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ANALYSIS OF A DYNAMIC VISCOELASTIC-VISCOPLASTIC PIEZOELECTRIC CONTACT PROBLEM

Marco Campo[1], Jose R. Fernández[2], Ángel D. Rodríguez-Arós[3] and José M. Rodríguez[3]

Abstract. In this paper, we study, from both variational and numerical points of view, a dynamic contact problem between a viscoelastic-viscoplastic piezoelectric body and a deformable obstacle. The contact is modelled using the classical normal compliance contact condition. The variational formulation is written as a nonlinear ordinary differential equation for the stress field, a nonlinear hyperbolic variational equation for the displacement field and a linear variational equation for the electric potential field. An existence and uniqueness result is proved using Gronwall's lemma, adequate auxiliary problems and fixed-point arguments. Then, fully discrete approximations are introduced using an Euler scheme and the finite element method, for which some *a priori* error estimates are derived, leading to the linear convergence of the algorithm under suitable additional regularity conditions. Finally, some two-dimensional numerical simulations are presented to show the accuracy of the algorithm and the behaviour of the solution.

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1. Introduction

Dynamic contact problems for viscoelastic materials have been studied in numerous publications. For instance, we could refer the papers [12,18–20,31,32,35–37,40,41] where these problems were considered assuming different friction laws and types of contact (deformable and rigid obstacles or bilateral contact, Coulomb's friction law, slip dependent friction, *etc.*). Moreover, the numerical approximation of these problems were also done, including effects as the adhesion, the piezoelectricity or the damage (see, *e.g.*, [1,2,7,8,13,14,21,22,26]).

These viscoelastic materials have been utilized in many engineering applications since they can be customized to meet a desired performance while maintain low cost. An important issue concerning such materials is that they may exhibit time-dependent and inelastic deformations. The viscoelastic strain component consists of a recoverable-reversible part (elastic strain) and a recoverable-dissipative deformation part (inelastic strain).

Keywords and phrases. Viscoelasticity, viscoplasticity, piezoelectricity, existence and uniqueness, *a priori* error estimates, numerical simulations.

[1] Centro Universitario de la Defensa, Escuela Naval Militar, Plaza de España s/n, 36920 Marín, Spain.

[2] Departamento de Matemática Aplicada I, Universidade de Vigo, Escola de Enxeñería de Telecomunicación, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain. jose.fernandez@uvigo.es

[3] Departamento de Métodos Matemáticos e Representación, Universidade de A Coruña, A Coruña, Spain.

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When an inelastic strain is assumed to depend only on the magnitude of the stress or strain, the term plastic strain is used. When the plastic deformation also changes with time, like in the viscous component of the viscoelastic part, the term viscoplastic strain is used. Therefore, a combined viscoelastic-viscoplastic constitutive relationships should be considered. Recently, new models coupling both viscoplastic and viscoelastic effects have been proposed (see, for instance, [3, 10, 15, 16, 27, 29, 33, 39, 42]).

Piezoelectricity is the ability of certain cristals, like the quartz (also ceramics (BaTiO3, KNbO3, LiNbO3, PZT-5A, *etc.*) and even the human mandible or the human bone), to produce a voltage when they are subjected to mechanical stress. This effect is characterized by the coupling between the mechanical and the electrical properties of the material. We note that this kind of materials appears usually in the industry as switches in radiotronics, electroacoustics or measuring equipments. Since the first studies by Toupin [47, 48] and Mindlin [43, 44] a large number of papers have been published dealing with related models (see, for instance, [11, 30, 45] and the references cited therein). During the last ten years, numerous contact problems involving this piezoelectric effect have been studied from the variational and numerical points of view (see, *i.e.*, [4, 6, 7, 37, 38, 41]).

In this paper, the contact is assumed to be with a deformable obstacle and so, the classical normal compliance contact condition is used ([34, 40]). Moreover, a viscoelastic-viscoplastic material is considered including, for the sake of generality in the modelling, piezoelectric effects. Both variational and numerical analyses are then performed, providing the existence of a unique weak solution to the continuous problem and an *a priori* error analysis for the fully discrete approximations. Finally, some numerical simulations are presented in two-dimensional examples to demonstrate the accuracy of the approximation and the behaviour of the solution.

The outline of this paper is as follows. In Section 2, we describe the mathematical problem and derive its variational formulation. An existence and uniqueness result is proved in Section 3. Then, fully discrete approximations are introduced in Section 4 by using the finite element method for the spatial approximation and an Euler scheme for the discretization of the time derivatives. An error estimate result is proved from which the linear convergence is deduced under suitable regularity assumptions. Finally, in Section 5 some two-dimensional numerical examples are shown to demonstrate the accuracy of the algorithm and the behaviour of the solution.

2. Mechanical problem and its variational formulation

Denote by $\mathbb{S}^d$, $d = 1, 2, 3$, the space of second order symmetric tensors on $\mathbb{R}^d$ and by "$\cdot$" and $\|\cdot\|$ the inner product and the Euclidean norms on $\mathbb{R}^d$ and $\mathbb{S}^d$.

Let $\varOmega \subset \mathbb{R}^d$ denote a domain occupied by a viscoelastic-viscoplastic piezoelectric body with a Lipschitz boundary $\varGamma = \partial\varOmega$ decomposed into three measurable parts $\varGamma_D$, $\varGamma_F$, $\varGamma_C$, on one hand, and on two measurable parts $\varGamma_A$ and $\varGamma_B$, on the other hand, such that meas $(\varGamma_D) > 0$, meas $(\varGamma_A) > 0$, and $\varGamma_C \subseteq \varGamma_B$. Let $[0, T]$, $T > 0$, be the time interval of interest. The body is being acted upon by a volume force with density $\boldsymbol{f}_0$, it is clamped on $\varGamma_D$ and surface tractions with density $\boldsymbol{f}_F$ act on $\varGamma_F$. Moreover, an electrical potential is prescribed on $\varGamma_A$ and electric charges are applied on $\varGamma_B$. Finally, we assume that the body may come in contact with a deformable obstacle on the boundary part $\varGamma_C$, which is located, in its reference configuration, at a distance s, measured along the outward unit normal vector $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$.

Let $\boldsymbol{x} \in \varOmega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, some times we do not indicate the dependence of various functions and unknowns on $\boldsymbol{x}$ and t. Moreover, a dot above a variable represents its first derivative with respect to the time variable and two dots indicate derivative of the second order.

Let $\boldsymbol{u} = (u_i)_{i=1}^d \in \mathbb{R}^d$, $\varphi \in \mathbb{R}$, $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^d \in \mathbb{S}^d$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^d \in \mathbb{S}^d$ denote the displacements, the electric potential, the stress tensor and the linearized strain tensor, respectively. We recall that

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \ldots, d.$$

The body is assumed to be made of a viscoelastic-viscoplastic piezoelectric material and it satisfies the following constitutive law (see, for instance, [23,30]),

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\boldsymbol{x},t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(\boldsymbol{x},s),\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},s)))\,\mathrm{d}s - \mathcal{E}^*\mathbf{E}(\varphi(\boldsymbol{x},t)), \tag{2.1}$$

where $\mathcal{A} = (a_{ijkl})$ and $\mathcal{B} = (b_{ijkl})$ are the fourth-order viscous and elastic tensors, respectively, $\mathcal{G}$ is a viscoplastic function whose properties will be detailed later, $\mathbf{E}(\varphi) = (E_i(\varphi))_{i=1}^d$ represents the electric field defined by

$$E_i(\varphi) = -\frac{\partial\varphi}{\partial x_i}, \quad i = 1,\ldots,d,$$

and $\mathcal{E}^* = (e^*_{ijk})^d_{i,j,k=1}$ denotes the transpose of the third-order piezoelectric tensor $\mathcal{E} = (e_{ijk})^d_{i,j,k=1}$. We recall that

$$e^*_{ijk} = e_{kij}, \quad \text{for all} \quad i,j,k = 1,\ldots,d.$$

Following [11] the following constitutive law is satisfied for the electric potential,

$$\mathcal{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\mathbf{E}(\varphi),$$

where $\mathcal{D} = (D_i)^d_{i=1}$ is the electric displacement field and $\beta = (\beta_{ij})^d_{i,j=1}$ is the electric permittivity tensor.

We turn now to describe the boundary conditions.

On the boundary part Γ_D we assume that the body is clamped and thus the displacement field vanishes there (and so $\boldsymbol{u} = 0$ on $\Gamma_D \times (0,T)$). Moreover, since the density of traction forces $\boldsymbol{f}_F$ is applied on the boundary part Γ_F, it follows that $\boldsymbol{\sigma\nu} = \boldsymbol{f}_F$ on $\Gamma_F \times (0,T)$.

The contact is assumed with a deformable obstacle and so, the well-known normal compliance contact condition is employed for its modelling (see [34,40]); that is, the normal stress $\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$ on Γ_C is given by

$$-\sigma_\nu = p(u_\nu - s),$$

where $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$ denotes the normal displacement, in such a way that, when $u_\nu > s$, the difference $u_\nu - s$ represents the interpenetration of the body's asperities into those of the obstacle. The normal compliance function p is prescribed and it satisfies $p(r) = 0$ for $r \le 0$, since then there is no contact. As an example, one may consider

$$p(r) = c_p\, r_+,$$

where $c_p > 0$ represents a deformability constant (that is, it denotes the stiffness of the obstacle), and $r_+ = \max\{0,r\}$. Formally, the classical Signorini nonpenetration conditions are obtained in the limit $c_p \to \infty$. We also assume that the contact is frictionless, *i.e.* the tangential component of the stress field, denoted by $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$, vanishes on the contact surface.

