|\mathbf{E}_h^n\|_0)\sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^\infty)}\Big]\\
&\quad\le c_{\nu,\varepsilon}c_{w,s}\sqrt{\Delta t}\big[\alpha_{32}h\|\mathbf{C}\|_{H^1(t^{n-1},t^{n+1};H^2)}+\sqrt{\Delta t}(\|\overline{D}_{\Delta t}\mathbf{E}_h^{n+1}\|_0+\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0)\\
&\qquad+(\alpha_{32}h\|\mathbf{C}^n\|_2+\|\mathbf{E}_h^n\|_0)\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^2)}\big]\\
&\quad\le c'_{\nu,\varepsilon}c'_{w,s}\sqrt{\Delta t}\Big[\sqrt{\Delta t}(\|\overline{D}_{\Delta t}\mathbf{E}_h^{n+1}\|_0+\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0)+(\Delta t+h)\|\mathbf{C}\|_{H^1(t^{n-1},t^{n+1};H^2)}\Big],
\end{aligned}
\tag{5.29}
$$

where we have used the estimates

$$
\begin{aligned}
&\mathbf{C}_h^n-\mathbf{C}_h^{n-1}=(\mathbf{E}_h^n-\boldsymbol{\Xi}^n+\mathbf{C}^n)-(\mathbf{E}_h^{n-1}-\boldsymbol{\Xi}^{n-1}+\mathbf{C}^{n-1})=\Delta t\overline{D}_{\Delta t}\mathbf{E}_h^n-(\boldsymbol{\Xi}^n-\boldsymbol{\Xi}^{n-1})+(\mathbf{C}^n-\mathbf{C}^{n-1}),\\
&\|\boldsymbol{\Xi}^n-\mathbf{E}_h^n\|_{0,\infty}=\|\mathbf{C}^n-\mathbf{C}_h^n\|_{0,\infty}\le\|\mathbf{C}^n\|_{0,\infty}+\|\mathbf{C}_h^n\|_{0,\infty}\le 2\|\mathbf{C}\|_{C(L^\infty)}+1\qquad\text{(by Thm. 4.7)}.
\end{aligned}
$$

Combining (5.29) with (5.26) with $i = 4$, we have

$$\begin{aligned}
&\Delta t \sum_{n=1}^{m} \langle \mathbf{r}_{h4}^n, \overline{D}_{\Delta t}\mathbf{e}_h^n \rangle \\
&\le \frac{\nu}{4}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2 + c_{\nu,\varepsilon} c_{w,s}\Big\{ (\Delta t^2 + h^2) \\
&\quad + \sum_{n=1}^{m-1}\Big[\sqrt{\Delta t}(\|\overline{D}_{\Delta t}\mathbf{E}_h^{n+1}\|_0 + \|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0) + (\Delta t + h)\|\mathbf{C}\|_{H^1(t^{n-1},t^{n+1};H^2)}\Big]\sqrt{\Delta t}\|\mathrm{D}(\mathbf{e}_h^n)\|_0 \Big\} \\
&\le \frac{\nu}{4}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2 + c_{\nu,\varepsilon} c_{w,s}\Big\{ (\Delta t^2 + h^2) + \|\mathbf{e}_h\|_{\ell^2(H^1)}^2 + (\Delta t^2 + h^2)\|\mathbf{C}\|_{H^1(H^2)}^2 + \|\overline{D}_{\Delta t}\mathbf{E}_h\|_{\ell^2(L^2)}^2 \Big\} \\
&\le \frac{\nu}{4}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2 + c'_{\nu,\varepsilon} c'_{w,s}(\Delta t^2 + h^2) \qquad \text{(by Thm. 4.7)},
\end{aligned} \tag{5.30}$$

which is the second inequality of (5.25b). □

5.6. Proof of Theorem 4.8

Let $p_h^0 = [\Pi_h^{\mathrm{SP}}(\mathbf{u}^0, 0, \mathbf{C}^0)]_2$, which leads to $(\mathbf{u}_h^0, p_h^0, \mathbf{C}_h^0) = [\Pi_h^{\mathrm{SP}}(\mathbf{u}^0, 0, \mathbf{C}^0)]$. Substituting $(\overline{D}_{\Delta t}\mathbf{e}_h^n, 0) \in V_h \times Q_h$ into $(\mathbf{v}_h, q_h)$ in (5.5a) and using

$$\frac{\mathbf{e}_h^n - \mathbf{e}_h^{n-1}\circ X_1^n}{\Delta t} = \overline{D}_{\Delta t}\mathbf{e}_h^n + \frac{\mathbf{e}_h^{n-1} - \mathbf{e}_h^{n-1}\circ X_1^n}{\Delta t},$$

we have

$$\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0^2 + \nu a_u(e_h^n, \overline{D}_{\Delta t}\mathbf{e}_h^n) + b(\overline{D}_{\Delta t}\mathbf{e}_h^n, \epsilon_h^n) = \langle \mathbf{r}_h^n, \overline{D}_{\Delta t}\mathbf{e}_h^n\rangle - \frac{1}{\Delta t}(\mathbf{e}_h^{n-1} - \mathbf{e}_h^{n-1}\circ X_1^n, \overline{D}_{\Delta t}\mathbf{e}_h^n). \tag{5.31}$$

