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SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHODS BASED ON UPWIND-BIASED FLUXES FOR 1D LINEAR HYPERBOLIC EQUATIONS*

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Abstract. In this paper, we study superconvergence properties of the discontinuous Galerkin method using upwind-biased numerical fluxes for one-dimensional linear hyperbolic equations. A $(2k+1)$th order superconvergence rate of the DG approximation at the numerical fluxes and for the cell average is obtained under quasi-uniform meshes and some suitable initial discretization, when piecewise polynomials of degree k are used. Furthermore, surprisingly, we find that the derivative and function value approximation of the DG solution are superconvergent at a class of special points, with an order $k+1$ and $k+2$, respectively. These superconvergent points can be regarded as the generalized Radau points. All theoretical findings are confirmed by numerical experiments.

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1. Introduction

In this paper, we study and analyze the discontinuous Galerkin (DG) method for the following one-dimensional linear hyperbolic conservation laws

$$\begin{aligned} &u_t + u_x = 0, && (x,t) \in [0, 2\pi] \times (0, T], \\ &u(x,0) = u_0(x), && x \in R, \end{aligned} \tag{1.1}$$

where u_0 is sufficiently smooth. We will consider both the periodic boundary condition $u(0,t) = u(2\pi, t)$ and the Dirichlet boundary condition $u(0,t) = g(t)$.

Keywords and phrases. Discontinuous Galerkin methods, superconvergence, generalized Radau points, upwind-biased fluxes.

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DG methods are a class of finite element methods using completely discontinuous piecewise polynomial space. During the past several decades, it has gained more popularity in solving various differential equations (see, *e.g.*, [9–14]). It is well known that to guarantee the stability of DG methods, a suitable choice of numerical fluxes is of great significance. Traditionally, purely upwind numerical fluxes are used for DG methods applied to the linear hyperbolic equation. However, when it comes to the complex systems or the nonlinear problems, purely upwind numerical fluxes may be difficult to construct. Therefore, study of the more general numerical fluxes, *e.g.*, upwind-biased fluxes, becomes very necessary and significant. Very recently, Meng *et al.* in [17] studied DG methods using upwind-biased numerical fluxes for linear conservation laws. They proved an optimal convergence order $k+1$ for the semi-discrete DG schemes.

This paper is devoted to the study of superconvergence of DG methods using upwind-biased fluxes for the hyperbolic conservation laws (1.1). Note that many superconvergence studies of DG methods for hyperbolic equations are based on the purely upwind fluxes, see, *e.g.*, [1–4, 7, 8, 16, 18–20]. The contribution of this paper is to establish the superconvergence theory for upwind-biased fluxes by providing a rigorous mathematical proof, and set a solid theoretical foundation on the fact that all upwind-biased fluxes share the same superconvergence properties with the purely upwind fluxes in designing DG schemes for linear hyperbolic equations. By doing so, we present a full picture for superconvergence properties of the DG method using upwind-biased fluxes, and thus extend the superconvergence results in [6] to more general cases. Furthermore, our current work is also part of an ongoing effort to develop superconvergence of DG methods using upwind-biased or Lax–Friedrichs fluxes for nonlinear hyperbolic equations.

The main idea in our superconvergence analysis is to design some special correction functions. Motivated by the successful applications of the correction idea to the DG methods for hyperbolic and parabolic equations (see, *e.g.* [4–6]), we will construct a special correction function w, which vanishes or is of high order at some special points, to correct the error between the DG solution and some particular projection of the exact solution. To this end, the first key ingredient in our superconvergence analysis is to construct some suitable projection Pu (for the periodic boundary condition) or $\tilde{P}u$ (for the Dirichlet boundary condition). Different from those in [4–6], the particular projection should be globally defined so as to eliminate the error on the inter-element boundary when the upwind-biased fluxes are concerned. The second key ingredient is to analyze the superconvergent approximation properties of the globally defined projection. However, the nonlocality of the projection makes the study of superconvergent approximation properties of the global projection more complicated than that in [4–6]. The latter is the Gauss–Radau projection, which is a local operator and is superconvergent at left and right Radau points. Based on the projection Pu or $\tilde{P}u$, our final step is to construct a correction function w to correct the error between DG solution and the globally defined projection Pu or $\tilde{P}u$.

Thanks to the correction function, we establish the supercloseness between the DG solution and the specially constructed interpolation function $u_I = Pu - w$ or $\tilde{P}u - w$. It is this supercloseness that gives us desired superconvergence results of the DG solution. To be more precise, we prove a $(2k+1)$th order superconvergence rate of the DG approximation at the numerical fluxes and for the cell average. As a by-product, we also prove that the DG solution is $(k+2)$th order superconvergent for the error to the particular globally defined projection Pu or $\tilde{P}u$. An unexpected discovery is that the derivative and function value approximations of the DG solution are superconvergent with an order $k+1$ and $k+2$ at a class of special points, which can be viewed as the generalized Radau points. As we may recall, all the results are similar to that for the purely upwind fluxes in [6]. In some special case, the superconvergence results established in this paper are reduced to that in [6]. Even though we study the problem in one space dimension only, the extension to the 2D case can be obtained follow the same analysis in [4] and this will be discussed in the future.

The rest of the paper is organized as follows. In Section 2, we present semi-discrete DG schemes using upwind-biased numerical fluxes for linear conservation laws under the periodic and Dirichlet boundary conditions. Section 3 is devoted to the superconvergence analysis for the periodic boundary condition, where we first study the superconvergence properties of the global prejection Pu, and then design the correction function to correct the error between DG solution and the globally defined projection Pu, and finally prove the supercloseness of the DG solution towards a particular interpolation function of the exact solution, which gives superconvergence results of the DG approximation. Extensions of the analysis to the Dirichlet boundary case are carried out in Section 4, and the same superconvergence results are obtained. In Section 5, we provide some numerical examples to support our theoretical findings. Finally, some concluding remarks are presented in Section 6.

2. DG schemes

Let $\Omega = [0, 2\pi]$ and $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}}$ be $N+1$ distinct points on the interval Ω. For all positive integers r, we define $\mathbb{Z}_r = \{1, \ldots, r\}$ and denote by

$$\tau_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right), \quad x_j = \frac{1}{2}\left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right), \ j \in \mathbb{Z}_N$$

the cells and cell centers, respectively. Let $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $\overline{h}_j = h_j/2$ and $h = \max_j h_j$. We assume that the mesh is quasi-uniform, *i.e.*, there exists a constant c such that

$$h \le ch_j, j \in \mathbb{Z}_N.$$

Define

$$V_h = \{v : \ v|_{\tau_j} \in \mathbb{P}_k(\tau_j), \ j \in \mathbb{Z}_N\}$$

to be the finite element space, where $\mathbb{P}_k$ denotes the space of polynomials of degree at most k with coefficients as functions of t. The DG scheme for (1.1) reads as: find $u_h \in V_h$ such that for any $v \in V_h$

$$(u_{ht}, v)_j - (u_h, v_x)_j + \hat{u}_h|_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{u}_h|_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0, \tag{2.1}$$

where $(u, v)_j = \int_{\tau_j} uv \mathrm{d}x$, $v^-_{j+\frac{1}{2}}$ and $v^+_{j+\frac{1}{2}}$ denote the left and right limits of v at the point $x_{j+\frac{1}{2}}$, respectively, and $\hat{u}_h$ is the numerical flux. In this paper, instead of using the purely upwind flux, we adopt the upwind-biased flux. That is, we choose

$$\hat{u}_h\big|_{j+\frac{1}{2}} = \left(\theta u_h^- + (1-\theta) u_h^+\right)_{j+\frac{1}{2}}, \quad j = 0, \ldots, N \tag{2.2}$$

for the periodic boundary condition, and

$$\hat{u}_h\big|_{j+\frac{1}{2}} = \begin{cases} \left(\theta u_h^- + \left(1-\theta\right) u_h^+\right)_{j+\frac{1}{2}}, & j \in \mathbb{Z}_{N-1}, \\ (u_h)^-_{\frac{1}{2}} = g(t), & j = 0, \\ (u_h)^-_{N+\frac{1}{2}}, & j = N \end{cases} \tag{2.3}$$

for the Dirichlet boundary condition. Here and in what follows, we take $\theta > \frac{1}{2}$.

