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MIXED FINITE ELEMENT METHODS FOR LINEAR ELASTICITY AND THE STOKES EQUATIONS BASED ON THE HELMHOLTZ DECOMPOSITION *

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Abstract. This paper introduces new mixed finite element methods (FEMs) of degree ≥1 for the equations of linear elasticity and the Stokes equations based on Helmholtz decompositions. These FEMs are robust with respect to the incompressible limit and also allow for mixed boundary conditions. Adaptive algorithms driven by efficient and reliable residual-based error estimators are introduced and proved to converge with optimal rate in the case of the Stokes equations with pure Dirichlet boundary.

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1. Introduction

Given the simply connected, bounded, polygonal Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ with closed Dirichlet boundary Γ_D and Neumann boundary $\Gamma_N = \partial\Omega \setminus \Gamma_D$ and the volume force $f \in L^2(\Omega;\mathbb{R}^2)$, the Navier−Lamé equations of linear elasticity seek the displacement $u \in H^1(\Omega;\mathbb{R}^2)$ and the stress $\sigma \in H(\operatorname{div},\Omega;\mathbb{R}^{2\times 2})$ with

$$\begin{aligned} -\operatorname{div}\sigma &= f && \text{in } \Omega,\\ \sigma &= \mathbb{C}\varepsilon(u) && \text{in } \Omega,\\ (\sigma\nu)|_{\Gamma_N} &= 0 && \text{on } \Gamma_N,\\ u &= 0 && \text{on } \Gamma_D. \end{aligned} \tag{1.1}$$

The linear Green strain is defined by $\varepsilon(u) := (Du + Du^\top)/2$ and the fourth-order elasticity tensor acts as $\mathbb{C}A = 2\mu A + \lambda \operatorname{tr}(A) I_{2\times 2}$ for Lamé parameters $\mu > 0$ and $\lambda > 0$. In contrast to the Poisson problem, discretizations of these equations have to deal with two difficulties. First, for almost incompressible materials, *i.e.*, $\lambda \to \infty$, standard primal low order conforming finite element methods (FEMs) exhibit the so-called locking behaviour, *i.e.*, the approximation properties in the energy norm suffer from a large preasymptotic regime. Second, the symmetry of the stress excludes low order ansatz spaces for mixed FEMs and leads to ansatz spaces in the Arnold–Winther FEM [2, 4] of polynomial degree ≥3 in two dimensions and ≥4 in three space dimensions,

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while the approach in the PEERS FEM [1] is to enforce the symmetry in a week sense. The symmetry of the stress also restricts the non-conforming FEM of [12] based on the Crouzeix–Raviart finite element space [22] to Dirichlet boundary conditions due to the absence of a discrete version of Korn's inequality.

This paper introduces a new formulation and its discretizations for the Navier−Lamé equations based on a Helmholtz decomposition of [16]. The discretizations are proved to be robust in the incompressible limit $\lambda \to \infty$. The formulation of this paper incorporates the symmetry of the stress through a saddle-point formulation and thus also allows for mixed boundary conditions. A reliable and efficient residual based error estimator is suggested for an adaptive algorithm. Numerical experiments show its quasi-optimal convergence. The discretizations also include a generalization of the non-conforming FEM of Kouhia and Stenberg [29] to higher polynomial degrees.

For the Poisson problem, a similar approach generalizes the non-conforming FEMs of Crouzeix–Raviart [22] and Morley [31] to higher polynomial degrees [34–36]. The discretization proposed in this paper does not generalize the non-conforming FEM of [12]. In fact, the discretization of this paper also allows for Neumann boundary conditions, while the non-conforming FEM of [12] is only suitable for the pure Dirichlet problem.

The formal limit for $\lambda \to \infty$ of the equations of linear elasticity are the Stokes equations

$$-\operatorname{div} \varepsilon(u) + \nabla p = f \quad \text{and} \quad \operatorname{div} u = 0 \qquad \text{in } \Omega.$$

The new discretizations for linear elasticity carry over to these equations and lead to approximations of $\varepsilon(u)$ with local mass conservation. For the pure Dirichlet problem, the Stokes equations simplify to

$$-\Delta u + \nabla p = f \quad \text{and} \quad \operatorname{div} u = 0 \qquad \text{in } \Omega.$$

A discretization for these equations based on a Helmholtz decomposition generalizes the non-conforming FEM of Crouzeix and Raviart [22] to higher polynomial degrees with local mass conservation. An adaptive algorithm suggested in Section 8 converges with optimal rates. This is the first proof of optimal convergence rates for a higher-order FEM for the Stokes equations. Indeed the existing proofs are restricted to the first-order non-conforming FEM of Crouzeix and Raviart [6, 20, 28, 30] and to a mixed Pseudostress approach [19].

The idea of the new method is as follows. The paper assumes that a function $\varphi \in H(\operatorname{div}, \Omega; \mathbb{R}^{2\times 2})$ is at hand which satisfies $-\operatorname{div} \varphi = f$ (plus some boundary conditions for the Neumann problem and symmetry for the linear elasticity problem and the Stokes equations with symmetric gradient). At least for the pure Dirichlet problem, this function can be defined by a pure integration. A Helmholtz-type decomposition then decomposes φ in an admissible stress part and a Curl part,

$$\varphi = \mathbb{C}\varepsilon(\widetilde{u}) + \operatorname{Curl} \alpha.$$

Since $-\operatorname{div} \operatorname{Curl} \alpha = 0$, the stress $\sigma = \mathbb{C}\varepsilon(\widetilde{u})$ also satisfies the equilibrium equation and, hence, $(\widetilde{u}, \sigma)$ is the solution of (1.1). While σ is determined by f only, the function α depends on the particular choice of φ.

After some preliminary notation and remarks in Section 2, Sections 3 and 4 introduce the novel weak formulation for the Navier−Lamé equations based on a Helmholtz decomposition from [16] and its discretizations. Section 5 is devoted to the *a posteriori* error analysis of the discretizations. Section 6 discusses the approximation of the symmetric part of the gradient in the Stokes problem with Neumann boundary conditions. Section 7 introduces the new approximation of the Stokes equations with pure Dirichlet boundary conditions, and Section 8 introduces an adaptive algorithm and discusses its optimal convergence rates. Section 9 concludes the paper with numerical experiments.

2. Preliminaries

Notation. Throughout this paper $\Omega \subseteq \mathbb{R}^2$ is a simply connected, bounded, polygonal Lipschitz domain. Standard notation on Lebesgue and Sobolev spaces and their norms is employed with L^2 scalar product $(\bullet, \bullet)_{L^2(\Omega)}$. Given a Hilbert space X, let $L^2(\Omega; X)$ resp. $H^k(\Omega; X)$ denote the space of functions with values in X whose components are in $L^2(\Omega)$ resp. $H^k(\Omega)$ and let $L^2_0(\Omega)$ denote the subset of $L^2(\Omega)$ of functions with vanishing integral mean. Let $H^1_\Gamma(\Omega;X)$ denote the subspace of $H^1(\Omega;X)$ of functions with vanishing trace on $\Gamma\subseteq\partial\Omega$ and let $H^1_0(\Omega;X):=H^1_{\partial\Omega}(\Omega;X)$ and $H^1_\Gamma(\Omega):=H^1_\Gamma(\Omega;\mathbb{R})$ and $H^1_0(\Omega):=H^1_0(\Omega;\mathbb{R})$. The space of L^2 functions whose weak divergence exists and is in L^2 is denoted with $H(\mathrm{div},\Omega)$ and $H(\mathrm{div},\Omega;X)$ is the space of functions with values in X, whose rows are in $H(\mathrm{div},\Omega)$. The space of infinitely differentiable functions reads $C^\infty(\Omega)$ and the subspace of functions with compact support in Ω is denoted with $C^\infty_c(\Omega)$. The piecewise action of differential operators is denoted with a subscript NC. The formulae $A\lesssim B$ and $B\gtrsim A$ represent an inequality $A\le CB$ for some mesh-size independent, positive generic constant C; $A\approx B$ abbreviates $A\lesssim B\lesssim A$. By convention, all generic constants $C\approx 1$ do neither depend on the mesh-size nor on the level of a triangulation nor on the Lamé parameter $\lambda\gtrsim 1$ but may depend on the fixed coarse triangulation $\mathcal{T}_0$ and its interior angles. The Curl operator in two dimensions is defined by $\mathrm{Curl}\,\beta:=(\partial\beta/\partial x_2,-\partial\beta/\partial x_1)$ for sufficiently smooth β, while $\mathrm{curl}\,\beta:=-\partial\beta_1/\partial x_2+\partial\beta_2/\partial x_1$ for $\beta:\Omega\to\mathbb{R}^2$.

A shape-regular triangulation $\mathcal{T}$ of $\Omega\subseteq\mathbb{R}^2$ is a set of closed triangles $T\in\mathcal{T}$ such that $\overline{\Omega}=\bigcup\mathcal{T}$ and any two distinct triangles are either disjoint or share exactly one common edge or one vertex. Let $\mathcal{E}(T)$ denote the edges of a triangle T and $\mathcal{E}:=\mathcal{E}(\mathcal{T}):=\bigcup_{T\in\mathcal{T}}\mathcal{E}(T)$ the set of edges in $\mathcal{T}$. Any edge $E\in\mathcal{E}$ is associated with a fixed orientation of the unit normal ν_E on E (and $\tau_E=(0,-1;1,0)\nu_E$ denotes the unit tangent on E). On the boundary, ν_E is the outer unit normal of Ω, while for interior edges $E\not\subseteq\partial\Omega$, the orientation is fixed through the choice of the triangles $T_+\in\mathcal{T}$ and $T_-\in\mathcal{T}$ with $E=T_+\cap T_-$ and $\nu_E:=\nu_{T_+}|_E$ is then the outer normal of T_+ on E. In this situation, $[v]_E:=v|_{T_+}-v|_{T_-}$ denotes the jump across E. For an edge $E\subseteq\partial\Omega$ on the boundary, the jump across E reads $[v]_E:=v$. For $T\in\mathcal{T}$ and $X\subseteq\mathbb{R}^n$, let

$$P_k(T;X):=\left\{v:T\to X\;\middle|\;\begin{array}{l}\text{each component of } v \text{ is a polynomial}\\ \text{of total degree}\le k\end{array}\right\};$$
$$P_k(\mathcal{T};X):=\{v:\Omega\to X\mid\forall T\in\mathcal{T}:\ v|_T\in P_k(T;X)\}$$

denote the set of piecewise polynomials and $P_k(\mathcal{T}):=P_k(\mathcal{T};\mathbb{R})$. Given a subspace $X\subseteq L^2(\Omega;\mathbb{R}^n)$, let $\Pi_X:L^2(\Omega;\mathbb{R}^n)\to X$ denote the L^2 projection onto X and let Π_k abbreviate $\Pi_{P_k(\mathcal{T};\mathbb{R}^n)}$. Given a triangle $T\in\mathcal{T}$, let $h_T:=(\mathrm{meas}_2(T))^{1/2}$ denote the square root of the area of T and let $h_\mathcal{T}\in P_0(\mathcal{T})$ denote the piecewise constant mesh-size with $h_\mathcal{T}|_T:=h_T$ for all $T\in\mathcal{T}$. For a set of triangles $\mathcal{M}\subseteq\mathcal{T}$, let $\|\bullet\|_\mathcal{M}$ abbreviate

$$\|\bullet\|_\mathcal{M}:=\sqrt{\sum_{T\in\mathcal{M}}\|\bullet\|^2_{L^2(T)}}.$$

Given an initial triangulation $\mathcal{T}_0$, an admissible triangulation is a regular triangulation which can be created from $\mathcal{T}_0$ by newest-vertex bisection [38]. The set of admissible triangulations is denoted by $\mathbb{T}$.

Let $\mathbb{R}^{2\times2}_{\mathrm{dev}}$ denote the 2×2 matrices with vanishing trace, *i.e.*,

$$\mathbb{R}^{2\times2}_{\mathrm{dev}}:=\{A\in\mathbb{R}^{2\times2}\mid\mathrm{tr}(A)=0\}\tag{2.1}$$

and define the deviatoric part $\mathrm{dev}A$ of $A\in\mathbb{R}^{2\times2}$ as $\mathrm{dev}A=A-(1/2)\mathrm{tr}(A)I_{2\times2}$. The following proposition proves a lower bound for the L^2 norm of the deviatoric part of Curls.

Proposition 2.1. *Any $\beta\in H^1(\Omega;\mathbb{R}^2)$ with $\int_\Omega\beta\,\mathrm{d}x=0$ and $\int_\Omega\mathrm{curl}\,\beta\,\mathrm{d}x=0$ satisfies*

$$\|\mathrm{Curl}\,\beta\|_{L^2(\Omega)}\lesssim\|\mathrm{dev}\,\mathrm{Curl}\,\beta\|_{L^2(\Omega)}.$$

Proof. Since

$$\int_\Omega\mathrm{tr}(\mathrm{Curl}\,\beta)\,\mathrm{d}x=\int_\Omega\mathrm{curl}\,\beta\,\mathrm{d}x=0,$$

the tr-dev-div lemma ([14], Prop. 3.1 in Sect. IV.3) leads to

$$\|\mathrm{tr}(\mathrm{Curl}\,\beta)\|_{L^2(\Omega)}\lesssim\|\mathrm{dev}\,\mathrm{Curl}\,\beta\|_{L^2(\Omega)}+\|\mathrm{div}\,\mathrm{Curl}\,\beta\|_{L^2(\Omega)}.$$

The orthogonality $\nabla H^1_0(\Omega)\perp_{L^2(\Omega)}\mathrm{Curl}\,H^1(\Omega)$ implies $\mathrm{div}\,\mathrm{Curl}\,\beta=0$. $\square$

3. Weak formulation

For the ease of reading, this and the following two sections consider the pure Dirichlet problem. Remark 3.5 explains how Neumann boundary conditions enter in the new formulation.