Let Ω be subject to a prescribed electric potential on Γ_A and to a density of surface electric charges q_F on Γ_B, that is,

$$\varphi = \varphi_A \quad \text{on} \quad \Gamma_A \times (0,T), \quad \mathcal{D}\cdot\boldsymbol{\nu} = q_F \quad \text{on} \quad \Gamma_B \times (0,T).$$

For the sake of simplicity, we assume that no electric potential is imposed on the boundary Γ_A (*i.e.* $\varphi_A = 0$), and that $q_F = 0$ on Γ_C; that is, the foundation is supposed to be insulator. We note that it is straightforward to extend the results presented below to more general situations by decomposing Γ in a different way and by introducing some modifications in the analyses shown in the following sections.

The mechanical problem of the dynamic deformation of a viscoelastic-viscoplastic piezoelectric body in contact with a deformable obstacle is then written as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \overline{\Omega} \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \overline{\Omega} \times [0,T] \to \mathbb{S}^d$, *an electric potential field* $\varphi : \overline{\Omega} \times (0,T) \to \mathbb{R}$ *and an electric displacement field* $\mathcal{D} : \overline{\Omega} \times (0,T) \to \mathbb{R}^d$ *such that,*

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\boldsymbol{x},t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(\boldsymbol{x},s), \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},s)))\,\mathrm{d}s - \mathcal{E}^*\mathbf{E}(\varphi(\boldsymbol{x},t)) \quad \text{for a.e.} \quad \boldsymbol{x} \in \Omega,\ t \in (0,T), \tag{2.2}$$

$$\mathcal{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\mathbf{E}(\varphi) \quad \text{in} \quad \Omega \times (0,T), \tag{2.3}$$

$$\rho\ddot{\boldsymbol{u}} - \operatorname{Div}\boldsymbol{\sigma} = \boldsymbol{f}_0 \quad \text{in} \quad \Omega \times (0,T), \tag{2.4}$$

$$\operatorname{div}\mathcal{D} = q_0 \quad \text{in} \quad \Omega \times (0,T), \tag{2.5}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_D \times (0,T), \tag{2.6}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0,T), \tag{2.7}$$

$$\boldsymbol{\sigma}_\tau = 0, \quad -\sigma_\nu = p(u_\nu - s) \quad \text{on} \quad \Gamma_C \times (0,T), \tag{2.8}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_A \times (0,T), \tag{2.9}$$

$$\mathcal{D}\cdot\boldsymbol{\nu} = q_F \quad \text{on} \quad \Gamma_B \times (0,T), \tag{2.10}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \quad \text{in} \quad \Omega. \tag{2.11}$$

Here, $\rho > 0$ is the density of the material (which is assumed constant for simplicity), and $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ represent initial conditions for the displacement and velocity fields, respectively. $\boldsymbol{f}_0$ is the density of the body forces acting in Ω and q_0 is the volume density of free electric charges. Moreover, Div and div represent the divergence operators for tensor and vector-valued functions, respectively.

In order to obtain the variational formulation of Problem P, let us denote by $H = [L^2(\Omega)]^d$ and define the variational spaces V, W and Q as follows,

$$V = \{\boldsymbol{w} \in [H^1(\Omega)]^d\,;\, \boldsymbol{w} = 0 \quad \text{on} \quad \Gamma_D\},$$
$$W = \{\psi \in H^1(\Omega)\,;\, \psi = 0 \quad \text{on} \quad \Gamma_A\},$$
$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d}\ ;\ \tau_{ij} = \tau_{ji},\ i,j = 1,\dots,d\}.$$

Remark 2.1. We could assume ρ to be more general. To do that we should follow a standard procedure (see for example [46], p. 105). If we assume

$$\rho \in L^\infty(\Omega),\ \rho(\boldsymbol{x}) \geq \rho^* > 0 \text{ a.e. } \boldsymbol{x} \in \Omega, \tag{2.12}$$

where ρ^* is a constant, we shall use a modified inner product in H, given by

$$((\boldsymbol{u},\boldsymbol{v}))_H = \int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \quad \forall \boldsymbol{u},\boldsymbol{v} \in H.$$

Let $|||\cdot|||_H$ denote the associated norm, *i.e.*,

$$|||\boldsymbol{u}|||_H^2 = ((\boldsymbol{u},\boldsymbol{u}))_H^{\frac{1}{2}} \quad \forall \boldsymbol{u} \in H.$$

By (2.12), the norm $|||\cdot|||_H$ is equivalent to the usual L^2-norm.

We will make the following assumptions on the problem data.

The viscosity tensor $\mathcal{A}(\boldsymbol{x}) = (a_{ijkl}(\boldsymbol{x}))_{i,j,k,l=1}^d : \boldsymbol{\tau} \in \mathbb{S}^d \to \mathcal{A}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

$$\begin{aligned} &\text{(a) } a_{ijkl} = a_{klij} = a_{jikl} \quad \text{for} \quad i,j,k,l = 1,\dots,d.\\ &\text{(b) } a_{ijkl} \in L^\infty(\Omega) \quad \text{for} \quad i,j,k,l = 1,\dots,d.\\ &\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that } \mathcal{A}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geq m_{\mathcal{A}}\,\|\boldsymbol{\tau}\|^2 \quad \forall \boldsymbol{\tau} \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega. \end{aligned} \tag{2.13}$$

The elastic tensor $\mathcal{B}(\boldsymbol{x}) = (b_{ijkl}(\boldsymbol{x}))_{i,j,k,l=1}^{d} : \boldsymbol{\tau} \in \mathbb{S}^d \to \mathcal{B}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

$$\begin{array}{l}\text{(a) } b_{ijkl} = b_{klij} = b_{jikl} \quad \text{for} \quad i,j,k,l = 1,\dots,d.\\ \text{(b) } b_{ijkl} \in L^\infty(\Omega) \quad \text{for} \quad i,j,k,l = 1,\dots,d.\\ \text{(c) There exists } m_{\mathcal{B}} > 0 \text{ such that } \mathcal{B}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geq m_{\mathcal{B}}\,\|\boldsymbol{\tau}\|^2 \quad \forall\,\boldsymbol{\tau} \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.\end{array} \tag{2.14}$$

The piezoelectric tensor $\mathcal{E}(\boldsymbol{x}) = (e_{ijk}(\boldsymbol{x}))_{i,j,k=1}^{d} : \boldsymbol{\tau} \in \mathbb{S}^d \to \mathcal{E}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{R}^d$ satisfies:

$$\begin{array}{l}\text{(a) } e_{ijk} = e_{ikj} \quad \text{for} \quad i,j,k = 1,\dots,d.\\ \text{(b) } e_{ijk} \in L^\infty(\Omega) \quad \text{for} \quad i,j,k = 1,\dots,d.\end{array} \tag{2.15}$$

The permittivity tensor $\beta(\boldsymbol{x}) = (\beta_{ij}(\boldsymbol{x}))_{i,j=1}^{d} : \boldsymbol{w} \in \mathbb{R}^d \to \beta(\boldsymbol{x})(\boldsymbol{w}) \in \mathbb{R}^d$ verifies:

$$\begin{array}{l}\text{(a) } \beta_{ij} = \beta_{ji} \quad \text{for} \quad i,j = 1,\dots,d.\\ \text{(b) } \beta_{ij} \in L^\infty(\Omega) \quad \text{for} \quad i,j = 1,\dots,d.\\ \text{(c) There exists } m_\beta > 0 \text{ such that } \beta(\boldsymbol{x})\boldsymbol{w}\cdot\boldsymbol{w} \geq m_\beta\,\|\boldsymbol{w}\|^2 \quad \forall\,\boldsymbol{w} \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.\end{array} \tag{2.16}$$

The normal compliance function $p : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}^+$ satisfies:

$$\begin{array}{l}\text{(a) There exists } L_p > 0 \text{ such that}\\ \qquad |p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2)| \leq L_p\,|r_1 - r_2| \quad \forall\, r_1,r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_C.\\ \text{(b) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x},r) \text{ is Lebesgue measurable on } \Gamma_C,\\ \qquad \forall r \in \mathbb{R}.\\ \text{(c) } (p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2))\cdot(r_1 - r_2) \geq 0 \quad \forall\, r_1,r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_C.\\ \text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x},r) = 0 \quad \text{for all } r \leq 0.\end{array} \tag{2.17}$$

The viscoplastic function $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathcal{G}(\boldsymbol{x})(\boldsymbol{\tau},\boldsymbol{\varepsilon}) \in \mathbb{S}^d$ satisfies:

$$\begin{array}{l}\text{(a) There exists } L_{\mathcal{G}} > 0 \text{ such that}\\ \qquad |\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{G}}\,(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2|)\\ \qquad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2 \in \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega.\\ \text{(b) The function } \boldsymbol{x} \to \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable.}\\ \text{(c) The mapping } \boldsymbol{x} \to \mathcal{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \text{ belongs to } Q.\end{array} \tag{2.18}$$

The following regularity is assumed on the density of volume forces and tractions:

$$\begin{array}{l}\boldsymbol{f}_0 \in C([0,T];H), \quad \boldsymbol{f}_F \in C([0,T];[L^2(\Gamma_F)]^d),\\ q_0 \in C([0,T];L^2(\Omega)), \quad q_F \in C([0,T];L^2(\Gamma_B)).\end{array} \tag{2.19}$$

Finally, we assume that the initial displacement and velocity satisfy

$$\boldsymbol{u}_0, \boldsymbol{v}_0 \in V. \tag{2.20}$$

Remark 2.2. We can replace (2.20) by asking a less restrictive condition $\boldsymbol{u}_0 \in V$, $\boldsymbol{v}_0 \in H$.