We denote

$$a_j(u, v) = (u_t, v)_j - (u, v_x)_j + \hat{u}v^-|_{j+\frac{1}{2}} - \hat{u}v^+|_{j-\frac{1}{2}}$$

and

$$a(u, v) = \sum_{j=1}^{N} a_j(u, v),$$

then (2.1) can be rewritten as

$$a_j(u_h, v) = 0, \quad \forall v \in V_h, \quad \forall j \in \mathbb{Z}_N.$$

A direct calculation from integration by parts yields

$$a(v,v) = (v_t, v) + \left(\theta - \frac{1}{2}\right)\sum_{j=1}^{N}[v]^2_{j+\frac{1}{2}} \tag{2.4}$$

for flux (2.2), and

$$a(v,v) = (v_t, v) + \left(\theta - \frac{1}{2}\right)\sum_{j=1}^{N-1}[v]^2_{j+\frac{1}{2}} + \frac{1}{2}[v]^2_{\frac{1}{2}} + \frac{1}{2}\left(v^-_{N+\frac{1}{2}}v^-_{N+\frac{1}{2}} - v^-_{\frac{1}{2}}v^-_{\frac{1}{2}}\right) \tag{2.5}$$

for flux (2.3).

3. Analysis for the periodic boundary condition

Following the superconvergence technique in [5,6], our goal here is to construct a special interpolation function u_I of u such that the DG solution is superclose to the specially designed function u_I. By doing this, the superconvergence analysis of the DG solution is reduced to the study of the superconvergent approximation properties of u_I. Therefore, in the rest of this section, we first design the special function u_I, and then analyze its superconvergence behavior.

We begin with some preliminaries.

3.1. Priliminaries

For any integer $m > 0$, let $W^{m,p}(D)$ be the standard Sobolev spaces on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$.

We denote by $L_m(s), s \in [-1,1]$ the standard Legendre polynomial of degree m on the interval $[-1,1]$, and $(L_{k+1} - \alpha L_k)(s)$ the generalized Radau polynomial of degree $k+1$ for some constant α. Noticing that when $\alpha = 1$ and $\alpha = -1$, $(L_{k+1} - \alpha L_k)(s)$ is reduced to the standard right and left Radau polynomials separately. For any positive α, we call $(L_{k+1} - \alpha L_k)(s)$ to be the generalized right Radau polynomial of degree $k+1$ and the zeros the generalized right Radau points of degree $k+1$. Moreover, we call the zeros of $D_s(L_{k+1} - \alpha L_k)(s)$ to be the generalized derivative right Radau points of degree k. The following lemma shows the properties of the generalized Radau points and generalized derivative Radau points. For simplicity, we consider $\alpha > 0$ only, and the case for $\alpha < 0$ can be obtained following the same line.

Lemma 3.1. *Let $r_1 < r_2 < \ldots < r_{k+1} = 1$ be the standard right Radau points on the interval $[-1,1]$,* i.e., *$r_i, i \in \mathbb{Z}_{k+1}$ are zeros of $(L_{k+1} - L_k)(s)$. Then for any positive constant α, the zeros of $(L_{k+1} - \alpha L_k)(s)$ and $D_s(L_{k+1} - \alpha L_k)(s)$ are all simple, and there are at least k zeros of $(L_{k+1} - \alpha L_k)(s)$ and $k-1$ zeros of $D_s(L_{k+1} - \alpha L_k)(s)$ in $[-1,1]$. Moreover, there exists exactly one zero of $(L_{k+1} - \alpha L_k)(s)$ in each subinterval $[r_i, r_{i+1}], i = 1,\ldots,k$. The position of the left one zero is dependent on the the choice of α:*

- *If $0 < \alpha \le 1$, the left one zero of $(L_{k+1} - \alpha L_k)(s)$ lies in the interval $[-1, r_1]$.*
- *If $\alpha > 1$, we extend the domain of Legendre polynomial from $[-1,1]$ to the domain $[-1,\infty)$, then there exists one zero of $(L_{k+1} - \alpha L_k)(s)$ in the interval $(1,\infty)$.*

Proof. We first prove that there are k zeros of $(L_{k+1} - \alpha L_k)(s)$ in $[-1,1]$ for all constant α, and each of the interior subinterval $[r_i, r_{i+1}], i = 1,\ldots,k$ contains at least one zero of $L_{k+1} - \alpha L_k$. To this end, it is sufficient to prove

$$(L_{k+1} - \alpha L_k)(r_i)(L_{k+1} - \alpha L_k)(r_{i+1}) \le 0.$$

We denote by $\phi_i, i \in \mathbb{Z}_{k+1}$ the Lagrange basis polynomials corresponding to the points $r_i, i \in \mathbb{Z}_{k+1}$, that is,

$$\phi_i(r_j) = \delta_{i,j}.$$

Obviously, each $\phi_i \in \mathbb{P}_k$ and

$$\phi_i(x) = c_i x^k + v(x), \quad v \in \mathbb{P}_{k-1},$$

where $c_i = 1/(\Pi_{j=1}^{i-1}(r_i - r_j)\Pi_{j=i+1}^{k+1}(r_i - r_j))$. A direct calculation leads to

$$c_i c_{i+1} < 0.$$

On the other hand, noticing that $L_k \phi_i \in \mathbb{P}_{2k}$, we have, from Gauss–Radau quadrature and orthogonal properties of Legerdre polynomials,

$$L_k(r_i) = \frac{1}{w_i} \int_{-1}^{1} L_k(x)\phi_i(x)\mathrm{d}x = \frac{c_i}{w_i} \int_{-1}^{1} L_k(x)x^k \mathrm{d}x.$$

Here $w_i > 0$ is the weight of Gauss–Radau quadrature. Then

$$L_k(r_i)L_k(r_{i+1}) = \frac{c_i c_{i+1}}{w_i w_{i+1}} \left(\int_{-1}^{1} L_k(x)x^k \mathrm{d}x \right)^2 < 0.$$

Since $L_{k+1}(r_i) = L_k(r_i), i \in \mathbb{Z}_{k+1}$, we have

$$(L_{k+1} - \alpha L_k)(r_i)(L_{k+1} - \alpha L_k)(r_{i+1}) = (1-\alpha)^2 L_k(r_i)L_k(r_{i+1}) \leq 0. \tag{3.1}$$

Therefore, for any α, there are at least one zero of $L_{k+1} - \alpha L_k$ in each subinterval $[r_i, r_{i+1}], i = 1, \ldots, k$.

When $\alpha \leq 1$, we shall prove that there exists at least one zero of $L_{k+1} - \alpha L_k$ in the subinterval $[r_0, r_1]$. Actually, by (3.1),

$$(L_{k+1} - \alpha L_k)(r_1)(L_{k+1} - \alpha L_k)(r_{k+1})(-1)^k > 0.$$

Note that

$$(L_{k+1} - \alpha L_k)(r_0)(L_{k+1} - \alpha L_k)(r_{k+1}) = (1-\alpha^2)(-1)^{k+1},$$

we have

$$(L_{k+1} - \alpha L_k)(r_0)(L_{k+1} - \alpha L_k)(r_1) \leq 0,$$

which implies $[-1, r_1]$ contains at least one zero of $L_{k+1} - \alpha L_k$.

Now we consider the case $\alpha > 1$. Noticing that

$$(L_{k+1} - \alpha L_k)(1) = 1 - \alpha < 0,$$

On the other hand, by the properties of Legendre polynomials, there exists a point $r_{k+2} \in (1, \infty)$ such that

$$(L_{k+1} - \alpha L_k)(r_{k+2}) \geq 0.$$

Then there is at least one zero of $L_{k+1} - \alpha L_k$ in $(1, \infty)$. Since $L_{k+1} - \alpha L_k \in \mathbb{P}_{k+1}$, each subinterval contains exactly one zero of $(L_{k+1} - \alpha L_k)(s)$.

