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A CONTACT PROBLEM IN THERMOVISCOELASTIC DIFFUSION THEORY WITH SECOND SOUND *, **

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Abstract. We consider a contact problem in thermoviscoelastic diffusion theory in one space dimension with second sound. The contact is modeled by the Signorini's condition and the stress-strain constitutive equation is of Kelvin−Voigt type. The thermal and diffusion disturbances are modeled by Cattaneo's law for heat and diffusion equations to remove the physical paradox of infinite propagation speed in the classical theory within Fourier's law. The system of equations is a coupling of a hyperbolic equation with four parabolic equations. It poses some new mathematical difficulties due to the nonlinear boundary conditions and the lack of regularity. We prove that the viscoelastic term provides additional regularity leading to the existence of weak solutions. Then, fully discrete approximations to a penalized problem are considered by using the finite element method. A stability property is shown, which leads to a discrete version of the energy decay property. *A priori* error analysis is then provided, from which the linear convergence of the algorithm is derived. Finally, we give some computational results.

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1. Introduction

The classical model for the propagation of heat turns into the well-known Fourier's law

$$q + \kappa \nabla \theta = 0,$$

where θ is the temperature (difference to a fixed constant reference temperature), q is the heat conduction vector and κ is the coefficient of thermal conductivity. The model using classic Fourier's law inhibits the physical paradox of infinite propagation speed of signals. To eliminate this paradox a generalized thermoelasticity theory has been developed subsequently. The development of this theory was accelerated by the advent of the second sound effects observed experimentally in materials at a very low temperature. In heat transfer problems involving

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very short time intervals and/or very high heat fluxes, it has been revealed that the inclusion of the second sound effects to the original theory yields results which are realistic and very much different from those obtained with classic Fourier's law. The first theory was developed by Lord and Shulman [21]. In this theory, a modified law of heat conduction, the Cattaneo's law given by

$$\tau_0 q_t + q + \kappa \nabla\theta = 0,$$

replaces the classical Fourier's law. The heat equation is associated with this hyperbolic one and, hence, automatically eliminates the paradox of infinite speeds. The positive parameter τ_0 is the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature.

The research conducted in the development of high technologies after the second world war confirmed that the field of diffusion in solids can not be ignored. The problems connected with the diffusion of matter in thermoelastic bodies and the interaction of mechano-diffusion processes have become the subject of research by many authors. At high and low temperatures, the processes of heat and mass transfer play a decisive role in many satellite problems, returning space vehicles, and landing on water or land. Diffusion can be defined as the random walk, of an ensemble of particles, from regions of high concentration to regions of lower concentration. In integrated circuit fabrication, diffusion is used to introduce "dopants" in controlled amounts into the semiconductor substrate. Thermodiffusion in an elastic solid is due to the coupling of the fields of temperature, diffusion and that of strain. Oil companies are interested in the process of thermodiffusion for more efficient extraction of oil from oil deposits. The theory of thermoviscoelastic diffusion of Kelvin−Voigt type was introduced by Kubik (see [18,19]). Sherief *et al.* [24] derived the theory of thermoelastic diffusion with second sound. Other thermoelastic diffusion theories are derived in different contexts (see [1–3,6,8]) with some qualitative results. An overview of different models of this theory is represented in a survey article [5].

Thermoelastic contact problems arise in applications such as the manufacturing of castings and pistons. One of the most important being sliding systems such as brakes, clutches, and seals. Also they are found in industrial processes and everyday life such as train wheels with the rails, a shoe with the floor, tectonic plates, the car's braking system, *etc.* (see Shi and Shillor [25] for more details and references).

Concerning the thermoelastic contact problems under Fourier's law there are a great number of papers available, whereas there are few papers with second sound. In particular, a finite element approximation to a contact problem in thermoelasticity with second sound under the Green–Lindsay model was proposed by Copetti [13] and Copetti and Aouadi [14] with analytical and numerical results. Sprenger [26] investigated the existence and stability of a thermoelastic contact problem in $\mathbb{R}^n$ with second sound under the Lord–Shulman model with existence, uniqueness and stability results. Munoz–Rivera and Racke [22] studied the same problem within Fourier's law and derived existence and stability under the condition of radial symmetry. The mathematical treatment of the same problem in $\mathbb{R}$ was discussed by Elliott and Tang [15], where more complicated boundary conditions are considered. In a recent work, Fernandez Sare and Racke [16] showed that, for a damped Timoshenko beam, exponential stability is lost when substituting the Fourier's Law of heat conduction by Cattaneo's law. Recently, a nonlinear dynamic thermoviscoelastic contact problem with second sound for a beam between two stops, was investigated analytically and numerically [10]. To the best of our knowledge, the only paper considering the diffusion effects in thermoelasticity contact problems is [4] under Fourier's law. No numerical experiments were performed in the latter paper. In this light, the investigation of the behaviour of this particular system under a transition from classical to hyperbolic heat conduction poses an interesting question.

Indeed, contact problems in thermoelasticity are not in general well understood and mathematical results are relatively rare. This lack of results is generally explained by the mathematical difficulties encountered in treating such problems. But we think that a part of these difficulties are caused by the absence of the diffusion effect and a realistic thermal model in the framework of thermoelastic contact problems. A natural question is to know what happens when the diffusion effect is considered with second sound in thermoelastic contact problems. This question is not only interesting from a mechanical, mathematical and numerical viewpoint, but especially economically. In fact, an estimated 0.5 percent of the US GNP is lost (see, *e.g.*, [23]) because of insufficient control of contact processes in machines, cars and mechanical equipment. By considering the Cattaneo's law and the possible damage caused by the diffusion conduction effect, we contribute to an accurate prediction of the evolution of contact processes and their control.

This work is a first step towards the understanding of the fundamental limits of intrinsic thermo-diffusion dissipations of elastic materials in contact problems arising naturally in many applications. The governing equations of our problem correspond to the coupling of a hyperbolic and four parabolic equations. This kind of coupling has not been considered previously in the study of contact problems and poses new mathematical and numerical difficulties. In particular, we shall see that this transition from classical to hyperbolic heat conduction leads to a loss in regularity which is not easily compensated. We prove that the viscoelastic term provides additional regularity leading to the convergence of some quadratic terms in norm necessary to prove the existence of weak solutions. Due to the relevant coupling of our system and the presence of two relaxation times, the determination of a prior error estimate becomes more complicated and requires a specific technique.

In this paper we restrict ourselves to the physical setting of a one-dimensional rod and investigate a fully dynamic contact problem which includes the thermal and diffusion effects under Cattaneo's law. The contact between the free end of the rod and a rigid obstacle is modeled by a Signorini's condition.

We now describe the remaining sections of this paper. In Section 2, we present the physical setting and the mathematical model. In Section 3, we introduce a penalized problem and prove an existence and uniqueness result using the Faedo−Galerkin method. Then, in Section 4, we show that the solution to the penalty problem converges to a solution of our original problem. In Section 5, we prove the exponential decay of solutions. In Section 6, fully discrete approximations of the penalized problem are introduced. Stability and energy decay properties are proved and an *a priori* error analysis is performed, from which the linear convergence of the approximation is derived. Finally, in Section 7, we show the results of some numerical simulations where the influence of diffusion, second sound and viscosity are illustrated in contact.

2. Basic equations and preliminaries

We present below a short description of the general three-dimensional theory of thermoviscoelastic diffusion of Kelvin−Voigt type introduced by Kubik (see [18, 19]). The evolution equations of the dynamic problem in the absence of body forces and heat sources are the following:

(i) The equation of motion:

$$\rho_0 \ddot{u}_i = \sigma_{ij,j}. \tag{2.1}$$

(ii) The stress-strain-temperature-diffusion relation:

$$\sigma_{ij} = 2(\mu e_{ij} + \mu' \dot{e}_{ij}) + \delta_{ij}(\lambda e_{kk} + \lambda' \dot{e}_{kk}) - \beta_1 \delta_{ij}(T - T_0) - \beta_2 \delta_{ij} C. \tag{2.2}$$

(iii) The displacement-strain relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.3}$$

(iv) The energy equation:

$$q_{i,i} = -\rho_0 T_0 \dot{S}. \tag{2.4}$$

(v) The Cattaneo's law for the temperature:

$$\left(\tau_0 \frac{\partial}{\partial t} + 1\right) q_i = -\kappa T_{,i}. \tag{2.5}$$

(vi) The entropy-strain-temperature-diffusion relation:

$$\rho_0 T_0 S = \rho_0 c_E (T - T_0) + \beta_1 T_0 e_{kk} + \nu T_0 C. \tag{2.6}$$

(vii) The equation of conservation of mass:

$$\eta_{i,i} = -\dot{C}. \tag{2.7}$$

(iix) The Cattaneo's law for the chemical potential:

$$\left(\tau\frac{\partial}{\partial t} + 1\right)\eta_i = -DP_{,i}. \tag{2.8}$$

(ix) The chemical-strain-temperature-diffusion relation:

$$P = -\beta_2 e_{kk} + \varrho C - \nu(T - T_0), \tag{2.9}$$

where $\beta_1 = (3\lambda + 2\mu)\alpha_t$ and $\beta_2 = (3\lambda + 2\mu)\alpha_c$, α_t and α_c are, respectively, the coefficients of linear thermal and diffusion expansion, λ and μ are Lamé's constants, while λ' and μ' are the damping coefficients. ρ_0 is the mass density of the material, T is the absolute temperature of the medium, T_0 is the reference uniform temperature of the body chosen such that $|(T - T_0)/T_0| \ll 1$. q_i is the heat conduction vector, κ is the coefficient of thermal conductivity, c_E is the specific heat at constant strain. σ_{ij} are the components of the stress tensor, u_i are the components of the displacement vector, e_{ij} are the components of the strain tensor, S is the entropy per unit mass, P is the chemical potential per unit mass, C is the concentration of the diffusive material in the elastic body, η_i is the diffusion conduction vector, D is the diffusion coefficient, ν is a measure of the thermodiffusion effect and ϱ is a measure of the diffusive effect. τ_0 is the thermal relaxation time, which will ensure that the heat conduction equation will predict finite speeds of heat propagation, and τ is the diffusion relaxation time, which will also predict finite speeds o propagation of matter from one medium to the other. Both of these relaxation times ensure finite speeds of propagation for heat and diffusion waves (see (7.4)). In fact the second sound affects the system of equations (2.1)–(2.9) by eliminating the physical paradox of the infinite propagation speed of the classical model.

In this paper we study a one-dimensional thermoviscoelastic diffusion contact problem. We consider the small longitudinal deformations along the x-axis of a rod of length ℓ that may come into contact with an obstacle. We shall use the following nondimensional variables:

$$x^* = \frac{x}{\ell}, \quad u^* = \delta\frac{u}{\ell}, \quad t^* = \frac{t}{\omega}, \quad \theta = \beta_1\frac{T - T_0}{(\lambda + 2\mu)\delta}, \quad C^* = \beta_2\frac{C}{(\lambda + 2\mu)\delta}, \quad P^* = \frac{P}{\beta_2\delta},$$

$$\sigma_{ij}^* = \frac{1}{\lambda + 2\mu}\sigma_{ij}, \quad \tau_0^* = \frac{\tau_0}{\omega}, \quad \tau^* = \frac{\tau}{\omega}, \quad q^* = \frac{\omega\beta_1}{\rho_0 c_E \delta\ell(\lambda + 2\mu)}q, \quad \eta^* = \frac{\ell}{D\beta_2\delta}\eta.$$

In terms of these nondimensional variables, (2.1)–(2.9) become in one-dimensional space (dropping the asterisks for convenience)

$$\begin{aligned}
\varsigma u_{tt} &= u_{xx} + \zeta u_{xxt} - \theta_x - C_x,\\
-q_x &= \theta_t + \breve{a} u_{xt} + \breve{a}\alpha_1 C_t,\\
\theta_x &= -(\tau_0\frac{\partial}{\partial t} + 1)q,\\
\eta_x &= -\alpha_2 C_t,\\
P_x &= -(\tau\frac{\partial}{\partial t} + 1)\eta,\\
\alpha_3 C_{xx} &= u_{xxx} + \alpha_1\theta_{xx} + \alpha_2(\tau\frac{\partial}{\partial t} + 1)C_t,\\
P &= -u_x - \alpha_1\theta + \alpha_3 C,
\end{aligned} \tag{2.10}$$

where

$$\omega=\frac{\rho_0 c_E \ell^2}{\kappa},\quad \delta=\sqrt{\frac{\lambda+2\mu}{\rho_0 c_E T_0}},\quad \varsigma=\frac{\rho_0\ell^2}{\omega^2(\lambda+2\mu)},\quad \zeta=\frac{\lambda'+2\mu'}{\omega(\lambda+2\mu)},$$
$$\breve{a}=\frac{T_0\beta_1^2}{\rho_0 c_E(\lambda+2\mu)},\quad \alpha_1=\frac{(\lambda+2\mu)\nu}{\beta_1\beta_2},\quad \alpha_2=\frac{(\lambda+2\mu)\ell^2}{D\beta_2^2\omega},\quad \alpha_3=\frac{(\lambda+2\mu)\varrho}{\beta_2^2}.$$

We shall now formulate a different alternative form that will be useful in the next sections. In this new formulation, we will use the chemical potential as a state variable instead of the concentration. The alternative form can be written by substituting the last equation of (2.10) into the others: we obtain the unidimensional thermoviscoelastic diffusion problem

$$\begin{aligned}\rho u_{tt}-\alpha u_{xx}-\varpi u_{xxt}+\gamma_1\theta_x+\gamma_2 P_x=0,&\quad \text{in }]0,1[\times\mathbb{R}^+,\\ c\theta_t+dP_t+q_x+\gamma_1 u_{xt}=0,&\quad \text{in }]0,1[\times\mathbb{R}^+,\\ \tau_0 q_t+q+\theta_x=0,&\quad \text{in }]0,1[\times\mathbb{R}^+,\\ \mathrm{d}\theta_t+rP_t+\hbar\eta_x+\gamma_2 u_{xt}=0,&\quad \text{in }]0,1[\times\mathbb{R}^+,\\ \tau\eta_t+\eta+P_x=0,&\quad \text{in }]0,1[\times\mathbb{R}^+,\end{aligned}\tag{2.11}$$

where

$$\rho=\breve{a}\varsigma,\quad \alpha=\breve{a}(1-\frac{1}{\alpha_3}),\quad \varpi=\zeta\breve{a},\quad \gamma_1=\breve{a}(1+\frac{\alpha_1}{\alpha_3}),\quad \gamma_2=\frac{\breve{a}}{\alpha_3},$$
$$c=1+\breve{a}\frac{\alpha_1^2}{\alpha_3},\quad d=\frac{\alpha_1\breve{a}}{\alpha_3},\quad r=\frac{\breve{a}}{\alpha_3},\quad \hbar=\frac{\breve{a}}{\alpha_2}$$

are physical positive constants. The constants γ_1 and γ_2 are, respectively, a measure of the mechanical-thermal and mechanical-chemical coupling present in the system. The constant α can be viewed as the small amplitude wave speed about an uniform temperature and chemical potential. r and c are material constants of the chemical and thermal process, while d is a measure of the thermal-chemical coupling present in the system. The positive constant ϖ is the coefficient of viscosity and $\hbar$ is the coefficient of chemical conductivity.

The system of equations is supplemented by the initial conditions

$$\begin{aligned}u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad \theta(x,0)=\theta_0(x),\quad x\in]0,1[,\\ q(x,0)=q_0(x),\quad P(x,0)=P_0(x),\quad \eta(x,0)=\eta_0(x),\quad x\in]0,1[,\end{aligned}\tag{2.12}$$

and adequate boundary conditions. We let the rod fixed at the left end $x=0$,

$$u(0,t)=0,\tag{2.13}$$

and at the right end $x=1$ the rod is free to expand or to contract and to get in contact with a rigid obstacle, held at zero temperature and chemical potential and located at a distance $g>0$ from the position $x=1$ (see Fig. 1). At the region of contact, the stress is zero when there is no contact with the obstacle, otherwise it is compressive, *i.e.*

$$\sigma(1,t)\le 0,\quad u(1,t)\le g,\quad \sigma(1,t)(u(1,t)-g)=0,\tag{2.14}$$

where

$$\sigma(x,t)=\alpha u_x+\varpi u_{xt}-\gamma_1\theta-\gamma_2 P.$$

This contact condition is known as Signorini condition. We assume the following boundary conditions

$$q(0,t)=\eta(0,t)=0,\quad \theta(1,t)=P(1,t)=0.\tag{2.15}$$

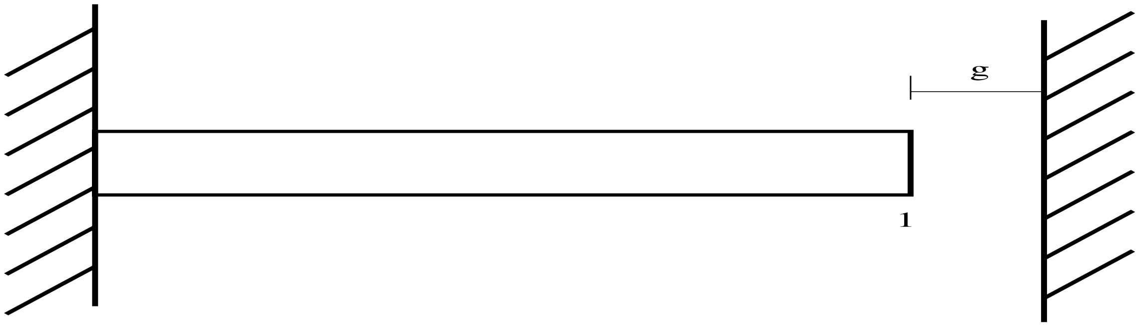

Figure 1. Contact with a rigid obstacle.

From the above physical constants, we have

$$cr - d^2 = \frac{\breve{a}}{\alpha_3} = \left(\frac{\beta_1\beta_2}{\lambda+2\mu}\right)^2\left(\frac{T_0}{\rho c_E \varrho}\right) > 0. \tag{2.16}$$

Note that this condition implies that

$$c\theta^2 + 2\mathrm{d}\theta P + rP^2 > 0.$$

Condition (2.16) is needed to stabilize the thermoelastic diffusion system (see [2] for more information on this). By virtue of (2.16) we deduce that $d/c < r/d$. Let, then, ζ be a number chosen in such a way that $d/c < \zeta < r/d$. Thus, Young's inequality leads to

$$2\int_0^1 |\mathrm{d}\theta P|\mathrm{d}x \le \frac{\mathrm{d}}{\zeta}\int_0^1 \theta^2\mathrm{d}x + \mathrm{d}\zeta\int_0^1 P^2\mathrm{d}x. \tag{2.17}$$

For the sake of simplicity, we will employ the same symbol C for different constants, even in the same formula. We will analyze functions in the variational spaces

$$\begin{aligned}\mathcal{V} =& \{w \in H^1(0,1);\quad w(0)=0\},\\ \mathcal{K} =& \{w \in \mathcal{V};\quad w(1) \le g\},\\ \mathcal{M} =& \{w \in H^1(0,1);\quad w(1)=0\}.\end{aligned}$$

Now we define what we will understand for weak solution to problem (2.11)-(2.15). Throughout the paper, we set $I = [0,T]$, with $T > 0$.