Moreover, we denote by V' the dual space of V. We identify H with its dual and consider the Gelfand triple

$$V \subset H \subset V'.$$

We use the notation $\langle \cdot,\cdot \rangle_{V'\times V}$ to denote the duality product and, in particular, we have

$$\langle \boldsymbol{v},\boldsymbol{u} \rangle_{V'\times V} = (\boldsymbol{v},\boldsymbol{u})_H \quad \forall \boldsymbol{u} \in V,\ \boldsymbol{v} \in H.$$

Using Riesz' theorem, from (2.19) we can define the elements $\boldsymbol{f}(t) \in V'$ and $q(t) \in W$ given by

$$\langle \boldsymbol{f}(t), \boldsymbol{w} \rangle_{V' \times V} = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_F} \boldsymbol{f}_F(t) \cdot \boldsymbol{w} \, \mathrm{d}\Gamma \quad \forall \boldsymbol{w} \in V,$$
$$(q(t), \psi)_W = \int_\Omega q_0(t) \psi \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_B} q_F(t) \psi \, \mathrm{d}\Gamma \quad \forall \psi \in W,$$

and then $\boldsymbol{f} \in C([0,T]; V')$ and $q \in C([0,T]; W)$. Now, let us define the contact functional $j : V \times V \to \mathbb{R}$ by

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} p(u_\nu - s) \, v_\nu \, \mathrm{d}\Gamma \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V,$$

where we let $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ for all $\boldsymbol{v} \in V$. Moreover, from properties (2.17) let us conclude that

$$|j(\boldsymbol{u}, \boldsymbol{w}) - j(\boldsymbol{v}, \boldsymbol{w})| \leq C \|\boldsymbol{u} - \boldsymbol{v}\|_V \|\boldsymbol{w}\|_V \ \forall \, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \tag{2.21}$$

Plugging (2.2) into (2.4), (2.3) into (2.5) and using the previous boundary conditions, applying a Green's formula we derive the following variational formulation of Problem P, written in terms of the velocity field $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t)$ and the electric potential $\varphi(t)$.

Problem 2.3. Find a velocity field $\boldsymbol{v} : [0,T] \to V$, a stress field $\boldsymbol{\sigma} : [0,T] \to Q$ and an electric potential field $\varphi : [0,T] \to W$ such that $\boldsymbol{v}(0) = \boldsymbol{v}_0$ and for a.e. $t \in (0,T)$ and for all $\boldsymbol{w} \in V$ and $\psi \in W$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))) \, \mathrm{d}s - \mathcal{E}^* \mathbf{E}(\varphi(t)), \tag{2.22}$$

$$\begin{aligned} &\langle \rho \dot{\boldsymbol{v}}(t), \boldsymbol{w} \rangle_{V' \times V} + \left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))) \, \mathrm{d}s, \boldsymbol{\varepsilon}(\boldsymbol{w}) \right)_Q \\ &\qquad + (\mathcal{E}^* \mathbf{E}(\varphi(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{u}(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w} \rangle_{V' \times V}, \end{aligned} \tag{2.23}$$

$$(\beta \nabla \varphi(t), \nabla \psi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla \psi)_H = (q(t), \psi)_W, \tag{2.24}$$

where the displacement field $\boldsymbol{u}(t)$ is given by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s) \, \mathrm{d}s + \boldsymbol{u}_0. \tag{2.25}$$

3. An existence and uniqueness result

Theorem 3.1. *Assume (2.13)–(2.20) hold. Then, there exists a unique solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi)$ to Problem 2.3. Moreover, the solution satisfies*

$$\boldsymbol{u} \in H^1(0,T;V) \cap C^1([0,T];H), \quad \ddot{\boldsymbol{u}} \in L^2(0,T;V'), \tag{3.1}$$
$$\boldsymbol{\sigma} \in L^2(0,T;Q), \ \mathrm{Div}\, \boldsymbol{\sigma} \in L^2(0,T;V'), \tag{3.2}$$
$$\varphi \in C([0,T];W). \tag{3.3}$$

The proof of Theorem 3.1 will be carried in several steps. First, let $\boldsymbol{M} \in L^2(0,T;Q)$ and consider the auxiliary problem.

Problem 3.2. Find a velocity field $\boldsymbol{v}_M : [0,T] \to V$ and an electric field $\varphi_M : [0,T] \to W$ such that $\boldsymbol{v}_M(0) = \boldsymbol{v}_0$ and for a.e. $t \in (0,T)$ and for all $\boldsymbol{w} \in V$ and $\psi \in W$,

$$\begin{aligned} &\langle \rho \dot{\boldsymbol{v}}_M(t), \boldsymbol{w} \rangle_{V' \times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_M(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_M(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (\mathcal{E}^* \nabla \varphi_M(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{u}_M(t), \boldsymbol{w}) \\ &\qquad = \langle \boldsymbol{f}(t), \boldsymbol{w} \rangle_{V' \times V} - (\boldsymbol{M}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q, \end{aligned} \tag{3.4}$$

$$(\beta \nabla \varphi_M(t), \nabla \psi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_M(t)), \nabla \psi)_H = (q(t), \psi)_W, \tag{3.5}$$

where the displacement field $\boldsymbol{u}_M(t)$ is given by

$$\boldsymbol{u}_M(t) = \int_0^t \boldsymbol{v}_M(s)\,\mathrm{d}s + \boldsymbol{u}_0. \tag{3.6}$$

Theorem 3.3. *Assume (2.13)–(2.20) hold. Then, there exists a unique solution $(\boldsymbol{v}_M, \varphi_M)$ to Problem 3.2. Moreover, the following regularities hold:*

$$\boldsymbol{u}_M \in H^1(0,T;V) \cap C^1([0,T];H), \quad \ddot{\boldsymbol{u}}_M \in L^2(0,T;V'), \tag{3.7}$$

$$\varphi_M \in C([0,T];W). \tag{3.8}$$

To show the proof of this theorem we have to proceed in several steps as well. Let $\boldsymbol{\eta} \in L^2(0,T;V')$ be given and consider the following additional auxiliary problem.

Problem 3.4. Find a velocity field $\boldsymbol{v}_{M\eta} : [0,T] \to V$ such that $\boldsymbol{v}_{M\eta}(0) = \boldsymbol{v}_0$ and for a.e. $t \in (0,T)$ and for all $\boldsymbol{w} \in V$,

$$\langle \rho \dot{\boldsymbol{v}}_{M\eta}(t), \boldsymbol{w}\rangle_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_{M\eta}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = \langle \boldsymbol{f}(t) - \boldsymbol{\eta}(t) - \boldsymbol{M}(t), \boldsymbol{w}\rangle_{V'\times V}, \tag{3.9}$$

where the displacement field $\boldsymbol{u}_{M\eta}(t)$ is given by

$$\boldsymbol{u}_{M\eta}(t) = \int_0^t \boldsymbol{v}_{M\eta}(s)\,\mathrm{d}s + \boldsymbol{u}_0. \tag{3.10}$$

Remark 3.5. Note that in the right hand-side of variational equation (3.9) we make *un abus de langage*, since actually we are identifying $\boldsymbol{M}(t) \in Q$ with the corresponding $\boldsymbol{M}(t) \in V'$ such that $(\boldsymbol{M}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q =< \boldsymbol{M}(t), \boldsymbol{w} >_{V'\times V}$ for all $\boldsymbol{w} \in V$.

Now, we can apply a result proved in ([46], p. 107) which we can reformulate here as follows.

Proposition 3.6. *Assume (2.13)–(2.20) hold. Then, there exists a unique solution to Problem 3.4 and it has the regularity expressed in (3.7).*

Remark 3.7. The result in [46] is used in the framework of the study of a viscoelastic dynamic contact problem, based itself on an abstract result found in ([9], p. 140).

We now consider the auxiliary problem for the electric part.

Problem 3.8. Find an electric field $\varphi_{M\eta} : [0,T] \to W$ such that for a.e. $t \in (0,T)$ and for all $\psi \in W$,

$$(\beta\nabla\varphi_{M\eta}(t), \nabla\psi)_H = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{M\eta}(t)), \nabla\psi)_H + (q(t), \psi)_W. \tag{3.11}$$

We have the following result.

Proposition 3.9. *Assume (2.13)–(2.20) hold. Then, there exists a unique solution to Problem 3.8 and it has the regularity expressed in (3.8).*

Proof. We define a bilinear form $b(\cdot,\cdot) : W \times W \to \mathbb{R}$ such that

$$b(\varphi, \psi) = (\beta\nabla\varphi, \nabla\psi)_H \quad \forall \varphi, \psi \in W.$$

We use (2.16) to show that the bilinear form is continuous, symmetric and coercive on W. Moreover, using (3.11) and (2.15), the Riesz' representation theorem allows us to define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t), \psi)_W = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{M\eta}(t)), \nabla\psi)_H + (q(t), \psi)_W \quad \forall \psi \in W.$$

We apply the Lax−Milgram theorem to deduce that there exists a unique element $\varphi_{M\eta}(t)$ such that

$$b(\varphi_{M\eta}(t), \psi) = (q_\eta(t), \psi)_W \quad \forall \psi \in W.$$

We conclude that $\varphi_{M\eta}(t)$ is the solution to variational equation (3.11). Moreover, by using (2.19) and the regularity of $\boldsymbol{u}_{M\eta}$ and q, we conclude straightforwardly that $\varphi_{M\eta} \in C([0,T];W)$. □

Now, let $\Lambda\boldsymbol{\eta}(t)$ denote the element of V' defined by

$$\langle\rho\Lambda\boldsymbol{\eta}(t),\boldsymbol{w}\rangle_{V'\times V}=(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{M\eta}(t)),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q+(\mathcal{E}^*\nabla\varphi_{M\eta}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q+j(\boldsymbol{u}_{M\eta}(t),\boldsymbol{w}),\tag{3.12}$$

for all $\boldsymbol{w}\in V$ and $t\in[0,T]$. We have the following result.