By Rolle's theorem, there exist k zeros of $D_s(L_{k+1} - \alpha L_k)(s)$. Moreover, the zeros of $D_s(L_{k+1} - \alpha L_k)(s)$ are simple, and there exists one zero of $D_s(L_{k+1} - \alpha L_k)(s)$ between two consecutive zeros of $(L_{k+1} - \alpha L_k)(s)$. The proof is complete. □

Let $L_{j,m}$ be the standard Legendre polynomial of degree m on the interval $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. For smooth function u, we suppose u has the following Legendre expansion in each element τ_j

$$u = \sum_{m=0}^{\infty} u_{j,m} L_{j,m}, \quad u_{j,m} = \frac{2m+1}{h_j} \int_{\tau_j} u(x) L_{j,m}(x) \mathrm{d}x. \tag{3.2}$$

Denoting $\hat{u}(s) = u(x), s = (x - x_j)/\overline{h}_j \in [-1, 1]$, we have from the integration by parts and the property of Legendre polynomials

$$\begin{aligned} u_{j,m} &= \left(\frac{2m+1}{2}\right) \frac{1}{(2m)!} \int_{-1}^{1} \hat{u}(s) D_s^m (s^2-1)^m \mathrm{d}s \\ &= \left(\frac{2m+1}{2}\right) \frac{(-1)^i}{(2m)!} \int_{-1}^{1} D_s^i \hat{u}(s) D_s^{m-i} (s^2-1)^m \mathrm{d}s, \quad i \le m. \end{aligned} \tag{3.3}$$

Noticing that $D_s^i \hat{u} = \left(\bar{h}_j\right)^i D_x^i u$, then

$$|u_{j,m}| \lesssim h^{i-\frac{1}{p}} \|u\|_{i+1,p,\tau_j}, \quad 0 \le i \le m. \tag{3.4}$$

Here and in the following, $A \lesssim B$ denotes that A can be bounded by B multiplied by a constant independent of the mesh size h. The above estimate will be frequently used in our later superconvergence analysis.

We define a special global projection $Pu \in V_h$ on function u as

$$(Pu, v)_j = (u, v)_j, \quad \forall v \in \mathbb{P}_{k-1}(\tau_j), \tag{3.5}$$

$$\widehat{Pu} = (\theta P u^- + (1-\theta) P u^+)|_{j+\frac{1}{2}} = \hat{u}_{j+\frac{1}{2}}, \quad j \in \mathbb{Z}_N. \tag{3.6}$$

It has been shown in [17] that the projection Pu is well defined and there holds

$$\|u - Pu\|_{0,\tau_j} + h^{\frac{1}{2}} \|u - Pu\|_{0,\infty,\tau_j} \lesssim h^{k+\frac{3}{2}} \|u\|_{k+1,\infty}. \tag{3.7}$$

For any function v, we define an integral projection D^{-1} by

$$D^{-1} v(x) = \frac{1}{\overline{h}_j} \int_{x_{j-\frac{1}{2}}}^{x} v(x) \mathrm{d}x = \int_{-1}^{s} v(s) \mathrm{d}s, \quad x \in \tau_j. \tag{3.8}$$

where $s = (x - x_j)/\overline{h}_j \in [-1, 1]$.

To end this section, we would like to introduce a class of functions $F_i(x), i \in \mathbb{Z}_k$, which will be used in the construction of the interpolation function. For all $i, j \in \mathbb{Z}_k$, we define

$$F_0(x)\big|_{\tau_j} = L_{j,k}(x), \quad F_i = PD^{-1} F_{i-1}. \tag{3.9}$$

Note that these functions $F_i, i \in \mathbb{Z}_k$ are different from those in [6], and the latter ones are locally defined.

Lemma 3.2. *Suppose A is an $N \times N$ circulant matrix with the first row $(\theta, (-1)^k(1-\theta), 0, \ldots, 0)$ and the last row $((-1)^k(1-\theta), 0, 0, \ldots, \theta)$, where $\frac{1}{2} < \theta \le 1$. Then A is non-singular. Moreover, For any vectors $X = (x_1, \ldots, x_N)^T, \quad b = (b_1, \ldots, b_N)^T$ satisfying $AX = b$, there holds*

$$|x_j| \lesssim \max_{1 \le l \le N} |b_l|, \quad \forall j \le N.$$

Here we omit the proof since the similar argument can be found in [17] (see, Lem. 2.6).

Now we study the properties of functions $F_i, i \in \mathbb{Z}_k$.

Lemma 3.3. *Suppose $F_i, i \in \mathbb{Z}_k$ are functions defined by (3.9). Then there hold*

$$\widehat{F}_i(x_{j+\frac{1}{2}}) = 0, \quad \forall j \in \mathbb{Z}_N, \tag{3.10}$$

and in each element $\tau_j, j \in \mathbb{Z}_N$

$$F_i\big|_{\tau_j} = \sum_{m=k-i}^{k} b_{j,m}^i L_{j,m}, \tag{3.11}$$

where the coefficients $b_{j,m}^i$ are some bounded constants independent of the mesh size h_j.

Proof. We will show (3.10)–(3.11) by induction. First, by the properties of Legendre polynomials,

$$D^{-1}F_0\left(x_{j+\frac{1}{2}}^-\right) = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} L_{j,k}\mathrm{d}x = 0 = D^{-1}F_0\left(x_{j+\frac{1}{2}}^+\right).$$

Then

$$\widehat{D^{-1}F_0}\left(x_{j+\frac{1}{2}}\right) = 0.$$

In light of (3.6), we have

$$\widehat{F_1}\left(x_{j+\frac{1}{2}}\right) = P\widehat{D^{-1}}F_0\left(x_{j+\frac{1}{2}}\right) = \widehat{D^{-1}F_0}\left(x_{j+\frac{1}{2}}\right) = 0, \quad \forall j \in \mathbb{Z}_N. \tag{3.12}$$

On the other hand, noticing that $F_i \in V_h$, we have the following representation in each element τ_j

$$F_i\big|_{\tau_j} = \sum_{m=0}^{k} b_{j,m}^i L_{j,m}, \quad i \in \mathbb{Z}_k.$$

Recalling the definition of the global projection P, we have

$$(F_1, v)_j = (D^{-1}F_0, v)_j = (D^{-1}L_{j,k}, v)_j, \quad \forall v \in \mathbb{P}_{k-1}.$$

Since

$$D^{-1}L_{j,m} = \frac{1}{2m+1}(L_{j,m+1} - L_{j,m-1}), \quad \forall m \geq 1, \tag{3.13}$$

we have, by choosing $v = L_{j,m}, m \leq k-1$,

$$b_{j,m}^1 = 0, \quad \forall m \leq k-2, \quad b_{j,k-1}^1 = -\frac{1}{2k+1}, \quad \forall j \in \mathbb{Z}_N.$$

Denoting

$$X = (b_{1,k}^1, \ldots, b_{N,k}^1)^T, \; b = -\sum_{m=0}^{k-1}(\theta b_{1,m}^1 + (-1)^m(1-\theta)b_{2,m}^1, \ldots, \theta b_{N,m}^1 + (-1)^m(1-\theta)b_{1,m}^1)^T,$$

then (3.12) can by rewritten as a linear system $AX = b$, where A is the same as in Lemma 3.2. By the conclusion of Lemma 3.2, we easily obtain

$$|b_{j,k}^1| \lesssim \max_{1\leq l\leq N} |b_{l,k-1}^1| \lesssim 1.$$

Consequently, both (3.10) and (3.11) are valid for $i = 1$. Suppose (3.11) is valid for all $i \le k-1$, we now prove that it also holds for $i+1$. Since

$$(F_{i+1}, v)_j = (D^{-1}F_i, v)_j = \sum_{r=k-i}^{k} b^i_{j,r}(D^{-1}L_{j,r}, v)_j, \quad v \in \mathbb{P}_{k-1},$$

we derive, by choosing $v = L_{j,m}, m \le k-1$ and using (3.13),

$$b^{i+1}_{j,m} = 0, \quad m \le k-i-2, \quad b^{i+1}_{j,m} = \frac{b^i_{j,m-1}}{2m-1} - \frac{b^i_{j,m+1}}{2m+3}, \; k-i-1 \le m \le k-1.$$

To estimate $b^{i+1}_{j,k}$, we first obtain, from the fact that $F_i \bot \mathbb{P}_{k-i-1}, i \le k-1$ and the orthogonal property of Legendre polynomials,

$$D^{-1}F_i\left(x^-_{j+\frac{1}{2}}\right) = D^{-1}F_i\left(x^+_{j-\frac{1}{2}}\right) = 0, \tag{3.14}$$

which yields

$$\widehat{F_{i+1}}\left(x_{j+\frac{1}{2}}\right) = \widehat{PD^{-1}F_i}\left(x_{j+\frac{1}{2}}\right) = \widehat{D^{-1}F_i}\left(x_{j+\frac{1}{2}}\right) = 0, \quad \forall j \in \mathbb{Z}_N.$$

Then by the same argument as what we did for $i = 1$, we obtain

$$|b^{i+1}_{j,k}| \lesssim \max_{1 \le l \le N} \sum_{m=0}^{k-1} |b^{i+1}_{j,m}| \lesssim 1.$$

Consequently, (3.10) and (3.11) are also valid for $i+1$. This finishes our proof. □

3.2. Construction of a special interpolation function

To construct the special interpolation function u_I, we first design a correction function w, which is used to correct the error between the DG solution and Pu. Note that the standard analysis only leads to the optimal error estimate (see [17])

$$\|u_h - Pu\|_0 \lesssim h^{k+1}.$$

Here we use the correction idea to achieve our goal, *i.e.* we design a correction function w such that

$$\|u_h - Pu + w\|_0 = \|u_h - u_I\|_0 \lesssim h^{k+l+1}$$

for some $l > 0$.