Let $\mathbb{S} := \{A \in \mathbb{R}^{2\times 2} \mid A = A^\top\}$ denote the set of symmetric 2×2 matrices and define the spaces

$$\begin{aligned}
\Sigma &:= L^2(\Omega;\mathbb{R}^{2\times 2}),\\
X &:= \left\{ v \in H^1(\Omega;\mathbb{R}^2) \,\middle|\, \int_\Omega v\,\mathrm{d}x = 0 \text{ and } \int_\Omega \operatorname{curl} v\,\mathrm{d}x = 0 \right\},\\
Z &:= \{v \in X \mid \operatorname{Curl} v \in L^2(\Omega;\mathbb{S})\} = \{v \in X \mid \operatorname{div} v = 0\},\\
L^2(\Omega;\mathbb{S})/\mathbb{R} &:= \left\{ \tau \in L^2(\Omega;\mathbb{S}) \,\middle|\, \int_\Omega \operatorname{tr}(\tau)\,\mathrm{d}x = 0 \right\}
\end{aligned} \tag{3.1}$$

and the scalar product $(\sigma,\tau)_{\mathbb{C}^{-1/2}} := (\sigma, \mathbb{C}^{-1}\tau)_{L^2(\Omega)}$ for $\sigma,\tau \in \Sigma$. Note that $\operatorname{Curl} v$ is symmetric if and only if $\operatorname{div} v = 0$.

The following theorem states a Helmholtz decomposition for symmetric vector fields and is proved by [16]. Recall that $\Omega \subseteq \mathbb{R}^2$ is a simply connected, bounded, polygonal Lipschitz domain.

Theorem 3.1 (Helmholtz decomposition for symmetric vector fields)**.** *It holds that*

$$L^2(\Omega;\mathbb{S})/\mathbb{R} = \varepsilon(H^1_0(\Omega;\mathbb{R}^2)) \oplus \operatorname{Curl} Z \tag{3.2}$$

and the sum is orthogonal with respect to the L^2 scalar product, or, equivalently,

$$L^2(\Omega;\mathbb{S})/\mathbb{R} = \mathbb{C}\varepsilon(H^1_0(\Omega;\mathbb{R}^2)) \oplus \operatorname{Curl} Z$$

and the sum is orthogonal with respect to $(\bullet,\bullet)_{\mathbb{C}^{-1/2}}$.

A conforming discretization of Z with piecewise smooth functions would involve the restriction $\operatorname{div}\bullet = 0$ pointwise. This leads to complicated finite element methods as in [27,37]. Therefore, it seems useful to include the divergence-free constraint in the mixed formulation. To this end, define the space

$$Y := L^2_0(\Omega),$$

the bilinear forms

$$\begin{aligned}
a(\tau,\sigma) &:= (\mathbb{C}^{-1}\tau,\sigma)_{L^2(\Omega)} && \text{for all } \tau,\sigma \in \Sigma,\\
b(\tau,\beta) &:= (\mathbb{C}^{-1}\tau,\operatorname{Curl}\beta)_{L^2(\Omega)} && \text{for all } \tau \in \Sigma, \beta \in X,\\
c(\xi,\beta) &:= (\xi,\operatorname{div}\beta)_{L^2(\Omega)} && \text{for all } \xi \in Y, \beta \in X,
\end{aligned} \tag{3.3}$$

and the norm $\|\bullet\|_{\mathbb{C}^{-1/2}} := \sqrt{a(\bullet,\bullet)}$. Let $\varphi \in H(\operatorname{div},\Omega;\mathbb{S})$ with $\int_\Omega \operatorname{tr}\varphi\,\mathrm{d}x = 0$ and $-\operatorname{div}\varphi = f$. Note that the constants hidden in $\lesssim$ (in particular those in the *a priori* and *a posteriori* analysis) do not depend on φ. Consider the problem: Seek $(\sigma,\alpha,\chi) \in \Sigma \times X \times Y$ with

$$\begin{aligned}
a(\tau,\sigma) + b(\tau,\alpha) &= (\varphi,\mathbb{C}^{-1}\tau) && \text{for all } \tau \in \Sigma,\\
b(\sigma,\beta) + c(\chi,\beta) &= 0 && \text{for all } \beta \in X,\\
c(\xi,\alpha) &= 0 && \text{for all } \xi \in Y.
\end{aligned} \tag{3.4}$$

Since $\|\operatorname{Curl}\beta\|_{L^2(\Omega)} = \|D\beta\|_{L^2(\Omega)}$, the inf-sup condition for c,

$$\|\xi\|_{L^2(\Omega)} \lesssim \sup_{\beta \in X\setminus\{0\}} \frac{c(\xi,\beta)}{\|\operatorname{Curl}\beta\|_{L^2(\Omega)}} \qquad \text{for all } \xi \in Y \tag{3.5}$$

is the standard inf-sup condition for the Stokes equations [26].

The following lemma proves an inf-sup condition for b.

Lemma 3.2 (Inf-sup condition for b)**.** *Any $\beta \in X$ satisfies*

$$\|\operatorname{Curl}\beta\|_{L^2(\Omega)} \lesssim \sup_{\tau\in\Sigma\setminus\{0\}} \frac{b(\tau,\beta)}{\|\tau\|_{\mathbb{C}^{-1/2}}}. \tag{3.6}$$

Proof. The crucial point in this inf-sup condition is that the constant hidden in $\lesssim$ is independent of the Lamé parameter λ. Let $\beta \in X$. Proposition 2.1 proves

$$\|\operatorname{Curl}\beta\|_{L^2(\Omega)} \lesssim \|\operatorname{dev}(\operatorname{Curl}\beta)\|_{L^2(\Omega)}.$$

A computation reveals

$$\mathbb{C}^{-1}A = \operatorname{dev}(A)/(2\mu) + \operatorname{tr}(A)I_{2\times 2}/(4(\lambda+\mu)). \tag{3.7}$$

This implies

$$\|\operatorname{Curl}\beta\|_{L^2(\Omega)} \lesssim \|\operatorname{Curl}\beta\|_{\mathbb{C}^{-1/2}}. \tag{3.8}$$

With the choice $\tau := \operatorname{Curl}\beta$, this proves the stated inf-sup condition. □

The following theorem proves the existence of a unique solution to (3.4) and its equivalence with (1.1).

Theorem 3.3. *Problem (3.4) has a unique solution $(\sigma,\alpha,\chi)\in\Sigma\times X\times Y$ and it holds that $\sigma = \mathbb{C}\varepsilon(u)$ for the solution $u \in H^1_0(\Omega;\mathbb{R}^2)$ of (1.1).*

Proof. The inf-sup conditions (3.6) and (3.5), the ellipticity of a (with respect to $\|\bullet\|_{\mathbb{C}^{-1/2}}$), the continuity of a, b, and c and a recursive application of Brezzi's splitting lemma [13] proves the unique existence of a solution $(\sigma,\alpha,\chi)\in\Sigma\times X\times Y$ to (3.4). Since the first equation of (3.4) is tested with all L^2 functions, it holds in fact $\sigma + \operatorname{Curl}\alpha = \varphi$. The third equation of (3.4) yields $\operatorname{div}\alpha = 0$, and therefore $\alpha \in L^2(\Omega;\mathbb{S})$. Hence, $\int_\Omega \operatorname{tr}(\operatorname{Curl}\alpha)\,\mathrm{d}x = \int_\Omega \operatorname{curl}\alpha\,\mathrm{d}x = 0$ implies $\operatorname{Curl}\alpha \in L^2(\Omega;\mathbb{S})/\mathbb{R}$. The assumption $\varphi \in L^2(\Omega;\mathbb{S})/\mathbb{R}$ then leads to $\sigma \in L^2(\Omega;\mathbb{S})/\mathbb{R}$. The Helmholtz decomposition from Theorem 3.1 therefore yield a decomposition of σ in

$$\sigma = \mathbb{C}\varepsilon(\widetilde{u}) + \operatorname{Curl}\gamma$$

for some $\widetilde{u} \in H^1_0(\Omega;\mathbb{R}^2)$ and some $\gamma \in Z$. However, the orthogonality $b(\sigma,\beta)=0$ for all $\beta\in Z$ implies $\gamma = 0$. Therefore, (3.4), the symmetry of φ and $\operatorname{Curl}\alpha$ and $-\operatorname{div}\varphi = f$ imply for all $v \in H^1_0(\Omega;\mathbb{R}^2)$ and $\tau = \mathbb{C}\varepsilon(v)$ that

$$\begin{aligned}(\varepsilon(v),\mathbb{C}\varepsilon(\widetilde{u}))_{L^2(\Omega)} &= a(\tau,\sigma) = (\varphi,\mathbb{C}^{-1}\tau)_{L^2(\Omega)} - b(\tau,\alpha)\\ &= (\varphi,\varepsilon(v))_{L^2(\Omega)} - (\varepsilon(v),\operatorname{Curl}\alpha)_{L^2(\Omega)}\\ &= (f,v)_{L^2(\Omega)} - (Dv,\operatorname{Curl}\alpha)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)}\end{aligned}$$

and, hence, $\widetilde{u}$ solves (1.1). □

Remark 3.4. The meaning of the variable χ is that $\chi = \operatorname{curl} u/2$ for the solution $u \in H^1_0(\Omega;\mathbb{R}^2)$ to (1.1): Since $\operatorname{div}\beta$ determines the antisymmetric part of $\operatorname{Curl}\beta$, the second equality of (3.4) reads

$$\left(\mathbb{C}^{-1}\sigma + \chi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \operatorname{Curl}\beta\right) = 0 \qquad \text{for all } \beta \in X,$$

which is equivalent to the fact that $\mathbb{C}^{-1}\sigma + \chi(0,1;-1,0)$ is a derivative and χ determines its antisymmetric part.

Remark 3.5 (More general boundary conditions)**.** Let $\partial\Omega = \Gamma_D \cup \Gamma_N$ with closed Dirichlet boundary $\Gamma_D \neq \emptyset$ and Neumann boundary $\Gamma_N = \partial\Omega \setminus \Gamma_D \neq \emptyset$. Define

$$\begin{aligned} X_{\Gamma_N} &:= \{v \in H^1(\Omega;\mathbb{R}^2) \mid v \text{ is constant on each connectivity component of } \Gamma_N\},\\ Z_{\Gamma_N} &:= \{v \in X_{\Gamma_N} \mid \operatorname{div} v = 0\}. \end{aligned}$$

Then the Helmholtz decomposition with mixed boundary conditions [16] reads

$$L^2(\Omega;\mathbb{R}^2) = \mathbb{C}\varepsilon H^1_{\Gamma_D}(\Omega;\mathbb{R}^2) \oplus \operatorname{Curl} Z_{\Gamma_N}.$$

Let $\varphi \in H(\operatorname{div},\Omega;\mathbb{S})$ with $-\operatorname{div}\varphi = f$ additionally fulfil $(\varphi\nu)|_{\Gamma_N} = g$. Then the linear elasticity problem with mixed boundary conditions is equivalent to

$$\begin{aligned} a(\tau,\sigma) + b(\tau,\alpha) &= (\varphi, \mathbb{C}^{-1}\tau)_{L^2(\Omega)} && \text{for all } \tau \in \Sigma,\\ b(\sigma,\beta) + c(\chi,\beta) &= (Du_D, \operatorname{Curl}\beta)_{L^2(\Omega)} && \text{for all } \beta \in X_{\Gamma_N},\\ c(\xi,\alpha) &= 0 && \text{for all } \xi \in Y. \end{aligned}$$

4. Discretizations

This section introduces robust discretizations of (3.4). Since the discrete inf-sup condition for the bilinear form c of (3.3) is the same as for a standard Stokes discretization, the following discretizations of X and Y employ well-known Stokes finite elements [9], namely the Mini FEM in Section 4.2, the Taylor−Hood FEM in Section 4.3, and the $P_2\,P_0$ FEM in Section 4.4. Section 4.5 discusses and generalizes the non-conforming FEM of Kouhia and Stenberg [29] in this context.

4.1. Abstract discretizations

Recall the bilinear forms a, b, c from (3.3). Let $\Sigma_h(\mathcal{T}) \subseteq \Sigma$, $X_h(\mathcal{T}) \subseteq X$, and $Y_h(\mathcal{T}) \subseteq Y$ be some (finite dimensional) closed subspaces of Σ, X, Y. Then the discretization of (3.4) seeks $(\sigma_h, \alpha_h, \chi_h) \in \Sigma_h(\mathcal{T}) \times X_h(\mathcal{T}) \times Y_h(\mathcal{T})$ such that

$$\begin{aligned} a(\tau_h,\sigma_h) + b(\tau_h,\alpha_h) &= (\varphi, \mathbb{C}^{-1}\tau_h)_{L^2(\Omega)} && \text{for all } \tau_h \in \Sigma_h(\mathcal{T}),\\ b(\sigma_h,\beta_h) + c(\chi_h,\beta_h) &= 0 && \text{for all } \beta_h \in X_h(\mathcal{T}),\\ c(\xi_h,\alpha_h) &= 0 && \text{for all } \xi_h \in Y_h(\mathcal{T}). \end{aligned} \tag{4.1}$$

Assume now that $\operatorname{Curl} X_h(\mathcal{T}) \subseteq \Sigma_h(\mathcal{T})$ and that $X_h(\mathcal{T})$ and $Y_h(\mathcal{T})$ fulfil the discrete inf-sup condition

$$\|\xi_h\|_{L^2(\Omega)} \lesssim \sup_{\beta_h \in X_h(\mathcal{T})\setminus\{0\}} \frac{c(\xi_h,\beta_h)}{\|\operatorname{Curl}\beta_h\|_{L^2(\Omega)}} \qquad \text{for all } \xi_h \in Y_h(\mathcal{T}). \tag{4.2}$$

Theorem 4.1 (*A priori* error estimate)**.** *Problem* (4.1) *has a unique solution* $(\sigma_h, \alpha_h, \xi_h) \in \Sigma_h(\mathcal{T}) \times X_h(\mathcal{T}) \times Y_h(\mathcal{T})$ *and it satisfies*

$$\begin{aligned} &\|\sigma - \sigma_h\|_{\mathbb{C}^{-1/2}} + \|\operatorname{Curl}(\alpha - \alpha_h)\|_{L^2(\Omega)} + \|\chi - \chi_h\|_{L^2(\Omega)}\\ &\quad\lesssim \inf_{\substack{\tau_h \in \Sigma_h(\mathcal{T}),\\ \beta_h \in X_h(\mathcal{T}),\\ \xi_h \in Y_h(\mathcal{T})}} \Big(\|\sigma - \tau_h\|_{\mathbb{C}^{-1/2}} + \|\operatorname{Curl}(\alpha - \beta_h)\|_{L^2(\Omega)} + \|\chi - \xi_h\|_{L^2(\Omega)}\Big). \end{aligned} \tag{4.3}$$

Proof. The bilinear form a is continuous and elliptic with respect to $\|\bullet\|_{\mathbb{C}^{-1/2}}$. The Cauchy inequality and $A : \mathbb{C}^{-1}A \lesssim A : A$ for all $A \in \mathbb{R}^{2\times 2}$ (*cf.* (3.7)) prove the continuity of b with respect to $\|\bullet\|_{\mathbb{C}^{-1/2}}$ and $\|\operatorname{Curl}\bullet\|_{L^2(\Omega)}$.