On the other hand, setting $\mathbf{v}_h = \mathbf{0} \in V_h$ in (5.5a), we have for $n = 1, \ldots, N_T$

$$b(\mathbf{e}_h^n, q_h) - \mathcal{S}_h(\epsilon_h^n, q_h) = 0, \ \ \forall q_h \in Q_h. \tag{5.32}$$

From the definitions of $(\mathbf{u}_h^0, p_h^0, \mathbf{C}_h^0)$ and $(\hat{\mathbf{u}}_h^0, \hat{p}_h^0, \hat{\mathbf{C}}_h^0)$ we have

$$b(\mathbf{e}_h^0, q_h) - \mathcal{S}_h(\epsilon_h^0, q_h) = b(\mathbf{u}_h^0, q_h) - \mathcal{S}_h(p_h^0, q_h) - \big\{b(\hat{\mathbf{u}}_h^0, q_h) - \mathcal{S}_h(\hat{p}_h^0, q_h)\big\} = b(\mathbf{u}^0, q_h) - b(\mathbf{u}^0, q_h) = 0, \ \forall q_h \in Q_h,$$

which implies that (5.32) holds also for $n = 0$. Hence, we get for $n = 1, \ldots, N_T$

$$b(\overline{D}_{\Delta t}\mathbf{e}_h^n, q_h) - \mathcal{S}_h(\overline{D}_{\Delta t}\epsilon_h^n, q_h) = 0, \quad \forall q_h \in Q_h,$$

which yields

$$b(\overline{D}_{\Delta t}\mathbf{e}_h^n, \epsilon_h^n) - \mathcal{S}_h(\overline{D}_{\Delta t}\epsilon_h^n, \epsilon_h^n) = 0 \tag{5.33}$$

by setting $q_h = \epsilon_h^n \in Q_h$. Subtracting (5.33) from (5.31), we have for $n = 1, \ldots, N_T$

$$\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0^2 + \nu a_u(e_h^n, \overline{D}_{\Delta t}\mathbf{e}_h^n) + \mathcal{S}_h(\overline{D}_{\Delta t}\epsilon_h^n, \epsilon_h^n) = \langle \mathbf{r}_h^n, \overline{D}_{\Delta t}\mathbf{e}_h^n\rangle - \left(\frac{\mathbf{e}_h^{n-1} - \mathbf{e}_h^{n-1}\circ X_1^n}{\Delta t}, \overline{D}_{\Delta t}\mathbf{e}_h^n\right). \tag{5.34}$$

From the estimates

$$\begin{aligned}
\nu a_u(\mathbf{e}_h^n, \overline{D}_{\Delta t}\mathbf{e}_h^n) &= \overline{D}_{\Delta t}\left(\frac{\nu}{2}a_u(\mathbf{e}_h^n, \mathbf{e}_h^n)\right) + \frac{\nu\Delta t}{2}a_u(\overline{D}_{\Delta t}\mathbf{e}_h^n, \overline{D}_{\Delta t}\mathbf{e}_h^n) \ge \overline{D}_{\Delta t}\left(\nu\|\mathrm{D}(\mathbf{e}_h^n)\|_0^2\right), \\
\mathcal{S}_h(\overline{D}_{\Delta t}\epsilon_h^n, \epsilon_h^n) &= \overline{D}_{\Delta t}\left(\frac{1}{2}\mathcal{S}_h(\epsilon_h^n, \epsilon_h^n)\right) + \frac{\Delta t}{2}\mathcal{S}_h(\overline{D}_{\Delta t}\epsilon_h^n, \overline{D}_{\Delta t}\epsilon_h^n) \ge \overline{D}_{\Delta t}\left(\frac{\delta_0}{2}|\epsilon_h^n|_h^2\right), \\
\left|\frac{1}{\Delta t}(\mathbf{e}_h^{n-1} - \mathbf{e}_h^{n-1}\circ X_1^n, \overline{D}_{\Delta t}\mathbf{e}_h^n)\right| &\le \frac{1}{\Delta t}\|\mathbf{e}_h^{n-1} - \mathbf{e}_h^{n-1}\circ X_1^n\|_0\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0 \le \alpha_{41}\|\mathbf{w}^n\|_{0,\infty}\|\mathbf{e}_h^{n-1}\|_1\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0 \\
&\le \alpha_{41}\|\mathbf{w}^n\|_{0,\infty}\alpha_1\|\mathrm{D}(\mathbf{e}_h^{n-1})\|_0\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0 \le c_w\|\mathrm{D}(\mathbf{e}_h^{n-1})\|_0^2 + \frac{1}{6}\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0^2,
\end{aligned}$$

equality (5.34) leads to, for $n = 1, \dots, N_T$,

$$\overline{D}_{\Delta t}\left(\nu\|\mathrm{D}(\mathbf{e}_h^n)\|_0^2 + \frac{\delta_0}{2}|\epsilon_h^n|_h^2\right) + \frac{5}{6}\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0^2 \le \langle \mathbf{r}_h^n, \overline{D}_{\Delta t}\mathbf{e}_h^n\rangle + c_w\|\mathrm{D}(\mathbf{e}_h^{n-1})\|_0^2. \tag{5.35}$$