Since $(u - Pu) \bot \mathbb{P}_{k-1}$, we have

$$(u - Pu)\big|_{\tau_j} = \tilde{u}_{j,k}L_{j,k} + \sum_{m=k+1}^{\infty} u_{j,m}L_{j,m}, \tag{3.15}$$

where $\tilde{u}_{j,k} = \frac{2k+1}{h_j}(u - Pu, L_{j,k})_j$ and $u_{j,m}$ is the same as in (3.2).

We construct the correction function w as follows. For any positive l, where $1 \le l \le k$, we define

$$w(x,t) = w^l(x,t) = \sum_{i=1}^{l} w_i(x,t), \quad w_i(x,t)|_{\tau_j} = (\bar{h}_j)^i \partial_t^i \tilde{u}_{j,k}(t) F_i(x). \tag{3.16}$$

Now we are ready to construct our special interpolation function u_I. We define, for all $l, 1 \le l \le k$,

$$u_I = u^l_I = Pu - w^l. \tag{3.17}$$

Theorem 3.4. *For any given l, where $1 \le l \le k$, suppose $u \in W^{k+l+2,\infty}$, and w^l, u_I^l are defined in (3.16) and (3.17), respectively. Then*

$$\hat{w}_i(x_{j+\frac{1}{2}}) = 0, \quad j \in \mathbb{Z}_N, \quad \|w_i\|_{0,\infty} \lesssim h^{k+i+1}\|u\|_{k+i+1,\infty}, \quad 1 \le i \le l, \tag{3.18}$$

and

$$a(u - u_I^l, v) \lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}\|v\|_0, \quad \forall v \in V_h. \tag{3.19}$$

Proof. As we can see, the first identity of (3.18) is a direct consequence of (3.10) and (3.16). By (3.7),

$$\partial_t^i \tilde{u}_{j,k} = \frac{2k+1}{h_j}\int_{\tau_j}(\partial_t^i u - P(\partial_t^i u))L_{j,k}\mathrm{d}x \lesssim h^{k+1}\|\partial_t^i u\|_{k+1,\infty}.$$

Then the second inequality of (3.18) follows from (3.11), (3.16) and the fact $\partial_t^i u_t = (-1)^i \partial_x^i u$.

Recalling the definitions of $a(\cdot,\cdot)$ and Pu, we obtain

$$\begin{aligned} a_j(u - Pu, v) &= ((u-Pu)_t, v)_j = \partial_t \tilde{u}_{j,k}(L_{j,k}, v)_j \\ &= -\bar{h}_j \partial_t \tilde{u}_{j,k}(D^{-1}L_{j,k}, v_x)_j = (w_1, v_x)_j, \quad \forall v \in V_h. \end{aligned}$$

Due to the special construction of w_i in (3.16),

$$\begin{aligned} (\partial_t w_i, v)_j - (w_{i+1}, v_x)_j &= \partial_t^{i+1}\tilde{u}_{j,k}(\bar{h}_j)^i\left((F_i, v)_j - \bar{h}_j(F_{i+1}, v_x)_j\right) \\ &= \partial_t^{i+1}\tilde{u}_{j,k}\left((\bar{h}_j)^{i+1}(D^{-1}F_i, v_x)_j - (F_{i+1}, v_x)_j\right) = 0, \end{aligned}$$

where in the second step, we have used the integration by parts and (3.14). Then

$$a_j(w^l, v) = ((w^l)_t, v)_j - (w^l, v_x)_j = (\partial_t w_l, v)_j - (w_1, v_x)_j.$$

Consequently,

$$a(u - u_I, v) = a(u - Pu, v) + a(w^l, v) = (\partial_t w_l, v), \quad \forall v \in V_h. \tag{3.20}$$

Then the desired result (3.19) follows from the fact $u_t = -u_x$ and (3.18). □

3.3. Approximation properties of the interpolation function u_I

The result of (3.18) indicates that w is of high order as the mesh size h decreases. Consequently, the term Pu in the formula of u_I is dominant. Then the analysis of the superconvergent approximation properties of u_I is reduced to that of Pu. However, as we may recall, Pu is a global projection, which makes the analysis more complicated. To overcome this difficulty, we first introduce a local projection $P_h u \in V_h$ of u, and then show the supercolseness between Pu and $P_h u$. Therefore, to achieve our ultimate goal, all we need to do is to analyze the approximation properties of the local projection $P_h u$, which is an easier task.

For any coefficients $\theta_j, j \in \mathbb{Z}_N$, where $\theta_j \ne \frac{1}{2}$, we define a local projection $P_h u \in V_h$ as

$$(P_h u, v)_j = (u, v)_j, \quad \forall v \in \mathbb{P}_{k-1}(\tau_j), \tag{3.21}$$

$$\theta_j P_h u\left(x_{j+\frac{1}{2}}^-\right) + (1-\theta_j)P_h u\left(x_{j-\frac{1}{2}}^+\right) = \theta_j u_{j+\frac{1}{2}}^- + (1-\theta_j)u_{j-\frac{1}{2}}^+, \quad j \in \mathbb{Z}_N. \tag{3.22}$$

Obviously, $P_h u$ is well defined when $\theta_j \ne \frac{1}{2}$. Moreover, we have the following approximation properties of $P_h u$.

Lemma 3.5. *Suppose $u \in W^{k+2,\infty}$ and $P_h u$ is the special projection of u defined in (3.21) and (3.22) with $\theta_j \neq \frac{1}{2}, j \in \mathbb{Z}_N$. Then*

$$|(u - P_h u)(R^r_{j,l})| \lesssim h^{k+2}\|u\|_{k+2,\infty}, \quad |\partial_x (u - P_h u)(R^d_{j,m})| \lesssim h^{k+1}\|u\|_{k+2,\infty}. \tag{3.23}$$

Here $R^r_{j,l} \in \tau_j, l \leq \mathbb{Z}_{k+1}$ and $R^d_{j,m} \in \tau_j, m \leq \mathbb{Z}_k$ are zeros of $L_{j,k+1} - \alpha_j L_{j,k}$ and $\partial_x(L_{j,k+1} - \alpha_j L_{j,k})$ respectively, where

$$\alpha_j = 2\theta_j - 1, \quad \text{if } k \text{ is even}, \qquad \alpha_j = 1/(2\theta_j - 1), \quad \text{if } k \text{ is odd}. \tag{3.24}$$

Proof. In light of (3.21), we obtain

$$P_h u\big|_{\tau_j} = \sum_{m=0}^{k-1} u_{j,m} L_{j,m} + \bar{u}_{j,k} L_{j,k},$$

where $u_{j,m}$ is the same as in (3.2) and $\bar{u}_{j,k}$ is a constant to be determined. To obtain $\bar{u}_{j,k}$, we have from (3.22)

$$\begin{aligned}(\theta_j + (-1)^k(1-\theta_j))\bar{u}_{j,k} &= \theta_j \sum_{m=k}^{\infty} u_{j,m} + (1-\theta_j)\sum_{m=k}^{\infty}(-1)^m u_{j,m} \\ &= (\theta_j + (-1)^k(1-\theta_j))u_{j,k} + \sum_{m=k+1}^{\infty}(\theta_j + (-1)^m(1-\theta_j))u_{j,m},\end{aligned}$$

which yields

$$(u - P_h u)\big|_{\tau_j} = (L_{j,k+1} - \alpha_j L_{j,k})u_{j,k+1} + \sum_{m=k+2}^{\infty}(L_{j,m} - \alpha_{j,m} L_{j,k})u_{j,m}. \tag{3.25}$$

Here $\alpha_{j,m} = (\theta_j + (-1)^m(1-\theta_j))/(\theta_j + (-1)^k(1-\theta_j))$. Then the desired result (3.23) follows from (3.4). □

Remark 3.6. The above lemma indicates that the function and derivative value approximations of $P_h u$ are superconvergent at the zeros of $L_{j,k+1} - \alpha_j L_{j,k}$ and $\partial_x(L_{j,k+1} - \alpha_j L_{j,k})$, respectively. These results can be regarded as the generalization of the superconvergence results of the standard Gauss–Radau projection $P_h^- u$ provided in [6]. In fact, if $\alpha_j = 1$ or $\theta_j = 1$, then $P_h u$ is reduced to the standard Gauss–Radau projection $P_h^- u$.