Definition 2.1. Assume that $u_0 \in \mathcal{K}$ and $u_1,\ \theta_0,\ q_0,\ P_0,\ \eta_0 \in L^2(0,1)$. We say that (u,θ,q,P,η) is a weak solution to problem (2.11)−(2.15) iff

$$\begin{gathered} u \in W^{1,\infty}(I;L^2(0,1)) \cap L^\infty(I;\mathcal{K}),\\ (\theta,P) \in L^\infty(I;L^2(0,1)\times L^2(0,1)) \cap L^2(I;\mathcal{M}\times\mathcal{M}),\\ (q,\eta) \in L^\infty(I;L^2(0,1)\times L^2(0,1)) \cap L^2(I;\mathcal{V}\times\mathcal{V}),\\ u_x(\cdot,T),\ u_t(\cdot,T),\ \theta(\cdot,T),\ q(\cdot,T),\ P(\cdot,T),\ \eta(\cdot,T) \in L^2(0,1), \end{gathered} \tag{2.18}$$

and $u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x),\ P(x,0) = P_0(x),\ \eta(x,0) = \eta_0(x)$ and satisfies :

$$\begin{aligned} &-\rho\int_0^T\int_0^1 u_t(w_t-u_t)\mathrm{d}x\mathrm{d}t + \rho\int_0^1 u_t(\cdot,T)[w(\cdot,T)-u(\cdot,T)]\mathrm{d}x - \rho\int_0^1 u_1[w(\cdot,0)-u_0]\mathrm{d}x\\ &\quad + \int_0^T\int_0^1 \Big(\alpha u_x + \varpi u_{xt} - \gamma_1\theta - \gamma_2 P\Big)(w_x-u_x)\mathrm{d}x\mathrm{d}t \ge 0, \end{aligned} \tag{2.19}$$

for any $w \in W^{1,\infty}(I; L^2(0,1)) \cap L^\infty(I; \mathcal{K})$,

$$-\int_0^T \int_0^1 (c\theta + dP)v_t \mathrm{d}x\mathrm{d}t + \int_0^1 (c\theta + dP + \gamma_1 u_x)(\cdot, T)v(\cdot, T)\mathrm{d}x - \int_0^T \int_0^1 (qv_x + \gamma_1 u_x v_t)\mathrm{d}x\mathrm{d}t = \int_0^1 (c\theta_0 + dP_0 + \gamma_1 u_{0x})v(\cdot, 0)\mathrm{d}x, \tag{2.20}$$

and

$$-\int_0^T \int_0^1 (\mathrm{d}\theta + rP)z_t \mathrm{d}x\mathrm{d}t + \int_0^1 (\mathrm{d}\theta + rP + \gamma_2 u_x)(\cdot, T)z(\cdot, T)\mathrm{d}x - \int_0^T \int_0^1 (\hbar\eta z_x + \gamma_2 u_x z_t)\mathrm{d}x\mathrm{d}t = \int_0^1 (\mathrm{d}\theta_0 + rP_0 + \gamma_2 u_{0x})z(\cdot, 0)\mathrm{d}x, \tag{2.21}$$

for all $(v, z) \in W^{1,\infty}(I; \mathcal{M} \times \mathcal{M})$,

$$-\tau_0 \int_0^T \int_0^1 q\phi_t \mathrm{d}x\mathrm{d}t + \tau_0 \int_0^1 q(\cdot, T)\phi(\cdot, T)\mathrm{d}x + \int_0^T \int_0^1 (q\phi - \theta\phi_x)\mathrm{d}x\mathrm{d}t = \tau_0 \int_0^1 q_0\phi(\cdot, 0)\mathrm{d}x, \tag{2.22}$$

and

$$\tau \int_0^T \int_0^1 \eta\psi_t \mathrm{d}x\mathrm{d}t + \tau \int_0^1 \eta(\cdot, T)\psi(\cdot, T)\mathrm{d}x + \int_0^T \int_0^1 (\eta\psi - P\psi_x)\mathrm{d}x\mathrm{d}t = \tau \int_0^1 \eta_0\psi(\cdot, 0)\mathrm{d}x, \tag{2.23}$$

for all $(\phi, \psi) \in W^{1,\infty}(I; \mathcal{V} \times \mathcal{V})$.

To prove the existence of a weak solution to the system (2.11)−(2.15), we first consider an auxiliary regularized problem and prove uniform *a priori* estimates.

3. A penalty problem

In this section, we introduce a penalized version of problem (2.11)–(2.15) by regularizing the Signorini contact condition with the well-known normal compliance condition. For any $\varepsilon > 0$ let us consider the following problem:

$$\begin{aligned} \rho u^\varepsilon_{tt} - \alpha u^\varepsilon_{xx} - \varpi u^\varepsilon_{xxt} + \gamma_1 \theta^\varepsilon_x + \gamma_2 P^\varepsilon_x &= 0, \quad \text{in }]0,1[\times\mathbb{R}^+, \\ c\theta^\varepsilon_t + \mathrm{d}P^\varepsilon_t + q^\varepsilon_x + \gamma_1 u^\varepsilon_{xt} &= 0, \quad \text{in }]0,1[\times\mathbb{R}^+, \\ \tau_0 q^\varepsilon_t + q^\varepsilon + \theta^\varepsilon_x &= 0, \quad \text{in }]0,1[\times\mathbb{R}^+, \\ \mathrm{d}\theta^\varepsilon_t + rP^\varepsilon_t + \hbar\eta^\varepsilon_x + \gamma_2 u^\varepsilon_{xt} &= 0, \quad \text{in }]0,1[\times\mathbb{R}^+ \\ \tau\eta^\varepsilon_t + \eta^\varepsilon + P^\varepsilon_x &= 0, \quad \text{in }]0,1[\times\mathbb{R}^+, \end{aligned} \tag{3.1}$$

with the initial conditions

$$\begin{aligned} &u^\varepsilon(x,0) = u^\varepsilon_0(x), \quad u^\varepsilon_t(x,0) = u^\varepsilon_1(x), \quad \theta^\varepsilon(x,0) = \theta^\varepsilon_0(x), \quad x \in]0,1[, \\ &q^\varepsilon(x,0) = q^\varepsilon_0(x), \quad P^\varepsilon(x,0) = P^\varepsilon_0(x), \quad \eta^\varepsilon(x,0) = \eta^\varepsilon_0(x), \quad x \in]0,1[, \end{aligned} \tag{3.2}$$

and the boundary conditions

$$u^\varepsilon(0,t)=0,\quad q^\varepsilon(0,t)=\eta^\varepsilon(0,t)=0,\quad \theta^\varepsilon(1,t)=P^\varepsilon(1,t)=0,\quad t\in[0,T], \tag{3.3}$$

$$\sigma^\varepsilon(1,t)=-\frac{1}{\varepsilon}[u^\varepsilon(1,t)-g]_+, \tag{3.4}$$

where

$$\sigma^\varepsilon(x,t)=\alpha u_x^\varepsilon+\varpi u_{xt}^\varepsilon-\gamma_1\theta^\varepsilon-\gamma_2 P^\varepsilon,$$

and $[u(1,t)-g]_+\equiv\max\{u(1,t)-g,0\}$. The normal compliance condition (3.4) can be interpreted as a regularization of Signorini's conditions and, in fact, it can be proved that condition (3.4) is an approximation of the Signorini's condition. The rigid obstacle is replaced by an elastic obstacle with stiffness constant $1/\varepsilon>0$. In the limit $\varepsilon\to 0$, the Signorini condition (2.14) is obtained.

Definition 3.1. Assume the initial data

$$u_0^\varepsilon\in\mathcal{K}\cap H^2(0,1),\quad u_1^\varepsilon\in H^2(0,1),\quad \theta_0^\varepsilon,P_0^\varepsilon\in\mathcal{M},\quad q_0^\varepsilon,\eta_0^\varepsilon\in\mathcal{V}. \tag{3.5}$$

Then $(u^\varepsilon,\theta^\varepsilon,q^\varepsilon,P^\varepsilon,\eta^\varepsilon)$ is a weak solution to problem (3.1)–(3.4) iff

$$\begin{gathered} u^\varepsilon\in W^{2,\infty}(I;L^2(0,1))\cap W^{1,\infty}(I;\mathcal{K}),\\ (\theta^\varepsilon,P^\varepsilon)\in W^{1,\infty}(I;L^2(0,1)\times L^2(0,1))\cap L^\infty(I;\mathcal{M}\times\mathcal{M}),\\ (q^\varepsilon,\eta^\varepsilon)\in W^{1,\infty}(I;L^2(0,1)\times L^2(0,1))\cap L^\infty(I;\mathcal{V}\times\mathcal{V}), \end{gathered} \tag{3.6}$$

and it satisfies, for all $t\in I$, $w,\phi,\psi\in\mathcal{V}$ and $\upsilon,z\in\mathcal{M}$,

$$\begin{aligned} \int_0^1\rho u_{tt}^\varepsilon w\mathrm{d}x+\int_0^1(\alpha u_x^\varepsilon+\varpi u_{xt}^\varepsilon-\gamma_1\theta^\varepsilon-\gamma_2P^\varepsilon)w_x\mathrm{d}x&=-\frac{1}{\varepsilon}[u^\varepsilon(1,t)-g]_+w(1),\\ \int_0^1(c\theta_t^\varepsilon+dP_t^\varepsilon+q_x^\varepsilon+\gamma_1u_{xt}^\varepsilon)\upsilon\mathrm{d}x&=0,\\ \int_0^1(\tau_0q_t^\varepsilon+q^\varepsilon+\theta_x^\varepsilon)\phi\mathrm{d}x&=0,\\ \int_0^1(rP_t^\varepsilon+\mathrm{d}\theta_t^\varepsilon+\hbar\eta_x^\varepsilon+\gamma_2u_{xt}^\varepsilon)z\mathrm{d}x&=0,\\ \int_0^1(\tau\eta_t^\varepsilon+\eta^\varepsilon+P_x^\varepsilon)\psi\mathrm{d}x&=0. \end{aligned} \tag{3.7}$$

The finite element method we propose in Section 6 is based on the above variational form.

We first prove an existence and uniqueness theorem to the penalized problem (3.1)–(3.4) and a convergence result to the solution to problem (2.11)–(2.15), leading to an existence theorem for that problem.

Theorem 3.2. *Assume the initial data (3.5). Then the penalized problem (3.1)–(3.4) admits a unique solution $(u^\varepsilon,\theta^\varepsilon,q^\varepsilon,P^\varepsilon,\eta^\varepsilon)$ such that*

$$\begin{gathered} u^\varepsilon\in W^{2,\infty}(I;L^2(0,1))\cap W^{1,\infty}(I;\mathcal{V}),\\ \theta^\varepsilon,P^\varepsilon\in L^\infty(I;\mathcal{M})\cap W^{1,\infty}(I;L^2(0,1)),\\ q^\varepsilon,\eta^\varepsilon\in L^\infty(I;\mathcal{V})\cap W^{1,\infty}(I;L^2(0,1)). \end{gathered}$$

Moreover, as $\varpi>0$ then $u_{xx}^\varepsilon\in L^\infty(I;L^2(0,1))$.

Proof. To construct a solution to the penalized problem we will use a Faedo−Galerkin method.
(i) **Step 1** (Faedo−Galerkin scheme): If $(v, \vartheta, \varphi, \wp, \xi)$ satisfy

$$v(\cdot,0) = v_t(\cdot,0) = \vartheta(\cdot,0) = \varphi(\cdot,0) = \wp(\cdot,0) = \xi(\cdot,0) = 0$$

and

$$\begin{aligned}
\int_0^1 \rho v_{tt} w \mathrm{d}x + \int_0^1 (\alpha v_x + \varpi v_{xt} - \gamma_1 \vartheta - \gamma_2 \wp) w_x \mathrm{d}x &= -\frac{1}{\varepsilon}[v(1,t) + u_0^\varepsilon(1) + t u_1^\varepsilon(1) - g]_+ w(1) \\
&\quad + \int_0^1 \xi_1 w \mathrm{d}x, \\
\int_0^1 (c\vartheta_t + d\wp_t + \varphi_x + \gamma_1 v_{xt}) \upsilon \mathrm{d}x &= \int_0^1 \xi_2 \upsilon \mathrm{d}x, \\
\int_0^1 (\tau_0 \varphi_t + \varphi + \vartheta_x) \phi \mathrm{d}x &= \int_0^1 \xi_3 \phi \mathrm{d}x, \\
\int_0^1 (r\wp_t + d\vartheta_t + \hbar \zeta_x + \gamma_2 v_{xt}) z \mathrm{d}x &= \int_0^1 \xi_4 z \mathrm{d}x, \\
\int_0^1 (\tau \zeta_t + \zeta + \wp_x) \psi \mathrm{d}x &= \int_0^1 \xi_5 \psi \mathrm{d}x,
\end{aligned} \tag{3.8}$$

with

$$\begin{aligned}
\xi_1 &= \alpha(u_{0xx}^\varepsilon + t u_{1xx}^\varepsilon) + \varpi u_{1xx}^\varepsilon - \gamma_1 \theta_{0x}^\varepsilon - \gamma_2 P_{0x}^\varepsilon, \\
\xi_2 &= -q_{0x}^\varepsilon - \gamma_1 u_{1x}^\varepsilon, \\
\xi_3 &= -q_0^\varepsilon - \theta_{0x}^\varepsilon, \\
\xi_4 &= -\hbar \eta_{0x}^\varepsilon - \gamma_2 u_{1x}^\varepsilon, \\
\xi_5 &= -\eta_0^\varepsilon - P_{0x}^\varepsilon,
\end{aligned} \tag{3.9}$$

then $u^\varepsilon := v + u_0^\varepsilon + t u_1^\varepsilon$, $\theta^\varepsilon := \vartheta + \theta_0^\varepsilon$, $q^\varepsilon := \varphi + q_0^\varepsilon$, $P^\varepsilon := \wp + P_0^\varepsilon$ and $\eta^\varepsilon := \zeta + \eta_0^\varepsilon$ are a solution to the penalized problem.

Let $\{w^p, \phi^p, \psi^p\}_{p=1}^\infty$ and $\{\upsilon^p, z^p\}_{p=1}^\infty$ be orthonormal bases of $\mathcal{V}$ and $\mathcal{M}$, respectively. We look for $(v^m, \vartheta^m, \varphi^m, \wp^m, \xi^m)$ solving the following set of equations in $[0,T]$, for $p = 1, \dots, m$,

$$\begin{aligned}
\int_0^1 \rho v_{tt}^m w^p \mathrm{d}x + \int_0^1 (\alpha v_x^m + \varpi v_{xt}^m - \gamma_1 \vartheta^m - \gamma_2 \wp^m) w_x^p \mathrm{d}x &= -\frac{1}{\varepsilon}[v^m(1,t) + u_0^\varepsilon(1) + t u_1^\varepsilon(1) - g]_+ w^p(1) \\
&\quad + \int_0^1 \xi_1 w^p \mathrm{d}x, \\
\int_0^1 (c\vartheta_t^m + d\wp_t^m + \gamma_1 v_{xt}^m) \upsilon^p \mathrm{d}x - \int_0^1 \varphi^m \upsilon_x^p \mathrm{d}x &= \int_0^1 \xi_2 \upsilon^p \mathrm{d}x, \\
\int_0^1 (\tau_0 \varphi_t^m + \varphi^m) \phi^p \mathrm{d}x - \int_0^1 \vartheta^m \phi_x^p \mathrm{d}x &= \int_0^1 \xi_3 \phi^p \mathrm{d}x, \\
\int_0^1 (r\wp_t^m + d\vartheta_t^m + \gamma_2 v_{xt}^m) z^p \mathrm{d}x + \hbar \int_0^1 \zeta^m z_x^p \mathrm{d}x &= \int_0^1 \xi_4 z^p \mathrm{d}x, \\
\int_0^1 (\tau \zeta_t^m + \zeta^m) \psi^p \mathrm{d}x - \int_0^1 \wp^m \psi_x^p \mathrm{d}x &= \int_0^1 \xi_5 \psi^p \mathrm{d}x,
\end{aligned} \tag{3.10}$$

with initial data

$$v^m(\cdot,0) = v_t^m(\cdot,0) = \vartheta^m(\cdot,0) = \varphi^m(\cdot,0) = \wp^m(\cdot,0) = \xi^m(\cdot,0) = 0, \tag{3.11}$$

where

$$v^m(t,x)=\sum_{p=1}^{m}a_p^m(t)w^p(x),\ \ \vartheta^m(t,x)=\sum_{p=1}^{m}b_p^m(t)\upsilon^p(x),\ \ \varphi^m(t,x)=\sum_{p=1}^{m}c_p^m(t)\phi^p(x),$$

$$\wp^m(t,x)=\sum_{p=1}^{m}d_p^m(t)z^p(x),\ \ \zeta^m(t,x)=\sum_{p=1}^{m}\mathrm{e}_p^m(t)\psi^p(x)$$

with unknown coefficients ($a_p^m,\ b_p^m,\ c_p^m,\ d_p^m,\ \mathrm{e}_p^m$). Then, (3.10)−(3.11) is a set of ordinary differential equations for ($a_p^m,\ b_p^m,\ c_p^m,\ d_p^m,\ \mathrm{e}_p^m$) possessing a local solution.

(ii) **Step 2** (Energy estimates): Multiplying $(3.10)_1$ by $\frac{\mathrm{d}}{\mathrm{d}t}a_p^m$, $(3.10)_2$ by b_p^m, $(3.10)_3$ by c_p^m, $(3.10)_4$ by d_p^m and $(3.10)_5$ by e_p^m we obtain, after summing from 1 to m,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^m(t)+\int_0^1\Big(\varpi(v_{xt}^m)^2+(\varphi^m)^2+\hbar(\zeta^m)^2\Big)\mathrm{d}x=\int_0^1 v_t^m\xi_1\mathrm{d}x+\int_0^1\vartheta^m\xi_2\mathrm{d}x+\int_0^1\varphi^m\xi_3\mathrm{d}x$$
$$+\int_0^1\wp^m\xi_4\mathrm{d}x+\hbar\int_0^1\zeta^m\xi_5\mathrm{d}x, \tag{3.12}$$

where

$$\mathcal{E}^m(t)=\frac{1}{2}\int_0^1\Big(\rho(v_t^m)^2+\alpha(v_x^m)^2+c(\vartheta^m)^2+2d\vartheta^m\wp^m+r(\wp^m)^2+\tau_0(\varphi^m)^2+\tau\hbar(\zeta^m)^2\Big)\mathrm{d}x$$
$$+\frac{1}{2\varepsilon}[v^m(1,.)+u_0^\varepsilon(1)+tu_1^\varepsilon(1)-g]_+^2. \tag{3.13}$$

Since all the material parameters are positive and (2.16)–(2.17) hold, after integrating (3.12) from 0 to t and using Gronwall's inequality, we obtain

$$\begin{aligned}\|v_t^m\|&\le C,\\ \|v_x^m\|&\le C,\\ \|\vartheta^m\|&\le C,\\ \|\wp^m\|&\le C,\\ \|\varphi^m\|&\le C,\\ \|\zeta^m\|&\le C,\\ \frac{1}{2\varepsilon}[v^m(1,.)+u_0^\varepsilon(1)+tu_1^\varepsilon(1)-g]_+^2&\le C,\\ \varpi\int_0^t\|v_{xt}^m\|^2\mathrm{d}s&\le C,\\ \int_0^t\|\varphi^m\|^2\mathrm{d}s&\le C,\\ \int_0^t\|\zeta^m\|^2\mathrm{d}s&\le C.\end{aligned}$$

(iii) **Step 3** (Regularity of solution): To get second order estimates, we differentiate equations (3.10) with respect to t. Then we multiply the first resulting equation by $\frac{\mathrm{d}^2a_p^m}{\mathrm{d}t^2}$, the second one by $\frac{\mathrm{d}b_p^m}{\mathrm{d}t}$, the third one by $\frac{dc_p^m}{\mathrm{d}t}$, the fourth one by $\frac{\mathrm{d}d_p^m}{\mathrm{d}t}$ and the last one by $\frac{\mathrm{d}e_p^m}{\mathrm{d}t}$, and we find that, after summation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}^m(t)+\int_0^1\Big(\varpi(v_{xtt}^m)^2+(\varphi_t^m)^2+\hbar(\zeta_t^m)^2\Big)\mathrm{d}x=-\int_0^1 v_{tt}^m\xi_{1t}\mathrm{d}x-\frac{1}{\varepsilon}B^m(t)v_{tt}^m(1,t), \tag{3.14}$$

where

$$\mathcal{K}^m(t) = \frac{1}{2}\int_0^1 \Big(\rho(v_{tt}^m)^2 + \alpha(v_{xt}^m)^2 + c(\vartheta_t^m)^2 + 2d\vartheta_t^m\wp_t^m + r(\wp_t^m)^2 + \tau_0(\varphi_t^m)^2 + \tau\hbar(\zeta_t^m)^2\Big)\mathrm{d}x,$$

and $B^m(t)$ is defined as the weak derivative

$$B^m(t) := \frac{\mathrm{d}}{\mathrm{d}t}[v^m(1,t) + u_0^\varepsilon(1) + tu_1^\varepsilon(1) - g]_+.$$

By means of Young and Sobolev inequalities and recalling that $|\frac{\mathrm{d}}{\mathrm{d}t}[f]_+| \le |\frac{\mathrm{d}}{\mathrm{d}t}f|$, we can estimate

$$\begin{aligned}\frac{1}{\varepsilon}|B^m(t)||v_{tt}^m(1,t)| &\le \frac{\varpi}{4}|v_{tt}^m(1,t)|^2 + C(\varepsilon)|B^m(t)|^2\\ &\le \frac{\varpi}{4}\int_0^1 (v_{xtt}^m)^2\mathrm{d}x + C(\varepsilon)\left(\int_0^1 (v_{xt}^m)^2\mathrm{d}x + |u_1^\varepsilon(1)|^2\right).\end{aligned} \tag{3.15}$$

According to (3.14), we obtain after using the Poincaré inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}^m(t) + \int_0^1 \Big(\varpi(v_{xtt}^m)^2 + (\varphi_t^m)^2 + (\zeta_t^m)^2\Big)\mathrm{d}x \le \frac{\varpi}{2}\int_0^1 (v_{xtt}^m)^2\mathrm{d}x + C(\varepsilon)\int_0^1 \Big((\xi_{1t})^2 + (v_{xt}^m)^2\Big)\mathrm{d}x + C(\varepsilon)|u_1^\varepsilon(1)|^2. \tag{3.16}$$

Now, we observe that letting $t = 0$ in (3.10) and recalling (3.11) we have $\mathcal{K}^m(0) \le C$ independently of m. Hence, for constant ε, using (2.17) and Gronwall's inequality we conclude that

$$\begin{aligned}&(v^m)_m \text{ is bounded in } W^{2,\infty}(I;L^2(0,1)) \cap W^{1,\infty}(I;\mathcal{V}),\\ &(\vartheta^m)_m \text{ and } (\wp^m)_m \text{ are bounded in } W^{1,\infty}(I;L^2(0,1))\\ &(\varphi^m)_m \text{ and } (\zeta^m)_m \text{ are bounded in } W^{1,\infty}(I;L^2(0,1)).\end{aligned} \tag{3.17}$$

It follows from (3.17) that, for fixed ε, there is a subsequence, again denoted by $(v^m, \vartheta^m, \varphi^m, \wp^m, \zeta^m)_m$, and $(v, \vartheta, \varphi, \wp, \zeta)$ such that, as $m \to \infty$,

$$\begin{aligned}(v^m)_m &\rightharpoonup v \text{ weakly} \star \text{ in } W^{2,\infty}(I;L^2(0,1)) \cap W^{1,\infty}(I;\mathcal{V}),\\ (\vartheta^m,\ \wp^m)_m &\rightharpoonup (\vartheta,\ \wp) \text{ weakly} \star \text{ in } W^{1,\infty}(I;L^2(0,1)),\\ (\varphi^m,\ \zeta^m)_m &\rightharpoonup (\varphi,\ \zeta) \text{ weakly} \star \text{ in } W^{1,\infty}(I;L^2(0,1)).\end{aligned} \tag{3.18}$$

With the help of Lemma 1.4 from [17] (essentially Gagliardo–Nirenberg type estimates), the convergence

$$[v^m(1,.) + u_0^\varepsilon(1) + tu_1^\varepsilon(1) - g]_+ \to [v(1,.) + u_0^\varepsilon(1) + tu_1^\varepsilon(1) - g]_+ \text{ in } C^0(I;L^2(0,1)),$$

as well as

$$v_t^m \to v_t \text{ in } C^0(I;L^2(0,1)),$$

follow. Using (3.18) and letting $m \to \infty$ in (3.10), we conclude that

$$u^\varepsilon := v + u_0^\varepsilon + tu_1^\varepsilon, \quad \theta^\varepsilon := \vartheta + \theta_0^\varepsilon, \quad q^\varepsilon := \varphi + q_0^\varepsilon, \quad P^\varepsilon := \wp + P_0^\varepsilon, \quad \eta^\varepsilon := \zeta + \eta_0^\varepsilon$$

satisfy problem (3.6)–(3.7).