Let m $(1 \le m \le N_T)$ be any integer. Summing up (5.35) for $n = 1, \dots, m$ and using Lemma 5.11, we have

$$\frac{\nu}{2}\|\mathrm{D}(\mathbf{e}_h^m)\|_0^2 + \frac{\delta_0}{2}|\epsilon_h^m|_h^2 + \frac{\Delta t}{2}\sum_{n=1}^{m}\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0^2 \le \frac{c_w}{\nu}\Delta t\sum_{n=0}^{m-1}\nu\|\mathrm{D}(\mathbf{e}_h^n)\|_0^2 + c_{\nu,\varepsilon}c_{w,s}(\Delta t^2 + h^2). \tag{5.36}$$

From Lemma 5.7 with

$$x_n = \frac{\nu}{2}\|\mathrm{D}(\mathbf{e}_h^n)\|_0^2 + \frac{\delta_0}{2}|\epsilon_h^n|_h^2, \qquad y_n = \frac{1}{2}\|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0^2, \qquad \alpha = \frac{2c_w}{\nu}, \qquad \beta = c_{\nu,\varepsilon}c_{w,s}(\Delta t^2 + h^2),$$

we have

$$\|\overline{D}_{\Delta t}\mathbf{e}_h\|_{\ell^2(L^2)} \le c'_{\nu,\varepsilon}c'_{w,s}(\Delta t + h). \tag{5.37}$$

The first inequality of (4.6) is obtained by combining the inequality above with the estimate

$$\begin{aligned}\left\|\overline{D}_{\Delta t}\mathbf{u}_h^n - \frac{\partial \mathbf{u}^n}{\partial t}\right\|_0 &\le \|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0 + \|\overline{D}_{\Delta t}\boldsymbol{\eta}^n\|_0 + \left\|\overline{D}_{\Delta t}\mathbf{u}^n - \frac{\partial \mathbf{u}^n}{\partial t}\right\|_0 \\ &\le \|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0 + \frac{\alpha_{31}h}{\nu\sqrt{\Delta t}}\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} + \sqrt{\frac{\Delta t}{3}}\left\|\frac{\partial^2\mathbf{u}}{\partial t^2}\right\|_{L^2(t^{n-1},t^n;L^2)}.\end{aligned}$$

The second inequality of (4.6) is proved as follows. We have

$$\begin{aligned}\|\epsilon_h^n\|_0 &\le \|(\mathbf{e}_h^n,\epsilon_h^n)\|_{V\times Q} \le \frac{1}{\nu\alpha_{30}}\sup_{(\mathbf{v}_h,q_h)\in V_h\times Q_h}\frac{\mathcal{A}_h((\mathbf{u}_h^n,\epsilon_h^n),(\mathbf{v}_h,q_h))}{\|(\mathbf{v}_h,q_h)\|_{V\times Q}} \\ &= \frac{1}{\nu\alpha_{30}}\sup_{(\mathbf{v}_h,q_h)\in V_h\times Q_h}\frac{\langle\mathbf{r}_h^n,\mathbf{v}_h\rangle - \frac{1}{\Delta t}(\mathbf{e}_h^n - \mathbf{e}_h^{n-1}\circ X_1^n,\mathbf{v}_h)}{\|(\mathbf{v}_h,q_h)\|_{V\times Q}} \\ &\le \frac{1}{\nu\alpha_{30}}\left[\|\mathbf{r}_{h1}^n\|_0 + \|\mathbf{r}_{h2}^n\|_0 + \|\mathbf{r}_{h3}^n\|_{-1} + \|\mathbf{r}_{h4}^n\|_{-1} + \|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0 + \frac{1}{\Delta t}\|\mathbf{e}_h^{n-1} - \mathbf{e}_h^{n-1}\circ X_1^n\|_0\right] \\ &\le \frac{c_s}{\nu\alpha_{30}}\Bigg[\sqrt{\Delta t}\big(\|\mathbf{u}\|_{Z^2(t^{n-1},t^n)} + \|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)}\big) + \frac{h}{\nu\sqrt{\Delta t}}\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} \\ &\qquad + \|\overline{D}_{\Delta t}\mathbf{e}_h^n\|_0 + \|\mathbf{e}_h^{n-1}\|_1 + \|\mathbf{E}_h^n\|_0 + \|\mathbf{E}_h^{n-1}\|_0 + h\Bigg] \quad \text{(by (5.10a)–(5.10d))},\end{aligned}$$

which implies the second inequality of (4.6) from Theorem 4.7, (5.37) and the estimate

$$\|p_h - p\|_{\ell^2(L^2)} \le \|\epsilon_h\|_{\ell^2(L^2)} + \|\hat{p}_h - p\|_{\ell^2(L^2)} \le \|\epsilon_h\|_{\ell^2(L^2)} + \sqrt{T}\frac{\alpha_{31}}{\nu}h\|(\mathbf{u},p)\|_{C(H^2\times H^1)}.$$

6. Numerical experiments

In this section we present numerical results by scheme (3.4) in order to confirm the theoretical convergence order. We refer to [35] for the detailed description of the algorithm that has been used to perform the numerical simulations. Further numerical experiments for linear scheme (3.4) as well as for the nonlinear scheme that has been discussed in our previous paper [29], Part I, can also be found in [35].