Remark 3.7. We would like to point out that the number of superconvergence points of the local projection $P_h u$ in each interval τ_j may depend upon the choice of θ_j or $\alpha_j, j \in \mathbb{Z}_N$. As indicated from Lemma 3.1, there are $k+1$ superconvergence points in each element τ_j for the function value approximation of $P_h u$ when $|\alpha_j| \leq 1$; and k superconvergence points when $|\alpha_j| > 1$. Similar results hold for the derivative approximation.

When choosing special $\theta_j, j \in \mathbb{Z}_N$ in (3.21)–(3.22), we can obtain the following superconvergence result for $P_h u - Pu$.

Lemma 3.8. *Let $u \in W^{k+2,\infty}$ and u have the Legendre polynomial (3.2) in each element τ_j. Suppose Pu and $P_h u$ are the projections of u defined in (3.5)–(3.6) and (3.21)–(3.22) with θ and θ_j satisfying*

$$\theta(h_j)^{k+1}\alpha_j + (-1)^k(1-\theta)(h_{j+1})^{k+1}\alpha_{j+1} = \theta(h_j)^{k+1} - (-1)^k(1-\theta)(h_{j+1})^{k+1}, \tag{3.26}$$

where α_j is given by (3.24). Then

$$\|P_h u - Pu\|_{0,\infty} \lesssim h^{k+2}\|u\|_{k+2,\infty}. \tag{3.27}$$

Consequently,

$$|(u - Pu)(R^r_{j,l})| \lesssim h^{k+2}\|u\|_{k+2,\infty}, \quad |\partial_x(u - Pu)(R^d_{j,l})| \lesssim h^{k+1}\|u\|_{k+2,\infty}, \tag{3.28}$$

where $R^r_{j,l}, R^d_{j,l}$ are the same as in (3.23).

Proof. Since (3.28) is a direct result of (3.23) and (3.27), we only prove (3.27) in the following.

By (3.5) and (3.21), we have $(Pu - P_h u)\perp\mathbb{P}_{k-1}$, which gives

$$(Pu - P_h u)\big|_{\tau_j} = \bar{u}_j L_{j,k}.$$

Here $\bar{u}_j$ is a constant to be determined. By (3.6) and (3.22),

$$\theta(Pu - P_h u)\left(x^-_{j+\frac{1}{2}}\right) + (1-\theta)(Pu - P_h u)\left(x^+_{j+\frac{1}{2}}\right) = b_j,$$

where

$$b_j = \theta(u - P_h u)\left(x^-_{j+\frac{1}{2}}\right) + (1-\theta)(u - P_h u)\left(x^+_{j+\frac{1}{2}}\right). \tag{3.29}$$

Then

$$\theta\bar{u}_j + (1-\theta)(-1)^k\bar{u}_{j+1} = b_j. \tag{3.30}$$

By denoting

$$X = (\bar{u}_1, \dots, \bar{u}_N)^T, \quad b = (b_1, \dots, b_N)^T,$$

(3.30) can be rewritten as a linear system $AX = b$, where A is the same as in Lemma 3.2. By the result of Lemma 3.2, we derive

$$|\bar{u}_j| \lesssim \max_{1\le l\le N} |b_l|.$$

We now estimate b_l. Substituting (3.25) into (3.29), we obtain

$$\begin{aligned} b_j &= \theta(1-\alpha_j)u_{j,k+1} - (-1)^k(1-\theta)(1+\alpha_{j+1})u_{j+1,k+1} \\ &\quad + \sum_{m=k+2}^{\infty}\left(\theta(1-\alpha_{j,m})u_{j,m} + (1-\theta)((-1)^m - \alpha_{j,m}(-1)^k)u_{j+1,m}\right), \end{aligned}$$

where $\alpha_{j,m}, m \ge k+2$ is the same as in (3.25). In light of (3.3) and the mean value theory of integrals, we have

$$\begin{aligned} u_{j,k+1} &= \left(\frac{2k+3}{2}\right)\frac{(-1)^{k+1}}{(2k+2)!}\int_{-1}^{1} D_s^{k+1}\hat{u}(s)(s^2-1)^{k+1}\mathrm{d}s \\ &= c_k(h_j)^{k+1}D_x^{k+1}u(\zeta_j)\int_{-1}^{1}(s^2-1)^{k+1}\mathrm{d}s = \bar{c}_k(h_j)^{k+1}D_x^{k+1}u(\zeta_j), \end{aligned}$$

where $\zeta_j \in \tau_j$ is some point and $c_k = \left(\frac{2k+3}{2}\right)\frac{(-1)^{k+1}}{(2k+2)!}$. Plugging the above identity into the formula of b_j and using the identity (3.26), we get

$$\begin{aligned} b_j &= \bar{c}_k\theta(1-\alpha_j)h_j^{k+1}\left(D_x^{k+1}u(\zeta_j) - D_x^{k+1}u(\zeta_{j+1})\right) \\ &\quad + \sum_{m=k+2}^{\infty}\left(\theta(1-\alpha_{j,m})u_{j,m} + (1-\theta)((-1)^m - \alpha_{j,m}(-1)^k)u_{j+1,m}\right), \end{aligned}$$

By (3.4) and the fact that $|D_x^{k+1}u(\zeta_j) - D_x^{k+1}u(\zeta_{j+1})| \le h\|u\|_{k+2,\infty,\Omega_j}$, where $\Omega_j = \tau_j \cup \tau_{j+1}$ with $\tau_{N+1} = \tau_1$, we obtain

$$\|P_h u - Pu\|_{0,\infty,\tau_j} \lesssim |\bar{u}_j| \lesssim h^{k+2,\infty}\|u\|_{k+2,\infty,\Omega_j}.$$

The proof is complete. ∎

As a direct consequence of Lemmas 3.5 and 3.8, we have the following superconvergence result of the specially designed function u_I.

Corollary 3.9. *Suppose $u \in W^{k+2,\infty}$ and u_I is the special interpolation function defined by (3.17), (3.16). There hold*

$$(u-\hat{u}_I)(x_{j+\frac{1}{2}})=0, \qquad j\in\mathbb{Z}_N, \tag{3.31}$$

and

$$|(u-u_I)(R^r_{j,l})| \lesssim h^{k+2}\|u\|_{k+2,\infty},\ \ |\partial_x(u-u_I)(R^d_{j,m})| \lesssim h^{k+1}\|u\|_{k+2,\infty}. \tag{3.32}$$

Here $R^r_{j,l}, l\le \mathbb{Z}_{k+1}$ and $R^d_{j,m}, m\le\mathbb{Z}_k$ are the same as in Lemma 3.5.

Remark 3.10. As we may see from Lemmas 3.5 and 3.8, the superconvergence points of u_I or $P_h u$ depend on the lengths of all the cells. In practice, to compute the interior superconvergence points, we need to find $\alpha_j' s$ by solving (3.26). Then the interior superconvergence points in cell τ_j are the zeros of the polynomial $L_{j,k+1}-\alpha_j L_{j,k}$.

3.4. Superconvergence of the DG approximation

We begin with the supercloseness between u_I and the DG solution u_h. Choosing $v=u_h-u_I^l$ in (2.4) and using (3.19), we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h-u_I^l\|_0^2 &\le a(u_h-u_I^l,u_h-u_I^l)=a(u-u_I^l,u_h-u_I^l)\\ &\lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}\|u_h-u_I^l\|_0.\end{aligned}$$

By the Gronwall inequality,

$$\|(u_I^l-u_h)(\cdot,t)\| \lesssim \|(u_I^l-u_h)(\cdot,0)\| + th^{k+l+1}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+l+2,\infty}. \tag{3.33}$$

The above inequality indicates that the initial dicretization should be carefully chosen to ensure the supercloseness between u_h and u_I^l. To obtain the desired superconvergence rate $k+l+1$ for $\|u_h-u_I\|_0$, the initial value $u_h(\cdot,0)$ should satisfy

$$\|(u_I^l-u_h)(\cdot,0)\| \lesssim h^{k+l+1}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+l+1,\infty}. \tag{3.34}$$

As a direct consequence of (3.33) and (3.18), we have the following superconvergence of u_h towards the global projection Pu.