Proposition 3.10. *For $\boldsymbol{\eta}\in L^2(0,T;V')$ it follows that $\Lambda\boldsymbol{\eta}\in C([0,T];V')$ and the operator $\Lambda: L^2(0,T;V')\to L^2(0,T;V')$ has a unique fixed point $\boldsymbol{\eta}^*$.*

Proof. The continuity of $\Lambda\boldsymbol{\eta}$ is a straightforward consequence of the continuity of $\varphi_{M\eta}$ and $\boldsymbol{u}_{M\eta}$. Let now $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in L^2(0,T;V')$ and $t\in[0,T]$. We use the shorter notation $\boldsymbol{u}_i=\boldsymbol{u}_{M\eta_i}$, $\boldsymbol{v}_i=\boldsymbol{v}_{M\eta_i}$, $\varphi_i=\varphi_{M\eta_i}$, for $i=1,2$. Then, taking $\boldsymbol{\eta}=\boldsymbol{\eta}_i$ for $i=1,2$ successively in (3.12) and subtracting the resulting equations, we have, for all $\boldsymbol{w}\in V$ and $t\in(0,T)$,

$$\begin{aligned}\langle\rho\Lambda\boldsymbol{\eta}_1(t)-\rho\Lambda\boldsymbol{\eta}_2(t),\boldsymbol{w}\rangle_{V'\times V}&=(\mathcal{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1(t))-\boldsymbol{\varepsilon}(\boldsymbol{u}_2(t))),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q\\&\quad+(\mathcal{E}^*\nabla(\varphi_1(t)-\varphi_2(t)),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q+j(\boldsymbol{u}_1(t),\boldsymbol{w})-j(\boldsymbol{u}_2(t),\boldsymbol{w}).\end{aligned}$$

By using (2.14), (2.15) and (2.21), we find that

$$\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_{V'}\le C(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V+\|\varphi_1(t)-\varphi_2(t)\|_W).\tag{3.13}$$

Here and below, C stands for a positive constant depending on data whose specific value may change from place to place. On the other hand, from (3.10) we know that

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V\le\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V\,\mathrm{d}s.\tag{3.14}$$

Also, from (3.11) and using (2.15) and (2.16) we deduce

$$\|\varphi_1(t)-\varphi_2(t)\|_W\le C\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V,$$

which, combined with (3.14), gives

$$\|\varphi_1(t)-\varphi_2(t)\|_W\le C\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{u}_2(s)\|_V\,\mathrm{d}s.\tag{3.15}$$

Thus, by using consecutively (3.15) and (3.14) in (3.13), we obtain

$$\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_{V'}\le C\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V\,\mathrm{d}s,$$

which implies

$$\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_{V'}^2\le C\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V^2\,\mathrm{d}s.\tag{3.16}$$

Taking $\boldsymbol{\eta}=\boldsymbol{\eta}_i$ for $i=1,2$ successively in (3.9) with $\boldsymbol{w}=\boldsymbol{v}_1(t)-\boldsymbol{v}_2(t)$ and subtracting the resulting expressions, we find

$$\begin{aligned}&\langle\rho\dot{\boldsymbol{v}}_1(t)-\rho\dot{\boldsymbol{v}}_2(t),\boldsymbol{v}_1(t)-\boldsymbol{v}_2(t)\rangle_{V'\times V}+(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_1(t))-\boldsymbol{\varepsilon}(\boldsymbol{v}_2(t))),\boldsymbol{\varepsilon}(\boldsymbol{v}_1(t))-\boldsymbol{\varepsilon}(\boldsymbol{v}_2(t)))_Q\\&\qquad=\langle\boldsymbol{\eta}_2(t)-\boldsymbol{\eta}_1(t),\boldsymbol{v}_1(t)-\boldsymbol{v}_2(t)\rangle_{V'\times V}.\end{aligned}$$

By integrating in time, using the ellipticity of $\mathcal{A}$, the fact that $\boldsymbol{v}_1(0) = \boldsymbol{v}_2(0) = \boldsymbol{v}_0$ and Korn's inequality, we find

$$m_{\mathcal{A}} \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V^2 \mathrm{d}s \leq \int_0^t \langle \boldsymbol{\eta}_2(s) - \boldsymbol{\eta}_1(s), \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\rangle_{V' \times V} \mathrm{d}s$$
$$\leq \frac{1}{m_{\mathcal{A}}} \int_0^t \|\boldsymbol{\eta}_2(s) - \boldsymbol{\eta}_1(s)\|_{V'}^2 \mathrm{d}s + \frac{m_{\mathcal{A}}}{4} \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V^2 \mathrm{d}s,$$

where we used several times Young's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2,\ a, b, \epsilon \in \mathbb{R},\ \epsilon > 0. \tag{3.17}$$

Plugging this into (3.16) we find that

$$\|\Lambda \boldsymbol{\eta}_1(t) - \Lambda \boldsymbol{\eta}_2(t)\|_{V'}^2 \leq C \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V'}^2 \, \mathrm{d}s,$$

and using a standard argument (see, for example, Lem. 4.7 in [46]), from the previous inequality and using the Banach's fixed point theorem we conclude that there exists a unique $\boldsymbol{\eta}^* \in L^2(0, T; V')$ such that $\Lambda \boldsymbol{\eta}* = \boldsymbol{\eta}*$. □

We can now give the Proof of Theorem 3.3.

Proof of Theorem 3.3. By using Proposition 3.10 there exists a unique $\boldsymbol{\eta}^* \in L^2(0, T; V')$ such that $\Lambda \boldsymbol{\eta}^* = \boldsymbol{\eta}^*$. We define $\boldsymbol{u}_M = \boldsymbol{u}_{M\eta^*}$, $\boldsymbol{v}_M = \boldsymbol{v}_{M\eta^*}$ and $\varphi_M = \varphi_{M\eta^*}$. By taking $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ in (3.11) we obtain (3.5). Also, from (3.12) we get

$$\langle \rho \boldsymbol{\eta}^*(t), \boldsymbol{w}\rangle_{V' \times V} = (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_M(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (\mathcal{E}^* \nabla \varphi_M(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{u}_M(t), \boldsymbol{w}),$$

for all $\boldsymbol{w} \in V$ and $t \in [0, T]$. Therefore, by taking $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ in (3.9) and using the previous equality, we obtain the variational equation (3.4). Finally, (3.6) follows from (3.10), and the regularities are a consequence of the regularities given by Propositions 3.6 and 3.9. □

Further, we define the operator $\Theta : C([0, T]; Q) \to C([0, T]; Q)$ by

$$\Theta \boldsymbol{F} = \mathcal{G}(\boldsymbol{\sigma}_M, \boldsymbol{\varepsilon}(\boldsymbol{u}_M)), \quad \text{where} \quad \boldsymbol{M}(t) = \int_0^t \boldsymbol{F}(s) \, \mathrm{d}s \quad \forall \boldsymbol{F} \in C([0, T]; Q),$$

and $\boldsymbol{u}_M$ is the displacement field solution to Problem 3.2 while $\boldsymbol{\sigma}_M$ is the stress field:

$$\boldsymbol{\sigma}_M = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_M) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_M) + \boldsymbol{M} + \mathcal{E}^* \nabla \varphi_M, \tag{3.18}$$

with φ_M being the electric potential solution to Problem 3.2. Note that since $\boldsymbol{M} \in C([0, T]; Q)$, it is straightforward that $\boldsymbol{\sigma}_M \in C([0, T]; Q)$. We obtain the following result.

Proposition 3.11. *The operator Θ has a unique fixed point $\boldsymbol{F}^* \in C([0, T]; Q)$.*

Proof. The continuity of $\Theta \boldsymbol{F}$ is a straightforward consequence of the continuity of $\boldsymbol{\sigma}_M$ and $\boldsymbol{u}_M$ and (2.18). Moreover, let $\boldsymbol{F}_1, \boldsymbol{F}_2 \in C([0, T]; Q)$ and let $\boldsymbol{M}_1, \boldsymbol{M}_2 \in C([0, T]; Q)$ be their corresponding integrals in time. For the sake of simplicity, we use the notation $\boldsymbol{u}_i = \boldsymbol{u}_{M_i}$, $\boldsymbol{v}_i = \boldsymbol{v}_{M_i}$, $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{M_i}$ and $\varphi_i = \varphi_{M_i}$ for $i = 1, 2$. Let $t \in [0, T]$. From (2.18) we find that

$$\|\Theta \boldsymbol{F}_1(t) - \Theta \boldsymbol{F}_2(t)\|_Q \leq C(\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_Q + \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V). \tag{3.19}$$

By using (3.18) in (3.4) successively for $\boldsymbol{M} = \boldsymbol{M}_i$, $i = 1, 2$, taking in both cases $\boldsymbol{w} = \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)$ and subtracting the resulting equations, we obtain

$$\langle \rho(\dot{\boldsymbol{v}}_1(t) - \dot{\boldsymbol{v}}_2(t)), \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\rangle_{V'\times V} + (\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t), \boldsymbol{\varepsilon}(\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)))_Q \\ + j(\boldsymbol{u}_1(t), \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)) - j(\boldsymbol{u}_2(t), \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)) = 0.$$

Integrating in time and using (2.21) and $\boldsymbol{v}_1(0) = \boldsymbol{v}_2(0)$ we deduce that

$$\frac{1}{2}\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_V^2 \leq C \int_0^t (\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q + \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V)\|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V \, \mathrm{d}s.$$

Also, from (3.18) it follows that

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_Q \leq C \left(\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_V + \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \|\varphi_1(t) - \varphi_2(t)\|_W + \int_0^t \|\boldsymbol{F}_1(s) - \boldsymbol{F}_2(s)\|_Q \, \mathrm{d}s \right). \tag{3.20}$$

Combining the last two inequalities, we get

$$\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_V^2 \leq C \int_0^t \Bigg(\|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V + \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \\ + \|\varphi_1(s) - \varphi_2(s)\|_W + \int_0^s \|\boldsymbol{F}_1(r) - \boldsymbol{F}_2(r)\|_Q \, \mathrm{d}r \Bigg) \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V \, \mathrm{d}s.$$

By using (3.14) and (3.15) and after some tedious calculations, we obtain

$$\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_V^2 \leq C \left(\int_0^t (\|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V^2 \, \mathrm{d}s + \int_0^t \|\boldsymbol{F}_1(s) - \boldsymbol{F}_2(s)\|_Q^2 \, \mathrm{d}s \right).$$

By using the Gronwall's Lemma, we find that

$$\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_V^2 \leq C \int_0^t \|\boldsymbol{F}_1(s) - \boldsymbol{F}_2(s)\|_Q^2 \, \mathrm{d}s.$$

This last inequality combined with (3.14), (3.15) and (3.20) allows us to have, from (3.19), the following estimate:

$$\|\Theta \boldsymbol{F}_1(t) - \Theta \boldsymbol{F}_2(t)\|_Q^2 \leq C \int_0^t \|\boldsymbol{F}_1(s) - \boldsymbol{F}_2(s)\|_Q^2 \, \mathrm{d}s.$$

Following a standard argument –see again Lemma 4.7 in [46]– from the previous inequality and using the Banach's fixed point theorem, we conclude that there exists a unique $\boldsymbol{F}^* \in C([0,T];Q)$ such that $\Theta \boldsymbol{F}^* = \boldsymbol{F}^*$. □

We can now give the Proof of Theorem 3.1.