For given $\beta_h \in X_h(\mathcal{T}) \setminus \{0\}$, define $\tau_h := \operatorname{Curl} \beta_h$. Since $\operatorname{Curl} X_h(\mathcal{T}) \subseteq \Sigma_h(\mathcal{T})$, this defines an element in $\Sigma_h(\mathcal{T})$ and

$$b(\tau_h, \beta_h)/\|\tau_h\|_{\mathbb{C}^{-1/2}} = \|\operatorname{Curl} \beta_h\|_{\mathbb{C}^{-1/2}} .$$

Proposition 2.1 and A: $\operatorname{dev} A \lesssim A : \mathbb{C}^{-1} A$ for all $A \in \mathbb{R}^{2\times 2}$ prove

$$\|\operatorname{Curl} \beta_h\|_{L^2(\Omega)} \lesssim \|\operatorname{dev} \operatorname{Curl} \beta_h\|_{L^2(\Omega)} \lesssim \|\operatorname{Curl} \beta_h\|_{\mathbb{C}^{-1/2}} .$$

This proves the discrete inf-sup condition for b with a constant independent of λ. The Cauchy inequality reveals the continuity of the bilinear form c. This, the inf-sup condition (4.2), and a recursive application of Brezzi's splitting lemma [13] yield the unique existence of a solution of (4.1). Standard arguments for conforming mixed FEMs lead to the *a priori* error estimate. □

4.2. Mini or stabilized discretization

Define the space of (cubic) bubble functions

$$\mathcal{B}(\mathcal{T};\mathbb{R}^2) := \left\{\psi \in P_3(\mathcal{T};\mathbb{R}^2) \cap H^1_0(\Omega;\mathbb{R}^2) \mid \forall T \in \mathcal{T} : \psi|_T \in H^1_0(T) \cap P_3(T)\right\}$$

and

$$\begin{aligned} \Sigma_h(\mathcal{T}) &:= P_0(\mathcal{T};\mathbb{R}^{2\times 2}) + \operatorname{Curl}(\mathcal{B}(\mathcal{T};\mathbb{R}^2)),\\ X_h(\mathcal{T}) &:= V_{\text{Mini}}(\mathcal{T}) := (P_1(\mathcal{T};\mathbb{R}^2) + \mathcal{B}(\mathcal{T};\mathbb{R}^2)) \cap X,\\ Y_h(\mathcal{T}) &:= P_1(\mathcal{T}) \cap H^1(\Omega) \cap L^2_0(\Omega). \end{aligned}$$

The Mini finite element discretization of (3.4) seeks $(\sigma_h, \alpha_h, \chi_h) \in \Sigma_h(\mathcal{T}) \times X_h(\mathcal{T}) \times Y_h(\mathcal{T})$ with (4.1). The inf-sup condition (4.2) for the bilinear form c from (3.3) is the same as the inf-sup condition for the Stokes equations and is proved in [3]. Since $\operatorname{Curl} X_h(\mathcal{T}) \subseteq \Sigma_h(\mathcal{T})$ by definition, Theorem 4.1 leads to the unique existence of solutions and the *a priori* error estimate (4.3).

4.3. Taylor−Hood discretization

The Taylor−Hood discretization of (3.4) employs the discrete spaces

$$\begin{aligned} \Sigma_h(\mathcal{T}) &:= P_k(\mathcal{T};\mathbb{R}^{2\times 2}),\\ X_h(\mathcal{T}) &:= P_{k+1}(\mathcal{T};\mathbb{R}^2) \cap X,\\ Y_h(\mathcal{T}) &:= P_k(\mathcal{T}) \cap H^1(\Omega) \cap L^2_0(\Omega) \end{aligned}$$

and seeks $(\sigma_h, \alpha_h, \chi_h) \in \Sigma_h(\mathcal{T}) \times X_h(\mathcal{T}) \times Y_h(\mathcal{T})$ with (4.1). The inf-sup condition (4.2) for the bilinear form c from (3.3) is the same as the inf-sup condition for the Stokes equations and is proved in [9]. Since $\operatorname{Curl} X_h(\mathcal{T}) \subseteq \Sigma_h(\mathcal{T})$, Theorem 4.1 yields the existence of unique solutions and the *a priori* error estimate (4.3).

4.4. $P_2\, P_0$ method

The $P_2\, P_0$ discretization of (3.4) considers the discrete spaces

$$\begin{aligned} \Sigma_h(\mathcal{T}) &:= P_1(\mathcal{T};\mathbb{R}^{2\times 2}),\\ X_h(\mathcal{T}) &:= P_2(\mathcal{T};\mathbb{R}^2) \cap X,\\ Y_h(\mathcal{T}) &:= P_0(\mathcal{T}) \cap L^2_0(\Omega) \end{aligned}$$

and seeks $(\sigma_h, \alpha_h, \chi_h) \in \Sigma_h(\mathcal{T}) \times X_h(\mathcal{T}) \times Y_h(\mathcal{T})$ with (4.1). The inf-sup condition (4.2) for the bilinear form c from (3.3) is the same as the inf-sup condition for the Stokes equations and is proved in [9]. Since $\operatorname{Curl} X_h(\mathcal{T}) \subseteq \Sigma_h(\mathcal{T})$, Theorem 4.1 proves the unique existence of discrete solutions and the *a priori* error estimate (4.3).

4.5. The non-conforming FEM of Kouhia and Stenberg

The non-conforming finite element space of Kouhia and Stenberg [29] reads

$$V_{\mathrm{KS}}(\mathcal{T}) := (P_1(\mathcal{T}) \cap H^1_0(\mathcal{T})) \times \mathrm{CR}^1_0(\mathcal{T})$$

with $\mathrm{CR}^1_0(\mathcal{T})$ defined by

$$\mathrm{CR}^1_0(\mathcal{T}) := \left\{ v_{\mathrm{CR}} \in P_1(\mathcal{T}) \,\middle|\, \begin{array}{l} v_{\mathrm{CR}} \text{ is continuous at midpoints of interior edges} \\ \text{and vanishes at midpoints of boundary edges} \end{array} \right\}.$$

Let $\varepsilon_{\mathrm{NC}}$ and $\mathrm{Curl}_{\mathrm{NC}}$ denote the piecewise versions of ε and Curl. The finite element method of Kouhia and Stenberg seeks $u_{\mathrm{KS}} \in V_{\mathrm{KS}}(\mathcal{T})$ such that

$$(\varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}}), \mathbb{C}\varepsilon_{\mathrm{NC}}(u_{\mathrm{KS}}))_{L^2(\Omega)} = (f, v_{\mathrm{KS}})_{L^2(\Omega)} \qquad \text{for all } v_{\mathrm{KS}} \in V_{\mathrm{KS}}(\mathcal{T}). \tag{4.4}$$

Define

$$\begin{aligned} \mathrm{CR}^1(\mathcal{T}) &:= \{ v_h \in P_1(\mathcal{T}) \mid v_h \text{ is continuous at midpoints of interior edges} \}, \\ \widetilde{V}_{\mathrm{KS}}(\mathcal{T}) &:= (\mathrm{CR}^1(\mathcal{T}) \cap L^2_0(\Omega)) \times (P_1(\mathcal{T}) \cap H^1(\Omega) \cap L^2_0(\Omega)), \\ Z_{\mathrm{KS}}(\mathcal{T}) &:= \{ v_{\mathrm{KS}} \in \widetilde{V}_{\mathrm{KS}}(\mathcal{T}) \mid \mathrm{Curl}_{\mathrm{NC}}\, v_{\mathrm{KS}} \in P_0(\mathcal{T};\mathbb{S}) \}. \end{aligned}$$

Then, the discrete Helmholtz decomposition

$$P_0(\mathcal{T};\mathbb{S}) = \mathbb{C}\varepsilon_{\mathrm{NC}}(V_{\mathrm{KS}}(\mathcal{T})) \oplus \mathrm{Curl}_{\mathrm{NC}}(Z_{\mathrm{KS}}(\mathcal{T}))$$

holds [18] and the sum is orthogonal with respect to $(\bullet,\bullet)_{\mathbb{C}^{-1/2}}$. If $\varphi \in H(\mathrm{div},\Omega;\mathbb{R}^{2\times 2})$ additionally allows for an integration by parts with Kouhia−Stenberg functions, *i.e.*,

$$(\varphi, D_{\mathrm{NC}} v_{\mathrm{KS}})_{L^2(\Omega)} = (f, v_{\mathrm{KS}})_{L^2(\Omega)} \qquad \text{for all } v_{\mathrm{KS}} \in V_{\mathrm{KS}}(\mathcal{T})$$

(*e.g.*, φ is a lowest-order Raviart−Thomas function [33]), then this implies that (4.4) is equivalent to the problem: Seek $(\sigma_h, \alpha_h) \in P_0(\mathcal{T};\mathbb{S}) \times Z_{\mathrm{KS}}(\mathcal{T})$ such that, for all $\tau_h \in P_0(\mathcal{T};\mathbb{S})$ and all $\beta_h \in Z_{\mathrm{KS}}(\mathcal{T})$

$$\begin{aligned} (\mathbb{C}^{-1}\tau_h, \sigma_h)_{L^2(\Omega)} + (\mathbb{C}^{-1}\tau_h, \mathrm{Curl}_{\mathrm{NC}}\, \alpha_h)_{L^2(\Omega)} &= (\varphi, \mathbb{C}^{-1}\tau_h)_{L^2(\Omega)}, \\ (\mathbb{C}^{-1}\sigma_h, \mathrm{Curl}_{\mathrm{NC}}\, \beta_h)_{L^2(\Omega)} &= 0. \end{aligned}$$

In contrast to the discretizations of Sections 4.2–4.4, this discretizes the space Z from (3.1) directly and the symmetry of σ_h is fulfilled pointwise. However, since $Z_{\mathrm{KS}}(\mathcal{T}) \not\subseteq X$, the approximation is non-conforming and Theorem 4.1 is not applicable; the *a priori* analysis requires techniques in the spirit of the Strang−Fix lemma [11] and is not further discussed here.

The Kouhia−Stenberg FEM can also be regarded as a conforming mixed FEM if the Helmholtz decomposition is applied in one component only. This allows the generalization to higher polynomial degrees. Define the spaces

$$\Sigma_{\mathrm{KS},k}(\mathcal{T}) := \left\{ \mathbb{C}\,\mathrm{sym}(q) \in P_k(\mathcal{T},\mathbb{S}) \,\middle|\, \begin{array}{l} q \in P_k(\mathcal{T};\mathbb{R}^{2\times 2}),\ \exists v_h \in P_{k+1}(\mathcal{T}) \cap H^1_0(\Omega) \\ \text{such that } q_{1,\bullet} = \nabla v_h \end{array} \right\},$$

where $q_{j,\bullet} = (q_{j,1}, q_{j,2})$ denotes the jth row of q. The Helmholtz decomposition leads to the discretization: Seek $(\sigma_h, \alpha_h) \in \Sigma_{\mathrm{KS},k}(\mathcal{T}) \times (P_{k+1}(\mathcal{T}) \cap H^1(\Omega)/\mathbb{R})$ such that, for all $\tau_h \in \Sigma_{\mathrm{KS},k}(\mathcal{T})$ and all $\beta_h \in (P_{k+1}(\mathcal{T}) \cap H^1(\Omega)/\mathbb{R})$

$$\begin{aligned} (\mathbb{C}^{-1}\tau_h, \sigma_h)_{L^2(\Omega)} + ((\mathbb{C}^{-1}\tau_h)_{2,\bullet}, \mathrm{Curl}\, \alpha_h)_{L^2(\Omega)} &= (\varphi, \mathbb{C}^{-1}\tau_h), \\ ((\mathbb{C}^{-1}\sigma_h)_{2,\bullet}, \mathrm{Curl}\, \beta_h)_{L^2(\Omega)} &= 0. \end{aligned}$$

This generalizes the Kouhia−Stenberg FEM of (4.4) to higher polynomial degrees.

5. *A posteriori* error analysis

This section introduces an error estimator and proves its efficiency and reliability. The results apply to all of the discretizations from Sections 4.2–4.4.

Let $\mathcal{T}_\ell \in \mathbb{T}$ be some regular triangulation. Given $T \in \mathcal{T}_\ell$, let $\|\bullet\|_{\mathbb{C}^{-1/2},T} := \sqrt{(\bullet, \mathbb{C}^{-1}\bullet)_{L^2(T)}}$ denote the $\mathbb{C}^{-1/2}$-norm on T. Define for $T \in \mathcal{T}_\ell$ the local error estimator contributions

$$\begin{aligned}\mu^2(T) &:= \|\varphi - \Pi_{\Sigma_h(\mathcal{T})}\varphi\|^2_{\mathbb{C}^{-1/2},T}\,,\\ \mathfrak{y}^2(\mathcal{T},T) &:= h_T^2\left\|\mathrm{curl}_{\mathrm{NC}}\,\mathbb{C}^{-1}\sigma_h + \nabla_{\mathrm{NC}}\chi_h\right\|^2_{L^2(T)} + \|\mathrm{div}\,\alpha_h\|^2_{L^2(T)}\\ &\quad + h_T\sum_{E\in\mathcal{E}(T)}\left\|\left[\mathbb{C}^{-1}\sigma_h + \chi_h\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]_E \tau_E\right\|^2_{L^2(E)}.\end{aligned} \tag{5.1}$$

Furthermore, define

$$\begin{aligned}\mathfrak{y}_\ell^2 &:= \mathfrak{y}^2(\mathcal{T}_\ell,\mathcal{T}_\ell) &&\text{with}\quad \mathfrak{y}^2(\mathcal{T}_\ell,\mathcal{M}) := \sum_{T\in\mathcal{M}}\mathfrak{y}^2(\mathcal{T}_\ell,T) &&\text{for any } \mathcal{M}\subseteq\mathcal{T}_\ell,\\ \mu_\ell^2 &:= \mu^2(\mathcal{T}_\ell) &&\text{with}\quad \mu^2(\mathcal{M}) := \sum_{T\in\mathcal{M}}\mu^2(T) &&\text{for any } \mathcal{M}\subseteq\mathcal{T}_\ell,\\ \eta_\ell^2 &:= \mathfrak{y}_\ell^2 + \mu_\ell^2.\end{aligned} \tag{5.2}$$

Theorem 5.1 (Efficiency and reliability)**.** *The estimator η_ℓ is reliable and efficient in the sense that*

$$\eta_\ell \approx \|\sigma - \sigma_h\|_{\mathbb{C}^{-1/2}} + \|\mathrm{Curl}(\alpha-\alpha_h)\|_{L^2(\Omega)} + \|\chi - \chi_h\|_{L^2(\Omega)}.$$

Proof. The proof is split into four steps.