Corollary 3.11. *Let $u\in W^{k+3,\infty}$ and u_h be the solutions of (1.1) and (2.1), respectively. Suppose the initial discretization is taken as $u_h(\cdot,0)=Pu_0(\cdot,0)$. Then*

$$\|u_h-Pu\|_0 \lesssim th^{k+2}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+3,\infty}. \tag{3.35}$$

3.4.1. Superconvergence at the numerical flux and for the cell-average

We denote by $e_{u,f}$ and $e_{u,c}$ the errors of $u-u_h$ at the numerical flux and for the cell-average, respectively. That is,

$$e_{u,f}=\left(\frac{1}{N}\sum_{j=1}^N\left((u-\hat{u}_h)(x_{j+\frac{1}{2}},t)\right)^2\right)^{\frac{1}{2}},\quad e_{u,c}=\left(\frac{1}{N}\sum_{j=1}^N\Big(\frac{1}{h_j}\int_{\tau_j}(u-u_h)(x)\mathrm{d}x\Big)^2\right)^{\frac{1}{2}}.$$

We have the following superconvergence results.

Theorem 3.12. *Suppose $u \in W^{2k+2,\infty}(\Omega)$ and $u_I = u_I^k$ is the specially constructed interpolation function defined in (3.17) and (3.16). Let u_h be the solution of (2.1) with the initial value chosen such that (3.34) satisfied with $l = k$. Then*

$$(e_{u,f} + e_{u,c}) \lesssim (1+t)h^{2k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{2k+2,\infty}. \tag{3.36}$$

Proof. Let $w = w^k$. By (3.11) and (3.16), we have

$$\int_{\tau_j} w_i = 0, \quad \forall i \le k-1.$$

Then

$$\int_{\tau_j} (u-u_h)\mathrm{d}x = \int_{\tau_j} (Pu-u_h)\mathrm{d}x = \int_{\tau_j} (u_I - u_h + w)\mathrm{d}x = \int_{\tau_j} (u_I - u_h + w_k)\mathrm{d}x.$$

Consequently, we obtain from Cauchy–Schwartz inequality and (3.18)

$$\left|\int_{\tau_j} (u-u_h)(x,t)\mathrm{d}x\right| \lesssim h^{\frac{1}{2}} \|u_I - u_h\|_{0,\tau_j} + h^{2k+2}\|u\|_{2k+1,\infty}.$$

Then a direct calculation yields

$$e_{u,c} \lesssim \|u_I - u_h\|_0 + h^{2k+1}\|u\|_{2k+1,\infty}.$$

On the other hand, by (3.31) and the inverse inequality,

$$\begin{aligned} e_{u,f} &= \left(\frac{1}{N}\sum_{j=1}^{N} \left(\hat{u}_I - \hat{u}_h\right)^2 \left(x_{j+\frac{1}{2}}, t\right)\right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{1}{N}\sum_{j=1}^{N} \|u_I - u_h\|^2_{0,\infty,\tau_j}\right)^{\frac{1}{2}} \lesssim \|u_I - u_h\|_0. \end{aligned}$$

Thanks to the special discretization and (3.33), we obtain

$$\|u_I - u_h\|_0 \lesssim (1+t)h^{2k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{2k+2,\infty}.$$

Then the desired result follows. □

3.4.2. Superconvergence at generalized Radau points

To study the supercovergence of DG solution at generalized Radau points, we first denote $e_{u,r}$ and $e_{u,d}$ the maximum error of $u - u_h$ at the generalized right Radau points $R^r_{j,l}, l \in \mathbb{Z}_{k+1}$ and generalized derivative right Radau points $R^d_{j,m}, m \in \mathbb{Z}_k$, respectively. Here $R^r_{j,l}$ and $R^d_{j,m}$ are the same as in Lemma 3.5. To be more precise,

$$e_{u,r} = \max_{j,l} |(u-u_h)(R^r_{j,l})|, \quad e_{u,d} = \max_{j,m} |(u-u_h)(R^d_{j,m})|.$$

Theorem 3.13. *Let $u \in W^{k+4,\infty}$ and u_h be the solution of (1.1) and (2.1), respectively. Suppose u_I is defined by (3.17), (3.16), and the initial discretization is chosen such that (3.34) is satisfied with $l = 2$. Then for all $(j,l,m) \in \mathbb{Z}_N \times \mathbb{Z}_{k+1} \times \mathbb{Z}_k$*

$$e_{u,r} \lesssim (1+t)h^{k+2} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+4,\infty}, \quad e_{u,d} \lesssim (1+t)h^{k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+4,\infty}. \tag{3.37}$$

Proof. Let $u_I = u_I^2$. Choosing $l = 2$ in (3.33) and (3.34), we obtain

$$\|u_h - u_I\|_0 \lesssim (1+t)h^{k+3} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+4,\infty}.$$

By the inverse inequality,

$$\|u_h - u_I\|_{0,\infty} \lesssim h^{-\frac{1}{2}}\|u_h - u_I\|_0 \lesssim (1+t)h^{k+\frac{5}{2}} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+4,\infty},$$
$$\|u_h - u_I\|_{1,\infty} \lesssim h^{-1}\|u_h - u_I\|_{0,\infty} \lesssim (1+t)h^{k+\frac{3}{2}} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+4,\infty}.$$

Then (3.37) follows from (3.32) and the triangle inequality. □

3.5. Initial dicretization

As we may observe, all our superconvergence results are based on the special initial condition (3.34). To obtain (3.34), a nature way of initial discretization is to choose $u_h(\cdot,0) = u_I^l(\cdot,0)$. Now we demonstrate how to calculate $u_I^l(x,0), 1 \le l \le k$ by u_0. Since $u_t + u_x = 0$, we have for all integers $i \ge 1$,

$$\partial_t^i u(x,0) = (-1)^i \partial_x^i u_0(x), \quad \partial_t^i Pu(x,0) = (-1)^i P\partial_x^i u_0(x).$$

By (3.15),

$$\partial_t^i \tilde{u}_{j,k+1}(0) = \frac{2k+1}{h_j}\int_{\tau_j} \partial_t^i (u - Pu)(\cdot,0)L_{j,k} = \frac{2k+1}{h_j}(-1)^i \int_{\tau_j} (\partial_x^i u_0 - P\partial_x^i u_0)L_{j,k}. \tag{3.38}$$

Now we divide the process into the following steps:

1. Calculate the global projection $(-1)^i P\partial_x^i u_0, i \in \mathbb{Z}_l$ by (3.5) and (3.6).
2. Compute $\partial_t^i \tilde{u}_{j,k+1}$ and $F_i, i \in \mathbb{Z}_l$ from (3.38) and (3.9), respectively.
3. Choose $w_i = (\overline{h}_j)^i F_i \partial_t^i \tilde{u}_{j,k+1}$ and $w^l = \sum_{i=1}^{l} w_i$.
4. Figure out $u_I^l(\cdot,0) = Pu_0 - w^l$.

4. Analysis for the Dirichlet boundary condition

The basic idea of the superconverngence analysis for Dirichlet boundary condition is similar to that for the periodic case, while it is easier than the latter.

We first modify the global projection Pu as follows.

$$(\tilde{P}u, v)_j = (u,v)_j, \quad \forall v \in \mathbb{P}_{k-1}(\tau_j), \tag{4.1}$$
$$\widehat{\tilde{P}u}\big|_{j+\frac{1}{2}} = (\theta \tilde{P}u(x^-_{j+\frac{1}{2}}) + (1-\theta)\tilde{P}u(x^+_{j+\frac{1}{2}})) = \hat{u}_{j+\frac{1}{2}}, \quad j \in \mathbb{Z}_{N-1}, \tag{4.2}$$
$$(\tilde{P}u)^-_{N+\frac{1}{2}} = u^-_{N+\frac{1}{2}} \tag{4.3}$$

with $\frac{1}{2} < \theta \le 1$. Noticing that when $\theta = 1$, the projection $\tilde{P}u$ is reduced to the Gauss–Radau projection $P_h^- u$.

The existence and optimal approximation property of $\tilde{P}u$ has been proved in [17]. Since $\tilde{P}u$ can be expressed explicitly, we can study its superconvergent approximation property directly.