Next we note that equation $(3.7)_1$ yields

$$u_{tt}^\varepsilon - \sigma_x^\varepsilon = 0$$

and, therefore, $\sigma_x^\varepsilon \in L^\infty(I;L^2(0,1))$. To prove H^2 regularity for u^ε, we observe that from the defining equations we have

$$\sigma^\varepsilon + \gamma_1\theta^\varepsilon + \gamma_2 P^\varepsilon = \alpha u_x^\varepsilon + \varpi u_{xt}^\varepsilon.$$

Let us multiply the above equation by the appropriate integrating factor. Since $\varpi > 0$, we can write it as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{\alpha}{\varpi}t}u_x^\varepsilon(\cdot,t)\right) = \frac{\mathrm{e}^{\frac{\alpha}{\nu}t}}{\varpi}\Big(\gamma_1\theta^\varepsilon(\cdot,t) + \gamma_2 P^\varepsilon(\cdot,t) + \sigma^\varepsilon(\cdot,t)\Big)$$

and integration in time gives

$$u_x^\varepsilon(\cdot,t) = \mathrm{e}^{-\frac{\alpha}{\varpi}t}\left[u_x^\varepsilon(\cdot,0) + \frac{1}{\varpi}\int_0^t \mathrm{e}^{\frac{\alpha}{\varpi}s}\Big(\gamma_1\theta^\varepsilon(\cdot,s) + \gamma_2 P^\varepsilon(\cdot,s) + \sigma^\varepsilon(\cdot,s)\Big)\mathrm{d}s\right].$$

Differentiating with respect to x, we find that $u_{xx}^\varepsilon(\cdot,t) \in L^2(0,1)$.

(iv) **Step 4** (Uniqueness): Let $(u_1^\varepsilon, \theta_1^\varepsilon, q_1^\varepsilon, P_1^\varepsilon, \eta_1^\varepsilon)$ and $(u_2^\varepsilon, \theta_2^\varepsilon, q_2^\varepsilon, P_2^\varepsilon, \eta_2^\varepsilon)$ be two solutions to the system (3.1)–(3.4). Let us denote by

$$U = u_1^\varepsilon - u_2^\varepsilon,\quad \Theta = \theta_1^\varepsilon - \theta_2^\varepsilon,\quad Q = q_1^\varepsilon - q_2^\varepsilon,\ \Phi = P_1^\varepsilon - P_2^\varepsilon,\quad \Psi = \eta_1^\varepsilon - \eta_2^\varepsilon,$$

then (U,Θ,Q,Φ,Ψ) satisfies

$$\begin{aligned}
\rho U_{tt} - \alpha U_{xx} - \varpi U_{xxt} + \gamma_1\Theta_x + \gamma_2\Phi_x &= 0 \qquad \text{in } (0,1)\times(0,T),\\
c\Theta_t + d\Phi_t + Q_x + \gamma_1 U_{xt} &= 0 \qquad \text{in } (0,1)\times(0,T),\\
\tau_0 Q_t + Q + \Theta_x &= 0, \qquad \text{in } (0,1)\times(0,T),\\
d\Theta_t + r\Phi_t + \hbar\Psi_x + \gamma_2 U_{xt} &= 0, \qquad \text{in } (0,1)\times(0,T),\\
\tau\Psi_t + \Psi + \Phi_x &= 0, \qquad \text{in } (0,1)\times(0,T),
\end{aligned} \tag{3.19}$$

together with

$$\sigma^1(1,t) - \sigma^2(1,t) = -\frac{1}{\varepsilon}\Big([u_1^\varepsilon(1,t)-g]_+ - [u_2^\varepsilon(1,t)-g]_+\Big),$$

$$U(0,t) = 0,\quad Q(0,t) = \Psi(0,t) = 0,\quad \Theta(1,t) = \Phi(1,t) = 0,$$
$$U(x,0) = 0,\quad \Theta(x,0) = 0,\quad Q(x,0) = 0,\quad \Phi(x,0) = 0,\quad \Psi(x,0) = 0.$$

Multiplying equations $(3.19)_{1,2,3,4,5}$ by $U_t \in L^2(1,0)$, $\Theta \in L^2(1,0)$, $Q \in L^2(1,0)$, $\Phi \in L^2(1,0)$ and $\Psi \in L^2(1,0)$, respectively, and integrating from 0 to 1, we deduce as before that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) + \int_0^1 \Big(\varpi U_{xt}^2 + Q^2 + \hbar\Psi^2\Big)\mathrm{d}x = -\frac{1}{\varepsilon}\Big([u_1^\varepsilon(1,t)-g]_+ - [u_2^\varepsilon(1,t)-g]_+\Big)U_t(1,t), \tag{3.20}$$

where

$$\mathcal{F}(t) = \frac{1}{2}\int_0^1 \Big(\rho U_t^2 + \alpha U_x^2 + c\Theta^2 + 2\mathrm{d}\theta\Phi + r\Phi^2 + \tau_0 Q^2 + \tau\hbar\Psi^2\Big)\mathrm{d}x.$$

Noting that $|f_+ - h_+| \leq |f - h|$ and using Young and Poincaré inequalities we arrive at

$$\left| -\frac{1}{\varepsilon}\left([u_1^\varepsilon(1,t) - g]_+ - [u_2^\varepsilon(1,t) - g]_+\right) U_t(1,t)\right| \leq \frac{\varpi}{2}\int_0^1 (U_{xt})^2 \mathrm{d}x + C\int_0^1 (U_x)^2 \mathrm{d}x. \tag{3.21}$$

From equations (3.20) and (3.21) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) \leq C\mathcal{F}(t),$$

for some constant $C > 0$. Applying Gronwall's Lemma to this inequality and keeping in mind that (2.16) holds and $\mathcal{F}(0) = 0$, we obtain $\mathcal{F}(t) = 0$ on $[0,T]$, which yields $(u_1^\varepsilon, \theta_1^\varepsilon, q_1^\varepsilon, P_1^\varepsilon, \eta_1^\varepsilon) = (u_2^\varepsilon, \theta_2^\varepsilon, q_2^\varepsilon, P_2^\varepsilon, \eta_2^\varepsilon)$. The proof of the theorem is now complete. □

4. Existence of weak solutions

The idea is to consider a sequence of approximate solutions (provided by Thm. 3.2) and to show their convergence (as $\varepsilon \to 0$) to a weak solution to problem (2.11)–(2.15). For the limit procedure the main difficulty is that the second energy level can no longer provide estimates on $(u^\varepsilon, \theta^\varepsilon, q^\varepsilon, P^\varepsilon, \eta^\varepsilon)$ as ε is no longer fixed constant. This difficulty is overcome by the additional regularity provided by the viscoelastic term which enables us to prove the existence of weak solutions.

Proposition 4.1. *There exists (u, θ, q, P, η) such that*

$$\begin{aligned} u^\varepsilon &\rightharpoonup u \text{ weakly} \star \text{ in } W^{1,\infty}(I; L^2) \cap L^\infty(I; \mathcal{V}), \\ (\theta^\varepsilon, P^\varepsilon) &\rightharpoonup (\theta, P) \text{ weakly} \star \text{ in } L^\infty(I; L^2 \times L^2), \\ (q^\varepsilon, \eta^\varepsilon) &\rightharpoonup (q, \eta) \text{ weakly} \star \text{ in } L^\infty(I; L^2 \times L^2), \end{aligned} \tag{4.1}$$

with $u \in \mathcal{K}$. Moreover, as $\varpi > 0$, then

$$u^\varepsilon \rightharpoonup u \text{ in } W^{1,2}(I; \mathcal{V}). \tag{4.2}$$

Proof. Keeping in mind the regularity of $(u^\varepsilon, \theta^\varepsilon, q^\varepsilon, P^\varepsilon, \eta^\varepsilon)$, we can choose $w = u_t^\varepsilon$, $\upsilon = \theta^\varepsilon$, $\phi = q^\varepsilon$, $z = P^\varepsilon$ and $\psi = \eta^\varepsilon$ in (3.7) and we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\varepsilon(t) + \int_0^1 \left(\varpi(u_{xt}^\varepsilon)^2 + (q^\varepsilon)^2 + \hbar(\eta^\varepsilon)^2\right)\mathrm{d}x = 0, \tag{4.3}$$

where

$$\mathcal{E}^\varepsilon(t) = \frac{1}{2}\int_0^1 \left(\rho(u_t^\varepsilon)^2 + \alpha(u_x^\varepsilon)^2 + c(\theta^\varepsilon)^2 + 2d\theta^\varepsilon P^\varepsilon + r(P^\varepsilon)^2 + \tau_0(q^\varepsilon)^2 + \tau\hbar(\eta^\varepsilon)^2\right)\mathrm{d}x + \frac{1}{2\varepsilon}[u^\varepsilon(1,t) - g]_+^2. \tag{4.4}$$

Since all the material parameters are positive and (2.16)–(2.17) hold, integrating (4.3) from 0 to t and using Gronwall's inequality, it follows the existence of a constant $C = C(\|u_0^\varepsilon\|, \|\theta_0^\varepsilon\|, \|q_0^\varepsilon\|, \|P_0^\varepsilon\|, \|\eta_0^\varepsilon\|)$ such that, for all $t > 0$,

$$
\begin{aligned}
\|u_t^\varepsilon\| &\le C,\\
\|u_x^\varepsilon\| &\le C,\\
\|\theta^\varepsilon\| &\le C,\\
\|P^\varepsilon\| &\le C,\\
\|q^\varepsilon\| &\le C,\\
\|\eta^\varepsilon\| &\le C,\\
\frac{1}{2\varepsilon}[u^\varepsilon(1,.)-g]_+^2 &\le C,\\
\varpi\int_0^t \|u_{xt}^\varepsilon\|^2 \mathrm{d}s &\le C,\\
\int_0^t \|q^\varepsilon\|^2 \mathrm{d}s &\le C,\\
\int_0^t \|\eta^\varepsilon\|^2 \mathrm{d}s &\le C.
\end{aligned}
$$

These estimates imply the desired convergence results. In addition we have that, as $\varepsilon \to 0$,

$$[u^\varepsilon(1,.)-g]_+ \to 0 \text{ strongly in } L^2(I)$$

and, on the other hand, the compactness of the trace operator from $H^1((0,1)\times I)$ to $L^2(I)$ yields

$$[u^\varepsilon(1,.)-g]_+ \to [u(1,.)-g]_+ \text{ strongly in } L^2(I).$$

Hence we conclude that $u \in \mathcal{K}$. □

Using the continuity in time of the solutions to the penalized problem (by Sobolev's Imbedding theorem), as a corollary of Proposition 4.1 we get

Corollary 4.2. *Let (u,θ,q,P,η) be the functions from Proposition 4.1. Then*

$$
\begin{aligned}
u^\varepsilon(\cdot,T) &\rightharpoonup u(\cdot,T) \ \textit{in}\ \mathcal{V},\\
u_t^\varepsilon(\cdot,T) &\rightharpoonup u_t(\cdot,T) \ \textit{in}\ L^2(0,1),\\
(\theta^\varepsilon,P^\varepsilon)(\cdot,T) &\rightharpoonup (\theta,P)(\cdot,T) \ \textit{in}\ L^2(0,1),\\
(q^\varepsilon,\eta^\varepsilon)(\cdot,T) &\rightharpoonup (q,\eta)(\cdot,T) \ \textit{in}\ L^2(0,1).
\end{aligned}
$$

Now, we are ready to give the main result of this section.

Theorem 4.3. *Let $u_0 \in \mathcal{K}$ and $u_1,\ \theta_0,\ q_0,\ P_0,\ \eta_0 \in L^2(0,1)$. There exists a solution (u,θ,q,P,η) to problem (2.11)–(2.15).*

Proof. Assume the initial data are defined such that, as $\varepsilon \to 0$,

$$
\begin{aligned}
u_0^\varepsilon &\to u_0 \text{ in } H^1(0,1),\\
u_1^\varepsilon &\to u_1 \text{ in } L^2(0,1),\\
\theta_0^\varepsilon &\to \theta_0 \text{ in } L^2(0,1),\\
q_0^\varepsilon &\to q_0 \text{ in } L^2(0,1),\\
P_0^\varepsilon &\to P_0 \text{ in } L^2(0,1),\\
\eta_0^\varepsilon &\to \eta_0 \text{ in } L^2(0,1).
\end{aligned}
$$

Let $(u^\varepsilon, \theta^\varepsilon, q^\varepsilon, P^\varepsilon, \eta^\varepsilon)$ be the solution to the penalized problem for each $\varepsilon > 0$ and (u, θ, q, P, η) be its limit from Proposition 4.1. Then (u, θ, q, P, η) satisfies (2.18).

Integrating $(3.7)_2$ and $(3.7)_4$ from 0 to T for $(\upsilon, z) \in W^{1,\infty}(I; \mathcal{M} \times \mathcal{M})$ we find that

$$\begin{aligned}
&-\int_0^T\int_0^1 (c\theta^\varepsilon + dP^\varepsilon)\upsilon_t \mathrm{d}x\mathrm{d}t + \int_0^1 (c\theta^\varepsilon + dP^\varepsilon + \gamma_1 u_x^\varepsilon)(\cdot,T)\upsilon(\cdot,T)\mathrm{d}x \\
&\qquad - \int_0^1 (c\theta_0^\varepsilon + dP_0^\varepsilon + \gamma_1 u_{0x}^\varepsilon)\upsilon(\cdot,0)\mathrm{d}x - \int_0^T\int_0^1 (q^\varepsilon \upsilon_x + \gamma_1 u_x^\varepsilon \upsilon_t)\mathrm{d}x\mathrm{d}t = 0, \\
&-\int_0^T\int_0^1 (\mathrm{d}\theta^\varepsilon + rP^\varepsilon)z_t \mathrm{d}x\mathrm{d}t + \int_0^1 (\mathrm{d}\theta^\varepsilon + rP^\varepsilon + \gamma_2 u_x^\varepsilon)(\cdot,T)z(\cdot,T)\mathrm{d}x \\
&\qquad - \int_0^1 (\mathrm{d}\theta_0^\varepsilon + rP_0^\varepsilon + \gamma_2 u_{0x}^\varepsilon)z(\cdot,0)\mathrm{d}x - \int_0^T\int_0^1 (\hbar\eta^\varepsilon z_x + \gamma_2 u_x^\varepsilon z_t)\mathrm{d}x\mathrm{d}t = 0.
\end{aligned}$$

By Proposition 4.1 and Corollary 4.2 and by taking the limit $\varepsilon \to 0$ we conclude that (u, θ, q, P, η) fulfill (2.20) and (2.21). Similarly, integrating $(3.7)_3$ and $(3.7)_5$ from 0 to T for $(\phi, \psi) \in W^{1,\infty}(I; \mathcal{V} \times \mathcal{V})$ we have

$$\begin{aligned}
&-\tau_0 \int_0^T\int_0^1 q^\varepsilon \phi_t \mathrm{d}x\mathrm{d}t + \tau_0 \int_0^1 q^\varepsilon(\cdot,T)\phi(\cdot,T)\mathrm{d}x - \tau_0 \int_0^1 q_0^\varepsilon \phi(\cdot,0)\mathrm{d}x + \int_0^T\int_0^1 (q^\varepsilon \phi - \theta^\varepsilon \phi_x)\mathrm{d}x\mathrm{d}t = 0, \\
&-\tau \int_0^T\int_0^1 \eta^\varepsilon \psi_t \mathrm{d}x\mathrm{d}t + \tau \int_0^1 \eta^\varepsilon(\cdot,T)\psi(\cdot,T)\mathrm{d}x - \tau \int_0^1 \eta_0^\varepsilon \psi(\cdot,0)\mathrm{d}x + \int_0^T\int_0^1 (\eta^\varepsilon \psi - P^\varepsilon \psi_x)\mathrm{d}x\mathrm{d}t = 0.
\end{aligned}$$

Again, taking the limit as $\varepsilon \to 0$ and using Proposition 4.1 and Corollary 4.2, we conclude that (u, θ, q, P, η) fulfill (2.22) and (2.23).

Now, let $w \in L^\infty(I; \mathcal{K}) \cap W^{1,\infty}(I; L^2(0,1))$. Replacing w in $(3.7)_1$ by $w - u^\varepsilon$ and integrating from 0 to T we obtain

$$\begin{aligned}
&\int_0^T (\rho\|u_t^\varepsilon\|^2 - \alpha\|u_x^\varepsilon\|^2)\mathrm{d}t - \rho\int_0^T\int_0^1 u_t^\varepsilon w_t \mathrm{d}x\mathrm{d}t + \rho\int_0^1 u_t^\varepsilon(\cdot,T)(w-u^\varepsilon)(\cdot,T)\mathrm{d}x - \rho\int_0^1 u_1^\varepsilon(w(\cdot,0) - u_0^\varepsilon)\mathrm{d}x \\
&- \int_0^T\int_0^1 (\gamma_1\theta^\varepsilon + \gamma_2 P^\varepsilon)(w_x - u_x^\varepsilon)\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^1 (\alpha u_x^\varepsilon + \varpi u_{xt}^\varepsilon)w_x \mathrm{d}x\mathrm{d}t - \frac{\varpi}{2}(\|u_x^\varepsilon\|^2 - \|u_{0x}^\varepsilon\|^2) \\
&= -\frac{1}{\varepsilon}\int_0^T [u^\varepsilon(1,t) - g]_+ (w(1,t) - u^\varepsilon(1,t))\mathrm{d}t \geq 0.
\end{aligned} \tag{4.5}$$

Using Proposition 4.1, Corollary 4.2 we can conclude that all the linear terms in $(u^\varepsilon, \theta^\varepsilon, q^\varepsilon, P^\varepsilon, \eta^\varepsilon)$ convergence to 0 as $\varepsilon \to 0$. However, the convergence of the quadratic terms, such as $\|u_t^\varepsilon\|$, $\|u_x^\varepsilon\|$ and $\int_0^1 (\gamma_1\theta^\varepsilon + \gamma_2 P^\varepsilon)u_x^\varepsilon \mathrm{d}x$, is not obvious because the weak $\star$ convergence (4.1) does not imply the convergence in norm. The same problem has been encountered by Rivera and Racke in [22] and circumvented by using the radially symmetry, by Sprenger in [26] and circumvented by the additional regularity provided by the presence of a viscoelastic term and by Aouadi and Copetti in [7] and circumvented by Lemma 4.3 of [20]. In this paper and as in [26], we use the additional regularity given by (4.2) and provided by the term ϖu_{xxt} to overcome this difficulty. This enable us to conclude the uniform convergence of u_t^ε as well as u_x^ε. It is then possible to take the limit $\varepsilon \to 0$ in (4.5) and deduce that (u, θ, q, P, η) will satisfy (2.19). This completes the proof of the theorem. □

5. Exponential decay of the weak solution

Our approach is based on the construction of a Lyapunov functional from inequality of the previous result. To save writing we omit the superindex ε in (u, θ, P, q, η) in this section.