TABLE 1. Symbols used in the figures.

$\mathbf{u}_h$		p_h		$\mathbf{C}_h$	
○	●	△	▲	□	■
$Er\,1$	$Er\,2$	$Er\,3$	$Er\,4$	$Er\,5$	$Er\,6$

Example 6.1. In problem (2.1) we set $\Omega = (0,1)^2$ and $T = 0.5$, and we consider three cases for the pair of ν and ε. Firstly we take both viscosities to be equal 10^{-1}, *i.e.*, $(\nu, \varepsilon) = (10^{-1}, 10^{-1})$. Secondly, we consider the case $(\nu, \varepsilon) = (10^{-1}, 10^{-3})$, since the elastic stress viscosity is typically much smaller than the fluid viscosity. Lastly, we set $(\nu, \varepsilon) = (1, 0)$. Although the non-diffusive case $\varepsilon = 0$ is out of the scope of theoretical analysis of this paper, we carry out the computation to see the performance of scheme (3.4). The functions $\mathbf{f}$, $\mathbf{F}$, $\mathbf{u}^0$ and $\mathbf{C}^0$ are given such that the exact solution to (2.1) is as follows:

$$\begin{aligned}
\mathbf{u}(x,t) &= \left(\frac{\partial\psi}{\partial x_2}(x,t), -\frac{\partial\psi}{\partial x_1}(x,t)\right), \quad p(x,t) = \sin\{\pi(x_1 + 2x_2 + t)\},\\
C_{11}(x,t) &= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+t)\} + 1,\\
C_{22}(x,t) &= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_2+t)\} + 1,\\
C_{12}(x,t) &= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+x_2+t)\} \ (= C_{21}(x,t)),\\
\psi(x,t) &= \frac{\sqrt{3}}{2\pi}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+x_2+t)\}.
\end{aligned} \tag{6.1}$$

Proposition 4.1 and Theorems 4.7 and 4.8 hold for any fixed positive constant δ_0. Here we simply fix $\delta_0 = 1$. Let N be the division number of each side of the square domain. We set $N = 16, 32, 64, 128$ and 256, and (re)define $h = 1/N$. The time increment is set as $\Delta t = h/2$. To solve Example we employ scheme (3.4) with $(\mathbf{u}_h^0, \mathbf{C}_h^0) = [\Pi_h^{\mathrm{SP}}(\mathbf{u}^0, 0, \mathbf{C}^0)]_{1,3}$.

For the solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (3.4) and the exact solution $(\mathbf{u}, p, \mathbf{C})$ given by (6.1) we define the relative errors $Er\,i$, $i = 1, \ldots, 6$, by

$$Er\,1 = \frac{\|\mathbf{u}_h - \Pi_h\mathbf{u}\|_{\ell^\infty(L^2)}}{\|\Pi_h\mathbf{u}\|_{\ell^\infty(L^2)}}, \qquad Er\,2 = \frac{\|\mathbf{u}_h - \Pi_h\mathbf{u}\|_{\ell^2(H^1)}}{\|\Pi_h\mathbf{u}\|_{\ell^2(H^1)}}, \qquad Er\,3 = \frac{\|p_h - \Pi_h p\|_{\ell^2(L^2)}}{\|\Pi_h p\|_{\ell^2(L^2)}},$$

$$Er\,4 = \frac{|p_h - \Pi_h p|_{\ell^2(|\cdot|_h)}}{\|\Pi_h p\|_{\ell^2(L^2)}}, \qquad Er\,5 = \frac{\|\mathbf{C}_h - \Pi_h\mathbf{C}\|_{\ell^\infty(L^2)}}{\|\Pi_h\mathbf{C}\|_{\ell^\infty(L^2)}}, \qquad Er\,6 = \frac{\|\mathbf{C}_h - \Pi_h\mathbf{C}\|_{\ell^2(H^1)}}{\|\Pi_h\mathbf{C}\|_{\ell^2(H^1)}},$$

where the same symbol Π_h has been employed as the scalar and vector versions of the Lagrange interpolation operator.

The values of the errors and the slopes are presented in the tables below, while the corresponding figures show the graphs of the errors *versus* h in logarithmic scale. Table 1 summarizes the symbols used in the figures. Tables & Figures 1, 2 and 3 present the results for the cases $(\nu, \varepsilon) = (10^{-1}, 10^{-1})$, $(10^{-1}, 10^{-3})$ and $(1, 0)$, respectively.

For all the cases it is confirmed that all the errors except $Er\,6$ for $(\nu, \varepsilon) = (1, 0)$ are almost of the first order in h. These results support Theorems 4.7 and 4.8. Since there is no diffusion for $\mathbf{C}$ in equation (2.1c) in the case $(\nu, \varepsilon) = (1, 0)$, it is natural that the slope of $Er\,6$ does not attain 1. Even if the theorems are not proved for $\varepsilon = 0$, scheme (3.4) has worked well in the numerical experiments.