Lemma 4.1. *Let $u \in W^{k+2,\infty}$ and u have the Legendre expansion (3.2) in each element τ_j. Suppose $\tilde{P}u$ is the projection of u defined in (4.1)–(4.3). Then for all $(l,m) \in \mathbb{Z}_{k+1} \times \mathbb{Z}_k$*

$$|(u-\tilde{P}u)(R^r_{j,l})| \lesssim h^{k+2}\|u\|_{k+2,\infty}, \quad |\partial_x(u-\tilde{P}u)(R^d_{j,m})| \lesssim h^{k+1}\|u\|_{k+2,\infty}, \tag{4.4}$$

where $R^r_{j,l}$ and $R^d_{j,l}$'s are zeros of $L_{j,k+1}-\alpha_j L_{j,k}$ and $\partial_x(L_{j,k+1}-\alpha_j L_{j,k})$ with

$$\alpha_N = 1, \quad \alpha_j = 1 + (-1)^{k+1}\frac{(1-\theta)u_{j+1,k+1}}{\theta u_{j,k+1}}(1+\alpha_{j+1}), \quad j \in \mathbb{Z}_{N-1}. \tag{4.5}$$

Proof. To prove (4.4), it is sufficiently to show

$$(u-\tilde{P}u)\big|_{\tau_j} = u_{j,k+1}(L_{j,k+1}-\alpha_j L_{j,k}) + O(h^{k+2}). \tag{4.6}$$

We will prove (4.6) by induction. First, by the definition of $\tilde{P}u$,

$$\tilde{P}u\big|_{\tau_N} = P_h^- u.$$

Consequently (see, [5]),

$$u - \tilde{P}u = u_{N,k+1}(L_{N,k+1}-L_{N,k}) + \sum_{m=k+2}^{\infty} u_{N,m}(L_{N,m}-L_{N,k}).$$

In light of the estimate in (3.4), (4.4) is valid for $j = N$ with $\alpha_N = 1$. Now we consider $j \le N-1$. We suppose (4.6) is valid for $j+1$. By (4.1)–(4.3), we obtain

$$\tilde{P}u\big|_{\tau_j} = \sum_{m=0}^{k-1} u_{j,m}L_{j,m} + \bar{u}_{j,k}L_{j,k}$$

with $\bar{u}_{j,k}$ some constant to be determined. Using the condition (4.2), we derive

$$\begin{aligned}
\theta\bar{u}_{j,k} &= \theta\sum_{m=k}^{\infty} u_{j,m} + (1-\theta)\left(u\left(x_{j+\frac12}\right) - \tilde{P}u\left(x^+_{j+\frac12}\right)\right)\\
&= \theta\sum_{m=k}^{\infty} u_{j,m} + (1-\theta)u_{j+1,k+1}\left(L_{j+1,k+1}-\alpha_{j+1}L_{j+1,k}\right)\left(x^+_{j+\frac12}\right) + O\left(h^{k+2}\right)\\
&= \theta u_{j,k} + \theta u_{j,k+1} + (1-\theta)u_{j+1,k+1}(1+\alpha_{j+1})(-1)^{k+1} + O\left(h^{k+2}\right).
\end{aligned}$$

Then for all $j \in \mathbb{Z}_{N-1}$,

$$\begin{aligned}
(u-\tilde{P}u)|_{\tau_j} &= u_{j,k+1}(L_{j,k+1} - \left(1+(-1)^{k+1}\frac{(1-\theta)u_{j+1,k+1}}{\theta u_{j,k+1}}(1+\alpha_{j+1})\right)L_{j,k}) + O(h^{k+2})\\
&= u_{j,k+1}(L_{j,k+1}-\alpha_j L_{j,k}) + O(h^{k+2}).
\end{aligned}$$

Consequently, (4.6) also holds true for $j-1$, which indicates that (4.6) is valid for all $j \in \mathbb{Z}_N$. Then (4.4) follows. ☐

The correction function for the Dirichlet boundary condition is similar to that for the periodic case. We define

$$F_0(x)\big|_{\tau_j} = L_{j,k}(x), \quad \tilde{F}_i = \tilde{P}D^{-1}F_{i-1}, \quad \forall i \in \mathbb{Z}_k, \tag{4.7}$$

and suppose

$$(u - \tilde{P}u)\big|_{\tau_j} = \bar{u}_{j,k} L_{j,k} + \sum_{m=k+1}^{\infty} u_{j,m} L_{j,m},$$

where $\bar{u}_{j,k} = \frac{2k+1}{h_j}(u - \tilde{P}u, L_{j,k})_j$ and $u_{j,m}$ is the same as in (3.2). Then we define by the correction function w

$$w(x,t) = w^l(x,t) = \sum_{i=1}^{l} w_i(x,t), \quad w_i(x,t)|_{\tau_j} = (\bar{h}_j)^i \partial_t^i \bar{u}_{j,k}(t) \tilde{F}_i(x),$$

and the interpolation function

$$u_I = u_I^l = \tilde{P}u - w^l. \tag{4.8}$$

In this case, (3.31) is still valid with the numerical flux chosen as (2.3). Then we take $v = u_I^l - u_h$ in (2.5) and follow the same argument as in Theorem 3.4 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_I^l - u_h\|_0^2 \lesssim \left|a(u - u_I^l, u_I^l - u_h)\right| \lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}\|u_I^l - u_h\|_0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u_I^l - u_h)_t\|_0^2 \lesssim \left|a((u - u_I^l)_t, (u_I^l - u_h)_t)\right| \lesssim h^{k+l+1}\|u\|_{k+l+3,\infty}\|(u_I^l - u_h)_t\|_0. \tag{4.9}$$

By the same argument as the periodic case, we can also prove that all the superconvergence results for the errors at numerical fluxes, of the cell-average, and at the generalized right Radau points with α_j given by (4.5), are valid in the Dirichlet boundary condition case. Furthermore, we have the following point-wise error estimate for the error at the numerical flux.

Theorem 4.2. *Suppose $u \in W^{2k+3,\infty}(\Omega)$ and $u_I = u_I^k$ is the specially constructed interpolation defined by (4.8). Let u_h be the solution of (2.1) with the initial value $u_h(\cdot,0) = u_I(\cdot,0)$. Then for flux (2.3),*

$$e_{u,f'} = \max_{j\in\mathbb{Z}_N}(u - \hat{u}_h)(x_{j+\frac{1}{2}}, t) \lesssim (1+t)h^{2k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{2k+3,\infty}. \tag{4.10}$$

Proof. First, we denote

$$e = u - u_h, \quad \xi = u_I - u_h = \tilde{P}u - w^k - u_h.$$

Choosing $v = 1$ in the equation $a_i(e,v) = 0$, we have

$$\hat{e}^-_{i+\frac{1}{2}} - \hat{e}^-_{i-\frac{1}{2}} = -\int_{\tau_i} e_t = -\int_{\tau_i} (\xi + w^k)_t = -\int_{\tau_i} (\xi + w_k)_t.$$

Here in the last step, we have used the fact that $(u - \tilde{P}u) \perp \mathbb{P}_0$ and the orthogonal property of $w_i, i \le k-1$. Summing up all i from 1 to j yields

$$\hat{e}_{j+\frac{1}{2}} - \hat{e}_{\frac{1}{2}} = -\sum_{i=1}^{j} \int_{\tau_i} (\xi_t + w_{kt}). \tag{4.11}$$

Note that

$$\hat{e}_{\frac{1}{2}} = e^-_{\frac{1}{2}} = 0, \quad \|w_{kt}\|_{0,\infty,\tau_j} \lesssim h^{2k+1}\|u\|_{2k+2,\infty}.$$

Then

$$\hat{e}_{j+\frac{1}{2}} \lesssim \|\xi_t\|_0 + h^{2k+1}\|u\|_{2k+2,\infty}.$$

Now we turn to estimating $\|\xi_t\|_0$. By (4.9), we easily obtain

$$\|\xi_t(\cdot,t)\|_0 = \|\xi_t(\cdot,0)\|_0 + th^{2k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{2k+2,\infty}.$$

At the initial time $t = 0$, due to the special initial discretization $u_h(x,0) = u_I(x,0)$,

$$\begin{aligned} 0 = a(e_t, v)\big|_{t=0} &= a((u-u_I)_t, v)\big|_{t=0} + a(\xi_t, v)\big|_{t=0} \\ &= a((u-u_I)_t, v)\big|_{t=0} + (\xi_t, v)\big|_{t=0}, \quad \forall v \in V_h. \end{aligned}$$

Choosing $v = \xi_t$ in the above identity and $l = k$ in the second inequality of (4.9), we get

$$\|\xi_t(\cdot,0)\|_0 \lesssim h^{2k+1}\|u_0\|_{2k+3,\infty}.$$

Consequently,

$$\|\xi_t\|_0 \lesssim (1+t)h^{2k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{2k+3,\infty}.$$

Then (4.10) follows. □

Remark 4.3. As we observe from Theorems 3.12 and 4.2, the regularity imposed on u for the point-wise error estimate is stronger (one order higher) than that for the average errors at numerical fluxes. The reason is that we have used the superconvergence result $\|(u_I - u_h)_t\|_0$ in our analysis instead of the estimate for $\|u_I - u_h\|_0$.