Lemma 5.1. *Let (u, θ, P, q, η) be a solution to the contact problem* (3.1)–(3.4). *Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(t) \leq -\frac{\alpha}{2}\int_0^1 u_x^2 \mathrm{d}x + \rho\int_0^1 u_t^2 \mathrm{d}x + \frac{\gamma_1^2}{\alpha}\int_0^1 \theta^2 \mathrm{d}x + \frac{\gamma_2^2}{\alpha}\int_0^1 P^2 \mathrm{d}x - \frac{1}{\varepsilon}[u(1,t) - g]_+^2, \tag{5.1}$$

where

$$\mathcal{K}(t) := \int_0^1 \left(\rho u_t u + \frac{\varpi}{2} u_x^2\right) \mathrm{d}x.$$

Proof. Taking $w = u$ in $(3.7)_1$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\rho u_t u + \frac{\varpi}{2}u_x^2\right)\mathrm{d}x &= \int_0^1 (\rho u_t^2 + \rho u u_{tt} + \varpi u_x u_{xt})\mathrm{d}x \\ &= \rho\int_0^1 u_t^2 \mathrm{d}x - \alpha\int_0^1 u_x^2 \mathrm{d}x + \gamma_1\int_0^1 \theta u_x \mathrm{d}x + \gamma_2\int_0^1 P u_x \mathrm{d}x \\ &\quad - \frac{1}{\varepsilon}[(u(1,t) - g)^+]u(1,t).\end{aligned} \tag{5.2}$$

If $u(1,t) \geq 0$, then $u(1,t) \geq [(u(1,t) - g)]_+$, so we have

$$-\frac{1}{\varepsilon}[u(1,t) - g]_+ u(1,t) \leq -\frac{1}{\varepsilon}[u(1,t) - g]_+^2. \tag{5.3}$$

If $u(1,t) < 0$, then $u(1,t) < [u(1,t) - g]_+ = 0$. Using Young inequality

$$2\gamma_1\int_0^1 \theta u_x \mathrm{d}x \leq \frac{\alpha}{2}\int_0^1 u_x^2 \mathrm{d}x + 2\frac{\gamma_1^2}{\alpha}\int_0^1 \theta^2 \mathrm{d}x \quad \text{and} \quad 2\gamma_2\int_0^1 P u_x \mathrm{d}x \leq \frac{\alpha}{2}\int_0^1 u_x^2 \mathrm{d}x + 2\frac{\gamma_2^2}{\alpha}\int_0^1 P^2 \mathrm{d}x,$$

we get (5.1). □

Our next objective is to introduce certain functionals which provide estimates on terms of the form $-\int \theta^2 \mathrm{d}x$ and $-\int P^2 \mathrm{d}x$.

Lemma 5.2. *Let (u, θ, P, q, η) be a solution to the contact problem* (3.1)–(3.4). *Then there exist two positive embedding constants M and M' such that*

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\tau_0\psi q + \tau\phi\eta)\mathrm{d}x \leq \left(2\tau_0 + \frac{M}{2c}\right)\int_0^1 q^2 \mathrm{d}x + \left(2\tau + \frac{M'}{2r}\right)\int_0^1 \eta^2 \mathrm{d}x + \left(\frac{\gamma_1^2}{4\tau_0} + \frac{\gamma_2^2}{4\tau}\right)\int_0^1 u_t^2 \mathrm{d}x \\ - \hat{c}\int_0^1 \theta^2 \mathrm{d}x - \hat{r}\int_0^1 P^2 \mathrm{d}x,\end{aligned} \tag{5.4}$$

where

$$\psi(x,t) := \int_0^x \Big[c\theta(y,t) + dP(y,t)\Big]\mathrm{d}y \quad \textit{and} \quad \phi(x,t) := \int_0^x \Big[rP(y,t) + d\theta(y,t)\Big]\mathrm{d}y,$$

and

$$2\hat{c} = c - \frac{d^2}{r} - d\left(\zeta + \frac{1}{\zeta}\right) > 0 \quad \textit{and} \quad 2\hat{r} = r - \frac{d^2}{c} - d\left(\zeta + \frac{1}{\zeta}\right) > 0$$

where ζ a positive constant that will be fixed later.

Proof. Integrating the second equation of (3.1) over $(0, x)$ and taking into account boundary conditions (3.3), we get

$$\psi_t + q + \gamma_1 u_t = 0. \tag{5.5}$$

Multiplying (5.5) by q and integrating with respect to x, we obtain after applying Young inequality

$$-\tau_0 \int_0^1 \psi_t q \mathrm{d}x = \tau_0 \int_0^1 (q^2 + \gamma_1 q u_t) \mathrm{d}x \leq 2\tau_0 \int_0^1 q^2 \mathrm{d}x + \frac{\gamma_1^2}{4\tau_0} \int_0^1 u_t^2 \mathrm{d}x. \tag{5.6}$$

Taking $\phi = \psi$ in $(3.7)_3$, we get

$$\tau_0 \int_0^1 \psi q_t \mathrm{d}x + \int_0^1 q\psi \mathrm{d}x + \int_0^1 \theta_x \psi \mathrm{d}x = 0,$$

and, by integration by parts and the definition of ψ, we have

$$\int_0^1 \theta_x \psi \mathrm{d}x = -c \int_0^1 \theta^2 \mathrm{d}x - d \int_0^1 \theta P \mathrm{d}x.$$

Using Young inequality, there exist two positive constants ϵ and $\zeta > 0$ such that

$$-\tau_0 \int_0^1 \psi q_t \mathrm{d}x \leq \frac{\epsilon}{2} \int_0^1 q^2 \mathrm{d}x + \frac{1}{2\epsilon} \int_0^1 \psi^2 \mathrm{d}x - c \int_0^1 \theta^2 \mathrm{d}x + \frac{d}{2\zeta} \int_0^1 \theta^2 \mathrm{d}x + \frac{\mathrm{d}\zeta}{2} \int_0^1 P^2 \mathrm{d}x.$$

Let $M > 0$ denote an embedding constant. It easy to see that

$$\int_0^1 \psi^2 \mathrm{d}x \leq M \left(\int_0^1 \theta^2 \mathrm{d}x + \frac{d^2}{c^2} \int_0^1 P^2 \mathrm{d}x \right).$$

By choosing $\epsilon = \dfrac{M}{c}$, it follows

$$-\tau_0 \int_0^1 \psi q_t \mathrm{d}x \leq \frac{M}{2c} \int_0^1 q^2 \mathrm{d}x - \frac{1}{2} \left(c - \frac{\mathrm{d}}{\zeta} \right) \int_0^1 \theta^2 \mathrm{d}x + \frac{d}{2} \left(\frac{d}{c} + \zeta \right) \int_0^1 P^2 \mathrm{d}x. \tag{5.7}$$

Combining (5.6) and (5.7), we reach

$$\begin{aligned} - \tau_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \psi q \mathrm{d}x \leq & \left(2\tau_0 + \frac{M}{2c} \right) \int_0^1 q^2 \mathrm{d}x + \frac{\gamma_1^2}{4\tau_0} \int_0^1 u_t^2 \mathrm{d}x - \frac{1}{2} \left(c - \frac{\mathrm{d}}{\zeta} \right) \int_0^1 \theta^2 \mathrm{d}x \\ & + \frac{d}{2} \left(\frac{d}{c} + \zeta \right) \int_0^1 P^2 \mathrm{d}x. \end{aligned} \tag{5.8}$$

Using the same argument, one can obtain

$$\begin{aligned} - \tau \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \phi \eta \mathrm{d}x \leq & \left(2\tau + \frac{M'}{2r} \right) \int_0^1 \eta^2 \mathrm{d}x + \frac{\gamma_2^2}{4\tau} \int_0^1 u_t^2 \mathrm{d}x - \frac{1}{2} \left(r - \frac{\mathrm{d}}{\zeta} \right) \int_0^1 P^2 \mathrm{d}x \\ & + \frac{d}{2} \left(\frac{d}{r} + \zeta \right) \int_0^1 \theta^2 \mathrm{d}x. \end{aligned} \tag{5.9}$$

Then (5.4) holds by adding (5.8) and (5.9). To fix $\hat{c}$ and $\hat{r}$ positive, it suffices to choose

$$\zeta + \frac{1}{\zeta} < \frac{cr - d^2}{d} \min \left\{ \frac{1}{c} \, ; \, \frac{1}{r} \right\}. \qquad \square$$

Let

$$\aleph(t) = N\mathcal{E}(t) + \epsilon\mathcal{K}(t) - \tau_0 \int_0^1 \psi q \mathrm{d}x - \tau \int_0^1 \phi\eta \mathrm{d}x,$$

where $\epsilon \in (0,1)$ and $\mathcal{E}(t)$ is defined by (4.4). We easily see that, for an appropriately large number N, there exist $C_3, C_4 > 0$, depending on N, such that

$$C_3\mathcal{E}(t) \le \aleph(t) \le C_4\mathcal{E}(t). \tag{5.10}$$

Remark 5.3. Since $\psi(x,t) = \int_0^x \Big[c\theta(y,t) + dP(y,t)\Big]\mathrm{d}y$ and $\phi(x,t) = \int_0^x \Big[rP(y,t) + \mathrm{d}\theta(y,t)\Big]\mathrm{d}y$, we can directly show that there exists a constant $C > 0$ such that $|\aleph(t)| \le C\mathcal{E}(t)$.

Based on the above lemmas, we will show the main result of this section.

Theorem 5.4. *Let (u, θ, P, q, η) be a solution to the contact problem (3.1)–(3.4), then the associated energy decays exponentially,* i.e.,

$$\exists c_0 > 0, \quad \exists C_0 > 0, \quad \mathcal{E}(t) \le C_0\mathcal{E}(0)\mathrm{e}^{-c_0 t}, \quad \forall t \ge 0.$$

Bounds for c_0 and C_0 can be given explicitly in terms of the physical coefficients.

Proof. Using (4.3), (5.1), (5.4) and Poincaré inequality, we conclude that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\aleph(t) \le{}& \left(\epsilon\rho + \frac{\gamma_1^2}{4\tau_0} + \frac{\gamma_2^2}{4\tau} - NC\varpi\right)\int_0^1 u_t^2 \mathrm{d}x + \left(2\tau_0 + \frac{M}{2c} - NC\right)\int_0^1 q^2\mathrm{d}x \\
&+ \left(2\tau + \frac{M'}{2r} - NC\hbar\right)\int_0^1 \eta^2\mathrm{d}x - \left(\hat{c} - \epsilon\frac{\gamma_1^2}{\alpha}\right)\int_0^1 \theta^2\mathrm{d}x \\
&- \left(\hat{r} - \epsilon\frac{\gamma_2^2}{\alpha}\right)\int_0^1 P^2\mathrm{d}x - \epsilon\frac{\alpha}{2}\int_0^1 u_x^2\mathrm{d}x - \frac{\epsilon}{\varepsilon}[u(1,t) - g]_+^2.
\end{aligned}$$

Taking ϵ small enough

$$\epsilon \le \min\left\{\frac{\hat{c}\alpha}{\gamma_1^2}\ ;\ \frac{\hat{r}\alpha}{\gamma_2^2}\right\}$$

and N sufficiently large

$$N \ge \sup\left\{\frac{\epsilon\rho + \frac{\gamma_1^2}{4\tau_0} + \frac{\gamma_2^2}{4\tau}}{C\varpi}\ ;\ \frac{2\tau_0 + \frac{M}{2c}}{C}\ ;\ \frac{2\tau + \frac{M'}{2r}}{C\hbar}\right\},$$

there exits a constant $C' > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}(t) \le -C'\mathcal{E}(t).$$

Using (5.10), this proves our theorem. □

Using the lower semicontinuity of the norm in $H^1 \times L^2 \times L^2 \times L^2$, as a corollary of Theorem 4.1 we have

Corollary 5.5. *Under the conditions of Theorem 4.1, the weak solution to problem (2.11)–(2.15) decays exponentially as $t \to \infty$; that is, we have*

$$E(t) \le C_0E(0)\mathrm{e}^{-c_0 t},$$

where

$$E(t) = \frac{1}{2}\int_0^1 \Big(\rho(u_t)^2 + \alpha(u_x)^2 + c(\theta)^2 + 2\mathrm{d}\theta P + r(P)^2 + \tau_0(q)^2 + \tau\hbar(\eta)^2\Big)\mathrm{d}x.$$

6. Numerical approximation of the penalized problem

In this section we describe a fully discrete finite element method to numerically approximate problem (2.11)–(2.15). Let $0 = x_0 < x_1 < \ldots < x_s = 1$ be a uniform partition of the interval $J =]0,1[$ into subintervals $J_j =]x_{j-1}, x_j[$, $j = 1, 2, \ldots s$, of length $h = 1/s$ and, for a given final time $T > 0$ and a positive integer N, let $\Delta t = T/N$ be the time step. We denote by S_h and V_h the finite element spaces:

$$S_h = \{\chi \in C^0(\bar{J}) \ / \ \chi_{|J_j} \in P_1(J_j), \ \chi(0) = 0\},$$

$$V_h = \{\chi \in C^0(\bar{J}) \ / \ \chi_{|J_j} \in P_1(J_j), \ \chi(1) = 0\},$$

where $P_1(J_j)$ is the space of polynomials with degree less or equal to one. Using the implicit Euler scheme, problem (3.7) is approximated as follows. For $n = 1, \ldots, N$, find $(v_h^n, \theta_h^n, P_h^n, q_h^n, \eta_h^n) \in S_h \times V_h \times V_h \times S_h \times S_h$ satisfying, for all $(w_h, \vartheta_h, \phi_h, z_h, \psi_h) \in S_h \times V_h \times V_h \times S_h \times S_h$,

$$\begin{aligned}
&\int_0^1 \frac{\rho}{\Delta t}(v_h^n - v_h^{n-1}) w_h \mathrm{d}x + \int_0^1 \Big(\alpha u_{hx}^n + \varpi v_{hx}^n - \gamma_1 \theta_h^n - \gamma_2 P_h^n\Big) w_{hx} \mathrm{d}x + \frac{1}{\varepsilon}[u_h^n(1) - g]_+ w_h(1) = 0,\\
&\int_0^1 \left(\frac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \frac{d}{\Delta t}(P_h^n - P_h^{n-1}) + q_{hx}^n + \gamma_1 v_{hx}^n\right) \vartheta_h \mathrm{d}x = 0,\\
&\int_0^1 \left(\frac{\tau_0}{\Delta t}(q_h^n - q_h^{n-1}) + q_h^n + \theta_{hx}^n\right)\phi_h \mathrm{d}x = 0,\\
&\int_0^1 \left(\frac{r}{\Delta t}(P_h^n - P_h^{n-1}) + \frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \hbar \eta_{hx}^n + \gamma_2 v_{hx}^n\right) z_h \mathrm{d}x = 0,\\
&\int_0^1 \left(\frac{\tau}{\Delta t}(\eta_h^n - \eta_h^{n-1}) + \eta_h^n + P_{hx}^n\right) \psi_h \mathrm{d}x = 0,
\end{aligned} \tag{6.1}$$

where u_h^0, v_h^0, θ_h^0, P_h^0, q_h^0 and η_h^0 are approximations of the initial conditions u_0, u_1, θ_0, P_0, q_0 and η_0, respectively. Here v_h^n is an approximation to the velocity $v(t_n) = u_t(t_n)$ and the discrete displacement is defined by

$$u_h^n = u_h^{n-1} + \Delta t v_h^n.$$

Theorem 6.1. *The sequences $\{v_h^n, u_h^n, \theta_h^n, P_h^n, q_h^n, \eta_h^n\}$ generated by problem (6.1) satisfy the stability estimate:*

$$\int_0^1 \left(\rho (v_h^n)^2 + \alpha (u_{hx}^n)^2 + \left(c - \frac{\mathrm{d}}{\zeta}\right)(\theta_h^n)^2 + (r - \mathrm{d}\zeta)(P_h^n)^2 + \tau_0 (q_h^n)^2 + \tau\hbar(\eta_h^n)^2\right) \mathrm{d}x + \frac{1}{\varepsilon}[u_h^n(1) - g]_+^2 \le C,$$

where C is a positive constant and $\zeta > 0$ is assumed to be such that $d/c < \zeta < r/d$.

Proof. Taking $w_h = v_h^n$ as a test function in $(6.1)_1$ we obtain

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\int_0^1 \Big((v_h^n - v_h^{n-1})^2 + (v_h^n)^2 - (v_h^{n-1})^2\Big)\mathrm{d}x + \frac{\alpha}{2\Delta t}\int_0^1 \Big((u_{hx}^n - u_{hx}^{n-1})^2 + (u_{hx}^n)^2 - (u_{hx}^{n-1})^2\Big)\mathrm{d}x\\
&\quad + \frac{\varpi}{2}\int_0^1 (v_{hx}^n)^2 \mathrm{d}x - \int_0^1 (\gamma_1 \theta_h^n + \gamma_2 P_h^n) v_{hx}^n \mathrm{d}x + \frac{1}{\varepsilon}[u_h^n(1) - g]_+ v_h^n(1) = 0,
\end{aligned} \tag{6.2}$$

and choosing $\vartheta_h = \theta_h^n$ and $\phi_h = q_h^n$ as test functions in $(6.1)_2$ and $(6.1)_3$, respectively, we have

$$\frac{c}{2\Delta t}\int_0^1 \Big((\theta_h^n - \theta_h^{n-1})^2 + (\theta_h^n)^2 - (\theta_h^{n-1})^2\Big)\mathrm{d}x + \int_0^1 \left(\frac{d}{\Delta t}(P_h^n - P_h^{n-1}) + q_{hx}^n + \gamma_1 v_{hx}^n\right)\theta_h^n \mathrm{d}x = 0, \tag{6.3}$$

$$\frac{\tau_0}{2\Delta t}\int_0^1 \Big((q_h^n - q_h^{n-1})^2 + (q_h^n)^2 - (q_h^{n-1})^2\Big)\mathrm{d}x + \int_0^1 (q_h^n)^2 \mathrm{d}x + \int_0^1 \theta_{hx}^n q_h \mathrm{d}x = 0. \tag{6.4}$$

Next, setting $z_h = P_h^n$ in $(6.1)_4$ and $\psi_h = \eta_h^n$ in $(6.1)_5$ it follows that

$$\frac{r}{\Delta t}\int_0^1 \left((P_h^n - P_h^{n-1})^2 + (P_h^n)^2 - (P_h^{n-1})^2\right)\mathrm{d}x + \int_0^1 \left(\frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \hbar\eta_{hx}^n + \gamma_2 v_{hx}^n\right) P_h^n \mathrm{d}x = 0, \tag{6.5}$$

$$\frac{\tau}{\Delta t}\int_0^1 \left((\eta_h^n - \eta_h^{n-1})^2 + (\eta_h^n)^2 - (\eta_h^{n-1})^2\right)\mathrm{d}x + \int_0^1 (\eta_h^n)^2\mathrm{d}x + \int_0^1 P_{hx}^n \eta_h \mathrm{d}x = 0. \tag{6.6}$$