Proof of Theorem 3.1. By using Proposition 3.11 there exists a unique $\boldsymbol{F}^* \in C([0,T];Q)$ such that $\Theta \boldsymbol{F}^* = \boldsymbol{F}^*$. We define

$$\boldsymbol{M}^*(t) = \int_0^t \boldsymbol{F}^*(s) \, \mathrm{d}s.$$

We also define $\boldsymbol{u} = \boldsymbol{u}_{M^*}$, $\boldsymbol{v} = \boldsymbol{v}_{M^*}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{M^*}$ and $\varphi = \varphi_{M^*}$. By taking $\boldsymbol{M} = \boldsymbol{M}^*$ in (3.4) we obtain (2.23), because

$$\boldsymbol{M}(t) = \boldsymbol{M}^*(t) = \int_0^t \boldsymbol{F}^*(s) \, \mathrm{d}s = \int_0^t \Theta \boldsymbol{F}^*(s) \, \mathrm{d}s = \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))) \, \mathrm{d}s.$$

Finally, from (3.5) it follows (2.24), and (3.18) leads to (2.22). □

4. Fully discrete approximations and an *a priori* error analysis

In this section, we introduce a finite element algorithm for approximating solutions to variational problem 2.3. Its discretization is done in two steps. First, we consider the finite element spaces $V^h \subset V$, $Q^h \subset Q$ and $W^h \subset W$ given by

$$V^h = \{\boldsymbol{v}^h \in [C(\overline{\Omega})]^d \ ; \ \boldsymbol{v}^h_{|_T} \in [P_1(T)]^d,\, T \in \mathcal{T}^h, \quad \boldsymbol{v}^h = 0 \ \text{ on } \ \Gamma_D\}, \tag{4.1}$$

$$Q^h = \{\boldsymbol{\tau}^h \in Q \ ; \ \boldsymbol{\tau}^h_{|_T} \in [P_0(T)]^{d\times d},\, T \in \mathcal{T}^h\}, \tag{4.2}$$

$$W^h = \{\psi^h \in C(\overline{\Omega}) \ ; \ \psi^h_{|_T} \in P_1(T),\, T \in \mathcal{T}^h, \quad \psi^h = 0 \ \text{ on } \ \Gamma_A\}, \tag{4.3}$$

where Ω is assumed to be a polyhedral domain, $\mathcal{T}^h$ denotes a triangulation of $\overline{\Omega}$ compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_D, Γ_N and Γ_C on one hand, and into Γ_A and Γ_B on the other hand, and $P_q(T)$, $q = 0, 1$, represents the space of polynomials of global degree less or equal to q in T. Here, $h > 0$ denotes the spatial discretization parameter.

Secondly, the time derivatives are discretized by using a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, and let k be the time step size, $k = T/N$. Moreover, for a continuous function $f(t)$ we denote $f_n = f(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Using a hybrid combination of the forward and backward Euler schemes, the fully discrete approximation of Problem 2.3 is the following.

Problem 4.1. Find a discrete velocity field $\boldsymbol{v}^{hk} = \{\boldsymbol{v}^{hk}_n\}_{n=0}^N \subset V^h$, a discrete stress field $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}^{hk}_n\}_{n=0}^N \subset Q^h$ and a discrete electric potential field $\varphi^{hk} = \{\varphi^{hk}_n\}_{n=0}^N \subset W^h$ such that $\boldsymbol{v}^{hk}_0 = \boldsymbol{v}^h_0$ and for $n = 1, \ldots, N$ and for all $\boldsymbol{w}^h \in V^h$ and $\psi^h \in W^h$,

$$\boldsymbol{\sigma}^{hk}_n = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^{hk}_n) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_n) + k\sum_{j=0}^{n-1} \mathcal{G}(\boldsymbol{\sigma}^{hk}_j, \boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_j)) - \mathcal{E}^*\mathbf{E}(\varphi^{hk}_{n-1}), \tag{4.4}$$

$$\begin{aligned}(\rho\delta\boldsymbol{v}^{hk}_n, \boldsymbol{w}^h)_H &+ \left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^{hk}_n) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_n) + k\sum_{j=0}^{n-1} \mathcal{G}(\boldsymbol{\sigma}^{hk}_j, \boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_j)), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q \\ &= \langle \boldsymbol{f}_n, \boldsymbol{w}^h\rangle_{V'\times V} - \left(\mathcal{E}^*\mathbf{E}(\varphi^{hk}_{n-1}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q - j(\boldsymbol{u}^{hk}_{n-1}, \boldsymbol{w}^h),\end{aligned} \tag{4.5}$$

$$(\beta\nabla\varphi^{hk}_n, \nabla\psi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_n), \nabla\psi^h)_H = (q_n, \psi^h)_W, \tag{4.6}$$

where the discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}^{hk}_n\}_{n=0}^N \subset V^h$ is given by

$$\boldsymbol{u}^{hk}_n = k\sum_{j=1}^{n} \boldsymbol{v}^{hk}_j + \boldsymbol{u}^h_0, \tag{4.7}$$

and the artificial discrete initial condition φ^{hk}_0 is the solution to the following problem:

$$(\beta\nabla\varphi^{hk}_0, \nabla\psi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^h_0), \nabla\psi^h)_H = (q_0, \psi^h)_W \quad \forall\psi^h \in W^h. \tag{4.8}$$

Here, we note that the discrete initial conditions, denoted by $\boldsymbol{u}^h_0$ and $\boldsymbol{v}^h_0$ are given by

$$\boldsymbol{u}^h_0 = \mathcal{P}^h\boldsymbol{u}_0, \quad \boldsymbol{v}^h_0 = \mathcal{P}^h\boldsymbol{v}_0, \tag{4.9}$$

where $\mathcal{P}^h$ is the $[L^2(\Omega)]^d$-projection operator over the finite element space V^h.

Using assumptions (2.13)–(2.20) and the classical Lax–Milgram lemma, it is easy to prove that Problem 4.1 has a unique discrete solution $(\boldsymbol{v}^{hk}, \varphi^{hk}, \boldsymbol{\sigma}^{hk}) \subset V^h \times W^h \times Q^h$.

Our aim in this section is to derive some *a priori* error estimates for the numerical errors $\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}$, $\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}$, $\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}$ and $\varphi_n - \varphi_n^{hk}$. Therefore, we assume that the solution to Problem 2.3 has the following regularity:

$$\boldsymbol{u} \in C^1([0,T];V) \cap C^2([0,T];H), \quad \boldsymbol{\sigma} \in C([0,T];Q), \quad \varphi \in C([0,T];W). \tag{4.10}$$

Thus, we have the following result.

Theorem 4.2. *Let assumptions (2.13)–(2.20) and the additional regularity (4.10) hold. If we denote by $(\boldsymbol{v}, \varphi, \boldsymbol{\sigma})$ and $(\boldsymbol{v}^{hk}, \varphi^{hk}, \boldsymbol{\sigma}^{hk})$ the respective solutions to problems 2.3 and 4.1, then there exists a positive constant $C > 0$, independent of the discretization parameters h and k, such that, for all $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$ and $\psi^h = \{\psi_j^h\}_{j=0}^N \subset W^h$,*

$$\begin{aligned}
&\max_{0\le n\le N} \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \max_{0\le n\le N} \|\varphi_n - \varphi_n^{hk}\|_W^2 \\
&\quad + Ck \sum_{j=0}^N \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + Ck \sum_{j=0}^N \|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 \\
&\le Ck \sum_{j=1}^N \Big(\|\dot{\boldsymbol{v}}_j - \delta \boldsymbol{v}_j\|_H^2 + \|\dot{\boldsymbol{u}}_j - \delta \boldsymbol{u}_j\|_V^2 + k^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\varphi_j - \psi_j^h\|_W^2 + I_j^2 \Big) \\
&\quad + C \max_{0\le n\le N} \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + C\Big(\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + C\|\varphi_0 - \psi_0^h\|_W^2 \Big) \\
&\quad + \frac{C}{k} \sum_{j=1}^{N-1} \|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2,
\end{aligned} \tag{4.11}$$

where the integration error I_n is defined as

$$I_n = \left\| \int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, \mathrm{d}s - k \sum_{j=0}^{n-1} \mathcal{G}(\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}(\boldsymbol{u}_j)) \right\|_Q. \tag{4.12}$$

Proof. First, we obtain some estimates on the stress field. Subtracting equations (2.22), at time $t = t_n$, and (4.4), taking into account assumptions (2.13)–(2.20) we easily find that

$$\begin{aligned}
\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q \le C\Big(&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + I_n + k \\
&+ k \sum_{j=0}^{n-1} \Big[\|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V + \|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q + \|\varphi_j - \varphi_j^{hk}\|_W \Big] \Big),
\end{aligned} \tag{4.13}$$

where the integration error I_n is defined in (4.12).