5. Numerical experiments

In this section, we use numerical examples to verify the theorems in Sections 3.4 and 4. If not otherwise stated, the initial discretization is given by the same way as in Section 3.5. Since all previous numerical tests in the literature (see, *e.g.*, [19]) are performed for lower order polynomials, *e.g.*, $k = 1$ and $k = 2$, in order not to repeat, we only provide data for $k = 3$ and $k = 4$ in our numerical experiments. Moreover, in this section, we take $\theta = 0.9$ in the numerical experiments.

Example 1. We solve the following problem

$$\begin{aligned} &u_t + u_x = 0, && (x,t) \in [0,2\pi] \times (0, 0.1], \\ &u(x,0) = \sin(x), && x \in [0, 2\pi], \end{aligned} \tag{5.1}$$

with the periodic boundary condition

$$u(0,t) = u(2\pi, t).$$

Clearly, the exact solution is

$$u(x,t) = \sin(x-t).$$

We use the ninth order SSP Runge–Kutta discretization in time [15] and take $\Delta t = 0.01h_{\min}$ to reduce the time error. Non-uniform meshes of n cells are obtained by randomly and independently perturbing each node in the x axes of a uniform mesh by up to 20%. The example is tested by using $\mathbb{P}_k$ polynomials with $k = 3, 4$. We compute the numerical solution at $t = 0.1$. In Table 1, we compute several errors between the numerical approximation and the exact solution, which are given in Theorems 3.12, 3.13, and 4.2.

Table 1 demonstrates superconvergence rates of $(2k+1)$th order for the numerical cell average and numerical flux ($e_{u,c}$ and $e_{u,f}$), $(k+2)$th order for the numerical solution at the generalized right Radau points ($e_{u,r}$). Moreover, the derivative of the error is $(k+1)$th order superconvergent at the generalized derivative right Radau points ($e_{u,d}$). All convergent rates in Table 1 match our theoretical error bounds in Theorems 3.12 and 3.13.

TABLE 1. Various errors with periodic boundary condition for $k = 3, 4$, $t = 0.1$.

		$e_{u,c}$		$e_{u,f}$		$e_{u,r}$		$e_{u,d}$	
k	n	error	order	error	order	error	order	error	order
3	40	9.70e-13	–	3.25e-12	–	4.97e-08	–	1.75e-06	–
	80	7.97e-15	6.93	3.31e-14	6.62	1.85e-09	4.74	1.27e-07	3.78
	160	5.40e-17	7.21	2.13e-16	7.28	7.72e-11	4.58	9.19e-09	3.79
	320	5.40e-19	6.64	1.80e-18	6.88	2.91e-12	4.73	6.53e-10	3.81
	640	3.85e-21	7.13	1.40e-20	7.01	8.93e-14	5.03	4.26e-10	3.94
4	40	9.04e-17	–	2.55e-16	–	1.08e-10	–	1.00e-08	–
	80	1.97e-19	8.84	4.36e-19	9.19	1.60e-12	6.08	3.30e-10	4.92
	160	3.21e-22	9.26	8.85e-22	8.94	2.82e-14	5.83	8.48e-12	5.28
	320	8.69e-25	8.53	1.98e-24	8.81	4.34e-16	6.02	3.13e-13	4.76
	640	1.48e-27	9.19	3.50e-27	9.14	6.80e-18	6.00	8.86e-15	5.14

TABLE 2. Various errors with periodic boundary condition for $k = 3, 4$, $t = 2\pi$.

		$e_{u,c}$		$e_{u,f}$		$e_{u,r}$		$e_{u,d}$	
k	n	error	order	error	order	error	order	error	order
3	40	1.13e-11	–	2.33e-11	–	4.69e-08	–	1.68e-06	–
	80	8.37e-14	7.08	1.70e-13	7.09	1.86e-09	4.66	1.27e-07	3.72
	160	6.49e-16	7.01	1.32e-15	7.01	8.16e-11	4.51	9.71e-09	3.71
	320	5.15e-18	6.98	1.04e-17	6.98	3.12e-12	4.71	6.99e-10	3.80
	640	4.05e-20	6.99	8.19e-20	6.99	9.02e-14	5.11	4.30e-11	4.02
4	40	7.11e-16	–	1.46e-15	–	1.07e-10	–	9.90e-09	–
	80	1.21e-18	9.20	2.46e-18	9.21	1.58e-12	6.09	3.29e-10	4.91
	160	2.32e-21	9.03	4.72e-21	9.03	2.85e-14	5.79	8.36e-12	5.30
	320	4.72e-24	8.94	9.57e-24	8.95	4.25e-16	6.07	3.07e-13	4.77
	640	9.29e-27	8.99	1.88e-26	8.99	6.80e-18	5.97	8.84e-15	5.12

Moreover, we also choose the final time $t = 2\pi$, and the results are given in Table 2. Similar superconvergence rates can be observed.

Example 2. We solve the following problem

$$\begin{aligned} &u_t + u_x = 0, && (x,t) \in [0, 2\pi] \times (0, 0.1], \\ &u(x,0) = \sin(x), && x \in [0, 2\pi], \end{aligned} \tag{5.2}$$

with the Dirichlet boundary condition

$$u(0,t) = \sin(-t).$$

Clearly, the exact solution is

$$u(x,t) = \sin(x - t).$$

We use the fourth order SSP multi-step discretization in time [15] and take $\Delta t = 0.1h_{\min}^{2.25}$ to reduce the time error. The same quantities as in Example 1 on the same kind of random meshes of n cells are computed. The example is tested by using $\mathbb{P}_k$ polynomials with $k = 3, 4$. We compute the numerical solution at $t = 0.1$. The computational results are given in Table 3.

From Table 3, we observe similar phenomena as in Example 1, especially, all convergent rates match those predicted by our theory. In this sense, our theoretical error bounds are sharp.

TABLE 3. Various errors with Dirichlet boundary condition for $k = 3, 4$.

		$e_{u,c}$		$e_{u,f'}$		$e_{u,r}$		$e_{u,d}$	
k	n	error	order	error	order	error	order	error	order
3	40	2.96e-12	–	1.00e-11	–	4.97e-08	–	1.75e-06	–
	80	2.10e-14	7.14	7.95e-14	6.98	1.85e-09	4.75	1.27e-07	3.78
	160	2.01e-16	6.71	6.35e-16	6.97	7.72e-11	4.58	9.19e-09	3.79
	320	2.06e-18	6.60	4.74e-18	7.07	2.91e-12	4.73	6.53e-10	3.81
4	40	1.30e-14	–	3.73e-14	–	1.08e-10	–	1.00e-08	–
	80	2.72e-17	8.90	5.30e-17	9.46	1.60e-12	6.08	3.30e-10	4.92
	160	5.08e-20	9.06	9.88e-20	9.07	2.82e-14	5.83	8.48e-12	5.28
	320	1.02e-22	8.96	1.99e-22	8.96	4.34e-16	6.02	3.13e-13	4.76

6. Concluding remarks

We have studied the superconvergence behavior of the DG solution for linear 1D hyperbolic equations using upwind-biased fluxes. We prove that, with suitable initial discretization, the error between the DG solution and the exact solution converges with the rate of $(2k+1)$th order (comparing with the standard optimal global rate of $(k+1)$th order) for the cell averages and the numerical fluxes, and with rate of $(k+2)$th order at all interior generalized right Radau points. Moreover, we prove that the derivative of the error converges with the rate of $(k+1)$th order at the generalized derivative right Radau points (comparing with the standard optimal global rate of kth order). Numerical experiments demonstrate that all aforementioned error bounds are sharp.

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