Remark 6.2. In the above the difference of $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ and $(\Pi_h\mathbf{u}, \Pi_h p, \Pi_h\mathbf{C})$ are computed. For the difference of $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ and $(\mathbf{u}, p, \mathbf{C})$ see Appendix A.2.

h	$Er\,1$	slope	$Er\,2$	slope
1/16	6.29×10^{-2}	–	7.94×10^{-2}	–
1/32	2.21×10^{-2}	1.51	3.14×10^{-2}	1.34
1/64	8.98×10^{-3}	1.30	1.32×10^{-2}	1.25
1/128	4.07×10^{-3}	1.14	6.35×10^{-3}	1.05
1/256	1.95×10^{-3}	1.07	2.86×10^{-3}	1.15
h	$Er\,3$	slope	$Er\,4$	slope
1/16	2.02×10^{-1}	–	1.70×10^{-1}	–
1/32	7.11×10^{-2}	1.50	4.99×10^{-2}	1.77
1/64	2.67×10^{-2}	1.41	1.86×10^{-2}	1.42
1/128	1.11×10^{-2}	1.27	8.39×10^{-3}	1.15
1/256	5.01×10^{-3}	1.15	3.69×10^{-3}	1.19
h	$Er\,5$	slope	$Er\,6$	slope
1/16	2.80×10^{-2}	–	1.22×10^{-1}	–
1/32	1.14×10^{-2}	1.30	4.41×10^{-2}	1.47
1/64	4.90×10^{-3}	1.21	1.72×10^{-2}	1.35
1/128	2.30×10^{-3}	1.09	7.64×10^{-3}	1.17
1/256	1.11×10^{-3}	1.05	3.59×10^{-3}	1.09

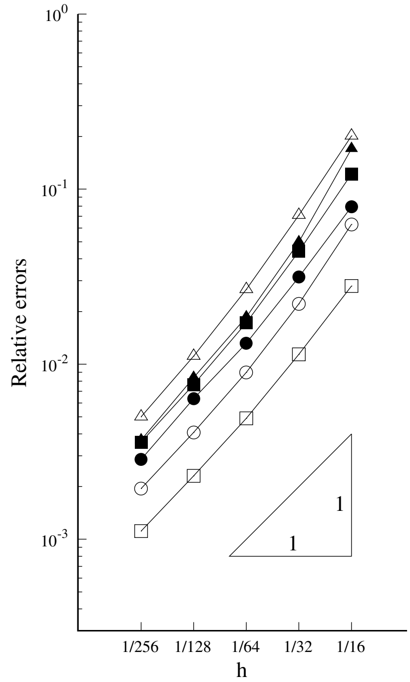

FIGURE 1. Errors and slopes for $(\nu, \varepsilon) = (10^{-1}, 10^{-1})$.

h	$Er\,1$	slope	$Er\,2$	slope
1/16	6.14×10^{-2}	–	7.29×10^{-2}	–
1/32	1.97×10^{-2}	1.64	2.91×10^{-2}	1.33
1/64	7.68×10^{-3}	1.36	1.21×10^{-2}	1.26
1/128	3.36×10^{-3}	1.19	5.93×10^{-3}	1.03
1/256	1.58×10^{-3}	1.09	2.66×10^{-3}	1.15
h	$Er\,3$	slope	$Er\,4$	slope
1/16	2.50×10^{-1}	–	2.06×10^{-1}	–
1/32	9.14×10^{-2}	1.45	6.08×10^{-2}	1.76
1/64	3.31×10^{-2}	1.46	2.11×10^{-2}	1.53
1/128	1.28×10^{-2}	1.37	8.78×10^{-3}	1.26
1/256	5.48×10^{-3}	1.23	3.74×10^{-3}	1.23
h	$Er\,5$	slope	$Er\,6$	slope
1/16	5.01×10^{-2}	–	5.38×10^{-1}	–
1/32	1.92×10^{-2}	1.38	2.54×10^{-1}	1.08
1/64	7.53×10^{-3}	1.35	1.05×10^{-1}	1.27
1/128	3.28×10^{-3}	1.20	3.88×10^{-2}	1.44
1/256	1.53×10^{-3}	1.10	1.35×10^{-2}	1.52

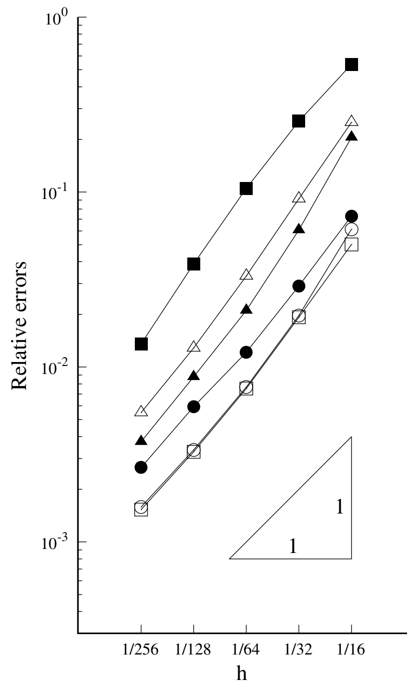

FIGURE 2. Errors and slopes for $(\nu, \varepsilon) = (10^{-1}, 10^{-3})$.