Step 1. (Equivalence to residuals). Define for all $\tau\in\Sigma$, $\beta\in X$, and $\xi\in Y$ the residuals

$$\begin{aligned}\mathrm{Res}_1(\sigma_h,\alpha_h;\tau) &:= a(\tau,\sigma_h) + b(\tau,\alpha_h) - (\varphi,\mathbb{C}^{-1}\tau)_{L^2(\Omega)},\\ \mathrm{Res}_2(\sigma_h,\chi_h;\beta) &:= b(\sigma_h,\beta) + c(\chi_h,\beta),\\ \mathrm{Res}_3(\alpha_h;\xi) &:= c(\xi,\alpha_h).\end{aligned}$$

The abstract theory of [15] proves the (λ independent) equivalence

$$\begin{aligned}&\|\sigma - \sigma_h\|_{\mathbb{C}^{-1/2}} + \|\mathrm{Curl}(\alpha-\alpha_h)\|_{L^2(\Omega)} + \|\chi-\chi_h\|_{L^2(\Omega)}\\ &\qquad\approx \|\mathrm{Res}_1(\sigma_h,\alpha_h;\bullet)\|_{\Sigma^\star} + \|\mathrm{Res}_2(\sigma_h,\chi_h;\bullet)\|_{X^\star} + \|\mathrm{Res}_3(\alpha_h;\bullet)\|_{Y^\star},\end{aligned}$$

where $\|\bullet\|_{\Sigma^\star}$ denotes the dual norm with respect to $\|\bullet\|_{\mathbb{C}^{-1/2}}$, *i.e.*,

$$\|\mathrm{Res}_1(\sigma_h,\alpha_h;\bullet)\|_{\Sigma^\star} := \sup_{\tau\in\Sigma\setminus\{0\}}\frac{\mathrm{Res}_1(\sigma_h,\alpha_h;\tau)}{\|\tau\|_{\mathbb{C}^{-1/2}}}.$$

Step 2. (Efficiency and reliability of Res_1). Since

$$\mathrm{Res}_1(\sigma_h,\alpha_h;\tau) = (\Pi_{\Sigma_h(\mathcal{T})}\varphi - \varphi, \mathbb{C}^{-1}\tau)_{L^2(\Omega)} \qquad \text{for all } \tau\in\Sigma,$$

the Cauchy inequality implies

$$\|\mathrm{Res}_1(\sigma_h,\alpha_h;\bullet)\|_{\Sigma^\star} = \|\varphi - \Pi_{\Sigma_h(\mathcal{T})}\varphi\|_{\mathbb{C}^{-1/2}}.$$

Step 3. (Efficiency and reliability of Res_2). Let I_h denote a quasi interpolant [21] with approximation and stability properties

$$\|h_{\mathcal{T}}^{-1}(\beta - I_h\beta)\|_{L^2(\Omega)} + \|\mathrm{Curl}(\beta - I_h\beta)\|_{L^2(\Omega)} \lesssim \|\mathrm{Curl}\,\beta\|_{L^2(\Omega)}\,. \tag{5.3}$$

Since $P_1(\mathcal{T};\mathbb{R}^2)\cap H^1(\Omega;\mathbb{R}^2) \subseteq X_h(\mathcal{T})$ for all discretizations from Section 4.2–4.4, the discrete problem (4.1) and a piecewise integration by parts lead for all $\beta \in X$ to

$$\begin{aligned}\mathrm{Res}_2(\sigma_h,\chi_h;\beta) &= (\mathbb{C}^{-1}\sigma_h, \mathrm{Curl}(\beta - I_h\beta))_{L^2(\Omega)} + (\chi_h, \mathrm{div}(\beta - I_h\beta))_{L^2(\Omega)}\\ &= (\mathrm{curl}_{\mathrm{NC}}\,\mathbb{C}^{-1}\sigma_h + \nabla_{\mathrm{NC}}\chi_h, \beta - I_h\beta)_{L^2(\Omega)}\\ &\quad + \sum_{E\in\mathcal{E}}\int_E (\beta - I_h\beta)\cdot\left(\left[\mathbb{C}^{-1}\sigma_h + \chi_h\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]_E \tau_E\right)\mathrm{d}s.\end{aligned}$$

The Cauchy and the trace inequality and the approximation properties of the quasi-interpolation (5.3) yield

$$\begin{aligned}\mathrm{Res}_2(\sigma_h,\chi_h;\beta) \lesssim \Bigg(&\|h_{\mathcal{T}}(\mathrm{curl}_{\mathrm{NC}}\,\mathbb{C}^{-1}\sigma_h + \nabla_{\mathrm{NC}}\chi_h)\|_{L^2(\Omega)}\\ &+ \sqrt{\sum_{E\in\mathcal{E}} h_T \left\|\left[\mathbb{C}^{-1}\sigma_h + \chi_h\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]_E \tau_E\right\|^2_{L^2(E)}}\,\Bigg)\|\mathrm{Curl}\,\beta\|_{L^2(\Omega)}\,.\end{aligned}$$

The efficiency follows with the standard bubble function technique of [40].

Step 4. (Efficiency and reliability of Res_3). The Cauchy inequality implies

$$\mathrm{Res}_3(\alpha_h,\xi) = (\xi, \mathrm{div}\,\alpha_h)_{L^2(\Omega)} \leq \|\xi\|_{L^2(\Omega)}\ \|\mathrm{div}\,\alpha_h\|_{L^2(\Omega)}\,.$$

This and $\mathrm{div}\,X_h(\mathcal{T}) \subseteq Y$ yield

$$\|\mathrm{Res}_3(\alpha_h,\xi)\|_{Y^\star} = \|\mathrm{div}\,\alpha_h\|_{L^2(\Omega)}\,.$$

□

6. Stokes equations with symmetric gradient

The Stokes equations can be interpreted as the incompressible limit $\lambda\to\infty$ of (1.1). The symmetric formulation is favourable for the discretization of Neumann boundary conditions (*cf.* [9], Rem. 8.1.3 and Remark 6.1 below). Let $\Omega \subseteq \mathbb{R}^2$ with closed Dirichlet boundary $\emptyset \neq \Gamma_D \subseteq \partial\Omega$ and Neumann boundary $\Gamma_N = \partial\Omega\setminus\Gamma_D \neq \emptyset$. The Stokes problem with Neumann boundary conditions then seeks $(u,p)\in H^1_{\Gamma_D}(\Omega;\mathbb{R}^2)\times L^2(\Omega)$ with

$$\left.\begin{aligned}-\mathrm{div}\,\varepsilon(u) + \nabla p &= f\\ \mathrm{div}\,u &= 0\end{aligned}\right\}\ \text{in } \Omega \quad\text{and}\quad (\varepsilon(u) + pI_{2\times 2})|_{\Gamma_N}\nu = 0. \tag{6.1}$$

In the presence of these boundary conditions, a weak formulation has to involve the symmetric part of the gradient $\varepsilon(u)$ (see Rem. 6.1 below).

6.1. Weak formulation

The classical weak formulation of (6.1) seeks $(u,p)\in H^1_{\Gamma_D}(\Omega;\mathbb{R}^2)\times L^2(\Omega)$ such that, for all $v \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^2)$ and for all $q\in L^2(\Omega)$,

$$\begin{aligned}(\varepsilon(v),\varepsilon(u))_{L^2(\Omega)} - (p,\mathrm{div}\,v)_{L^2(\Omega)} &= (f,v)_{L^2(\Omega)},\\ (q,\mathrm{div}\,u)_{L^2(\Omega)} &= 0.\end{aligned} \tag{6.2}$$

Remark 6.1 (Symmetric *vs.* non-symmetric approximation)**.** For the pure Dirichlet problem, an integration by parts proves for all $u, v \in \{w \in H^1_0(\Omega;\mathbb{R}^2) \mid \operatorname{div} w = 0\}$ that

$$2(\varepsilon(u), \varepsilon(v))_{L^2(\Omega)} = (Du, Dv)_{L^2(\Omega)} + (\operatorname{div} u, \operatorname{div} v)_{L^2(\Omega)} = (Du, Dv)_{L^2(\Omega)}.$$

This shows equivalence of (6.2) and the Stokes problem without symmetric gradient (7.2) below. However, this equivalence does no longer hold if $\Gamma_D \neq \partial\Omega$. Problem (6.2) covers also the Neumann boundary condition $(\varepsilon(u) + pI_{2\times 2})|_{\Gamma_N}\nu = 0$.

Define $\mathbb{R}^{2\times 2}_{\mathrm{dev,sym}} := \mathbb{R}^{2\times 2}_{\mathrm{dev}} \cap \mathbb{S}$ with $\mathbb{R}^{2\times 2}_{\mathrm{dev}}$ from (2.1) and the spaces

$$\begin{aligned}
\Sigma^N &:= L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{dev}}),\\
Z^{\mathfrak{s}} &:= \{v \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^2) \mid \operatorname{div} v = 0\},\\
X^N &:= \left\{\beta \in H^1(\Omega;\mathbb{R}^2) \mid \beta \text{ is constant on each connectivity component of } \Gamma_N\right\},\\
X^{\mathfrak{s}} &:= \left\{\beta \in X^N \;\middle|\; \operatorname{Curl}\beta \in L^2(\Omega;\mathbb{S})\right\},\\
Y^N &:= L^2(\Omega)
\end{aligned}$$

and bilinear forms $a : \Sigma^N \times \Sigma^N \to \mathbb{R}$ and $b : \Sigma^N \times X^N \to \mathbb{R}$ by

$$\begin{aligned}
a(\sigma,\tau) &:= (\sigma,\tau)_{L^2(\Omega)} && \text{for all } \sigma,\tau \in \Sigma^N,\\
b(\tau,\alpha) &:= (\tau, \operatorname{dev}\operatorname{Curl}\alpha)_{L^2(\Omega)} && \text{for all } \tau \in \Sigma^N, \alpha \in X^N
\end{aligned} \tag{6.3}$$

and recall c from (3.3). A computation reveals for $\Omega \subseteq \mathbb{R}^2$

$$X^{\mathfrak{s}} = \left\{\beta \in X^N \mid \operatorname{div}\beta = 0\right\}.$$

Theorem 6.2 (Helmholtz decomposition for deviatoric and symmetric functions)**.** *It holds*

$$L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{dev,sym}}) = \varepsilon(Z^{\mathfrak{s}}) \oplus \operatorname{dev}(\operatorname{Curl} X^{\mathfrak{s}})$$

and the sum is L^2 orthogonal.

Proof. Let $\phi \in L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{dev,sym}})$. Korn's inequality [10], the inf-sup condition for the bilinear form $(\bullet, \operatorname{div}\bullet)_{L^2(\Omega)}$ on $L^2(\Omega) \times H^1_{\Gamma_D}(\Omega;\mathbb{R}^2)$ [26], and Brezzi's splitting lemma [13] guarantee the existence of a solution $(u,p) \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^2) \times L^2(\Omega)$ of

$$\begin{aligned}
(\varepsilon(v), \varepsilon(u))_{L^2(\Omega)} - (p, \operatorname{div} v)_{L^2(\Omega)} &= (\phi, \varepsilon(v))_{L^2(\Omega)} && \text{for all } v \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^2),\\
(q, \operatorname{div} u)_{L^2(\Omega)} &= 0 && \text{for all } q \in L^2(\Omega).
\end{aligned}$$

This implies $u \in Z^{\mathfrak{s}}$ and $(\phi - \varepsilon(u) - pI_{2\times 2}) \perp_{L^2(\Omega)} \nabla H^1_{\Gamma_D}(\Omega;\mathbb{R}^2)$. The Helmholtz decomposition $L^2(\Omega;\mathbb{R}^2) = \nabla H^1_{\Gamma_D}(\Omega) \oplus \operatorname{Curl}\{\beta \in H^1(\Omega) \mid \beta \text{ is constant on each connectivity component of } \Gamma_N\}$ (applied row-wise) yields the existence of $\alpha \in X^N$ with $\phi - \varepsilon(u) - pI_{2\times 2} = \operatorname{Curl}\alpha$. Since $u \in Z^{\mathfrak{s}}$, this implies

$$\operatorname{dev}\operatorname{Curl}\alpha = \operatorname{dev}(\phi - \varepsilon(u) - pI_{2\times 2}) = \phi - \varepsilon(u).$$

Any $A \in \mathbb{R}^{2\times 2}$ is symmetric if and only if $\operatorname{dev} A$ is symmetric. Since $\phi - \varepsilon(u) \in L^2(\Omega;\mathbb{S})$ is symmetric, it follows $\alpha \in X^{\mathfrak{s}}$. $\square$

Let $\varphi \in H(\operatorname{div},\Omega;\mathbb{S})$ with $-\operatorname{div}\varphi = f$ and $\varphi|_{\Gamma_N}\nu = 0$. Consider the problem: Seek $(\sigma,\alpha,\chi) \in \Sigma^N \times X^N \times Y^N$ such that, for all $(\tau,\beta,\xi) \in \Sigma^N \times X^N \times Y^N$,

$$\begin{aligned}
a(\sigma,\tau) + b(\tau,\alpha) &= (\varphi,\tau)_{L^2(\Omega)},\\
b(\sigma,\beta) + c(\chi,\beta) &= 0,\\
c(\xi,\alpha) &= 0.
\end{aligned} \tag{6.4}$$

Lemma 6.3 (Inf-sup condition)**.** *Any $\beta \in X^N$ satisfies*

$$\|\operatorname{Curl}\beta\|_{L^2(\Omega)} \lesssim \sup_{\tau\in\Sigma\setminus\{0\}} \frac{b(\tau,\beta)}{\|\tau\|_{L^2(\Omega)}}.$$