Combining equations (6.2)–(6.6) and using integration by parts we find that

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\int_0^1 \left((v_h^n)^2 - (v_h^{n-1})^2\right)\mathrm{d}x + \frac{\alpha}{2\Delta t}\int_0^1 \left((u_{hx}^n)^2 - (u_{hx}^{n-1})^2\right)\mathrm{d}x + \frac{\varpi}{2}\int_0^1 (v_{hx}^n)^2\mathrm{d}x + \frac{1}{\varepsilon}[u_h^n(1) - g]_+ v_h^n(1) \\
&+\frac{c}{2\Delta t}\int_0^1 \left((\theta_h^n - \theta_h^{n-1})^2 + (\theta_h^n)^2 - (\theta_h^{n-1})^2\right)\mathrm{d}x + \int_0^1 \frac{d}{\Delta t}(P_h^n - P_h^{n-1})\theta_h^n\mathrm{d}x + \int_0^1 \frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1})P_h^n\mathrm{d}x \\
&+\frac{\tau_0}{2\Delta t}\int_0^1 \left((q_h^n)^2 - (q_h^{n-1})^2\right)\mathrm{d}x + \int_0^1 (q_h^n)^2\mathrm{d}x + \frac{r}{\Delta t}\int_0^1 \left((P_h^n - P_h^{n-1})^2 + (P_h^n)^2 - (P_h^{n-1})^2\right)\mathrm{d}x \\
&+\frac{\tau\hbar}{\Delta t}\int_0^1 \left((\eta_h^n - \eta_h^{n-1})^2 + (\eta_h^n)^2 - (\eta_h^{n-1})^2\right)\mathrm{d}x + \hbar\int_0^1 (\eta_h^n)^2 \leq 0.
\end{aligned}$$

Observing that

$$\int_0^1 \left((P_h^n - P_h^{n-1})\theta_h^n + (\theta_h^n - \theta_h^{n-1})P_h^n\right)\mathrm{d}x = \int_0^1 \left(P_h^n\theta_h^n - P_h^{n-1}\theta_h^{n-1} + (\theta_h^n - \theta_h^{n-1})(P_h^n - P_h^{n-1})\right)\mathrm{d}x$$

and that, from (2.16),

$$\int_0^1 \left(\frac{c}{2}(\theta_h^n - \theta_h^{n-1})^2 + \frac{r}{2}(P_h^n - P_h^{n-1})^2 + d(\theta_h^n - \theta_h^{n-1})(P_h^n - P_h^{n-1})\right)\mathrm{d}x > 0,$$

we end up with

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\int_0^1 \left((v_h^n)^2 - (v_h^{n-1})^2\right)\mathrm{d}x + \frac{\alpha}{2\Delta t}\int_0^1 \left((u_{hx}^n)^2 - (u_{hx}^{n-1})^2\right)\mathrm{d}x + \frac{c}{2\Delta t}\int_0^1 \left((\theta_h^n)^2 - (\theta_h^{n-1})^2\right)\mathrm{d}x \\
&+\frac{\tau_0}{2\Delta t}\int_0^1 \left((q_h^n)^2 - (q_h^{n-1})^2\right)\mathrm{d}x + \frac{r}{\Delta t}\int_0^1 \left((P_h^n)^2 - (P_h^{n-1})^2\right)\mathrm{d}x + \frac{\tau\hbar}{\Delta t}\int_0^1 \left((\eta_h^n)^2 - (\eta_h^{n-1})^2\right)\mathrm{d}x \\
&+\frac{d}{\Delta t}(P_h^n\theta_h^n - P_h^{n-1}\theta_h^{n-1}) + \frac{1}{\varepsilon}[u_h^n(1) - g]_+ v_h^n(1) \leq 0.
\end{aligned} \tag{6.7}$$

Noting that

$$\begin{aligned}
\Delta t[u_h^n(1) - g]_+ v_h^n(1) &= [u_h^n(1) - g]_+(u_h^n(1) - g + g - u_h^{n-1}(1)) \\
&= [u_h^n(1) - g]_+^2 - [u_h^n(1) - g]_+(u_h^{n-1}(1) - g) \\
&\geq [u_h^n(1) - g]_+^2 - [u_h^n(1) - g]_+[u_h^{n-1}(1) - g]_+ \geq \frac{1}{2}([u_h^n(1) - g]_+^2 - [u_h^{n-1}(1) - g]_+^2), \\
\int_0^1 [c(\theta_h^n)^2 + r(P_h^n)^2 + 2dP_h^n\theta_h^n]\mathrm{d}x &\geq \int_0^1 \left[\left(c - \frac{\mathrm{d}}{\zeta}\right)(\theta_h^n)^2 + (r - \mathrm{d}\zeta)(P_h^n)^2\right]\mathrm{d}x,
\end{aligned}$$

for $\zeta > 0$ assumed to be such that $d/c < \zeta < r/d$, and summing (6.7) over n, we prove the result. □

The next result, which is a discrete version of the energy decay property that holds for the solution of the continuous problem, follows from the above proof.

Corollary 6.2. *The discrete energy function* E^n *defined by*

$$E^n = \frac{1}{2}\int_0^1 \Big(\rho(v_h^n)^2 + \alpha(u_{hx}^n)^2 + c(\theta_h^n)^2 + r(P_h^n)^2 + 2dP_h^n\theta_h^n + \tau_0(q_h^n)^2 + \tau\hbar(\eta_h^n)^2\Big)\mathrm{d}x + \frac{1}{2\varepsilon}[u_h^n(1)-g]_+^2,$$

where $\zeta > 0$ *is assumed to be such that* $d/c < \zeta < r/d$*, satisfies*

$$\frac{E^n - E^{n-1}}{\Delta t} \le 0.$$

We turn now to obtain *a priori* error estimates. Thus, assume that the continuous solution to problem (3.1)–(3.4) has the following additional regularity:

$$\begin{aligned} &u^\varepsilon \in C^2([0,T];L^2(0,1)) \cap C^1([0,T];H^1(0,1)),\\ &\theta^\varepsilon, P^\varepsilon, q^\varepsilon, \eta^\varepsilon \in C([0,T];H^1(0,1)) \cap C^1([0,T];L^2(0,1)). \end{aligned} \tag{6.8}$$

First, we estimate the terms involving the temperature and the heat conduction. For a continuous function $\chi(t)$, let $\chi^n(t) = \chi(t_n)$ and, for simplicity, drop the superscript ε in (u,θ,P,q,η). Then, we subtract variational equation $(3.7)_2$, at time $t = t_n$ and for $v = \vartheta_h \in V_h$, and the discrete variational equation $(6.1)_2$ to obtain

$$\int_0^1 \left(c\theta_t^n - \frac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + dP_t^n - \frac{d}{\Delta t}(P_h^n - P_h^{n-1}) + q_x^n - q_{hx}^n + \gamma_1(v_x^n - v_{hx}^n)\right)\vartheta_h \mathrm{d}x = 0.$$

Therefore,

$$\begin{aligned} \int_0^1 (q_x^n - q_{hx}^n)(\theta^n - \theta_h^n)\,\mathrm{d}x = &-\int_0^1 \left(c\theta_t^n - \frac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + dP_t^n - \frac{d}{\Delta t}(P_h^n - P_h^{n-1}) + \gamma_1(v_x^n - v_{hx}^n)\right)\\ &\times(\theta^n - \theta_h^n)\mathrm{d}x + \int_0^1 \Big(c\theta_t^n - \frac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + dP_t^n - \frac{d}{\Delta t}(P_h^n - P_h^{n-1}) + q_x^n - q_{hx}^n\\ &+ \gamma_1(v_x^n - v_{hx}^n)\Big)(\theta^n - \vartheta_h)\mathrm{d}x, \quad \forall \vartheta_h \in V_h. \end{aligned} \tag{6.9}$$

Now, we subtract variational equation $(3.7)_3$, at time $t = t_n$ and for $\phi = \phi_h \in V_h$, and the discrete variational inequality $(6.1)_3$ to find

$$\int_0^1 \Big(\tau_0 q_t^n - \frac{\tau_0}{\Delta t}(q_h^n - q_h^{n-1}) + q^n - q_h^n + \theta_x^n - \theta_{hx}^n\Big)\phi_h \mathrm{d}x = 0.$$

Thus, we have

$$\begin{aligned} &\int_0^1 \Big(\tau_0 q_t^n - \frac{\tau_0}{\Delta t}(q_h^n - q_h^{n-1}) + q^n - q_h^n + \theta_x^n - \theta_{hx}^n\Big)(q^n - q_h^n)\mathrm{d}x\\ &\quad = \int_0^1 \Big(\tau_0 q_t^n - \frac{\tau_0}{\Delta t}(q_h^n - q_h^{n-1}) + q^n - q_h^n + \theta_x^n - \theta_{hx}^n\Big)(q^n - \phi_h)\mathrm{d}x, \quad \forall \phi_h \in V_h. \end{aligned}$$

Using the relation

$$\begin{aligned} &\int_0^1 (q_x^n - q_{hx}^n)(\theta^n - \theta_h^n)\,\mathrm{d}x + \int_0^1 (q_x^n - q_{hx}^n)(\theta^n - \vartheta_h)\,\mathrm{d}x\\ &\quad = -\int_0^1 (q^n - q_h^n)(\theta_x^n - \theta_{hx}^n)\,\mathrm{d}x - \int_0^1 (q^n - q_h^n)(\theta_x^n - \vartheta_{hx})\,\mathrm{d}x \end{aligned}$$

and (6.9) in the previous equation we get, for all $\phi_h, \vartheta_h \in V_h$,

$$
\begin{aligned}
&\int_0^1 \Big(\tau_0 q_t^n - \frac{\tau_0}{\Delta t}(q_h^n - q_h^{n-1}) + q^n - q_h^n\Big)(q^n - q_h^n)\mathrm{d}x \\
&\quad + \int_0^1 \Big(c\theta_t^n - \frac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + dP_t^n - \frac{d}{\Delta t}(P_h^n - P_h^{n-1}) + \gamma_1(v_x^n - v_{hx}^n)\Big)(\theta^n - \theta_h^n)\mathrm{d}x \\
&= \int_0^1 \Big(\tau_0 q_t^n - \frac{\tau_0}{\Delta t}(q_h^n - q_h^{n-1}) + q^n - q_h^n\Big)(q^n - \phi_h)\mathrm{d}x - \int_0^1 (\theta^n - \theta_h^n)(q_x^n - \phi_{hx})\mathrm{d}x \\
&\quad + \int_0^1 \Big(c\theta_t^n - \frac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + dP_t^n - \frac{d}{\Delta t}(P_h^n - P_h^{n-1}) + \gamma_1(v_x^n - v_{hx}^n)\Big)(\theta^n - \vartheta_h)\mathrm{d}x \\
&\quad - \int_0^1 (q^n - q_h^n)(\theta_x^n - \vartheta_{hx})\mathrm{d}x.
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
&\int_0^1 \Big(c\theta_t^n - \frac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1})\Big)(\theta^n - \theta_h^n)\mathrm{d}x \\
&\quad = \int_0^1 \Big(c\theta_t^n - \frac{c}{\Delta t}(\theta^n - \theta^{n-1})\Big)(\theta^n - \theta_h^n)\mathrm{d}x \\
&\qquad + \frac{c}{2\Delta t}\Big(\|\theta^n - \theta^{n-1} - (\theta_h^n - \theta_h^{n-1})\|^2 + \|\theta^n - \theta_h^n\|^2 - \|\theta^{n-1} - \theta_h^{n-1}\|^2\Big), \\
&\int_0^1 \Big(\tau_0 q_t^n - \frac{\tau_0}{\Delta t}(q_h^n - q_h^{n-1})\Big)(q^n - q_h^n)\mathrm{d}x \\
&\quad = \int_0^1 \Big(\tau_0 q_t^n - \frac{\tau_0}{\Delta t}(q^n - q^{n-1})\Big)(q^n - q_h^n)\mathrm{d}x \\
&\qquad + \frac{\tau_0}{2\Delta t}\Big(\|q^n - q^{n-1} - (q_h^n - q_h^{n-1})\|^2 + \|q^n - q_h^n\|^2 - \|q^{n-1} - q_h^{n-1}\|^2\Big),
\end{aligned}
$$

applying Cauchy−Schwarz and Cauchy's inequalities we have

$$
\begin{aligned}
&\frac{c}{2\Delta t}\Big(\|\theta^n - \theta_h^n\|^2 - \|\theta^{n-1} - \theta_h^{n-1}\|^2 + \|\theta^n - \theta^{n-1} - (\theta_h^n - \theta_h^{n-1}\|^2\Big) \\
&\quad + \frac{\tau_0}{2\Delta t}\Big(\|q^n - q_h^n\|^2 - \|q^{n-1} - q_h^{n-1}\|^2 + \|q^n - q^{n-1} - (q_h^n - q_h^{n-1})\|^2\Big) \\
&\quad - \frac{d}{\Delta t}\int_0^1 \Big((P^n - P_h^n) - (P^{n-1} - P_h^{n-1})\Big)(\theta^n - \theta_h^n)\,\mathrm{d}x \\
&\leq C\Big(\|P_t^n - \frac{1}{\Delta t}(P^n - P^{n-1})\|^2 + \|\theta_t^n - \frac{1}{\Delta t}(\theta^n - \theta^{n-1})\|^2 + \|\theta^n - \theta_h^n\|^2 \\
&\quad + \|q_t^n - \frac{1}{\Delta t}(q^n - q^{n-1})\|^2 + \|q^n - q_h^n\|^2 + \|q^n - \phi_h\|^2_{H^1(0,1)} + \|\theta^n - \vartheta_h\|^2_{H^1(0,1)} \\
&\quad + \frac{1}{\Delta t}\int_0^1 \Big(\theta^n - \theta_h^n - (\theta^{n-1} - \theta_h^{n-1})\Big)(\theta^n - \vartheta_h)\,\mathrm{d}x + \epsilon\|v_x^n - v_{hx}^n\|^2 \\
&\quad + \frac{1}{\Delta t}\int_0^1 \Big(P^n - P_h^n - (P^{n-1} - P_h^{n-1})\Big)(\theta^n - \vartheta_h)\,\mathrm{d}x \\
&\quad + \frac{1}{\Delta t}\int_0^1 \Big(q^n - q_h^n - (q^{n-1} - q_h^{n-1})\Big)(q^n - \phi_h)\,\mathrm{d}x\Big), \quad \forall \phi_h, \vartheta_h \in V_h
\end{aligned} \tag{6.10}
$$

where $\epsilon > 0$ is assumed to be small enough.

Secondly, we will estimate the terms involving the chemical potential and the diffusion conduction. Then, we subtract variational equation (3.7)$_4$, at time $t = t_n$ and for $z = z_h \in S_h$, and discrete variational equation (6.1)$_4$ to get

$$\int_0^1 \Big(rP_t^n - \frac{r}{\Delta t}(P_h^n - P_h^{n-1}) + \mathrm{d}\theta_t^n - \frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \hbar\eta_x^n - \hbar\eta_{hx}^n + \gamma_2(v_x^n - v_{hx}^n)\Big) z_h \mathrm{d}x = 0.$$

Hence, we obtain

$$\begin{aligned}
&\int_0^1 \Big(rP_t^n - \frac{r}{\Delta t}(P_h^n - P_h^{n-1}) + d\theta_t^n - \frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \hbar\eta_x^n - \hbar\eta_{hx}^n + \gamma_2(v_x^n - v_{hx}^n)\Big)(P^n - P_h^n)\mathrm{d}x \\
&\quad = \int_0^1 \Big(rP_t^n - \frac{r}{\Delta t}(P_h^n - P_h^{n-1}) + \mathrm{d}\theta_t^n - \frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \hbar\eta_x^n - \hbar\eta_{hx}^n + \gamma_2(v_x^n - v_{hx}^n)\Big) \\
&\qquad \times (P^n - z_h)\mathrm{d}x, \quad \forall z_h \in S_h,
\end{aligned}$$

which leads to the following

$$\begin{aligned}
&\int_0^1 \Big(\hbar\eta_x^n - \hbar\eta_{hx}^n\Big)(P^n - P_h^n)\mathrm{d}x \\
&\quad = -\int_0^1 \Big(rP_t^n - \frac{r}{\Delta t}(P_h^n - P_h^{n-1}) + \mathrm{d}\theta_t^n - \frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \gamma_2(v_x^n - v_{hx}^n)\Big)(P^n - P_h^n)\mathrm{d}x \\
&\qquad + \int_0^1 \Big(rP_t^n - \frac{r}{\Delta t}(P_h^n - P_h^{n-1}) + \mathrm{d}\theta_t^n - \frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \hbar\eta_x^n - \hbar\eta_{hx}^n \\
&\qquad + \gamma_2(v_x^n - v_{hx}^n)\Big)(P^n - z_h)\mathrm{d}x, \quad \forall z_h \in S_h.
\end{aligned} \tag{6.11}$$

Now, subtracting variational equation (3.7)$_5$, at time $t = t_n$ and for $\psi = \psi_h \in S_h$, and discrete variational equation (6.1)$_5$ we find that

$$\int_0^1 \Big(\tau\eta_t^n - \frac{\tau}{\Delta t}(\eta_h^n - \eta_h^{n-1}) + \eta^n - \eta_h^n + P_x^n - P_{hx}^n\Big)\psi_h \mathrm{d}x = 0,$$

and so,

$$\begin{aligned}
&\hbar\int_0^1 \Big(\tau\eta_t^n - \frac{\tau}{\Delta t}(\eta_h^n - \eta_h^{n-1}) + \eta^n - \eta_h^n + P_x^n - P_{hx}^n\Big)(\eta^n - \eta_h^n)\mathrm{d}x \\
&\quad = \hbar\int_0^1 \Big(\tau\eta_t^n - \frac{\tau}{\Delta t}(\eta_h^n - \eta_h^{n-1}) + \eta^n - \eta_h^n + P_x^n - P_{hx}^n\Big)(\eta^n - \psi_h)\mathrm{d}x, \quad \forall \psi_h \in S_h.
\end{aligned}$$

Using now the relation

$$\begin{aligned}
&\int_0^1 \Big(\hbar\eta_x^n - \hbar\eta_{hx}^n\Big)(P^n - P_h^n)\mathrm{d}x + \int_0^1 \Big(\hbar\eta_x^n - \hbar\eta_{hx}^n\Big)(P^n - z_h)\mathrm{d}x \\
&\quad = -\int_0^1 \Big(\hbar\eta^n - \hbar\eta_h^n\Big)(P_x^n - P_{hx}^n)\mathrm{d}x - \int_0^1 \Big(\hbar\eta^n - \hbar\eta_h^n\Big)(P_x^n - z_{hx})\mathrm{d}x,
\end{aligned}$$

and keeping in mind (6.11) it follows that, for all $\psi_h, z_h \in S_h$,

$$\begin{aligned}
&\hbar\int_0^1 \Big(\tau\eta_t^n - \frac{\tau}{\Delta t}(\eta_h^n - \eta_h^{n-1}) + \eta^n - \eta_h^n\Big)(\eta^n - \eta_h^n)\mathrm{d}x \\
&\quad + \int_0^1 \Big(rP_t^n - \frac{r}{\Delta t}(P_h^n - P_h^{n-1}) + \mathrm{d}\theta_t^n - \frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \gamma_2(v_x^n - v_{hx}^n)\Big)(P^n - P_h^n)\mathrm{d}x \\
&= \hbar\int_0^1 \Big(\tau\eta_t^n - \frac{\tau}{\Delta t}(\eta_h^n - \eta_h^{n-1}) + \eta^n - \eta_h^n\Big)(\eta^n - \psi_h)\mathrm{d}x - \hbar\int_0^1 (P^n - P_h^n)(\eta_x^n - \psi_{hx})\mathrm{d}x \\
&\quad + \int_0^1 \Big(rP_t^n - \frac{r}{\Delta t}(P_h^n - P_h^{n-1}) + \mathrm{d}\theta_t^n - \frac{d}{\Delta t}(\theta_h^n - \theta_h^{n-1}) + \gamma_2(v_x^n - v_{hx}^n)\Big)(P^n - z_h)\mathrm{d}x \\
&\quad -\hbar\int_0^1 (\eta^n - \eta_h^n)(P_x^n - z_{hx})\mathrm{d}x.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&\int_0^1 \Big(\tau\eta_t^n - \frac{\tau}{\Delta t}(\eta_h^n - \eta_h^{n-1})\Big)(\eta^n - \eta_h^n)\mathrm{d}x \\
&\quad = \int_0^1 \Big(\tau\eta_t^n - \frac{\tau}{\Delta t}(\eta^n - \eta^{n-1})\Big)(\eta^n - \eta_h^n)\mathrm{d}x \\
&\qquad + \frac{\tau}{2\Delta t}\Big(\|\eta^n - \eta^{n-1} - (\eta_h^n - \eta_h^{n-1})\|^2 + \|\eta^n - \eta_h^n\|^2 - \|\eta^{n-1} - \eta_h^{n-1}\|^2\Big), \\
&\int_0^1 \Big(rP_t^n - \frac{r}{\Delta t}(P_h^n - P_h^{n-1})\Big)(P^n - P_h^n)\mathrm{d}x \\
&\quad = \int_0^1 \Big(rP_t^n - \frac{r}{\Delta t}(P^n - P^{n-1})\Big)(P^n - P_h^n)\mathrm{d}x \\
&\qquad + \frac{r}{2\Delta t}\Big(\|P^n - P^{n-1} - (P_h^n - P_h^{n-1})\|^2 + \|P^n - P_h^n\|^2 - \|P^{n-1} - P_h^{n-1}\|^2\Big),
\end{aligned}$$

applying Cauchy−Schwarz and Cauchy's inequalities it leads

$$\begin{aligned}
&\frac{\tau}{2\Delta t}\Big(\|\eta^n - \eta_h^n\|^2 - \|\eta^{n-1} - \eta_h^{n-1}\|^2 + \|\eta^n - \eta^{n-1} - (\eta_h^n - \eta_h^{n-1})\|^2\Big) \\
&\quad + \frac{r}{2\Delta t}\Big(\|P^n - P_h^n\|^2 - \|P^{n-1} - P_h^{n-1}\|^2 + \|P^n - P^{n-1} - (P_h^n - P_h^{n-1})\|^2\Big) \\
&\quad - \frac{d}{\Delta t}\int_0^1 \Big(\theta^n - \theta_n^h - (\theta^{n-1} - \theta_h^{n-1})\Big)(P_n - P_n^h)\,\mathrm{d}x \\
&\quad \leq C\Big(\|P_t^n - \frac{1}{\Delta t}(P^n - P^{n-1})\|^2 + \|P^n - P_h^n\|^2 + \|\theta_t^n - \frac{1}{\Delta t}(\theta^n - \theta^{n-1})\|^2 \\
&\quad + \|\eta_t^n - \frac{1}{\Delta t}(\eta^n - \eta^{n-1})\|^2 + \|\eta^n - \eta_h^n\|^2 + \|\eta^n - \psi_h\|_{H^1(0,1)}^2 + \|P_n - z_h\|_{H^1(0,1)}^2 \\
&\quad + \frac{1}{\Delta t}\int_0^1 \Big(\eta^n - \eta_h^n - (\eta^{n-1} - \eta_h^{n-1})\Big)(\eta^n - \psi_h)\,\mathrm{d}x + \epsilon\|v_x^n - v_{hx}^n\|^2 \\
&\quad + \frac{1}{\Delta t}\int_0^1 \Big(\theta^n - \theta_h^n - (\theta^{n-1} - \theta_h^{n-1})\Big)(P^n - z_h)\,\mathrm{d}x \\
&\quad + \frac{1}{\Delta t}\int_0^1 \Big(P^n - P_h^n - (P^{n-1} - P_h^{n-1})\Big)(P^n - z_h)\,\mathrm{d}x\Big), \quad \forall\psi_h, z_h \in S_h.
\end{aligned} \tag{6.12}$$