Secondly, we obtain the estimates on the electric potential field. We subtract variational equation (2.24), at time $t = t_n$ and for $\psi = \psi^h \in W^h$, and discrete variational equation (4.6) to get, for all $\psi^h \in W^h$,

$$(\beta \nabla(\varphi_n - \varphi_n^{hk}), \nabla \psi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla \psi^h)_H = 0 \quad \forall \psi^h \in W^h.$$

Therefore, it follows that, for all $\psi^h \in W^h$,

$$\begin{aligned}
&(\beta \nabla(\varphi_n - \varphi_n^{hk}), \nabla(\varphi_n - \varphi_n^{hk}))_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\varphi_n - \varphi_n^{hk}))_H \\
&\quad = (\beta \nabla(\varphi_n - \varphi_n^{hk}), \nabla(\varphi_n - \psi^h))_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\varphi_n - \psi^h))_H.
\end{aligned}$$

Using again assumptions (2.13)–(2.20) and several times Young's inequality (3.17), we find that

$$\|\varphi_n - \varphi_n^{hk}\|_W^2 \le C\Big(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\varphi_n - \psi^h\|_W^2 \Big) \quad \forall \psi^h \in W^h. \tag{4.14}$$

Finally, we obtain the estimates on the velocity and displacement fields. To do that, we subtract variational equation (2.23), at time $t = t_n$ and for $\boldsymbol{w} = \boldsymbol{w}^h \in V^h$, and discrete variational equation (4.5) to get, for all $\boldsymbol{w}^h \in V^h$,

$$
\begin{aligned}
&(\rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}), \boldsymbol{w}^h)_H + \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\big)_Q \\
&\quad + \left(\int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s - k\sum_{j=1}^{n-1} \mathcal{G}(\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}(\boldsymbol{u}_j)), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q \\
&\quad + \left(k\sum_{j=0}^{n-1} [\mathcal{G}(\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}(\boldsymbol{u}_j)) - \mathcal{G}(\boldsymbol{\sigma}_j^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}))], \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q \\
&\quad - \big(\mathcal{E}^*\mathbf{E}(\varphi_n) - \mathcal{E}^*\mathbf{E}(\varphi_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\big)_Q + j(\boldsymbol{u}_n, \boldsymbol{w}^h) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}^h) = 0.
\end{aligned}
$$

Therefore, we find that, for all $\boldsymbol{w}^h \in V^h$,

$$
\begin{aligned}
&(\rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk})\big)_Q \\
&\quad + \left(\int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s - k\sum_{j=0}^{n-1} \mathcal{G}(\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}(\boldsymbol{u}_j)), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk})\right)_Q \\
&\quad + \left(k\sum_{j=0}^{n-1} [\mathcal{G}(\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}(\boldsymbol{u}_j)) - \mathcal{G}(\boldsymbol{\sigma}_j^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}))], \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk})\right)_Q \\
&\quad - \big(\mathcal{E}^*\mathbf{E}(\varphi_n) - \mathcal{E}^*\mathbf{E}(\varphi_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk})\big)_Q \\
&\quad + j(\boldsymbol{u}_n, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) \\
&= (\rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}), \boldsymbol{v}_n - \boldsymbol{w}^h)_H + \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}^h)\big)_Q \\
&\quad + \left(\int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s - k\sum_{j=0}^{n-1} \mathcal{G}(\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}(\boldsymbol{u}_j)), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}^h)\right)_Q \\
&\quad + \left(k\sum_{j=0}^{n-1} [\mathcal{G}(\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}(\boldsymbol{u}_j)) - \mathcal{G}(\boldsymbol{\sigma}_j^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}))], \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}^h)\right)_Q \\
&\quad - \big(\mathcal{E}^*\mathbf{E}(\varphi_n) - \mathcal{E}^*\mathbf{E}(\varphi_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}^h)\big)_Q \\
&\quad + j(\boldsymbol{u}_n, \boldsymbol{v}_n - \boldsymbol{w}^h) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h).
\end{aligned}
$$

Keeping in mind assumptions (2.13) and (2.14) it follows that

$$
\begin{aligned}
&\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk})\big)_Q \geq C\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2, \\
&\big(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk})\big)_Q \geq \big(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n)\big)_Q \\
&\qquad + \frac{C}{2k}\left\{\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 - \|\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}\|_V^2\right\},
\end{aligned}
$$

where $\delta\boldsymbol{u}_n = (\boldsymbol{u}_n - \boldsymbol{u}_{n-1})/k$ and we used (4.7). Moreover, since

$$
(\rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H \geq (\rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \frac{C}{2k}\left\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right\},
$$

where $\delta \boldsymbol{v}_n = (\boldsymbol{v}_n - \boldsymbol{v}_{n-1})/k$, using again several times Young's inequality (3.17) and assumptions (2.13)–(2.20), we have

$$\begin{aligned}
&\frac{1}{2k}\left[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right] + \frac{1}{2k}\left[\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 - \|\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}\|_V^2\right] + C\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2\\
&\quad \le C\Big(\|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H^2 + \|\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n\|_V^2 + k^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \|\varphi_{n-1} - \varphi_{n-1}^{hk}\|_W^2\\
&\qquad + I_n^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + \rho(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H\Big) \quad \forall \boldsymbol{w}^h \in V^h.
\end{aligned}$$

Therefore, by induction we find that, for all $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^n \subset V^h$,

$$\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + Ck\sum_{j=1}^n \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2\\
&\quad \le Ck\sum_{j=1}^n \Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_H^2 + \|\dot{\boldsymbol{u}}_j - \delta\boldsymbol{u}_j\|_V^2 + k^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_H^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\varphi_{j-1} - \varphi_{j-1}^{hk}\|_W^2\\
&\qquad + I_j^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2 + \rho(\delta\boldsymbol{v}_j - \delta\boldsymbol{v}_j^{hk}, \boldsymbol{v}_j - \boldsymbol{w}_j^h)_H\Big) + C\left(\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2\right).
\end{aligned} \tag{4.15}$$

Now, combining (4.13), (4.14) and (4.15) it follows that, for all $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^n \subset V^h$ and $\psi^h = \{\psi_j^h\}_{j=0}^n \subset W^h$,

$$\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + Ck\sum_{j=1}^n \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + Ck\sum_{j=1}^n \|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2\\
&\quad \le Ck\sum_{j=1}^n \Bigg(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_H^2 + \|\dot{\boldsymbol{u}}_j - \delta\boldsymbol{u}_j\|_V^2 + k^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_H^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\varphi_{j-1} - \varphi_{j-1}^{hk}\|_W^2\\
&\qquad + I_j^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2 + \|\varphi_j - \psi_j^h\|_W^2 + \rho(\delta\boldsymbol{v}_j - \delta\boldsymbol{v}_j^{hk}, \boldsymbol{v}_j - \boldsymbol{w}_j^h)_H + k\sum_{l=1}^j \|\boldsymbol{\sigma}_l - \boldsymbol{\sigma}_l^{hk}\|_Q^2\Bigg)\\
&\qquad + C\left(\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2\right).
\end{aligned}$$

Now, taking into account that

$$\begin{aligned}
k\sum_{j=1}^n \rho(\delta\boldsymbol{v}_j - \delta\boldsymbol{v}_j^{hk}, \boldsymbol{v}_j - \boldsymbol{w}_j^h)_H &= \sum_{j=1}^n \rho(\boldsymbol{v}_j - \boldsymbol{v}_j^{hk} - (\boldsymbol{v}_{j-1} - \boldsymbol{v}_{j-1}^{hk}), \boldsymbol{v}_j - \boldsymbol{w}_j^h)_H\\
&= \rho(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}_n^h)_H + \rho(\boldsymbol{v}_0^h - \boldsymbol{v}_0, \boldsymbol{v}_1 - \boldsymbol{w}_1^h)_H\\
&\quad + \sum_{j=1}^{n-1} \rho(\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}, \boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h))_H,
\end{aligned}$$

using once again Young's inequality (3.17) we have, for all $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^n \subset V^h$ and $\psi^h = \{\psi_j^h\}_{j=0}^n \subset W^h$,

$$\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + Ck\sum_{j=1}^n \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + Ck\sum_{j=1}^n \|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 \\
&\quad \le Ck\sum_{j=1}^n \Big(\|\dot{\boldsymbol{v}}_j - \delta \boldsymbol{v}_j\|_H^2 + \|\dot{\boldsymbol{u}}_j - \delta \boldsymbol{u}_j\|_V^2 + k^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_H^2 + I_j^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 \\
&\qquad + \|\varphi_{j-1} - \varphi_{j-1}^{hk}\|_W^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2 + \|\varphi_j - \psi_j^h\|_W^2 + k\sum_{l=1}^j \|\boldsymbol{\sigma}_l - \boldsymbol{\sigma}_l^{hk}\|_Q^2 \Big) \\
&\qquad + C\left(\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{v}_1 - \boldsymbol{w}_1^h\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2\right) \\
&\qquad + \frac{C}{k}\sum_{j=1}^{n-1} \|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2.
\end{aligned}$$

From the regularity (4.10) we conclude that $\varphi(0)$ is the solution to the following problem:

$$(\beta\nabla\varphi(0), \nabla\psi_0)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(0)), \nabla\psi_0)_H = (q(0), \psi_0)_W \quad \forall \psi_0 \in W,$$

and so, proceeding as in the proof of estimates (4.14), we easily find that

$$\|\varphi_0 - \varphi_0^{hk}\|_W^2 \le C\Big(\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_W^2 + \|\varphi_0 - \psi_0^h\|_W^2\Big) \quad \forall \psi_0^h \in W^h.$$

Finally, using a discrete version of Gronwall's inequality (see, for instance, [13]) we derive the *a priori* error estimates (4.11). □

We note that from estimates (4.11) we can derive the convergence order under suitable additional regularity conditions. For instance, if we assume that the continuous solution has the additional regularity:

$$\boldsymbol{u} \in H^2(0,T;V) \cap H^3(0,T;H) \cap C^1([0,T];[H^2(\Omega)]^d), \quad \varphi \in C([0,T];H^2(\Omega)), \tag{4.16}$$

then we have the following result.