Proof. Given $\beta \in X^N$, the choice of $\tau := \operatorname{dev}\operatorname{Curl}\beta$ leads to

$$b(\tau,\beta) = \|\operatorname{dev}\operatorname{Curl}\beta\|^2_{L^2(\Omega)}.$$

The boundary conditions for X^N lead to $(\operatorname{Curl}\beta)|_{\Gamma_N}\nu = 0$. Since $\operatorname{div}\operatorname{Curl}\beta = 0$, this and the tr-dev-div lemma ([16], Lem. 4.1) prove

$$\|\operatorname{Curl}\beta\|_{L^2(\Omega)} \lesssim \|\operatorname{dev}\operatorname{Curl}\beta\|_{L^2(\Omega)}. \tag{6.5}$$

This yields the assertion. □

Proposition 6.4. *Problem* (6.4) *has a unique solution $(\sigma,\alpha,\chi) \in \Sigma \times X^{\mathfrak{s}} \times Y^N$. The solution $u \in Z^{\mathfrak{s}}$ to* (6.2) *satisfies $\sigma = \varepsilon(u)$.*

Proof. The existence of a unique solution follows from a recursive application of Brezzi's splitting lemma [13] and the inf-sup conditions from Lemmas 6.3 and (3.5) as in the proof of Theorem 3.3. The second equation of (6.4) and the Helmholtz decomposition of Theorem 6.2 guarantee the existence of $\widetilde{u} \in Z^{\mathfrak{s}}$ with $\sigma = \varepsilon(\widetilde{u})$. Let $v \in Z^{\mathfrak{s}}$. Then, (6.4) and an integration by parts imply

$$(D\widetilde{u}, Dv)_{L^2(\Omega)} = (\varphi, Dv)_{L^2(\Omega)} - (Dv, \operatorname{dev}\operatorname{Curl}\alpha)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)}.$$

This yields $\widetilde{u} = u$. □

Remark 6.5 (Pressure)**.** Define $p := -\operatorname{tr}(\varphi - \operatorname{Curl}\alpha)/2$. Since $\sigma + \operatorname{dev}\operatorname{Curl}\alpha = \operatorname{dev}\varphi$ and $\operatorname{Curl}\alpha \perp_{L^2(\Omega)} D\big(H^1_{\Gamma_D}(\Omega;\mathbb{R}^2)\big)$, the function p satisfies for all $v \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^2)$ that

$$(p, \operatorname{div} v)_{L^2(\Omega)} = (pI_{2\times 2}, Dv)_{L^2(\Omega)} = (\sigma, \varepsilon(v))_{L^2(\Omega)} - (\varphi, Dv)_{L^2(\Omega)}.$$

This, an integration by parts and $\varphi|_{\Gamma_N}\nu = 0$ lead to

$$\begin{aligned} -(p, \operatorname{div} v)_{L^2(\Omega)} &= (f,v)_{L^2(\Omega)} + \int_{\Gamma_N} \varphi\nu\cdot v\, ds - (\sigma,\varepsilon(v))_{L^2(\Omega)} \\ &= (f,v)_{L^2(\Omega)} - (\sigma,\varepsilon(v))_{L^2(\Omega)}. \end{aligned}$$

This implies that p is the pressure from (6.2).

6.2. Discretization

Let $\Sigma_h^N(\mathcal{T})$, $X_h(\mathcal{T}) \cap X^N \subseteq X^N$, and $Y_h^N(\mathcal{T}) \subseteq Y^N$ be one of the choices of discrete spaces from Sections 4.2–4.4 without the respective conditions $\int_\Omega \operatorname{tr}(\tau_h)\,\mathrm{d}x = 0$, $\int_\Omega \beta_h\,\mathrm{d}x = 0$, $\int_\Omega \operatorname{curl}\beta_h\,\mathrm{d}x = 0$ and $\int_\Omega \xi_h\,\mathrm{d}x = 0$, and define

$$\Sigma_h^{\mathrm{dev}}(\mathcal{T}) := \operatorname{dev}\Sigma_h^N(\mathcal{T}) \subseteq \Sigma^N \quad\text{and}\quad X_h^N(\mathcal{T}) := X_h(\mathcal{T}) \cap X^N.$$

Recall the bilinear forms a, b from (6.3) and c from (3.3). Then the discretization of (6.4) seeks $(\sigma_h,\alpha_h,\xi_h) \in \Sigma_h^{\mathrm{dev}}(\mathcal{T}) \times X_h^N(\mathcal{T}) \times Y_h^N(\mathcal{T})$ such that

$$\begin{aligned} a(\tau_h,\sigma_h) + b(\tau_h,\alpha_h) &= (\varphi,\tau_h)_{L^2(\Omega)} &&\text{for all } \tau_h \in \Sigma_h^{\mathrm{dev}}(\mathcal{T}),\\ b(\sigma_h,\beta_h) + c(\chi_h,\beta_h) &= 0 &&\text{for all } \beta_h \in X_h^N(\mathcal{T}),\\ c(\xi_h,\alpha_h) &= 0 &&\text{for all } \xi_h \in Y_h^N(\mathcal{T}). \end{aligned} \tag{6.6}$$

The following theorem proves an *a priori* error estimate similar to Theorem 4.1.

Theorem 6.6 (Best-approximation)**.** *The discrete problem* (6.6) *has a unique solution* $(\sigma_h, \alpha_h, \chi_h) \in \Sigma_h^{\mathrm{dev}}(\mathcal{T}) \times X_h^N(\mathcal{T}) \times Y_h^N(\mathcal{T})$*, which satisfies*

$$\begin{aligned}&\|\sigma-\sigma_h\|_{L^2(\Omega)} + \|\mathrm{Curl}(\alpha-\alpha_h)\|_{L^2(\Omega)} + \|\chi-\chi_h\|_{L^2(\Omega)}\\ &\lesssim \inf_{\substack{\tau_h\in\Sigma_h^{\mathrm{dev}}(\mathcal{T})\\ \beta_h\in X_h^N(\mathcal{T})\\ \xi_h\in Y_h^N(\mathcal{T})}} \Big(\|\sigma-\tau_h\|_{L^2(\Omega)} + \|\mathrm{Curl}(\alpha-\beta_h)\|_{L^2(\Omega)} + \|\chi-\xi_h\|_{L^2(\Omega)}\Big).\end{aligned}$$

Proof. The bilinear form a is continuous and elliptic with respect to $\|\bullet\|_{L^2(\Omega)}$. The Cauchy inequality implies that the bilinear forms b and c are continuous with respect to $\|\bullet\|_{L^2(\Omega)}$ and $\|\mathrm{Curl}\bullet\|_{L^2(\Omega)}$. The inf-sup condition

$$\|\mathrm{Curl}\,\beta_h\|_{L^2(\Omega)} \lesssim \sup_{\tau_h\in\Sigma_h^{\mathrm{dev}}(\mathcal{T})\setminus\{0\}} \frac{b(\tau_h,\beta_h)}{\|\tau_h\|_{L^2(\Omega)}} \qquad \text{for all } \beta_h\in X_h^N(\mathcal{T})$$

follows from $\mathrm{dev}\,\mathrm{Curl}\,X_h^N(\mathcal{T}) \subseteq \Sigma_h^{\mathrm{dev}}(\mathcal{T})$ and (6.5). The discrete inf-sup conditions for c for the choices of $X_h^N(\mathcal{T})$ and $Y_h^N(\mathcal{T})$ from above are proved in [3,9]. A recursive application of Brezzi's splitting lemma [13] and standard arguments for conforming mixed FEMs lead to the assertion. □

Remark 6.7 (Non-conforming approximation)**.** Since $(\varepsilon_{\mathrm{NC}}\bullet, \varepsilon_{\mathrm{NC}}\bullet)$ for the piecewise symmetric gradient $\varepsilon_{\mathrm{NC}}$ is not positive definite on $\mathrm{CR}_0^1(\mathcal{T})$, there is no obvious non-conforming approximation of the Stokes problem (6.2) and so no non-conforming approximation for the Neumann problem of the form (6.1).

6.3. *A posteriori* error analysis

Let $\mathcal{E}(\Gamma_N) := \{E\in\mathcal{E} \mid E\subseteq \overline{\Gamma}_N\}$ denote the edges on the Neumann boundary. Given a triangulation $\mathcal{T}_\ell$, define for all $T\in\mathcal{T}_\ell$ the local error estimator contributions by

$$\begin{aligned}\mathfrak{y}^2(\mathcal{T}_\ell,T) &:= \|h_{\mathcal{T}}(\mathrm{curl}_{\mathrm{NC}}\,\sigma_h + \nabla_{\mathrm{NC}}\chi_h)\|_{L^2(T)}^2 + \|\mathrm{div}\,\alpha_h\|_{L^2(T)}^2\\ &\quad + h_T \sum_{E\in\mathcal{E}(T)\setminus\mathcal{E}(\Gamma_N)} \left\| \left[\sigma_h + \chi_h\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]_E \tau_E\right\|_{L^2(E)}^2 \qquad (6.7)\\ \mu^2(T) &:= \|\mathrm{dev}(\varphi - \Pi_{\Sigma_h^{\mathrm{dev}}(\mathcal{T})}\varphi)\|_{L^2(T)}^2\end{aligned}$$

and the global error estimators by the formulae (5.2). The following theorem states efficiency and reliability of η_ℓ. The proof is similar to the proof of Theorem 5.1 and therefore omitted.

Theorem 6.8 (Efficiency, reliability)**.** *The error estimator* η_ℓ *is reliable and efficient in the sense that*

$$\eta_\ell^2 \approx \|\sigma-\sigma_h\|_{L^2(\Omega)}^2 + \|\mathrm{Curl}(\alpha-\alpha_h)\|_{L^2(\Omega)}^2 + \|\chi-\chi_h\|_{L^2(\Omega)}^2. \qquad \square$$

7. Stokes equations without symmetric gradient

This section considers the Stokes problem

$$-\Delta u + \nabla p = f \qquad \text{and} \qquad \mathrm{div}\,u = 0 \qquad \text{in } \Omega \tag{7.1}$$

with homogeneous Dirichlet boundary conditions, so that the weak form

$$\begin{aligned}(Du, Dv)_{L^2(\Omega)} - (p,\mathrm{div}\,v)_{L^2(\Omega)} &= (f,v)_{L^2(\Omega)} &&\text{for all } v\in H_0^1(\Omega;\mathbb{R}^2),\\ (q,\mathrm{div}\,u)_{L^2(\Omega)} &= 0 &&\text{for all } q\in L_0^2(\Omega)\end{aligned} \tag{7.2}$$

is suitable (see Rem. 6.1). Standard low-order conforming methods fulfil $\mathrm{div}\,u = 0$ in some weak sense only. In contrast, the non-conforming method of [22] allows piecewise divergence free approximations. In Sections 7.1 and 7.2 that method of [22] is generalized to higher polynomial degrees with deviatoric ansatz spaces for the approximation of Du. Section 7.3 defines an *a posteriori* error estimator and proves its efficiency and reliability.

7.1. Weak formulation

This section introduces the new weak formulation based on the Helmholtz decomposition from Theorem 7.1 below.

Recall the definition of dev and $\mathbb{R}^{2\times 2}_{\mathrm{dev}}$ from Section 2. Furthermore, define

$$
\begin{aligned}
&Z := \{v \in H^1_0(\Omega;\mathbb{R}^2) \mid \operatorname{div} v = 0\},\\
&X := \left\{v \in H^1(\Omega;\mathbb{R}^2) \,\middle|\, \int_\Omega v \,\mathrm{d}x = 0 \text{ and } \int_\Omega \operatorname{curl} v \,\mathrm{d}x = 0\right\},\\
&\Sigma := L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{dev}}).
\end{aligned}
\tag{7.3}
$$

Proposition 2.1 proves that $\|\operatorname{dev}\operatorname{Curl}\bullet\|_{L^2(\Omega)}$ defines a norm on X. The following Helmholtz decomposition is a continuous version of the discrete Helmholtz decomposition of [20].

Theorem 7.1 (Helmholtz decomposition for deviatoric functions)**.** *It holds*

$$\Sigma = DZ \oplus \operatorname{dev}\operatorname{Curl} X$$

and the sum is L^2-orthogonal.

Proof. The L^2-orthogonality follows from the L^2-orthogonality of $\nabla H^1_0(\Omega)$ and $\operatorname{Curl} H^1(\Omega)$. Proposition 2.1 implies that $(\operatorname{dev}\operatorname{Curl}\bullet, \operatorname{dev}\operatorname{Curl}\bullet)_{L^2(\Omega)}$ defines a scalar product on the complete space X. Given $\tau \in \Sigma$, this implies that there exists a unique solution $\beta \in X$ of

$$(\operatorname{dev}\operatorname{Curl}\beta, \operatorname{dev}\operatorname{Curl}\gamma)_{L^2(\Omega)} = (\tau, \operatorname{dev}\operatorname{Curl}\gamma)_{L^2(\Omega)} \qquad \text{for all } \gamma \in X.$$

Let $\widehat{\gamma} \in H^1(\Omega;\mathbb{R}^2)/\mathbb{R}^2$ be defined by $\widehat{\gamma}(y) := (1/2)(-y_2, y_1) - \fint_\Omega (1/2)(-x_2, x_1)\,\mathrm{d}x$. Then $\operatorname{curl}\widehat{\gamma} = 1$. Given $\gamma \in H^1(\Omega;\mathbb{R}^2)$ with $\int_\Omega \gamma\,\mathrm{d}x = 0$, define $\widetilde{\gamma} \in X$ by $\widetilde{\gamma} := \gamma - \int_\Omega \operatorname{curl}\gamma\,\mathrm{d}x\widehat{\gamma}$. Since $\operatorname{tr}(\tau) = 0$ and $\operatorname{dev}\operatorname{Curl}\gamma = \operatorname{dev}\operatorname{Curl}\widetilde{\gamma}$, the definition of β implies

$$(\tau - \operatorname{dev}\operatorname{Curl}\beta, \operatorname{Curl}\gamma)_{L^2(\Omega)} = (\tau - \operatorname{dev}\operatorname{Curl}\beta, \operatorname{dev}\operatorname{Curl}\widetilde{\gamma})_{L^2(\Omega)} = 0.$$

Thus, the Helmholtz decomposition $L^2(\Omega;\mathbb{R}^2) = \nabla H^1_0(\Omega) \oplus \operatorname{Curl} H^1(\Omega)$ (applied row-wise) implies the existence of $u \in H^1_0(\Omega;\mathbb{R}^2)$ with $\tau - \operatorname{dev}\operatorname{Curl}\beta = Du$. Then, $\operatorname{div} u = \operatorname{tr}(\tau - \operatorname{dev}\operatorname{Curl}\beta) = 0$ leads to $u \in Z$ and concludes the proof. $\square$

Let $\varphi \in H(\operatorname{div},\Omega;\mathbb{R}^{2\times 2})$ with $\int_\Omega \operatorname{tr}(\varphi)\,\mathrm{d}x = 0$ and $-\operatorname{div}\varphi = f$ and seek $(\sigma,\alpha) \in \Sigma \times X$ with

$$
\begin{aligned}
(\sigma,\tau)_{L^2(\Omega)} + (\tau, \operatorname{dev}\operatorname{Curl}\alpha)_{L^2(\Omega)} &= (\varphi,\tau)_{L^2(\Omega)} &&\text{for all } \tau \in \Sigma,\\
(\sigma, \operatorname{dev}\operatorname{Curl}\beta)_{L^2(\Omega)} &= 0 &&\text{for all } \beta \in X.
\end{aligned}
\tag{7.4}
$$

Define the bilinear forms $a : \Sigma \times \Sigma \to \mathbb{R}$ and $b : \Sigma \times X \to \mathbb{R}$ by

$$
\begin{aligned}
a(\sigma,\tau) &:= (\sigma,\tau)_{L^2(\Omega)} &&\text{for all } \sigma,\tau \in \Sigma,\\
b(\tau,\alpha) &:= (\tau, \operatorname{dev}\operatorname{Curl}\alpha)_{L^2(\Omega)} &&\text{for all } \tau \in \Sigma, \alpha \in X.
\end{aligned}
$$

The following inf-sup condition is employed in the proof of the existence and uniqueness of solutions from Proposition 7.3.