Finally, we will estimate the term related to the velocity field $v = u_t$. Writing the variational equation $(3.7)_1$, at time $t = t_n$ and for $w = w_h \in S_h$, and discrete variational equation $(6.1)_1$ we find that

$$\int_0^1 \Big(\rho v_t^n - \frac{\rho}{\Delta t}(v_h^n - v_h^{n-1})\Big) w_h \mathrm{d}x + \int_0^1 \Big(\alpha(u^n - u_{hx}^n) + \varpi(v_x^n - v_{hx}^n) - \gamma_1(\theta^n - \theta_h^n) \\ -\gamma_2(P^n - P_h^n)\Big) w_{hx} \mathrm{d}x + \Big(\frac{1}{\varepsilon}[u^n(1) - g]_+ - \frac{1}{\varepsilon}[u_h^n(1) - g]_+\Big) w_h(1) = 0.$$

Therefore, we have

$$\int_0^1 \Big(\rho v_t^n - \frac{\rho}{\Delta t}(v_h^n - v_h^{n-1})\Big)(v^n - v_h^n)\mathrm{d}x \\ + \int_0^1 \Big(\alpha(u^n - u_{hx}^n) + \varpi(v_x^n - v_{hx}^n) - \gamma_1(\theta^n - \theta_h^n) - \gamma_2(P^n - P_h^n)\Big)(v_x^n - v_{hx}^n)\mathrm{d}x \\ + \Big(\frac{1}{\varepsilon}[u^n(1) - g]_+ - \frac{1}{\varepsilon}[u_h^n(1) - g]_+\Big)(v^n(1) - v_h^n(1)) \\ = \int_0^1 \Big(\rho v_t^n - \frac{\rho}{\Delta t}(v_h^n - v_h^{n-1})\Big)(v^n - w_h)\mathrm{d}x \\ + \int_0^1 \Big(\alpha(u^n - u_{hx}^n) + \varpi(v_x^n - v_{hx}^n) - \gamma_1(\theta^n - \theta_h^n) - \gamma_2(P^n - P_h^n)\Big)(v_x^n - w_{hx})\mathrm{d}x \\ + \Big(\frac{1}{\varepsilon}[u^n(1) - g]_+ - \frac{1}{\varepsilon}[u_h^n(1) - g]_+\Big)(v^n(1) - w_h(1)), \quad \forall w_h \in S_h.$$

Taking into account that

$$\int_0^1 \Big(\rho v_t^n - \frac{\rho}{\Delta t}(v_h^n - v_h^{n-1})\Big)(v^n - v_h^n)\mathrm{d}x \\ = \int_0^1 \Big(\rho v_t^n - \frac{\rho}{\Delta t}(v^n - v^{n-1})\Big)(v^n - v_h^n)\mathrm{d}x \\ + \frac{\rho}{2\Delta t}\Big(\|v^n - v^{n-1} - (v_h^n - v_h^{n-1})\|^2 + \|v^n - v_h^n\|^2 - \|v^{n-1} - v_h^{n-1}\|^2\Big), \\ \int_0^1 (u_x^n - u_{hx}^n)(v_x^n - v_{hx}^n)\mathrm{d}x \geq \int_0^1 (u_x^n - u_{hx}^n)(v_x^n - \frac{1}{\Delta t}(u_x^n - u_x^{n-1}))\mathrm{d}x \\ + \frac{1}{2\Delta t}\Big(\|u^n - u_h^n\|_{H^1(0,1)}^2 - \|u^{n-1} - u_h^{n-1}\|_{H^1(0,1)}^2\Big), \\ \Big|\frac{1}{\varepsilon}[u^n(1) - g]_+ - \frac{1}{\varepsilon}[u_h^n(1) - g]_+\Big| \leq C|u^n(1) - u_h^n(1)|$$

and using Cauchy−Schwarz and Cauchy's inequalities it follows that

$$\frac{\rho}{2\Delta t}\Big(\|v^n - v^{n-1} - (v_h^n - v_h^{n-1})\|^2 + \|v^n - v_h^n\|^2 - \|v^{n-1} - v_h^{n-1}\|^2\Big) + \varpi\|v^n - v_h^n\|_{H^1(0,1)}^2 \\ + \frac{1}{2\Delta t}\Big(\|u^n - u_h^n\|_{H^1(0,1)}^2 - \|u^{n-1} - u_h^{n-1}\|_{H^1(0,1)}^2\Big) \\ \leq C\Big(\|\rho v_t^n - \frac{\rho}{\Delta t}(v^n - v^{n-1})\|^2 + \epsilon\|v^n - v_h^n\|_{H^1(0,1)}^2 + \|u^n - u_h^n\|_{H^1(0,1)}^2 + \|\theta^n - \theta_h^n\|^2 \\ + \frac{\rho}{\Delta t}\int_0^1 \Big(v^n - v_h^n - (v^{n-1} - v_h^{n-1})\Big)(v^n - w_h)\,\mathrm{d}x + \|P^n - P_h^n\|^2 \\ + \|v_x^n - \frac{1}{\Delta t}(u_x^n - u_x^{n-1})\|^2 + \|v^n - w_h\|_{H^1(0,1)}^2\Big), \quad \forall w_h \in S_h, \tag{6.13}$$

where $\epsilon > 0$ is assumed to be small enough.

Combining (6.10), (6.12) and (6.13), it follows that

$$
\begin{aligned}
&\frac{\tau}{2\Delta t}\Big(\|\eta^n-\eta_h^n\|^2-\|\eta^{n-1}-\eta_h^{n-1}\|^2+\|\eta^n-\eta^{n-1}-(\eta_h^n-\eta_h^{n-1})\|^2\Big)\\
&\quad+\frac{r}{2\Delta t}\Big(\|P^n-P_h^n\|^2-\|P^{n-1}-P_h^{n-1}\|^2+\|P^n-P^{n-1}-(P_h^n-P_h^{n-1})\|^2\Big)\\
&\quad-\frac{d}{\Delta t}\int_0^1\Big(\theta^n-\theta_n^h-(\theta^{n-1}-\theta_h^{n-1})\Big)(P_n-P_n^h)\,\mathrm{d}x\\
&\quad+\frac{c}{2\Delta t}\Big(\|\theta^n-\theta_h^n\|^2-\|\theta^{n-1}-\theta_h^{n-1}\|^2+\|\theta^n-\theta^{n-1}-(\theta_h^n-\theta_h^{n-1}\|^2\Big)\\
&\quad+\frac{\tau_0}{2\Delta t}\Big(\|q^n-q_h^n\|^2-\|q^{n-1}-q_h^{n-1}\|^2+\|q^n-q^{n-1}-(q_h^n-q_h^{n-1})\|^2\Big)\\
&\quad-\frac{d}{\Delta t}\int_0^1\Big((P^n-P_h^n)-(P^{n-1}-P_h^{n-1})\Big)(\theta^n-\theta_h^n)\,\mathrm{d}x\\
&\quad+\frac{\rho}{2\Delta t}\Big(\|v^n-v^{n-1}-(v_h^n-v_h^{n-1})\|^2+\|v^n-v_h^n\|^2-\|v^{n-1}-v_h^{n-1}\|^2\Big)+\varpi\|v^n-v_h^n\|^2_{H^1(0,1)}\\
&\quad+\frac{1}{2\Delta t}\Big(\|u^n-u_h^n\|^2_{H^1(0,1)}-\|u^{n-1}-u_h^{n-1}\|^2_{H^1(0,1)}\Big)\\
&\le C\Big(\|P_t^n-\frac{1}{\Delta t}(P^n-P^{n-1})\|^2+\|\theta_t^n-\frac{1}{\Delta t}(\theta^n-\theta^{n-1})\|^2+\|\theta^n-\theta_h^n\|^2\\
&\quad+\|q_t^n-\frac{1}{\Delta t}(q^n-q^{n-1})\|^2+\|q^n-q_h^n\|^2+\|q^n-\phi_h\|^2_{H^1(0,1)}+\|\theta^n-\vartheta_h\|^2_{H^1(0,1)}\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(\theta^n-\theta_h^n-(\theta^{n-1}-\theta_h^{n-1})\Big)(\theta^n-\vartheta_h)\,\mathrm{d}x+\|\eta^n-\psi_h\|^2_{H^1(0,1)}\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(q^n-q_h^n-(q^{n-1}-q_h^{n-1})\Big)(q^n-\phi_h)\,\mathrm{d}x+\|P^n-P_h^n\|^2+\|\eta^n-\eta_h^n\|^2\\
&\quad+\|\eta_t^n-\frac{1}{\Delta t}(\eta^n-\eta^{n-1})\|^2+\frac{\rho}{\Delta t}\int_0^1\Big(v^n-v_h^n-(v^{n-1}-v_h^{n-1})\Big)(v^n-w_h)\,\mathrm{d}x\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(\eta^n-\eta_h^n-(\eta^{n-1}-\eta_h^{n-1})\Big)(\eta^n-\psi_h)\,\mathrm{d}x+\|P_n-z_h\|^2_{H^1(0,1)}\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(P^n-P_h^n-(P^{n-1}-P_h^{n-1})\Big)(P^n-z_h)\,\mathrm{d}x\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(P^n-P_h^n-(P^{n-1}-P_h^{n-1})\Big)(\theta^n-\vartheta_h)\,\mathrm{d}x\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(\theta^n-\theta_h^n-(\theta^{n-1}-\theta_h^{n-1})\Big)(P^n-z_h)\,\mathrm{d}x\\
&\quad+\|\rho v_t^n-\frac{\rho}{\Delta t}(v^n-v^{n-1})\|^2+\epsilon\|v^n-v_h^n\|^2_{H^1(0,1)}+\|u^n-u_h^n\|^2_{H^1(0,1)}\\
&\quad+\|v^n-\frac{1}{\Delta t}(u^n-u^{n-1})\|^2_{H^1(0,1)}+\|v^n-w_h\|^2_{H^1(0,1)}\Big),
\end{aligned}
$$

for all $(w_h,\vartheta_h,\phi_h,z_h,\psi_h)\in S_h\times V_h\times V_h\times S_h\times S_h$.

Now, we note that

$$
\begin{aligned}
&\frac{d}{\Delta t}(\theta^n-\theta_h^n-(\theta^{n-1}-\theta_h^{n-1}),P^n-P_h^n)+\frac{d}{\Delta t}(P^n-P_h^n-(P^{n-1}-P_h^{n-1}),\theta^n-\theta_h^n)\\
&\quad=\frac{d}{\Delta t}(\theta^n-\theta_h^n,P^n-P_h^n)-\frac{d}{\Delta t}(\theta^{n-1}-\theta_h^{n-1},P^{n-1}-P_h^{n-1})\\
&\qquad+\frac{d}{\Delta t}(\theta^n-\theta^{n-1}-(\theta_h^n-\theta_h^{n-1}),P^n-P^{n-1}-(P^n-P_h^{n-1})),
\end{aligned}
$$

and using (2.16) we observe that

$$c\|\theta^n-\theta^{n-1}-(\theta_h^n-\theta_h^{n-1}\|^2+r\|P^n-P^{n-1}-(P^n-P_h^{n-1})\|^2 \\ +2d(\theta^n-\theta^{n-1}-(\theta_h^n-\theta_h^{n-1}),P^n-P^{n-1}-(P_h^n-P_h^{n-1}))\geq 0.$$

Then, keeping in mind that, from (2.16), we can choose ζ such that $d/c<\zeta<r/d$, and writing

$$-2d(\theta^n-\theta_h^n,P^n-P_h^n)\leq \frac{\mathrm{d}}{\zeta}\|\theta^n-\theta_h^n\|^2+\mathrm{d}\zeta\|P^n-P_h^n\|^2,$$

we can conclude that

$$c\|\theta^n-\theta_h^n\|^2+r\|P^n-P_h^n\|^2+2d(\theta^n-\theta_h^n,P^n-P_h^n)\geq\left(c-\frac{\mathrm{d}}{\zeta}\right)\|\theta^n-\theta_h^n\|^2+(r-\mathrm{d}\zeta)\|P^n-P_h^n\|^2.$$

Therefore, multiplying estimates (6.14) by Δt and summing up to n, we obtain, for all $\{w_h^j\}_{j=0}^n\subset S_h$, $\{\vartheta_h^j\}_{j=0}^n\subset V_h$, $\{\phi_h^j\}_{j=0}^n\subset V_h$, $\{z_h^j\}_{j=0}^n\subset S_h$ and $\{\eta_h^j\}_{j=0}^n\subset S_h$,

$$\begin{aligned}
&\|\eta^n-\eta_h^n\|^2+\|P^n-P_h^n\|^2+\|\theta^n-\theta_h^n\|^2+\|q^n-q_h^n\|^2+\|v^n-v_h^n\|^2\\
&\quad+\Delta t\sum_{j=1}^n\|v^j-v_h^j\|^2_{H^1(0,1)}+\|u^n-u_h^n\|^2_{H^1(0,1)}\\
&\leq C\Delta t\sum_{j=1}^n\Big(\|P_t^j-\frac{1}{\Delta t}(P^j-P^{j-1})\|^2+\|\theta_t^j-\frac{1}{\Delta t}(\theta^j-\theta^{j-1})\|^2+\|\theta^j-\theta_h^j\|^2\\
&\quad+\|q_t^j-\frac{1}{\Delta t}(q^j-q^{j-1})\|^2+\|q^j-\phi_h^j\|^2_{H^1(0,1)}+\|\theta^j-\vartheta_h^j\|^2_{H^1(0,1)}+\|P^j-P_h^j\|^2\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(\theta^j-\theta_h^j-(\theta^{j-1}-\theta_h^{j-1})\Big)(\theta^j-\vartheta_h^j)\,\mathrm{d}x\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(q^j-q_h^j-(q^{j-1}-q_h^{j-1})\Big)(q^j-\phi_h^j)\,\mathrm{d}x\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(\eta^j-\eta_h^j-(\eta^{j-1}-\eta_h^{j-1})\Big)(\eta^j-\psi_h^j)\,\mathrm{d}x\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(P^j-P_h^j-(P^{j-1}-P_h^{j-1})\Big)(P^j-z_h^j)\,\mathrm{d}x+\|\eta^j-\eta_h^j\|^2\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(P^j-P_h^j-(P^{j-1}-P_h^{j-1})\Big)(\theta^j-\vartheta_h^j)\,\mathrm{d}x+\|P^j-z_h^j\|^2_{H^1(0,1)}\\
&\quad+\frac{1}{\Delta t}\int_0^1\Big(\eta^j-\eta_h^j-(\eta^{j-1}-\eta_h^{j-1})\Big)(P^j-z_h^j)\,\mathrm{d}x+\|\eta^j-\psi_h^j\|^2_{H^1(0,1)}\\
&\quad+\frac{\rho}{\Delta t}\int_0^1\Big(v^j-v_h^j-(v^{j-1}-v_h^{j-1})\Big)(v^j-w_h^j)\,\mathrm{d}x+\|\eta_t^j-\frac{1}{\Delta t}(\eta^j-\eta^{j-1})\|^2\\
&\quad+\|\rho v_t^j-\frac{\rho}{\Delta t}(v^j-v^{j-1})\|^2+\|u^j-u_h^j\|^2_{H^1(0,1)}+\|v^j-w_h^j\|^2_{H^1(0,1)}\\
&\quad+\|v^j-\frac{1}{\Delta t}(u^j-u^{j-1})\|^2_{H^1(0,1)}\Big)+C\Big(\|q^0-q_h^0\|^2+\|u^0-u^0\|^2_{H^1(0,1)}\\
&\quad+\|\theta^0-\theta^0\|^2+\|\eta^0-\eta_h^0\|^2+\|P^0-P_h^0\|^2+\|v^0-v_h^0\|^2\Big).
\end{aligned}\tag{6.14}$$

Keeping in mind that (see [9])

$$\sum_{j=1}^{n}\int_0^1 \Big(\theta^j-\theta_h^j-(\theta^{j-1}-\theta_h^{j-1})\Big)(\theta^j-\vartheta_h^j)\,\mathrm{d}x=\int_0^1(\theta^n-\theta_h^n)(\theta^n-\vartheta_h^n)\,\mathrm{d}x$$

$$-\int_0^1(\theta^0-\theta_h^0)(\theta^1-\vartheta_h^1)\,\mathrm{d}x+\sum_{j=1}^{n-1}\int_0^1(\theta^j-\theta_h^j)(\theta^j-\vartheta_h^j-(\theta^{j+1}-\vartheta_h^{j+1}))\,\mathrm{d}x,$$

$$\sum_{j=1}^{n}\int_0^1 \Big(q^j-q_h^j-(q^{j-1}-q_h^{j-1})\Big)(q^j-\phi_h^j)\,\mathrm{d}x=\int_0^1(q^n-q_h^n)(q^n-\phi_h^n)\,\mathrm{d}x$$

$$-\int_0^1(q^0-q_h^0)(q^1-\phi_h^1)\,\mathrm{d}x+\sum_{j=1}^{n-1}\int_0^1(q^j-q_h^j)(q^j-\phi_h^j-(q^{j+1}-\phi_h^{j+1}))\,\mathrm{d}x,$$

$$\sum_{j=1}^{n}\int_0^1 \Big(\eta^j-\eta_h^j-(\eta^{j-1}-\eta_h^{j-1})\Big)(\eta^j-\psi_h^j)\,\mathrm{d}x=\int_0^1(\eta^n-\eta_h^n)(\eta^n-\psi_h^n)\,\mathrm{d}x$$

$$-\int_0^1(\eta^0-\eta_h^0)(\eta^1-\psi_h^1)\,\mathrm{d}x+\sum_{j=1}^{n-1}\int_0^1(\eta^j-\eta_h^j)(\eta^j-\psi_h^j-(\eta^{j+1}-\psi_h^{j+1}))\,\mathrm{d}x,$$

$$\sum_{j=1}^{n}\int_0^1 \Big(P^j-P_h^j-(P^{j-1}-P_h^{j-1})\Big)(P^j-z_h^j)\,\mathrm{d}x=\int_0^1(P^n-P_h^n)(P^n-z_h^n)\,\mathrm{d}x$$

$$-\int_0^1(P^0-P_h^0)(P^1-z_h^1)\,\mathrm{d}x+\sum_{j=1}^{n-1}\int_0^1(P^j-P_h^j)(P^j-z_h^j-(P^{j+1}-z_h^{j+1}))\,\mathrm{d}x,$$

$$\sum_{j=1}^{n}\int_0^1 \Big(P^j-P_h^j-(P^{j-1}-P_h^{j-1})\Big)(\theta^j-\vartheta_h^j)\,\mathrm{d}x=\int_0^1(P^n-P_h^n)(\theta^n-\vartheta_h^n)\,\mathrm{d}x$$

$$-\int_0^1(P^0-P_h^0)(\theta^1-\vartheta_h^1)\,\mathrm{d}x+\sum_{j=1}^{n-1}\int_0^1(P^j-P_h^j)(\theta^j-\vartheta_h^j-(\theta^{j+1}-\vartheta_h^{j+1}))\,\mathrm{d}x,$$

$$\sum_{j=1}^{n}\int_0^1 \Big(\theta^j-\theta_h^j-(\theta^{j-1}-\theta_h^{j-1})\Big)(P^j-z_h^j)\,\mathrm{d}x=\int_0^1(\theta^n-\theta_h^n)(P^n-z_h^n)\,\mathrm{d}x$$

$$-\int_0^1(\theta^0-\theta_h^0)(P^1-z_h^1)\,\mathrm{d}x+\sum_{j=1}^{n-1}\int_0^1(\theta^j-\theta_h^j)(P^j-z_h^j-(P^{j+1}-z_h^{j+1}))\,\mathrm{d}x,$$

$$\sum_{j=1}^{n}\int_0^1 \Big(v^j-v_h^j-(v^{j-1}-v_h^{j-1})\Big)(v^j-w_h^j)\,\mathrm{d}x=\int_0^1(v^n-v_h^n)(v^n-w_h^n)\,\mathrm{d}x$$

$$-\int_0^1(v^0-v_h^0)(v^1-w_h^1)\,\mathrm{d}x+\sum_{j=1}^{n-1}\int_0^1(v^j-v_h^j)(v^j-w_h^j-(v^{j+1}-w_h^{j+1}))\,\mathrm{d}x,$$

using a discrete version of Gronwall's inequality (see [11]), we obtain the following *a priori* error estimates result.