h	$Er\,1$	slope	$Er\,2$	slope
1/16	4.51×10^{-2}	–	5.83×10^{-2}	–
1/32	1.42×10^{-2}	1.67	2.36×10^{-2}	1.31
1/64	4.53×10^{-3}	1.65	9.85×10^{-3}	1.26
1/128	1.52×10^{-3}	1.58	4.89×10^{-3}	1.01
1/256	5.72×10^{-4}	1.41	2.10×10^{-3}	1.22
h	$Er\,3$	slope	$Er\,4$	slope
1/16	4.78×10^{-1}	–	3.16×10^{-1}	–
1/32	2.00×10^{-1}	1.26	9.18×10^{-2}	1.79
1/64	7.03×10^{-2}	1.51	2.95×10^{-2}	1.64
1/128	2.31×10^{-2}	1.60	1.17×10^{-2}	1.33
1/256	8.04×10^{-3}	1.52	5.01×10^{-3}	1.23
h	$Er\,5$	slope	$Er\,6$	slope
1/16	4.93×10^{-2}	–	7.97×10^{-1}	–
1/32	1.92×10^{-2}	1.36	6.05×10^{-1}	0.40
1/64	7.30×10^{-3}	1.39	5.32×10^{-1}	0.19
1/128	2.91×10^{-3}	1.33	4.04×10^{-1}	0.40
1/256	1.24×10^{-3}	1.22	2.74×10^{-1}	0.56

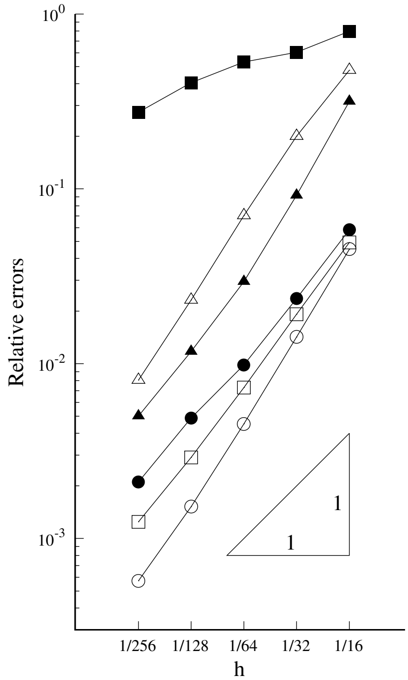

FIGURE 3. Errors and slopes for $(\nu, \varepsilon) = (1, 0)$.

7. CONCLUSIONS

In this paper we have presented a linear stabilized Lagrange–Galerkin scheme (3.4) for the Oseen-type diffusive Peterlin viscoelastic model. The scheme employs the conforming linear finite elements for all unknowns, velocity, pressure and conformation tensor, together with Brezzi–Pitkäranta's stabilization method. In Theorems 4.7 and 4.8 we have established error estimates with the optimal convergence order under mild conditions, $\Delta t = \mathcal{O}(1/\sqrt{1 + |\log h|})$ for $d = 2$ and $\Delta t = \mathcal{O}(\sqrt{h})$ for $d = 3$. These estimates hold in the standard norms not only for the velocity and the conformation tensor but also for the pressure. The theoretical convergence order has been confirmed by two-dimensional numerical experiments.

Although we have treated the stabilized scheme to reduce the number of degrees of freedom, the extension of the result to the combination of stable pairs for $(\mathbf{u}, p)$ and conventional elements for $\mathbf{C}$ is straightforward, *e.g.*, P2/P1/P2 element. In future we will extend this work to the Peterlin viscoelastic model with the nonlinear convective terms, and compare numerical results with other schemes in some benchmark problems.

We recall that in our previous paper [29], Part I, essentially unconditional stability and error estimates with the optimal convergence order were proved in two space dimensions. There, our analysis allowed to include also the case $\varepsilon = 0$.

APPENDIX

A.1. Proof of Lemma 5.9

We prove only (5.10c), (5.10d), (5.10h) and (5.10l), since (5.10a), (5.10b) and (5.10f) have been proved in Part I [29] and the other estimates are similarly obtained.

(5.10c), (5.10d) and (5.10h) are obtained as follows:

$$\begin{aligned}
\|\mathbf{r}_{h3}^n\|_{-1} &\leq \|(\operatorname{tr}\mathbf{C}^n)(\mathbf{C}^n - \mathbf{C}^{n-1} + \mathbf{\Xi}^{n-1} - \mathbf{E}_h^{n-1})\|_0 \leq c_s\big(\|\mathbf{C}^n - \mathbf{C}^{n-1}\|_0 + \|\mathbf{\Xi}^{n-1}\|_0 + \|\mathbf{E}_h^{n-1}\|_0\big)\\
&\leq c_s\big(\sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)} + \alpha_{32}h\|\mathbf{C}^{n-1}\|_2 + \|\mathbf{E}_h^{n-1}\|_0\big)\\
&\leq c_s'\big(\|\mathbf{E}_h^{n-1}\|_0 + \sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)} + h\big),\\
\|\mathbf{r}_{h4}^n\|_{-1} &\leq \|[\operatorname{tr}(\mathbf{\Xi}^n - \mathbf{E}_h^n)]\mathbf{C}_h^{n-1}\|_0 \leq c\|\mathbf{C}_h^{n-1}\|_{0,\infty}\|\operatorname{tr}(\mathbf{\Xi}^n - \mathbf{E}_h^n)\|_0\\
&\leq c'\|\mathbf{C}_h^{n-1}\|_{0,\infty}(\|\mathbf{\Xi}^n\|_0 + \|\mathbf{E}_h^n\|_0) \leq c'\|\mathbf{C}_h^{n-1}\|_{0,\infty}(\alpha_{32}h\|\mathbf{C}^n\|_2 + \|\mathbf{E}_h^n\|_0)\\
&\leq c_s\|\mathbf{C}_h^{n-1}\|_{0,\infty}(\|\mathbf{E}_h^n\|_0 + h),\\
\|\mathbf{R}_{h4}^n\|_0 &= 2\|(\nabla\mathbf{e}_h^n)\mathbf{C}_h^{n-1}\|_0 \leq 2d\|\mathbf{C}_h^{n-1}\|_{0,\infty}\|\nabla\mathbf{e}_h^n\|_0 \leq 2d\|\mathbf{C}_h^{n-1}\|_{0,\infty}\|\mathbf{e}_h^n\|_1,
\end{aligned}$$

where in the estimate of $\|\mathbf{R}_{h4}^n\|_0$ the inequality $\|AB\|_0 \leq d\|A\|_{0,\infty}\|B\|_0$ for $A \in L^\infty(\Omega)^{d\times d}$ and $B \in L^2(\Omega)^{d\times d}$ has been employed.

Finally, (5.10l) is proved as

$$\begin{aligned}
\|\mathbf{R}_{h8}^n\|_0 &= \|[\operatorname{tr}(\mathbf{C}_h^{n-1} + \hat{\mathbf{C}}_h^{n-1})](\operatorname{tr}\mathbf{E}_h^{n-1})\mathbf{C}^n\|_0 \leq c_s(\|\mathbf{C}_h^{n-1}\|_{0,\infty} + \|\hat{\mathbf{C}}_h^{n-1}\|_{0,\infty})\|\mathbf{E}_h^{n-1}\|_0\\
&\leq c_s'(\|\mathbf{C}_h^{n-1}\|_{0,\infty} + 1)\|\mathbf{E}_h^{n-1}\|_0,
\end{aligned}$$

where for the last inequality we have used the boundedness of $\|\hat{\mathbf{C}}_h^{n-1}\|_{0,\infty}$ obtained by the estimate

$$\begin{aligned}
\|\hat{\mathbf{C}}_h^{n-1}\|_{0,\infty} &\leq \|\hat{\mathbf{C}}_h^{n-1} - \Pi_h\mathbf{C}^{n-1}\|_{0,\infty} + \|\Pi_h\mathbf{C}^{n-1}\|_{0,\infty} \leq \alpha_{21}D(h)\|\hat{\mathbf{C}}_h^{n-1} - \Pi_h\mathbf{C}^{n-1}\|_1 + \|\mathbf{C}\|_{C(L^\infty)}\\
&\leq \alpha_{21}D(h)\big(\|\hat{\mathbf{C}}_h^{n-1} - \mathbf{C}^{n-1}\|_1 + \|\mathbf{C}^{n-1} - \Pi_h\mathbf{C}^{n-1}\|_1\big) + \|\mathbf{C}\|_{C(L^\infty)}\\
&\leq \alpha_{21}D(h)\big(\alpha_{32}h\|\mathbf{C}^{n-1}\|_2 + \alpha_{20}h\|\mathbf{C}^{n-1}\|_2\big) + \|\mathbf{C}\|_{C(L^\infty)}\\
&\leq \alpha_{21}hD(h)(\alpha_{20} + \alpha_{32})\|\mathbf{C}\|_{C(H^2)} + \|\mathbf{C}\|_{C(L^\infty)}\\
&\leq \alpha_{21}h_1D(h_1)(\alpha_{20} + \alpha_{32})\|\mathbf{C}\|_{C(H^2)} + \|\mathbf{C}\|_{C(L^\infty)} \leq c_s.
\end{aligned}$$

A.2. Difference of $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ and $(\mathbf{u}, p, \mathbf{C})$ in Example.

In Section 6 we have computed the difference of $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ and $(\Pi_h\mathbf{u}, \Pi_h p, \Pi_h\mathbf{C})$. Here, we give additional information on the error between $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ and $(\mathbf{u}, p, \mathbf{C})$. We introduce a numerical integration formula of degree five with seven quadrature points for each triangle, and we denote the norm derived by the formula by adding the prime to the corresponding norm,

$$\|\psi\|_{L^2(\Omega)'} = \left\{\sum_{K\in\mathcal{T}_h}\operatorname{meas}(K)\sum_{i=1}^{7}|\psi(a_i^K)|^2\,w_i\right\}^{1/2} \approx \|\psi\|_{L^2(\Omega)},$$