Lemma 7.2 (Inf-sup condition)**.** *Any $\beta \in X$ satisfies*

$$\|\operatorname{Curl}\beta\|_{L^2(\Omega)} \lesssim \sup_{\tau \in \Sigma\setminus\{0\}} \frac{b(\tau,\beta)}{\|\tau\|_{L^2(\Omega)}}.$$

Proof. The choice of $\tau := \operatorname{dev}\operatorname{Curl}\beta$ leads to $b(\tau,\beta) = \|\operatorname{dev}\operatorname{Curl}\beta\|^2_{L^2(\Omega)}$. Proposition 2.1 then yields the assertion. $\square$

Proposition 7.3. *Problem (7.4) admits a unique solution $(\sigma,\alpha)\in\Sigma\times X$ and it holds $\sigma = Du$ for the solution $u\in Z$ of (7.2).*

Proof. The existence of a unique solution follows from Brezzi's splitting lemma [13] and Lemma 7.2. The second equation of (7.4) and the Helmholtz decomposition of Theorem 7.1 guarantees the existence of $\widetilde{u}\in Z$ with $\sigma = D\widetilde{u}$. Then, (7.4), $-\operatorname{div}\varphi = f$ and the orthogonality $DH^1_0(\Omega;\mathbb{R}^2)\perp_{L^2(\Omega)}\operatorname{Curl}X$ imply

$$\begin{aligned}(D\widetilde{u},Dv)_{L^2(\Omega)} &= (\varphi,Dv)_{L^2(\Omega)} - (Dv,\operatorname{dev}\operatorname{Curl}\alpha)_{L^2(\Omega)}\\ &= (f,v)_{L^2(\Omega)} - (Dv,\operatorname{Curl}\alpha) = (f,v)_{L^2(\Omega)}\end{aligned}$$

for all $v\in Z$. This yields $\widetilde{u}=u$. $\square$

Remark 7.4 (Pressure)**.** Define $p := -(1/2)\operatorname{tr}(\varphi - \operatorname{Curl}\alpha)\in L^2_0(\Omega)$. Since $\operatorname{dev}(\varphi-\operatorname{Curl}\alpha) = \sigma$, it follows

$$\begin{aligned}(p,\operatorname{div}v)_{L^2(\Omega)} &= (Dv, pI_{2\times 2})_{L^2(\Omega)}\\ &= (Dv,\operatorname{dev}(\varphi-\operatorname{Curl}\alpha))_{L^2(\Omega)} - (Dv,\varphi-\operatorname{Curl}\alpha)_{L^2(\Omega)}\\ &= (\sigma,Dv)_{L^2(\Omega)} - (f,v)_{L^2(\Omega)}.\end{aligned}$$

Proposition 7.3 implies that $\sigma = Du$ for the solution $u\in Z$ of (7.2) and, hence, $(u,p)\in Z\times L^2_0(\Omega)$ fulfils (7.2).

7.2. Discretization

For $k\geq 0$, define

$$\begin{aligned}\Sigma_h(\mathcal{T}) &:= P_k(\mathcal{T};\mathbb{R}^{2\times 2}_{\mathrm{dev}})\subseteq\Sigma,\\ X_h(\mathcal{T}) &:= P_{k+1}(\mathcal{T};\mathbb{R}^2)\cap X\subseteq X.\end{aligned}$$

The discrete problem seeks $(\sigma_h,\alpha_h)\in\Sigma_h(\mathcal{T})\times X_h(\mathcal{T})$ with

$$\begin{aligned}(\sigma_h,\tau_h)_{L^2(\Omega)} + (\tau_h,\operatorname{dev}\operatorname{Curl}\alpha_h)_{L^2(\Omega)} &= (\varphi,\tau_h)_{L^2(\Omega)} &&\text{for all } \tau_h\in\Sigma_h(\mathcal{T}),\\ (\sigma_h,\operatorname{dev}\operatorname{Curl}\beta_h)_{L^2(\Omega)} &= 0 &&\text{for all } \beta_h\in X_h(\mathcal{T}).\end{aligned}\tag{7.5}$$

The following lemma proves a discrete inf-sup condition.

Lemma 7.5 (Discrete inf-sup condition)**.** *Any $\beta_h\in X_h(\mathcal{T})$ satisfies*

$$\|\operatorname{Curl}\beta_h\|_{L^2(\Omega)}\lesssim \sup_{\tau_h\in\Sigma_h(\mathcal{T})\setminus\{0\}}\frac{b(\tau_h,\beta_h)}{\|\tau_h\|_{L^2(\Omega)}}.$$

Proof. As in the proof of Lemma 7.2, the choice $\tau_h := \operatorname{dev}\operatorname{Curl}\beta_h\in X_h(\mathcal{T})$ and Proposition 2.1 yield the assertion. $\square$

The following corollary is a consequence of Brezzi's splitting lemma [13], Lemma 7.5, and the standard theory of mixed FEMs [9].

Corollary 7.6 (*A priori* error estimate)**.** *The discrete problem (7.5) has a unique solution $(\sigma_h,\alpha_h)\in\Sigma_h(\mathcal{T})\times X_h(\mathcal{T})$ and it satisfies*

$$\|\sigma-\sigma_h\|_{L^2(\Omega)} + \|\operatorname{Curl}(\alpha-\alpha_h)\|_{L^2(\Omega)}\lesssim \min_{\tau_h\in\Sigma_h(\mathcal{T})}\|\sigma-\tau_h\|_{L^2(\Omega)} + \min_{\beta_h\in X_h(\mathcal{T})}\|\operatorname{Curl}(\alpha-\beta_h)\|_{L^2(\Omega)}. \quad\square$$

Define

$$W_h(\mathcal{T}) := \{\tau_h \in \Sigma_h(\mathcal{T}) \mid \forall \beta_h \in X_h(\mathcal{T}):\ (\tau_h, \operatorname{dev}\operatorname{Curl}\beta_h)_{L^2(\Omega)} = 0\}. \tag{7.6}$$

The following lemma proves a projection property for $W_h(\mathcal{T})$. This is the key argument in the *a posteriori* and optimality analysis in Sections 7.3 and 8.

Lemma 7.7 (Projection property)**.** *Let $\tau \in \Sigma$ with $(\tau, \operatorname{dev}\operatorname{Curl}\beta)_{L^2(\Omega)} = 0$ for all $\beta \in X$ (that means that there exists $v \in Z$ with $\tau = Dv$). Then $\Pi_{\Sigma_h(\mathcal{T})}\tau \in W_h(\mathcal{T})$. If $\mathcal{T}_\star$ is an admissible refinement of $\mathcal{T}$ and $\tau_\star \in W_h(\mathcal{T}_\star)$, then $\Pi_{\Sigma_h(\mathcal{T})}\tau_\star \in W_h(\mathcal{T})$.*

Proof. This follows from $\operatorname{dev}\operatorname{Curl} X_h(\mathcal{T}) \subseteq \Sigma_h(\mathcal{T})$ and $X_h(\mathcal{T}) \subseteq X_h(\mathcal{T}_\star) \subseteq X$. □

Remark 7.8 (Equivalence with Crouzeix–Raviart FEM for $k = 0$)**.** The discrete Helmholtz decomposition of [20] proves

$$P_0(\mathcal{T}; \mathbb{R}^{2\times 2}_{\mathrm{dev}}) = D_{\mathrm{NC}} Z_{\mathrm{CR}}(\mathcal{T}) \oplus \operatorname{dev}\operatorname{Curl}(P_1(\mathcal{T}; \mathbb{R}^2) \cap H^1(\Omega; \mathbb{R}^2))$$

for

$$Z_{\mathrm{CR}}(\mathcal{T}) := \{v_h \in \mathrm{CR}^1_0(\mathcal{T}) \times \mathrm{CR}^1_0(\mathcal{T}) \mid \operatorname{div}_{\mathrm{NC}} v_h = 0\}.$$

If $k = 0$, this proves $\sigma_h = D_{\mathrm{NC}}\widetilde{u}_{\mathrm{CR}}$ for the solution $\sigma_h \in \Sigma_h(\mathcal{T})$ of (7.5) and some $\widetilde{u}_{\mathrm{CR}} \in Z_{\mathrm{CR}}(\mathcal{T})$. If the right-hand side φ is a Raviart−Thomas vector field [33], a piecewise integration by parts proves, for all $v_{\mathrm{CR}} \in Z_{\mathrm{CR}}(\mathcal{T})$,

$$(D_{\mathrm{NC}}\widetilde{u}_{\mathrm{CR}}, D_{\mathrm{NC}} v_{\mathrm{CR}})_{L^2(\Omega)} = (\varphi, D_{\mathrm{NC}} v_{\mathrm{CR}})_{L^2(\Omega)} = (f, v_{\mathrm{CR}})_{L^2(\Omega)}$$

and hence $\widetilde{u}_{\mathrm{CR}}$ is the solution of the P_1 non-conforming FEM of [22].

Let $\mathcal{E}(\Omega)$ (resp. $\mathcal{N}(\Omega)$) denote the interior edges (resp. nodes) of $\mathcal{T}$. A computation reveals

$$\begin{aligned} &\dim(W_h(\mathcal{T})) = 3\operatorname{card}(\mathcal{E}(\Omega)) + \operatorname{card}(\mathcal{N}(\Omega)) && \text{for } k = 1,\\ &\dim(W_h(\mathcal{T})) = 6\operatorname{card}(\mathcal{E}(\Omega)) + 1 && \text{for } k = 2, \end{aligned}$$

while the non-conforming piecewise quadratic finite element space with vanishing divergence of [24] has dimension $3\operatorname{card}(\mathcal{N}(\Omega)) + \operatorname{card}(\mathcal{E}(\Omega))$ and the non-conforming piecewise cubic finite element space with vanishing divergence of [23] has dimension $\operatorname{card}(\mathcal{N}) + 7\operatorname{card}(\mathcal{E}(\Omega)) + 1$. Therefore, these non-conforming FEMs are different from the discretization (7.5).

7.3. *A posteriori* error analysis

Given a triangulation $\mathcal{T}_\ell$, define for all $T \in \mathcal{T}_\ell$ the local error estimator contributions by

$$\begin{aligned} \eta^2(\mathcal{T}_\ell, T) &:= \|h_{\mathcal{T}} \operatorname{curl}_{\mathrm{NC}} \sigma_h\|^2_{L^2(T)} + h_T \sum_{E \in \mathcal{E}(T)} \|[\sigma_h \tau_E]_E\|^2_{L^2(E)},\\ \mu^2(T) &:= \|\operatorname{dev}(\varphi - \Pi_{\Sigma_h(\mathcal{T})}\varphi)\|^2_{L^2(T)} \end{aligned} \tag{7.7}$$

and the global error estimators by the formulae (5.2). The following theorem states efficiency and reliability of η.

Theorem 7.9 (Efficiency, reliability)**.** *There exist constants $C_{\mathrm{eff}}, C_{\mathrm{rel}} > 0$ with*

$$C_{\mathrm{eff}}^{-2}\eta_\ell^2 \le \|\sigma - \sigma_h\|^2_{L^2(\Omega)} + \|\operatorname{Curl}(\alpha - \alpha_h)\|^2_{L^2(\Omega)} \le C_{\mathrm{rel}}^2 \eta_\ell^2$$

Proof. The proof is similar to that of Theorem 5.1 and therefore omitted. □

8. Adaptive algorithm

This section defines an adaptive algorithm in Section 8.1 for the linear elasticity problem (1.1) and the Stokes equations (6.1) and (7.1) and proves its optimal convergence rates for the Stokes equations (7.1) without symmetric gradient.

8.1. Adaptive algorithm

Let $\mathfrak{y}$ and μ denote the error estimators for linear elasticity from Section 5, for the Stokes equations with symmetric gradient from Section 6.3, or for the Stokes equations without symmetric gradient from Section 7.3. The adaptive algorithm is driven by these two error estimators and runs the following loop.