Theorem 6.3. *Assume the initial data* (3.5) *and the regularity conditions* (6.8). *Let us denote by* (u,θ,q,P,η) *and* $\{(u_h^n,\theta_h^n,q_h^n,P_h^n,\eta_h^n)\}_{n=0}^N$ *the solutions to the penalized problem* (3.1)–(3.4) *and the fully discrete problem* (6.1), *respectively. Then, there exists a positive constant* $C>0$, *independent of the discretization parameters* *h and k, such that, for all* $(w_h, \vartheta_h, \phi_h, z_h, \psi_h) \in S_h \times V_h \times V_h \times S_h \times S_h$,

$$
\begin{aligned}
&\max_{0\le n\le N} \Big\{\|\eta^n-\eta_h^n\|^2+\|P^n-P_h^n\|^2+\|\theta^n-\theta_h^n\|^2+\|q^n-q_h^n\|^2+\|v^n-v_h^n\|^2 \\
&\qquad +\|u^n-u_h^n\|^2_{H^1(0,1)}\Big\}+\Delta t\sum_{j=1}^N \|v^j-v_h^j\|^2_{H^1(0,1)} \\
&\le C\Delta t\sum_{j=1}^N\Big(\|P_t^j-\frac{1}{\Delta t}(P^j-P^{j-1})\|^2+\|\theta_t^j-\frac{1}{\Delta t}(\theta^j-\theta^{j-1})\|^2 \\
&\quad +\|q_t^j-\frac{1}{\Delta t}(q^j-q^{j-1})\|^2+\|\eta_t^j-\frac{1}{\Delta t}(\eta^j-\eta^{j-1})\|^2 \\
&\quad +\|P^j-z_h^j\|^2_{H^1(0,1)}+\|\theta^j-\vartheta_h^j\|^2_{H^1(0,1)}+\|q^j-\phi_h^j\|^2_{H^1(0,1)}+\|\eta^j-\psi_h^j\|^2_{H^1(0,1)} \\
&\quad +\|\rho v_t^j-\frac{\rho}{\Delta t}(v^j-v^{j-1})\|^2+\|v^j-w_h^j\|^2_{H^1(0,1)}+\|v^j-\frac{1}{\Delta t}(u^j-u^{j-1})\|^2_{H^1(0,1)}\Big) \\
&\quad +\frac{C}{\Delta t}\sum_{j=1}^{N-1}\Big(\|\theta^j-\vartheta_h^j-(\theta^{j+1}-\vartheta^{j+1})\|^2+\|q^j-\phi_h^j-(q^{j+1}-\phi_h^{j+1})\|^2 \\
&\quad +\|\eta^j-\psi_h^j-(\eta^{j+1}-\psi_h^{j+1})\|^2+\|P^j-z_h^j-(P^{j+1}-z_h^{j+1})\|^2 \\
&\quad +\|v^j-w_h^j-(v^{j+1}-w^{j+1})\|^2\Big)+C\Big(\|q^0-q_h^0\|^2+\|u^0-u^0\|^2_{H^1(0,1)}+\|\theta^0-\theta^0\|^2 \\
&\quad +\|\eta^0-\eta_h^0\|^2+\|P^0-P_h^0\|^2+\|v^0-v_h^0\|^2\Big).
\end{aligned}
\tag{6.15}
$$

Error estimates (6.15) are the basis for the analysis of the convergence rate. As an example, let us assume the following additional regularity conditions:

$$
\begin{aligned}
&u^\varepsilon \in H^3(I;L^2(0,1))\cap W^{1,\infty}(I;H^2(0,1))\cap H^2(I;H^1(0,1)),\\
&\theta^\varepsilon, P^\varepsilon, \eta^\varepsilon, q^\varepsilon \in H^2(I;L^2(0,1))\cap H^1(I;H^1(0,1))\cap L^\infty(I;H^2(0,1)).
\end{aligned}
\tag{6.16}
$$

Using the classical properties on the approximation by the standard finite element spaces (see [12]) and taking into account that ([9])

$$
\begin{aligned}
&\frac{C}{\Delta t}\sum_{j=1}^{N-1}\Big(\|\theta^j-\vartheta_h^j-(\theta^{j+1}-\vartheta^{j+1})\|^2+\|q^j-\phi_h^j-(q^{j+1}-\phi_h^{j+1})\|^2 \\
&\quad +\|\eta^j-\psi_h^j-(\eta^{j+1}-\psi_h^{j+1})\|^2+\|P^j-z_h^j-(P^{j+1}-z_h^{j+1})\|^2+\|v^j-w_h^j-(v^{j+1}-w^{j+1})\|^2\Big) \\
&\quad \le Ch^2\Big(\|v\|^2_{H^1(I;H^1(0,1))}+\|\theta\|^2_{H^1(I;H^1(0,1))}+\|P\|^2_{H^1(0,T;H^1(0,1))}+\|\eta\|^2_{H^1(I;H^1(0,1))}+\|q\|^2_{H^1(I;H^1(0,1))}\Big),
\end{aligned}
$$

we derive the linear convergence of the algorithm obtained from (6.1), that we state in the following.

Corollary 6.4. *Under the assumptions of Theorem 6.3 and the additional regularity conditions (6.16), the linear convergence of the algorithm is achieved,* i.e. *there exists a positive constant* $C > 0$, *independent of the discretization parameters h and k, such that*

$$
\begin{aligned}
&\max_{0\le n\le N}\Big\{\|\eta^\varepsilon(t_n)-\eta_h^n\|+\|P^\varepsilon(t_n)-P_h^n\|+\|\theta^\varepsilon(t_n)-\theta_h^n\|+\|q^\varepsilon(t_n)-q_h^n\|+\|v^\varepsilon(t_n)-v_h^n\| \\
&\qquad +\|u^\varepsilon(t_n)-u_h^n\|_{H^1(0,1)}\Big\}\le C(h+k).
\end{aligned}
$$

7. Numerical experiments

In this final section, we present the numerical simulations performed.

Since problem (6.1) is nonlinear, we now propose an iterative algorithm to find its solution. Assuming that $(v_h^{n,l-1}, u_h^{n,l-1}\theta_h^{n,l-1}, P_h^{n,l-1}, q_h^{n,l-1}, \eta_h^{n,l-1})$ are known, we look for a solution of the following problem: find $(v_h^{n,l}, \theta_h^{n,l}, P_h^{n,l}, q_h^{n,l}, \eta_h^{n,l})$ such that

$$\begin{aligned}
&\int_0^1 \left(\frac{c}{\Delta t}(\theta_h^{n,l}-\theta_h^{n-1}) + \frac{d}{\Delta t}(P_h^{n,l}-P_h^{n-1}) + q_{hx}^{n,l} + \gamma_1 v_{hx}^{n,l-1}\right) v_h \mathrm{d}x = 0,\\
&\int_0^1 \left(\frac{\tau_0}{\Delta t}(q_h^{n,l}-q_h^{n-1}) + q_h^{n,l} + \theta_{hx}^{n,l}\right)\phi_h \mathrm{d}x = 0,\\
&\int_0^1 \left(\frac{r}{\Delta t}(P_h^{n,l}-P_h^{n-1}) + \frac{d}{\Delta t}(\theta_h^{n,l}-\theta_h^{n-1}) + \hbar\eta_{hx}^{n,l} + \gamma_2 v_{hx}^{n,l}\right) z_h \mathrm{d}x = 0,\\
&\int_0^1 \left(\frac{\tau}{\Delta t}(\eta_h^{n,l}-\eta_h^{n-1}) + \eta_h^{n,l} + P_{hx}^{n,l}\right)\psi_h \mathrm{d}x = 0,\\
&\int_0^1 \frac{\rho}{\Delta t}(v_h^{n,l}-v_h^{n-1}) w_h \mathrm{d}x + \int_0^1 \left(\alpha u_{hx}^{n,l} + \varpi v_{hx}^{n,l} - \gamma_1\theta_h^{n,l} - \gamma_2 P_h^{n,l}\right) w_{hx}\mathrm{d}x\\
&\qquad + \frac{1}{\varepsilon}[u_h^{n,l-1}(1)-g]_+ w_h(1) = 0,
\end{aligned}$$

with $u_h^{n,l} = u_h^{n-1} + \Delta t v_h^{n,l}$, $l = 1, 2, \ldots$. We set $v_h^{n,0} = v_h^{n-1}$, $u_h^{n,0} = u_h^{n-1}$, $\theta_h^{n,0} = \theta_h^{n-1}$, $P_h^{n,0} = P_h^{n-1}$, $q_h^{n,0} = q_h^{n-1}$ and $\eta_h^{n,0} = \eta_h^{n-1}$.

Writing

$$v_h^{n,l} = \sum_{i=1}^{s} d_i^{n,l}\chi_i, \qquad \theta_h^{n,l} = \sum_{i=0}^{s-1} b_i^{n,l}\xi_i, \qquad P_h^{n,l} = \sum_{i=0}^{s-1} a_i^{n,l}\xi_i, \qquad q_h^{n,l} = \sum_{i=1}^{s} c_i^{n,l}\chi_i, \qquad \eta_h^{n,l} = \sum_{i=1}^{s} r_i^{n,l}\chi_i$$

where $\{\xi_i\}_{i=0}^{s-1}$ and $\{\chi_i\}_{i=1}^{s}$ are the standard basis for V^h and S^h, respectively, it follows from (6.1) that we need to solve, at each iteration l, the following algebraic linear systems:

$$\begin{pmatrix} cM & dM & \Delta tK & 0\\ dM & rM & 0 & \Delta t\hbar K\\ -\Delta tK^t & 0 & (\tau_0+\Delta t)\tilde{M} & 0\\ 0 & -\Delta tK^t & 0 & (\tau+\Delta t)\tilde{M}\end{pmatrix}\begin{pmatrix}\underline{b}^{n,l}\\ \underline{a}^{n,l}\\ \underline{c}^{n,l}\\ \underline{r}^{n,l}\end{pmatrix} = \begin{pmatrix} M(c\underline{b}^{n-1}+d\underline{a}^{n-1}) - \gamma_1\Delta tK\underline{d}^{n,l-1}\\ M(d\underline{b}^{n-1}+r\underline{a}^{n-1}) - \gamma_2\Delta tK\underline{d}^{n,l-1}\\ \tau_0\tilde{M}\underline{c}^{n-1}\\ \tau\tilde{M}\underline{r}^{n-1}\end{pmatrix},$$

$$\left(\rho\tilde{M} + \Delta t\hat{K}(\varpi+\Delta t\alpha)\right)\underline{d}^{n,l} = \rho\tilde{M}\underline{d}^{n-1} - \Delta t\alpha\hat{K}\underline{y}^{n-1} + \Delta t\tilde{K}(\gamma_1\underline{b}^{n,l} + \gamma_2\underline{a}^{n,l}) + \frac{\Delta t}{\varepsilon}[u_h^{n,l-1}(1)-g]_+\underline{e},$$

where

$$M_{ij} = \int_0^1 \xi_i\xi_j\mathrm{d}x,\ K_{ij} = \int_0^1 \xi_i\chi_{jx}\mathrm{d}x,\ \tilde{M}_{ij} = \int_0^1 \chi_i\chi_j\mathrm{d}x,\ \hat{K}_{ij} = \int_0^1 \chi_{ix}\chi_{jx}\mathrm{d}x,$$

$$\tilde{K}_{ij} = \int_0^1 \chi_{ix}\xi_j\mathrm{d}x,\ u_h^{n-1} = \sum_{i=1}^{s} y_i^{n-1}\chi_i,\ \{\underline{e}\}_i = \chi_i(1).$$

The latter systems have unique solutions since the coefficient matrices have non-zero determinants. A tolerance of 10^{-7} was used to stop the iterative procedure.

The copper material was chosen for purposes of numerical evaluations (see [27]).

From Table 1, it was found that

$$\begin{aligned}
&\breve{a} = 0.0168,\quad \alpha_1 = 5.43,\quad \alpha_2 = 0.533,\quad \alpha_3 = 36.24,\quad \alpha = 0.0163,\quad \rho = 1.2298\times 10^{-17},\\
&\gamma_1 = 0.01931,\quad \gamma_2 = r = 4.635\times 10^{-4},\quad c = 1.0136,\quad d = 0.00251,\quad \hbar = 0.03058,\\
&\tau_0 = 0.02,\quad \tau = 0.2.
\end{aligned}\tag{7.1}$$

TABLE 1. The values of material parameters.

$T_0 = 293$ K	$\rho_0 = 8954$ kg/m^3	$c_E = 383.1$ J/(kg K)
$\kappa = 386$ W/(m K)	$\lambda = 7.76 \times 10^{10}$ kg/(m s^2)	$\mu = 3.86 \times 10^{10}$ kg/(m s^2)
$\alpha_t = 1.78 \times 10^{-5}$ K^{-1}	$\alpha_c = 1.98 \times 10^{-4}$ m^3/kg	$D = 0.85 \times 10^{-8}$ kg s/m^3
$\nu = 1.2 \times 10^4$ m^2/(s^2 K)	$\varrho = 0.9 \times 10^6$ m^5/(kg s^2)	$\ell = 1$ m

The viscous term ϖ assume values from 0 (thermoelastic diffusion case) to 1. We choose $g = 0.1$, $\varepsilon = 0.01$,

$$u_0(x) = q_0(x) = \eta_0(x) = 0, \quad u_1(x) = 10x,$$

and discretization parameters $h = 1/500$, $\Delta t = 10^{-4}$.

As no external forces act upon the system, changes in the temperature and in the chemical potential, generated by different initial conditions and viscosity parameters, may cause the rod′s contraction or expansion.

In the first experiment, the energy decay and the influence of ϖ is investigated. The initial temperature and the initial chemical potential are given by

$$\theta_0(x) = 40x(x-1)^2, \quad P_0(x) = 20x^2(x-1)^2.$$

We note that in this case the evolution of the system is towards the zero steady-state. For $\varpi = 0.1$, after a small time interval, the rod gets in contact with the obstacle and, after some time, it loses contact with $u(1,t)$ decreasing very slowly to zero. A fast decay of the energy is seen with θ, P, q and η vanishing before u. A strong dependence of u on the viscosity coefficient was noticed while the other quantities remained virtually the same. When we increased ϖ to $\varpi = 0.5$ and $\varpi = 1$, contact was not observed. Figures 2–4 show the results.

In the second experiment, the second sound effect is observed. Combining equation $(2.11)_2$ with $(2.11)_3$ and equation $(2.11)_4$ with $(2.11)_5$ we find that θ and P satisfy

$$\tau_0 c\theta_{tt} - \theta_{xx} + \tau_0 \mathrm{d}P_{tt} + c\theta_t + \mathrm{d}P_t + \gamma_1 u_{xt} + \tau_0\gamma_1 u_{xtt} = 0, \tag{7.2}$$

$$\tau r P_{tt} - \hbar P_{xx} + \tau \mathrm{d}\theta_{tt} + \mathrm{d}\theta_t + rP_t + \gamma_2 u_{xt} + \tau\gamma_2 u_{xtt} = 0. \tag{7.3}$$

Thus, waves propagating with speed equal to

$$v_\theta = 1/\sqrt{\tau_0 c}, \qquad v_P = \sqrt{\hbar/(r\tau)} \tag{7.4}$$

are expected for the temperature and the chemical potential. As initial condition for the temperature and the chemical potential we take single pulses, that is,

$$\theta_0(x) = P_0(x) = \begin{cases} 20, & 0 \le x \le 0.1, \\ 0, & x > 0.1, \end{cases}$$

and we observe how they move. The results presented in Figures 5 and 6 show that, for a small fixed t, the temperature θ and the heat flux q remain undisturbed after the wave fronts located at $x = 0.1 + tv_\theta$. Similar observations hold for P and η with wave fronts at $x = 0.1 + tv_P$. On the other hand, the displacement profiles are smooth. Note that in this experiment there is contact with the obstacle (see Fig. 6). For initial conditions that were pulses with amplitude 5, contact was not seen. For comparison, we show in Figure 7 the results of simulations when $\tau = \tau_0 = 0$ in equations (7.2) and (7.3).

Finally, we remark that the numerical oscillations seen in some graphs are due to large gradients and the fixed mesh used.

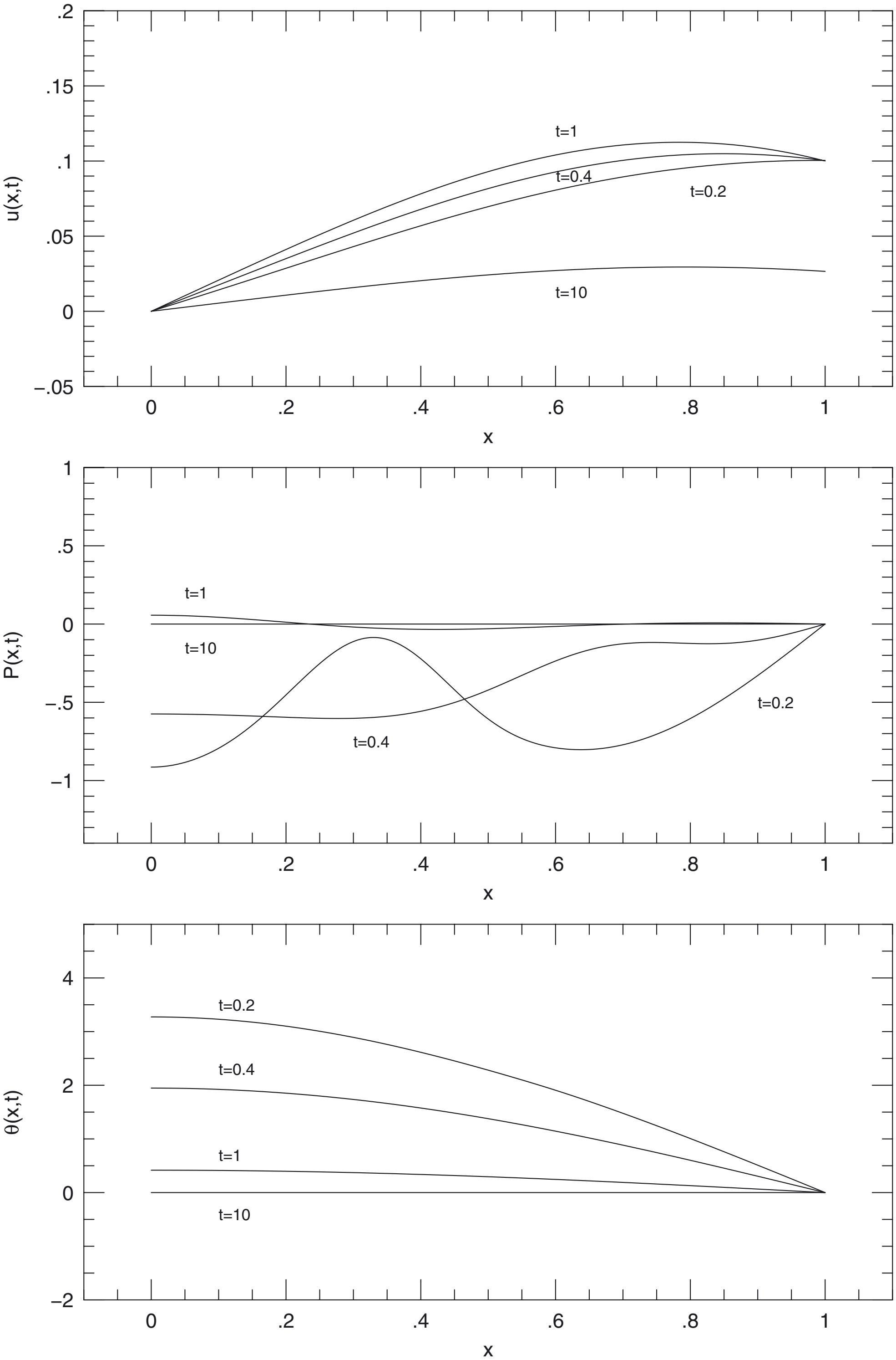

FIGURE 2. The time evolution of u, P and θ for the first experiment when $\varpi = 0.1$.