Corollary 4.3. *Let the assumptions of Theorem 4.2 still hold. Under the additional regularity conditions (4.16), it follows the linear convergence of the solution obtained by Problem 4.1; that is, there exists a positive constant C, independent of the discretization parameters h and k, such that*

$$\max_{0\le n\le N} \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H + \max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \max_{0\le n\le N} \|\varphi_n - \varphi_n^{hk}\|_W \le C(h+k).$$

Notice that this linear convergence is based on some well-known results concerning the approximation by the finite element method (see, for instance, [17]), the discretization of the time derivatives and the following result (see [5,14] for details),

$$\frac{1}{k}\sum_{j=1}^{N-1} \|\boldsymbol{v}_j - \mathcal{P}^h\boldsymbol{v}_j - (\boldsymbol{v}_{j+1} - \mathcal{P}^h\boldsymbol{v}_{j+1})\|_H^2 \le Ch^2\|\boldsymbol{u}\|_{H^2(0,T;V)}^2.$$

Moreover, from the approximation properties of operator $\mathcal{P}^h$, taking into account regularities (4.16) we can easily find that

$$\|\boldsymbol{u}_0 - \mathcal{P}^h\boldsymbol{u}_0\|_V^2 + \|\boldsymbol{v}_0 - \mathcal{P}^h\boldsymbol{v}_0\|_H^2 \le Ch^2.$$

5. Numerical results

In order to verify the behaviour of the numerical method analyzed in the previous section, some numerical experiments have been performed in two-dimensional problems. In all the examples presented, the elastic tensor was chosen as the $2D$ plane-stress elasticity tensor,

$$(\mathcal{B}\boldsymbol{\tau})_{\alpha\beta} = \frac{Er}{1-r^2}(\tau_{11}+\tau_{22})\delta_{\alpha\beta} + \frac{E}{1+r}\tau_{\alpha\beta} \quad \forall \boldsymbol{\tau} \in \mathbb{S}^2, \tag{5.1}$$

where $\alpha, \beta = 1, 2$, E and r are the Young's modulus and the Poisson's ratio, respectively, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. The viscous tensor $\mathcal{A}$ has a similar form but multiplied by a damping coefficient 10^{-2}, *i.e.* $\mathcal{A} = 10^{-2}\,\mathcal{B}$.

The viscoplastic function is a version of the Maxwell function given by

$$\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) = -\frac{1}{100}\Phi(\boldsymbol{\sigma}), \tag{5.2}$$

being Φ a truncation operator defined as

$$\forall \boldsymbol{\tau} = (\tau_{\alpha\beta})_{\alpha,\beta=1}^2, \quad (\Phi(\boldsymbol{\tau}))_{\alpha\beta} = \begin{cases} L & \text{if} \quad \tau_{\alpha\beta} > L, \\ \tau_{\alpha\beta} & \text{if} \quad \tau_{\alpha\beta} \in [-L, L], \\ -L & \text{if} \quad \tau_{\alpha\beta} < -L, \end{cases}$$

where value $L = 1000$ was taken.

Moreover, as piezoelectric and permittivity tensors, the following matricial forms were considered:

$$e_{ijk} \equiv e_{pq} = \begin{pmatrix} 0 & 0 & e_{13} \\ e_{21} & e_{22} & 0 \end{pmatrix}, \qquad \beta_{ij} = \begin{pmatrix} \beta_{11} & 0 \\ 0 & \beta_{22} \end{pmatrix}, \tag{5.3}$$

where we have used the notations e_{ijk} and e_{pq} in such a way that $p = i$ and $q = 1$ if $(j,k) = (1,1)$, $q = 2$ if $(j,k) = (2,2)$ and $q = 3$ if $(j,k) = (1,2)$ or $(j,k) = (2,1)$.

Finally, in all the examples the normal compliance function is defined as

$$p(r) = c_p \max\{0, r\},$$

where $c_p > 0$ is a deformability coefficient.

5.1. Numerical scheme

As a first step, the artificial discrete initial condition for the electric potential field is obtained by solving equation (4.8). This leads to a linear symmetric system solved by using classical Cholesky's method.

Secondly, being the solution $\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_{n-1}^{hk}$ and φ_{n-1}^{hk} known at time t_{n-1}, the velocity field is obtained by solving the discrete equation

$$\begin{aligned}(\rho\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H &+ k\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}) + k\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q = (\rho\boldsymbol{v}_{n-1}^{hk}, \boldsymbol{w}^h)_H + k\left\langle \boldsymbol{f}_n, \boldsymbol{w}^h\right\rangle_{V'\times V} - kj(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}^h) \\ &+ k\left(\mathcal{E}^*\boldsymbol{E}(\varphi_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q - \left(k\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q - k^2\left(\sum_{j=1}^{n-1}\mathcal{G}\left(\boldsymbol{\sigma}_j^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk})\right), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q,\end{aligned}$$

where the decomposition

$$\boldsymbol{u}_n^{hk} = \boldsymbol{u}_{n-1}^{hk} + k\boldsymbol{v}_n^{hk}, \tag{5.4}$$

has been used. Later, displacement field $\boldsymbol{u}_n^{hk}$ is updated through expression (5.4), and the electric potential field is obtained solving equation (4.6). We note that both numerical problems lead to linear symmetric systems and therefore, Cholesky's method is applied again for their solution.

The numerical scheme was implemented on a Intel Core $i5-3337U$ @ 1.80 GHz using FreeFEM++ (see [28] for details) and a typical run (100 step times and 10000 nodes) took about 3 min of CPU time.

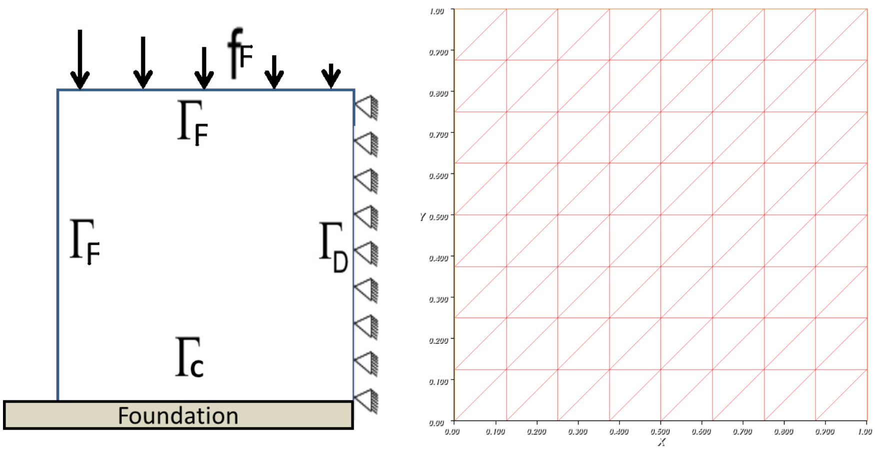

FIGURE 1. Example 1: Physical setting and mesh for $N_{\mathrm{el}} = 8$.

TABLE 1. Material constants.

Piezoelectric ($\mathrm{C/m^2}$)			Permittivity ($\mathrm{C^2/(N\ m^2)}$)	
e_{21}	e_{22}	e_{13}	β_{11}	β_{22}
−5.4	15.8	12.3	916	830

5.2. A first example: numerical convergence

As a first example, a sequence of numerical solutions, based on uniform partitions of both the time interval $[0,1]$ and the domain $\Omega = [0,1] \times [0,1]$, have been performed in order to check the behaviour of the numerical scheme.

The physical setting of the example is depicted in Figure 1 (*left-hand side*). The body is in initial contact with a deformable foundation on its lower part while $\Gamma_D = \Gamma_A$ is the right boundary $\{1\} \times [0,1]$ (and so both the displacement and electric potential fields vanish there). A surface force acts on the upper surface $[0,1] \times \{1\}$ and no electric charges are applied nor in the body or the surface.

The numerical solution corresponding to $N_{\mathrm{el}} = 512$ subdivisions on each outer side of the square (see the *right-hand side* of Figure 1 for the case $N_{\mathrm{el}} = 8$), and $k = 0.00078612$ has been considered as the "exact" solution in order to compute the numerical errors given by

$$E^{hk} = \max_{0 \le n \le N} \left\{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H + \|\varphi_n - \varphi_n^{hk}\|_W \right\}.$$

Both the piezoelectric and permittivity coefficients are depicted in Table 1. Moreover the following data have been employed in the simulations:

$$T = 1\,s, \quad \boldsymbol{f}_0(\boldsymbol{x},t) = 0\ \mathrm{N/m^3}, \quad \boldsymbol{f}_F(\boldsymbol{x},t) = \begin{cases} (0, -60(1-x_1)t)\ \mathrm{N/m^2} & \text{if } x_2{=}1, \\ 0\ \ \mathrm{N/m^2} & \text{elsewhere,} \end{cases}$$
$$E = 20\,000\ \ \mathrm{N/m^2}, \quad r = 0.3, \quad c_p = 10^5, \quad \rho = 1\ \mathrm{kg/m^3}, \quad \varphi_A = 0\ \ \mathrm{V}, \quad q_0 = 0\ \ \mathrm{C/m^3}, \quad q_F = 0\ \mathrm{C/m^2},$$
$$\boldsymbol{u}_0 = \boldsymbol{0}\ \ \mathrm{m}, \quad \boldsymbol{v}_0 = \boldsymbol{0}\ \ \mathrm{m/s}, \quad \varphi_0 = 0\ \ V.$$

In Table 2 the numerical errors obtained for some discretization parameters N_{el} and k are shown. As can be seen, the convergence of the numerical algorithm is clearly observed. The evolution of the error with respect to the parameter $k+h$ is plotted in Figure 2 (here, $h = \frac{\sqrt{2}}{N_{\mathrm{el}}}$). The linear convergence of the algorithm seems to be achieved.

TABLE 2. Example 1: Numerical errors (×100) for some $N_{\rm el}$ and k.