Algorithm 8.1 (AFEM)**.**

Input: Initial triangulation $\mathcal{T}_0$, parameters $0 < \theta_A \leq 1$, $0 < \rho_B < 1$, $0 < \kappa$.
for $\ell = 0, 1, 2, \dots$ **do**
Solve. Compute solution of (4.1), (6.6), or (7.5) with respect to $\mathcal{T}_\ell$.
Estimate. Compute local contributions of the error estimators $\left(\mathfrak{y}^2(\mathcal{T}_\ell, T)\right)_{T\in\mathcal{T}_\ell}$ and $\left(\mu^2(\mathcal{T}_\ell, T)\right)_{T\in\mathcal{T}_\ell}$.
if $\mu_\ell^2 \leq \kappa \mathfrak{y}_\ell^2$ **then**
Mark. The Dörfler marking chooses a minimal subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta_A \mathfrak{y}_\ell^2 \leq \mathfrak{y}_\ell^2(\mathcal{T}_\ell, \mathcal{M}_\ell)$.
Refine. Generate the smallest admissible refinement $\mathcal{T}_{\ell+1}$ of $\mathcal{T}_\ell$ in which at least all triangles in $\mathcal{M}_\ell$ are refined.
else
Mark. Compute a triangulation $\mathcal{T} \in \mathbb{T}$ with $\mu^2(\mathcal{T}) \leq \rho_B \mu_\ell^2$.
Refine. Generate the overlay $\mathcal{T}_{\ell+1}$ of $\mathcal{T}_\ell$ and $\mathcal{T}$.
end if
end for
Output: Sequences of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$, discrete solutions and error estimators $(\mathfrak{y}_\ell)_{\ell\in\mathbb{N}_0}$ and $(\mu_\ell)_{\ell\in\mathbb{N}_0}$. ♦

8.2. Remark on optimal convergence rates for the Stokes equations without symmetric gradient

Given an initial triangulation $\mathcal{T}_0$, recall the set of admissible triangulations $\mathbb{T}$ from Section 2. Let $\mathbb{T}(N)$ denote the subset of all admissible triangulations with at most $\mathrm{card}(\mathcal{T}_0) + N$ triangles. For $s > 0$ and $(\sigma, \alpha, \varphi) \in \Sigma \times X \times H(\mathrm{div}, \Omega; \mathbb{R}^{2\times 2})$ define the seminorm

$$\begin{aligned}|(\sigma,\alpha,\varphi)|_{\mathcal{A}_s} := \sup_{N\in\mathbb{N}_0} N^s \inf_{\mathcal{T}\in\mathbb{T}(N)} \Big(&\|\sigma - \Pi_{\Sigma_h(\mathcal{T})}\sigma\|_{L^2(\Omega)} \\ &+ \inf_{\beta_\mathcal{T}\in X_h(\mathcal{T})} \|\mathrm{Curl}(\alpha - \beta_\mathcal{T})\|_{L^2(\Omega)} + \|\mathrm{dev}(\varphi - \Pi_{\Sigma_h(\mathcal{T})}\varphi)\|_{L^2(\Omega)} \Big).\end{aligned} \tag{8.1}$$

Remark 8.2 (Pure local approximation class)**.** The result ([39], Thm. 2) proves

$$\min_{v_h\in P_{k+1}(\mathcal{T})\cap H^1(\Omega)} \|\nabla(v - v_h)\|_{L^2(\Omega)} \approx \|\nabla v - \Pi_{P_k(\mathcal{T};\mathbb{R}^2)} \nabla v\|_{L^2(\Omega)} \quad \text{for all } v \in H^1(\Omega)$$

and therefore the term

$$\inf_{\beta_\mathcal{T}\in X_h(\mathcal{T})} \|\mathrm{Curl}(\alpha - \beta_\mathcal{T})\|_{L^2(\Omega)}$$

in (8.1) can be replaced by the local term

$$\left\|\operatorname{Curl}\alpha - \Pi_{\Sigma_h(\mathcal{T})}\operatorname{Curl}\alpha\right\|_{L^2(\Omega)}. \qquad \square$$

We assume that the algorithm used in the marking step for $\mu_\ell^2 > \kappa\lambda_\ell^2$ is the algorithm `Approx` from [8, 17], *i.e.*, the thresholding second algorithm [7] followed by a completion algorithm. This ensures quasi optimal data approximation (axiom (B1) of [17]).

The following theorem states optimal convergence rates of Algorithm 8.1.

Theorem 8.3 (Optimal convergence rates of AFEM)**.** *Let $s > 0$. For $0 < \rho_B < 1$ and sufficiently small $0 < \kappa$ and $0 < \theta < 1$, Algorithm 8.1 for the Stokes problem (7.4) without symmetric gradient computes sequences of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ and discrete solutions $(\sigma_\ell, \alpha_\ell)_{\ell\in\mathbb{N}}$ for the right-hand side φ of optimal rate of convergence in the sense that*

$$(\operatorname{card}(\mathcal{T}_\ell) - \operatorname{card}(\mathcal{T}_0))^s \Big(\|\sigma - \sigma_\ell\|_{L^2(\Omega)} + \|\operatorname{Curl}(\alpha - \alpha_\ell)\|_{L^2(\Omega)}\Big) \lesssim |(\sigma, \alpha, \varphi)|_{\mathcal{A}_s}.$$

The proof follows from the abstract framework of [17] (see also [32]) as in [35, 36] for the Poisson problem and is therefore omitted. Details can be found in [34].

8.3. Remark on optimal convergence rates for an adaptive algorithm for linear elasticity and the Stokes equations with symmetric gradient

Optimal convergence rates of the adaptive algorithm 8.1 for linear elasticity or the Stokes equations with symmetric gradient do not follow from the abstract framework of [17] for two reasons. First, the local error estimator term $\|\operatorname{div}\alpha_h\|_{L^2(T)}$ does not involve the local mesh-size, and, hence, the reduction property does not follow in the usual way. On the one hand, this term can be bounded by [5]

$$h_T \sum_{E\in\mathcal{E}(T)\setminus\mathcal{E}(\partial\Omega)} \|[\nabla\alpha_h]\cdot\nu_E\|^2_{L^2(E)} \tag{8.2}$$

for the Taylor−Hood discretization from 4.3 for boundary edges $\mathcal{E}(\partial\Omega) := \{E \in \mathcal{E} \mid E \subseteq \partial\Omega\}$. On the other hand, (8.2) is only efficient for $\varphi - \sigma \in H(\operatorname{curl}, \Omega)$ and only up to oscillations of $\operatorname{curl}\operatorname{Curl}\alpha$.

Second, since the spaces

$$Z_h(\mathcal{T}) := \{\beta_h \in X_h(\mathcal{T}) \mid \forall\xi_h \in Y_h(\mathcal{T}) :\ (\xi_h, \operatorname{div}\beta_h)_{L^2(\Omega)} = 0\}$$

are not nested for nested triangulations, the discretizations of Section 4 lack a projection property. This disables the proof of the axiom (A3) quasi-orthogonality of [17] and, thus, the proof of optimal convergence rates of an adaptive algorithm. The same difficulties also arise for the approximation of the Stokes equations with Taylor−Hood FEMs: The work [25] states the quasi-orthogonality as an assumption without proof.

9. Numerical experiments

This section is devoted to numerical experiments for the Taylor−Hood discretization from Section 4.3 for $k = 1$ for linear elasticity in Sections 9.1 and 9.2 and the discretization (7.5) for the Stokes equations for $k = 0, 1$ in Sections 9.3 and 9.4. The experiments compare the errors and error estimators on a sequence of uniformly red-refined triangulations (that is, the midpoints of the edges of a triangle are connected; this generates four new triangles) with the errors and error estimators on a sequence of triangulations created by the adaptive algorithm 8.1. Furthermore, for linear elasticity the robustness of the method with respect to $\lambda \to \infty$ is compared with the conforming P_1 FEM defined in Section 9.1 below. The convergence history plots are logarithmically scaled and display the error $\|\sigma - \sigma_h\|_{L^2(\Omega)}$ against the number of degrees of freedom of the resulting linear system for the Schur complement.

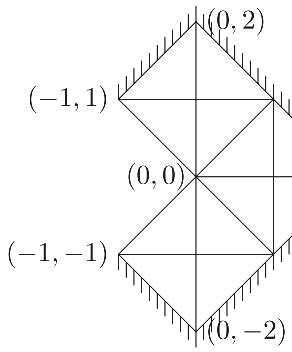

FIGURE 1. Initial triangulation of the L-shaped domain from Section 9.1.

9.1. Rotated L-shaped domain

Let $\Omega := \operatorname{conv}\{(-1,-1),(0,-2),(2,0),(1,1)\} \cup \operatorname{conv}\{(0,2),(-1,1),(0,0),(1,1)\}$ be the rotated L-shaped domain from Figure 1. The considered exact solution in radial components for the right-hand side $f = 0$ reads

$$u_r(r,\varphi) = \frac{r^\alpha}{2\mu}(-(\alpha+1)\cos((\alpha+1)\varphi) + (C_2-\alpha-1)\,C_1\cos((\alpha-1)\varphi)),$$
$$u_\varphi(r,\varphi) = \frac{r^\alpha}{2\mu}((\alpha+1)\sin((\alpha+1)\varphi) + (C_2+\alpha-1)\,C_1\sin((\alpha-1)\varphi))$$

with

$$C_1 := \frac{-\cos((\alpha+1)\,\omega)}{\cos((\alpha-1)\,\omega)}, \quad C_2 := \frac{2\,(\lambda+2\mu)}{\lambda+\mu},$$

and the positive solution $\alpha \approx 0.544483736782$ to

$$\alpha\sin(2\omega) + \sin(2\omega\alpha) = 0 \qquad \text{with} \qquad \omega = 3\pi/4.$$

Dirichlet boundary conditions are applied on

$$\Gamma_D := \operatorname{conv}\{(-1,-1),(0,-2)\} \cup \operatorname{conv}\{(0,-2),(2,0)\} \\ \cup \operatorname{conv}\{(2,0),(0,2)\} \cup \operatorname{conv}\{(0,2),(-1,1)\},$$

while $(\sigma\nu)|_{\Gamma_N} = g := 0$ on the Neumann boundary $\Gamma_N := \partial\Omega\backslash\Gamma_D$ (see Rem. 3.5 for mixed boundary conditions). Let $\varphi := 0 \in H(\operatorname{div},\Omega;\mathbb{S})$ and define $u_D := qu$ with

$$q(r,\theta) := \begin{cases} 0 & \text{if } r \le 1/2, \\ 16r^4 - 64r^3 + 88r^2 - 48r + 9 & \text{if } 1/2 \le r \le 1, \\ 1 & \text{if } r \ge 1. \end{cases}$$

Then $u_D \in H^2(\Omega;\mathbb{R}^2)$ and $u_D|_{\Gamma_D} = u|_{\Gamma_D}$.

The error estimator μ defined by (5.1) and (5.2) for non-homogeneous Dirichlet data is modified as

$$\mu^2(T) := \left\|(\varphi - \mathbb{C}Du_D) - \Pi_{\Sigma_h(T)}(\varphi - \mathbb{C}Du_D)\right\|^2_{\mathbb{C}^{-1/2},T}.$$

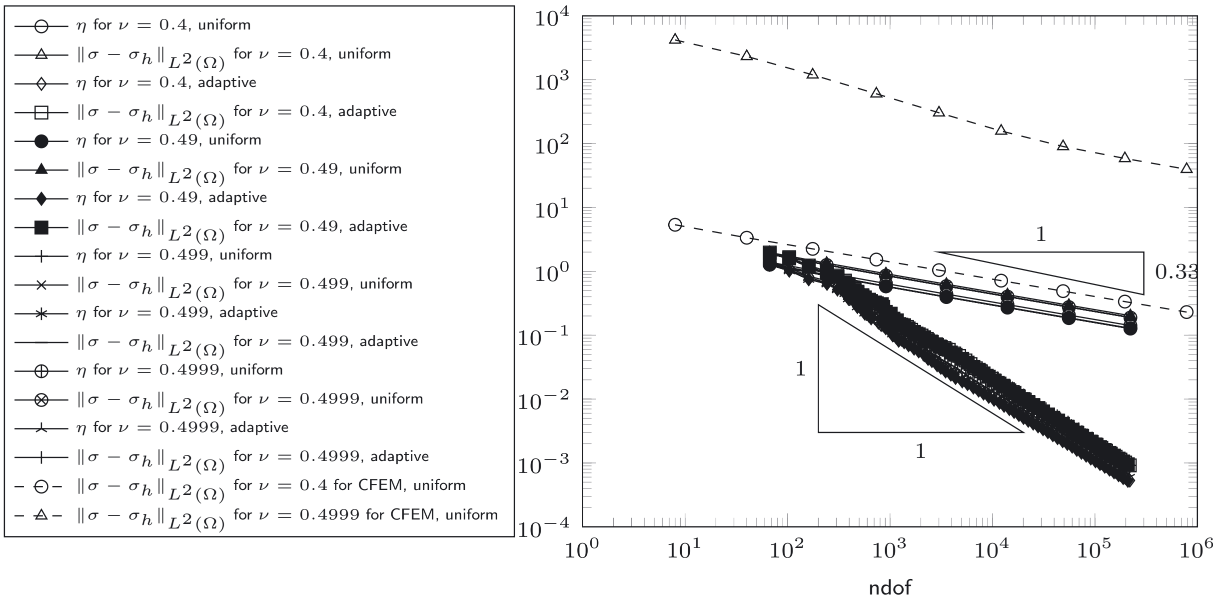

FIGURE 2. Errors and error estimators for the experiment on the rotated L-shaped domain from Section 9.1.

For edges on the Dirichlet boundary $E \in \mathcal{E}$, $E \subseteq \Gamma_D$, with the one adjacent triangle T_+, the jump reads

$$\left[\mathbb{C}^{-1}\sigma_h + \chi_h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]_E := \left(\mathbb{C}^{-1}\sigma_h + \chi_h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\Big|_{T_+} - Du_D.$$

The sum in the definition of the error estimator $\mathfrak{y}$ in (5.1) runs only over edges that do not lie on the Neumann boundary $\overline{\Gamma}_N$.

Let E denote Young's modulus and ν the Poisson ratio, *i.e.*,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \mu = \frac{E}{2(1+\nu)}$$

with $\lambda \to \infty$ if $\nu \to 1/2$.

The stress errors $\|\sigma - \sigma_h\|_{L^2(\Omega)}$ and the error estimators $\eta := \sqrt{\mu^2 + \mathfrak{y}^2}$ are computed on a sequence of uniformly red-refined triangulations and on a sequence of triangulations created by Algorithm 8.1 for Young's modulus $E = 10^5$ and Poisson ratios $\nu = 0.4$, 0.49, 0.499, and 0.4999. They are plotted against the number of degrees of freedom in Figure 2. For uniformly refined meshes, the convergence rates of $h^{2/3} \approx \texttt{ndof}^{-1/3}$ are suboptimal, while Algorithm 8.1 reveals the optimal convergence rate of $\texttt{ndof}^{-1}$.