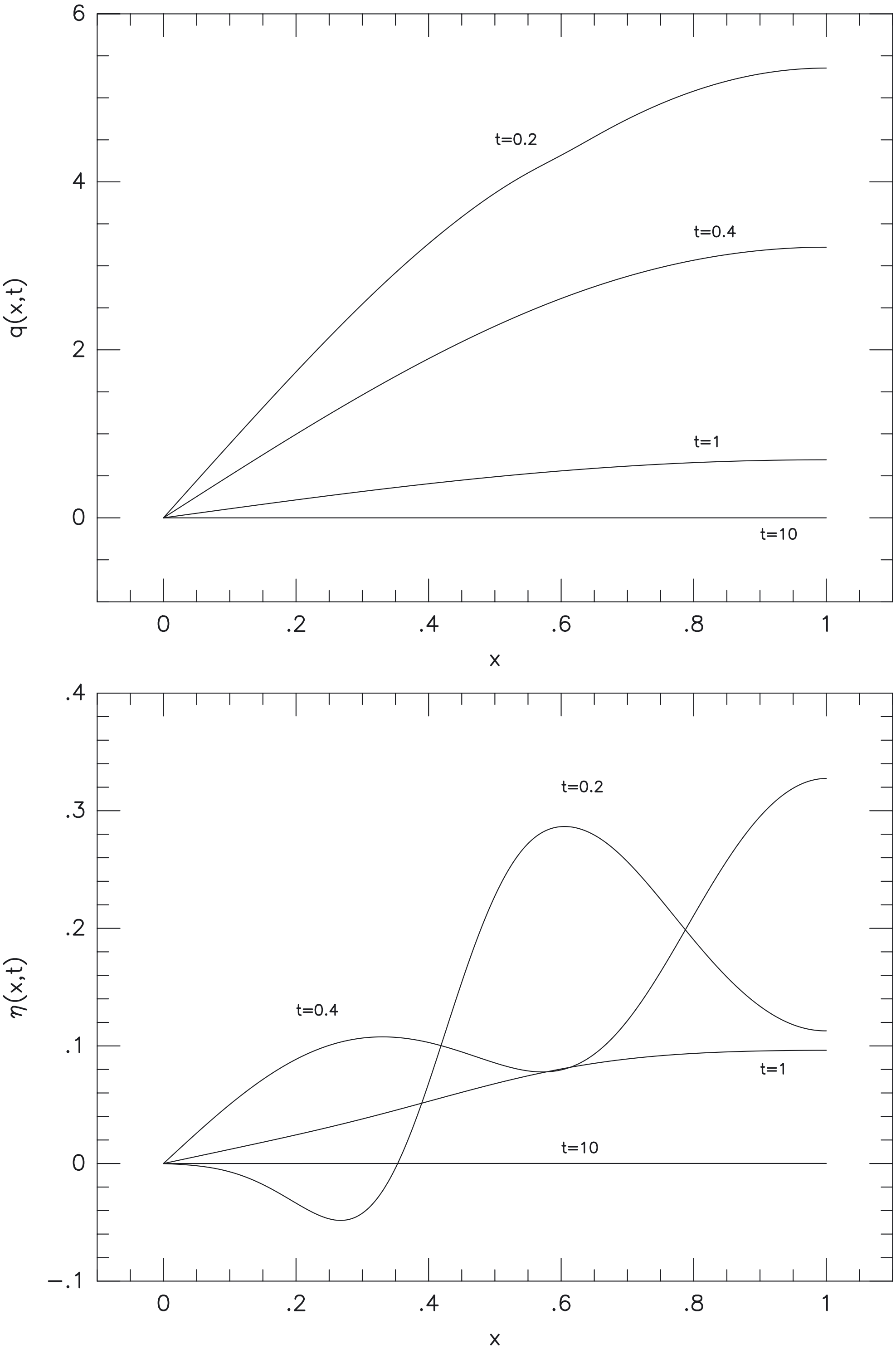

FIGURE 3. The time evolution of q and η for the first experiment when $\varpi = 0.1$.

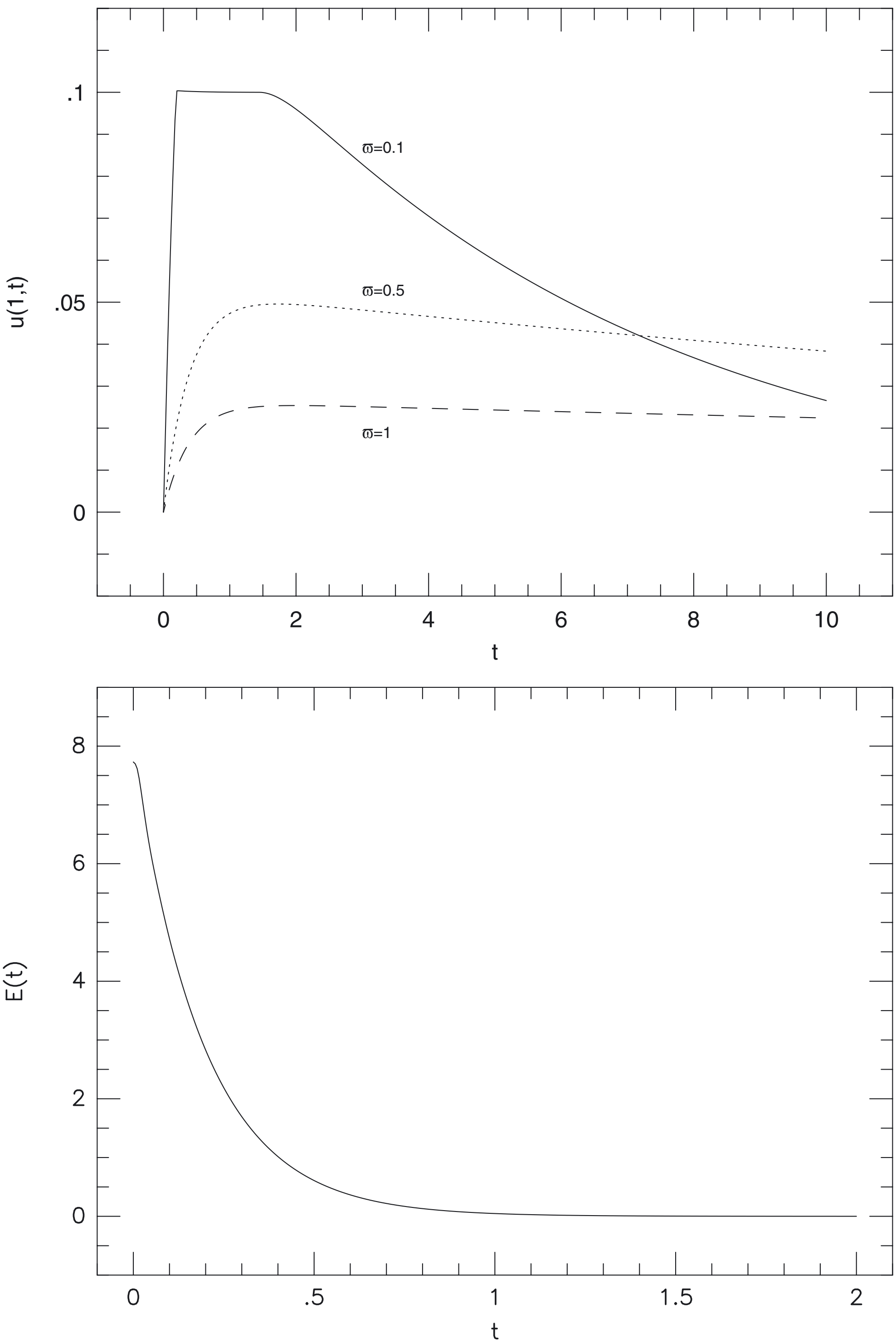

FIGURE 4. The time evolution of the energy and the displacement at the contact point for the first experiment.

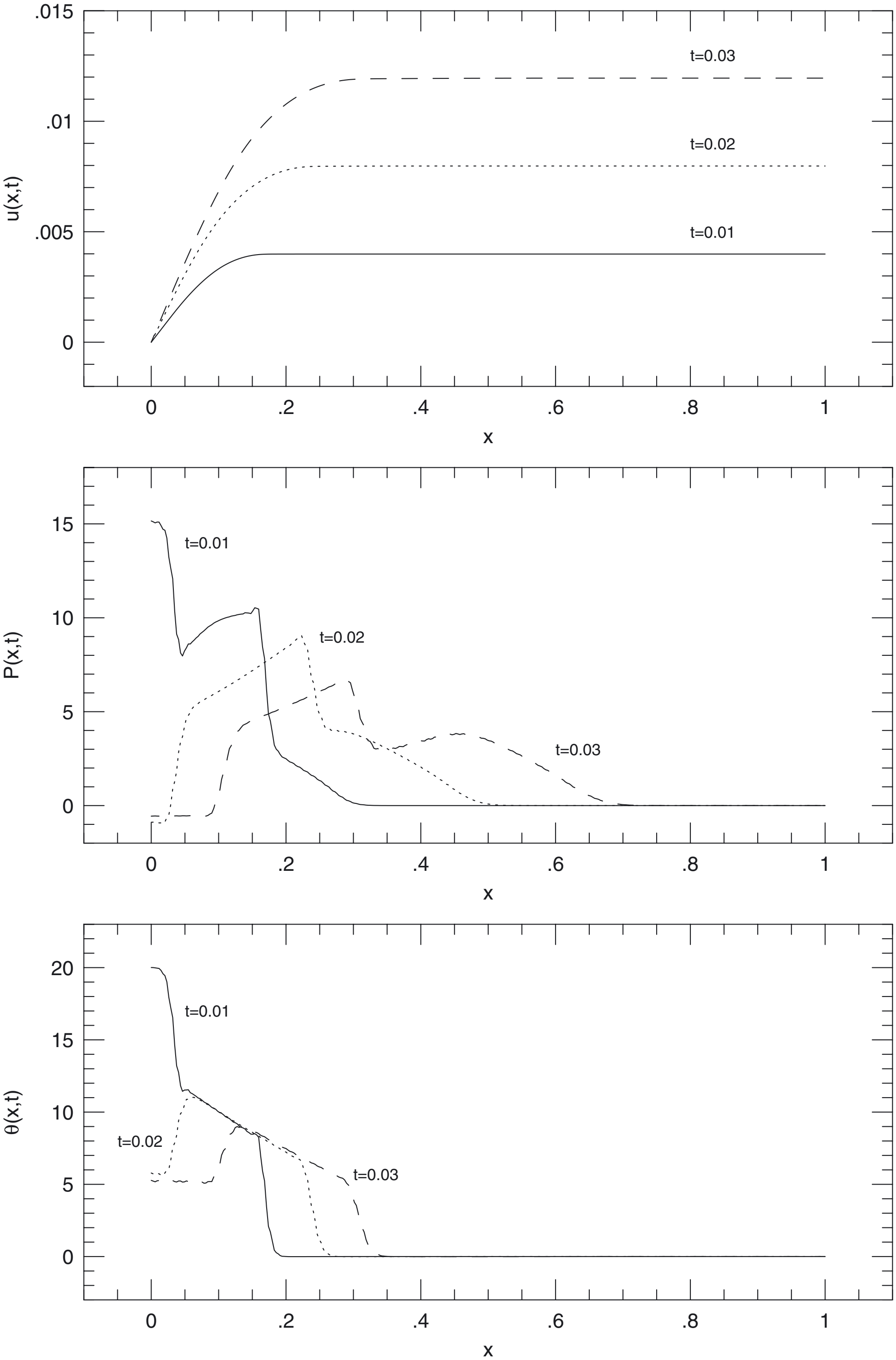

FIGURE 5. The time evolution of u, P and θ for the second experiment.

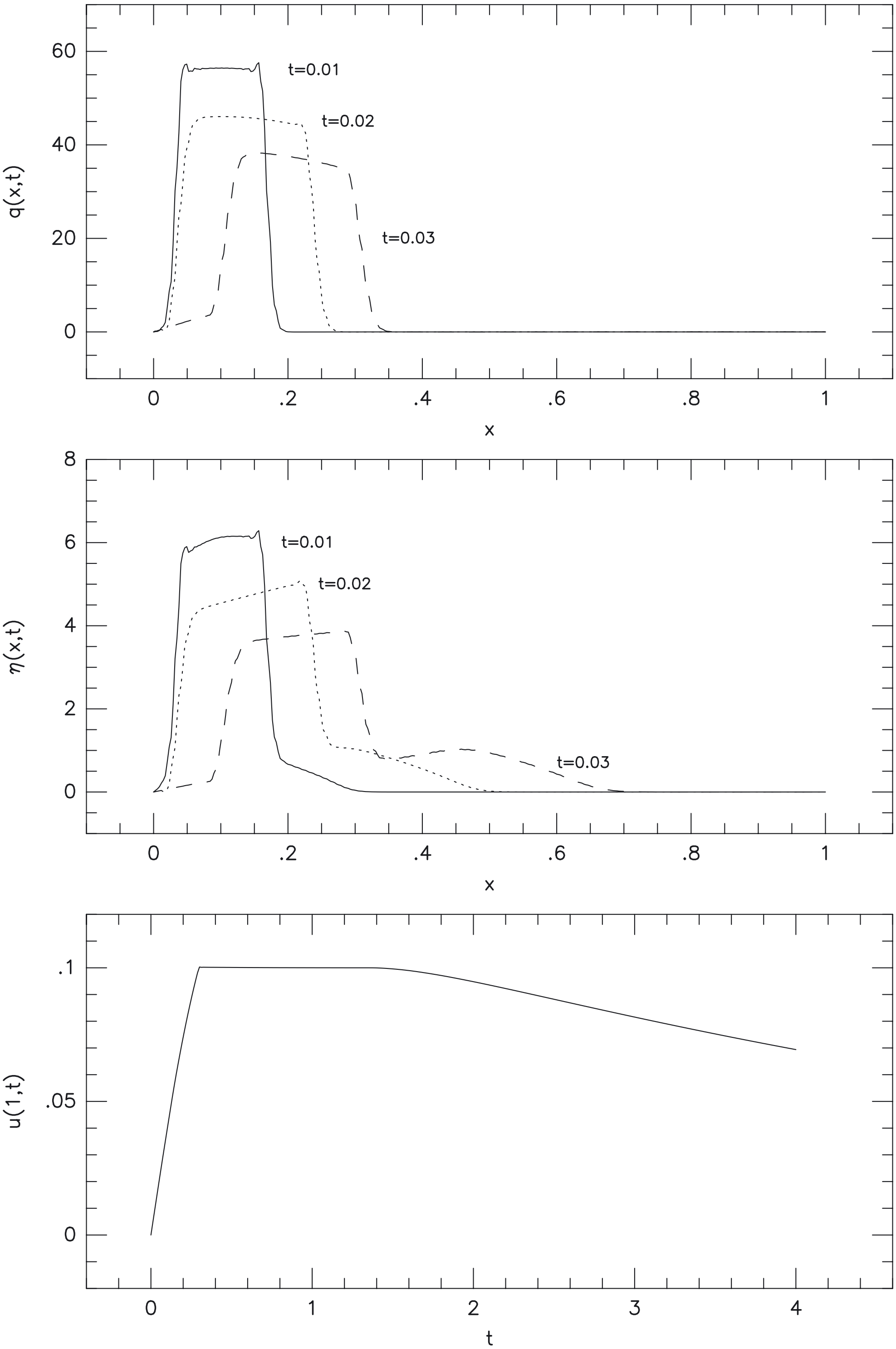

Figure 6. The time evolution of q, η and the displacement at the contact point $x = 1$ for the second experiment.

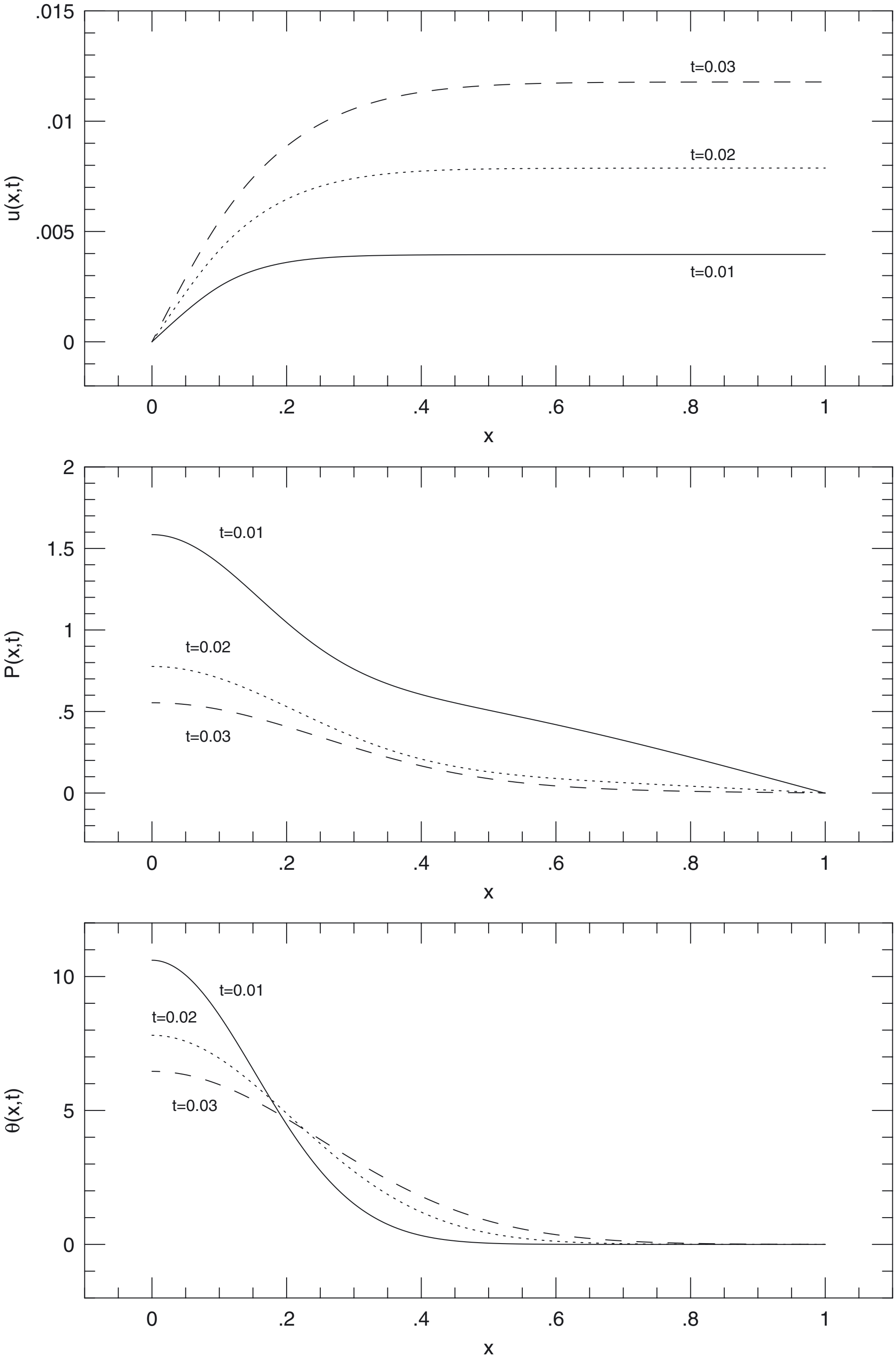

FIGURE 7. The time evolution of u, P and θ when $\tau = \tau_0 = 0$ for initial conditions of the second experiment.

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