where $\{(a_i^K, w_i)\}$ is a set of pairs of quadrature point and weight on $K \in \mathcal{T}_h$. When ψ is a function in P1 finite element space, it holds that $\|\psi\|_{L^2(\Omega)'} = \|\psi\|_{L^2(\Omega)}$. We abbreviate $\|\psi\|_{L^2(\Omega)'}$ as $\|\psi\|_{L^{2\prime}}$. In the following the symbol $\prime$ means that the numerical integration is used in place of the exact integration. We define the relative errors $Er\,k'$, $k = 1, \ldots, 6$, by

$$Er\,1' = \frac{\|\mathbf{u}_h - \mathbf{u}\|_{\ell^\infty(L^{2\prime})}}{\|\Pi_h\mathbf{u}\|_{\ell^\infty(L^2)}}, \qquad Er\,2' = \frac{\|\mathbf{u}_h - \mathbf{u}\|_{\ell^2(H^{1\prime})}}{\|\Pi_h\mathbf{u}\|_{\ell^2(H^1)}}, \qquad Er\,3' = \frac{\|p_h - p\|_{\ell^2(L^{2\prime})}}{\|\Pi_h p\|_{\ell^2(L^2)}},$$

$$Er\,4' = \frac{|p_h - p|_{\ell^2(|\cdot|_h')}}{\|\Pi_h p\|_{\ell^2(L^2)}}, \qquad Er\,5' = \frac{\|\mathbf{C}_h - \mathbf{C}\|_{\ell^\infty(L^{2\prime})}}{\|\Pi_h\mathbf{C}\|_{\ell^\infty(L^2)}}, \qquad Er\,6' = \frac{\|\mathbf{C}_h - \mathbf{C}\|_{\ell^2(H^{1\prime})}}{\|\Pi_h\mathbf{C}\|_{\ell^2(H^1)}}.$$

We deal with the case $(\nu, \varepsilon) = (10^{-1}, 10^{-1})$. Table 2 shows the comparison of the values of $Er\,k'$ with those of $Er\,k$, which reflects that convergence order of $Er\,k$ is almost same with those of $Er\,k'$, though the

TABLE 2. Comparison of $Er\,k$ with $Er\,k'$, $k = 1, \ldots, 6$, for $(\nu, \varepsilon) = (10^{-1}, 10^{-1})$.

h	$Er\,1$	slope	$Er\,1'$	slope	$Er\,2$	slope	$Er\,2'$	slope
1/16	6.29×10^{-2}	–	8.15×10^{-2}	–	7.94×10^{-2}	–	1.94×10^{-1}	–
1/32	2.21×10^{-2}	1.51	2.68×10^{-2}	1.60	3.14×10^{-2}	1.34	9.20×10^{-2}	1.08
1/64	8.98×10^{-3}	1.30	1.02×10^{-2}	1.39	1.32×10^{-2}	1.25	4.54×10^{-2}	1.02
1/128	4.07×10^{-3}	1.14	4.40×10^{-3}	1.22	6.35×10^{-3}	1.05	2.27×10^{-2}	1.00
1/256	1.95×10^{-3}	1.07	2.03×10^{-3}	1.12	2.86×10^{-3}	1.15	1.12×10^{-2}	1.02
h	$Er\,3$	slope	$Er\,3'$	slope	$Er\,4$	slope	$Er\,4'$	slope
1/16	2.02×10^{-1}	–	2.13×10^{-1}	–	1.70×10^{-1}	–	1.81×10^{-1}	–
1/32	7.11×10^{-2}	1.50	7.38×10^{-2}	1.53	4.99×10^{-2}	1.77	5.21×10^{-2}	1.80
1/64	2.67×10^{-2}	1.41	2.73×10^{-2}	1.43	1.86×10^{-2}	1.42	1.90×10^{-2}	1.46
1/128	1.11×10^{-2}	1.27	1.12×10^{-2}	1.28	8.39×10^{-3}	1.15	8.44×10^{-3}	1.17
1/256	5.01×10^{-3}	1.15	5.03×10^{-3}	1.16	3.69×10^{-3}	1.19	3.69×10^{-3}	1.19
h	$Er\,5$	slope	$Er\,5'$	slope	$Er\,6$	slope	$Er\,6'$	slope
1/16	2.80×10^{-2}	–	2.80×10^{-2}	–	1.22×10^{-1}	–	1.64×10^{-1}	–
1/32	1.14×10^{-2}	1.30	1.14×10^{-2}	1.30	4.41×10^{-2}	1.47	6.95×10^{-2}	1.24
1/64	4.90×10^{-3}	1.21	4.90×10^{-3}	1.21	1.72×10^{-2}	1.35	3.22×10^{-2}	1.11
1/128	2.30×10^{-3}	1.09	2.30×10^{-3}	1.09	7.64×10^{-3}	1.17	1.56×10^{-2}	1.04
1/256	1.11×10^{-3}	1.05	1.11×10^{-3}	1.05	3.59×10^{-3}	1.09	7.69×10^{-3}	1.02

values of $Er\,2'$ are about three to four times larger than $Er\,2$. Therefore, the computation of the difference of $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ and $(\Pi_h\mathbf{u}, \Pi_h p, \Pi_h\mathbf{C})$ is sufficient in order to observe the behavior of convergence of $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ to $(\mathbf{u}, p, \mathbf{C})$.

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