$N_{\rm el} \downarrow k \rightarrow$	0.0015625	0.003125	0.00625	0.0125	0.025	0.05	0.1
4	0.470744	0.470809	0.470941	0.471220	0.471838	0.473329	0.477455
8	0.255173	0.255208	0.255284	0.255468	0.255957	0.257490	0.263109
16	0.141647	0.141660	0.141698	0.141839	0.142445	0.144762	0.152973
32	0.080098	0.080101	0.080149	0.080420	0.081371	0.084702	0.096829
64	0.045528	0.045547	0.045663	0.046057	0.047449	0.052511	0.069826
128	0.025358	0.025405	0.025569	0.026165	0.028363	0.035951	0.058180
256	0.012722	0.012791	0.013062	0.014099	0.017720	0.028208	0.053679

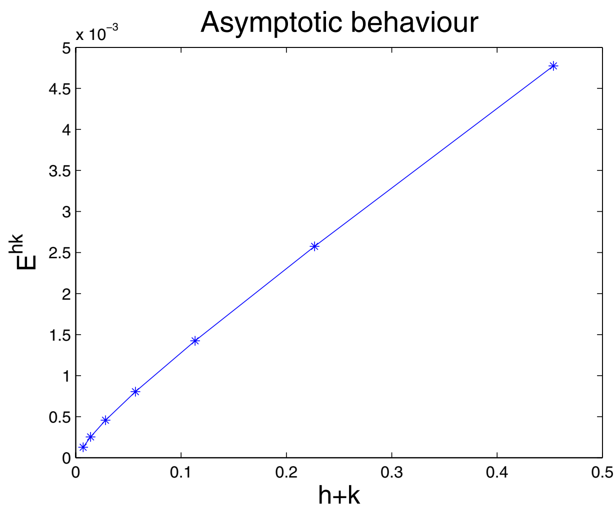

FIGURE 2. Example 1: Asymptotic behaviour of the numerical scheme.

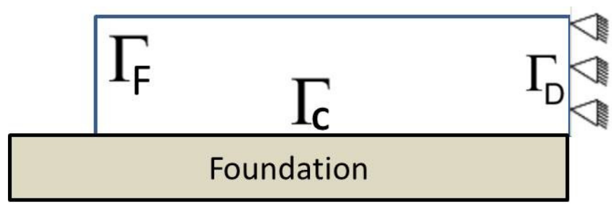

FIGURE 3. Example 2: Physical setting.

5.3. A second example: piezoelectric effect

As a second numerical example, in order to observe the effect of the piezoelectric properties of the material, a physical setting as the one depicted in Figure 3 is considered.

In this case, the body $\Omega = [0,4] \times [0,1]$ is clamped on its right end and it remains in initial contact with a deformable foundation on its lower boundary. No physical forces act on the body, but a constant surface electric charge ($q_F = 200\ C/M^2$) is applied on the lower part of the boundary, where contact is produced. Here, as in the previous example, we assume that $\Gamma_D = \Gamma_A$.

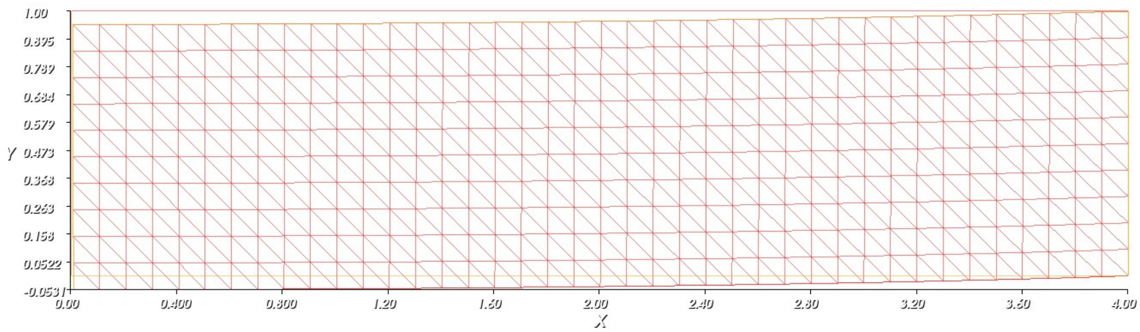

FIGURE 4. Example 2: Deformed configuration (× 5000) at final time.

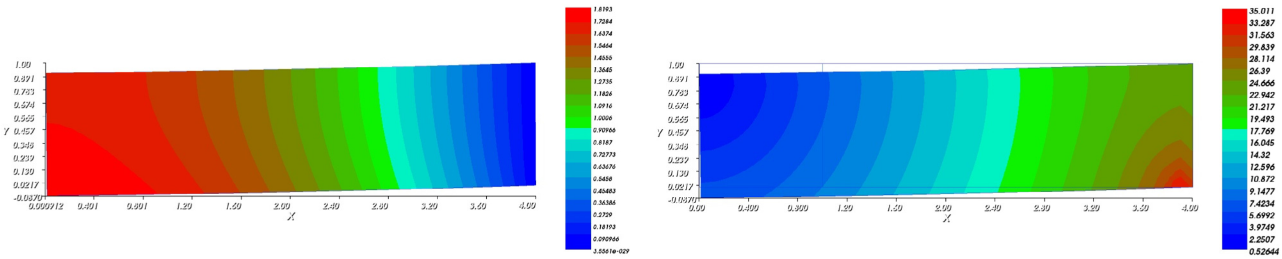

FIGURE 5. Example 2: Deformed configuration (× 5000) at final time.

The following data have been used

$$T = 1\,s, \quad \boldsymbol{f}_B(\boldsymbol{x},t) = 0 \ \text{N/m}^3, \quad \boldsymbol{f}_F = 0 \ \text{N/m}^2, \quad E = 2\times 10^6 \ \text{N/m}^2, \quad r = 0.3, \quad c_p = 10^5, \quad \rho = 1000\,\text{kg/m}^3,$$
$$\boldsymbol{u}_0 = 0 \ m, \quad \boldsymbol{v}_0 = 0 \ \text{m/s}, \quad \varphi_0 = 0 \ V, \quad \varphi_A = 0 \ \text{V}, \quad q_0 = 0 \ \text{C/m}^3, \quad q_F = \begin{cases} 200 \ \text{C/m}^3 \text{ if } x_2 = 0, \\ 0 \quad \text{elsewhere.} \end{cases}$$

We can see in Figure 4 that deformations appear due to the piezoelectric effect which, added to the mechanical restrictions, lead the body to a stress-state which can be observed in Figure 5 (*right-hand side*). In this figure (*left-hand side*), the electric potential field is shown at final time.

5.4. A third example: deformable contact of an L-shaped domain

As a final example, we consider an L-shaped body which is submitted to the action of traction forces on its upper horizontal boundary. The body is clamped on its lower horizontally boundary and an obstacle is assumed to be in initial contact, as can be observed in Figure 6.

The following data are used in the simulation:

$$T = 1\,s, \quad \boldsymbol{f}_B(\boldsymbol{x},t) = 0 \ \text{N/m}^3, \quad \boldsymbol{f}_F = \begin{cases} (0, -500(60 - x_1)t) \ \text{N/m}^2 \ \text{ if } x_2 = 50, \\ 0 \ \text{N/m}^2 \quad \text{elsewhere,} \end{cases}$$
$$E = 2.1\times 10^9 \ \text{N/m}^2, \quad r = 0.3, \quad c_p = 10^5, \quad \rho = 27\,000\,\text{kg/m}^3, \quad \varphi_A = 0 \ \text{V},$$
$$q_0 = 0 \ \text{C/m}^3, \quad q_F = 0 \ \text{C/m}^3, \quad \boldsymbol{u}_0 = 0 \ m, \quad \boldsymbol{v}_0 = 0 \ \text{m/s}, \quad \varphi_0 = 0 \ \text{V}.$$

Both electric potential and von Mises stress norm are plotted, over the final configuration of the body and at final time, in Figure 7. The area of maximum stress concentration is located near the contact boundary due to the bending movement, and it coincides with the region where the electric potential reaches its maximum value as it was expected.

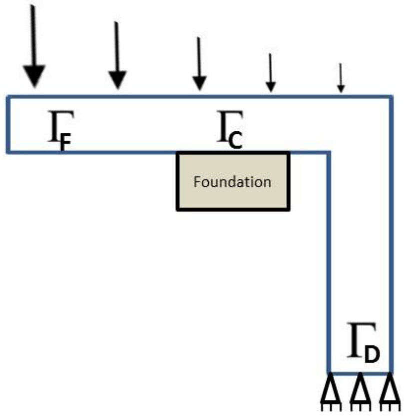

FIGURE 6. Example 3: Contact problem of an L-shaped domain.

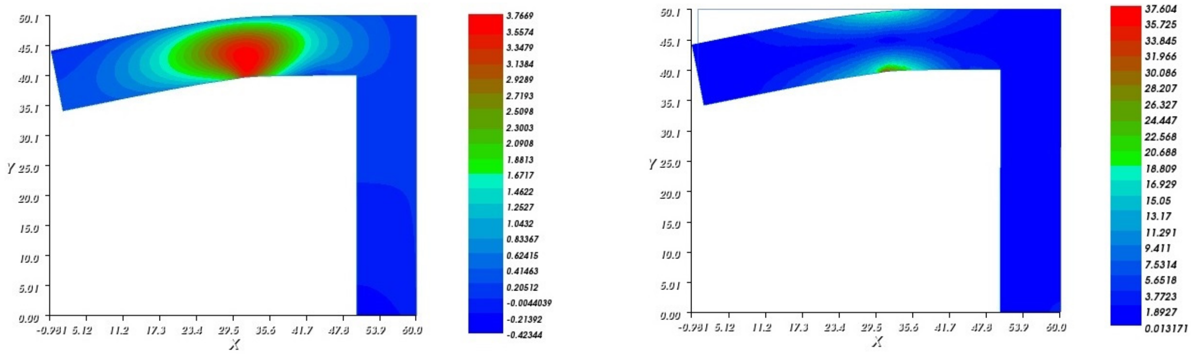

FIGURE 7. Example 3: Contact problem of an L-shaped domain.

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