In contrast to standard conforming low-order FEMs, the discretization (4.1), as predicted by Theorem 4.1, does not show a locking behaviour for $\lambda \to \infty$. To illustrate this, the errors $\|\sigma - \mathbb{C}\varepsilon(u_C)\|_{L^2(\Omega)}$ for the solution u_C of the conforming P_1 FEM are included in Figure 2 for comparison for Poisson ratios $\nu = 0.4$ and $\nu = 0.4999$ on a sequence of uniformly red-refined triangulations. While the size of the error of the conforming P_1 FEM has a strong dependence on λ, the errors of the discretization (4.1) are of the same size for all considered Poisson ratios.

Figure 3 depicts triangulations created by Algorithm 8.1 for Poisson ratios $\nu = 0.4$ and $\nu = 0.4999$ with approximately 1500 degrees of freedom. The singularity at $(0,0)$ leads to a strong refinement towards the re-entrant corner. For $\nu = 0.4$ and $\nu = 0.49$ all marking steps in Algorithm 8.1 used the Dörfler marking ($\mu_\ell^2 \le \kappa\mathfrak{y}_\ell^2$),

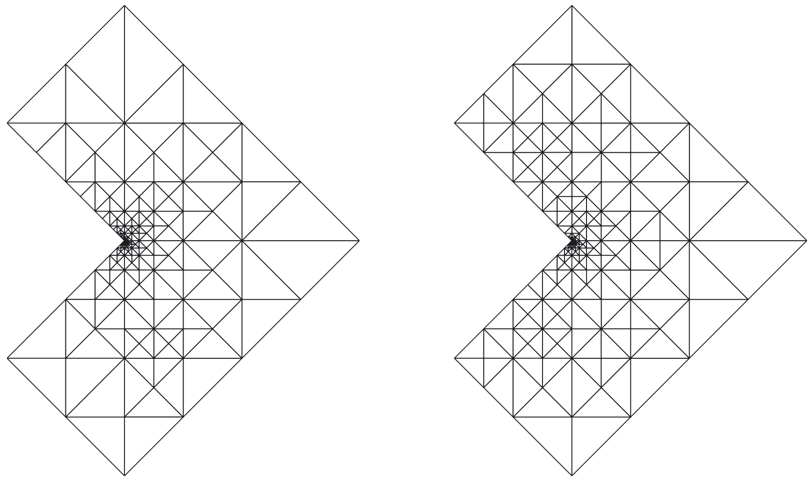

FIGURE 3. Adaptively refined triangulations for $\nu = 0.4$ and $\nu = 0.4999$ for the experiment on the rotated L-shaped domain from Section 9.1.

while for $\nu = 0.499$ the marking with respect to the data-approximation ($\mu_\ell^2 > \kappa\mathfrak{y}_\ell^2$ in Algorithm 8.1) was applied at level 54 (55 696 dofs) and level 62 (118 732 dofs) and for $\nu = 0.4999$ from level 11 onwards at approximately every third refinement.

9.2. L-shaped domain with piecewise constant f

This example considers the L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1] \times [-1,0])$ with pure Dirichlet boundary $\Gamma_D := \partial\Omega$. Define the piecewise constant volume force $f \in L^2(\Omega;\mathbb{R}^2)$ by

$$f(x,y) := \begin{cases} (0,0) & \text{if } x \le 0 \text{ and } y \ge 0, \\ (0,1) & \text{if } x, y \ge 0, \\ (1,0) & \text{if } x, y \le 0. \end{cases}$$

Define $\varphi = (\varphi_{11}, \varphi_{12}; \varphi_{21}, \varphi_{22}) \in H(\operatorname{div}, \Omega; \mathbb{S})$ with $-\operatorname{div}\varphi = f$ by

$$\begin{aligned} \varphi_{11} &:= \begin{cases} -x & \text{if } y \le 0, \\ 0 & \text{else,} \end{cases} \\ \varphi_{12} &:= \varphi_{21} := 0, \\ \varphi_{22} &:= \begin{cases} -y & \text{if } x \ge 0, \\ 0 & \text{else} \end{cases} \end{aligned}$$

and let $u_D = 0$.

The error estimators $\eta := \sqrt{\mu^2 + \mathfrak{y}^2}$ are plotted in Figure 4 on a sequence of uniformly red-refined triangulations and on a sequence of triangulations created by Algorithm 8.1 for Young's modulus $E = 10^5$ and Poisson ratios $\nu = 0.4$, 0.49, 0.499, and 0.4999. Uniform refinement yields a suboptimal convergence rate of $h^{2/3} \approx \texttt{ndof}^{-1/3}$, while Algorithm 8.1 recovers the optimal convergence rates of $\texttt{ndof}^{-1}$. For a comparison, the P_1 conforming FEM is computed. Since the exact solution is not known for this example, the efficient and reliable error estimator $\eta_{\mathrm{C}} = \eta_{\mathrm{C}}(\mathcal{T})$ (with efficiency and reliability constants that are independent of the Lamé

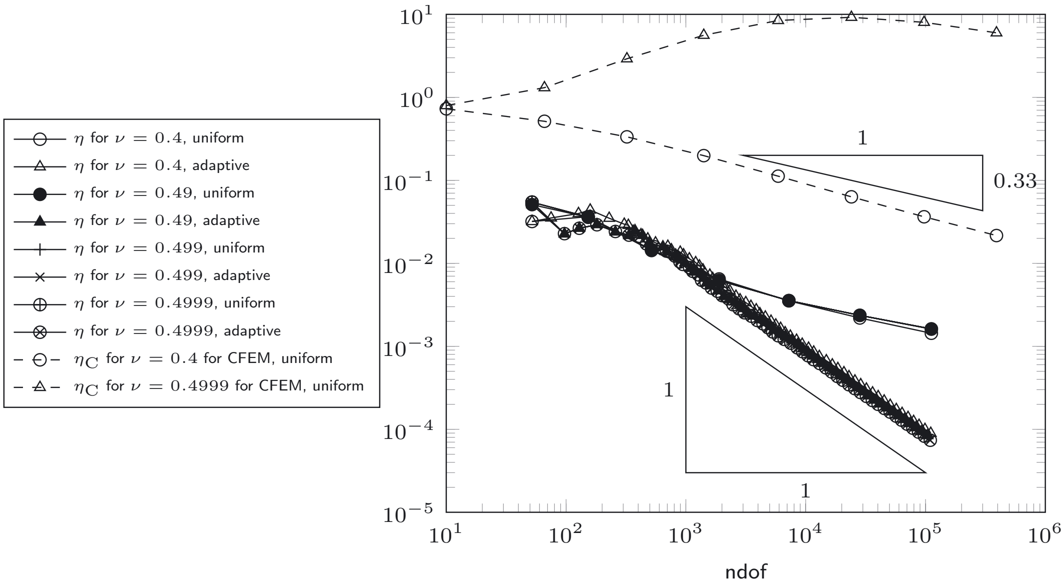

FIGURE 4. Error estimators for the experiment on the L-shaped domain from Section 9.2.

parameter λ [15]), defined by

$$\eta_{\mathrm{C}}^2(\mathcal{T},T) := \|h_{\mathcal{T}} f\|_{L^2(T)}^2 + h_T \sum_{E\in\mathcal{E}(T)\setminus\mathcal{E}(\Gamma_D)} \|[\mathbb{C}\varepsilon(u_{\mathrm{C}})]_E \nu_E\|_{L^2(E)}^2 \qquad \text{for all } T\in\mathcal{T},$$

$$\eta_{\mathrm{C}}(\mathcal{T}) := \sqrt{\sum_{T\in\mathcal{T}} \eta_{\mathrm{C}}^2(\mathcal{T},T)}$$

for $\mathcal{E}(\Gamma_D) := \{E\in\mathcal{E} \mid E\subseteq\Gamma_D\}$ the set of edges that lie on the Dirichlet boundary, is plotted in Figure 4. These error estimators are approximately 250 times larger for the Poisson ratio $\nu = 0.4999$ in comparison with the Poisson ratio $\nu = 0.4$ on a triangulation with 391170 dofs. The error estimators for the discretization from Section 4.3 are almost of the same size for all considered Poisson ratios. Figure 5 depicts triangulations created by Algorithm 8.1 with approximately 1500 degrees of freedom for Poisson ratios $\nu = 0.4$ and $\nu = 0.4999$. The strong refinement towards the singularity at the re-entrant corner is clearly visible. Since φ is piecewise affine, the error estimator μ with respect to the data approximation vanishes and only the Dörfler marking ($\mu_\ell^2 \le \kappa\mathfrak{y}_\ell^2$) was applied in Algorithm 8.1.

9.3. L-shaped domain for the Stokes equations

This subsection considers the Stokes problem (7.4) on the L-shaped domain $\Omega = ((-1,1)\times(-1,1))\setminus([0,1]\times[-1,0])$. The exact solution for the right-hand side $f=0$ and corresponding Dirichlet boundary conditions reads

$$u(r,\vartheta) = \begin{pmatrix} r^\alpha((1+\alpha)\sin(\vartheta)w(\vartheta) + \cos(\vartheta)w_\vartheta(\vartheta)) \\ r^\alpha(-(1+\alpha)\cos(\vartheta)w(\vartheta) + \sin(\vartheta)w_\vartheta(\vartheta)) \end{pmatrix}$$

in polar coordinates with $\alpha = 0.54448373$ and

$$\begin{aligned} w(\vartheta) =&(\sin((1+\alpha)\vartheta)\cos(\alpha\omega))/(1+\alpha) - \cos((1+\alpha)\vartheta) \\ &- (\sin((1-\alpha)\vartheta)\cos(\alpha\omega))/(1-\alpha) + \cos((1-\alpha)\vartheta). \end{aligned}$$

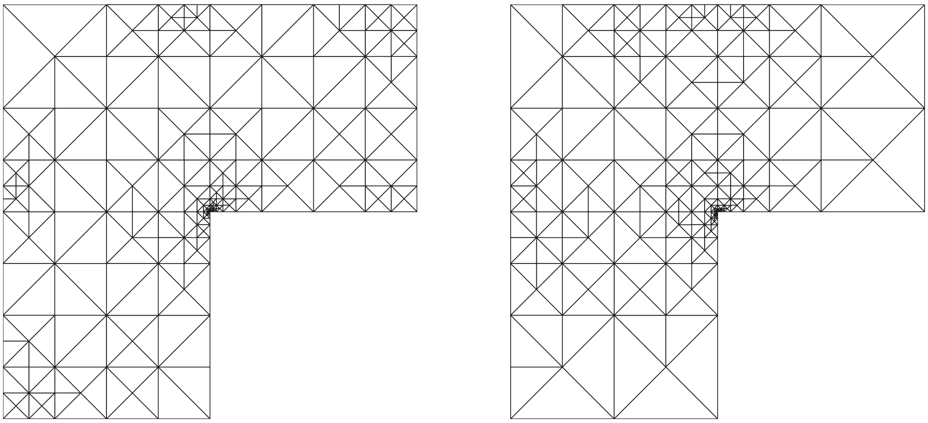

FIGURE 5. Adaptively refined triangulations for $\nu = 0.4$ and $\nu = 0.4999$ for the experiment on the L-shaped domain from Section 9.2.

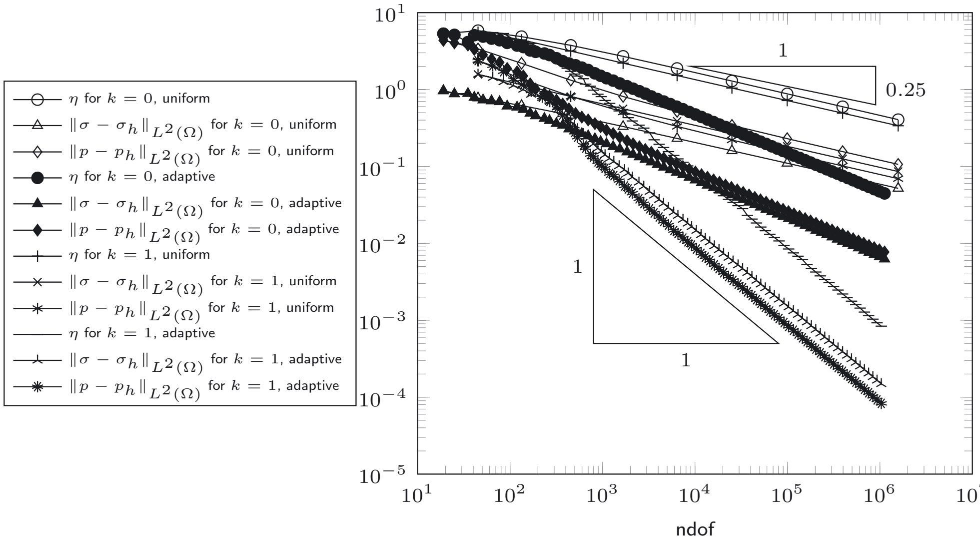

FIGURE 6. Errors and error estimators for the experiment on the L-shaped domain from Section 9.3.

Define

$$u_D(x, y) = \big(1 - x(x-1)(x+1)y(y-1)(y+1)\big)\, u(x, y)$$

and $\varphi = 0$. Then $u_D|_{\partial\Omega} = u|_{\partial\Omega}$.

The errors and error estimators for $k = 0, 1$ are plotted in Figure 6 against the number of degrees of freedom. For uniform refinement, the errors and error estimators show a convergence rate of $h^{1/2} \approx \texttt{ndof}^{1/4}$. The error $\|\sigma - \sigma_h\|_{L^2(\Omega)}$ for $k = 1$ lies even above the error $\|\sigma - \sigma_h\|_{L^2(\Omega)}$ for $k = 0$. Note that the saddle-point structure implies that this does not contradict the conformity of the method. The adaptive algorithm 8.1 with bulk parameter $\theta = 0.1$ and $\kappa = 0.5$ and $\rho = 0.75$ leads to an optimal convergence rate of $\texttt{ndof}^{(k+1)/2}$. Figure 7 depicts triangulations with approximately 1500 degrees of freedom created by the adaptive algorithm for $k = 0, 1$.

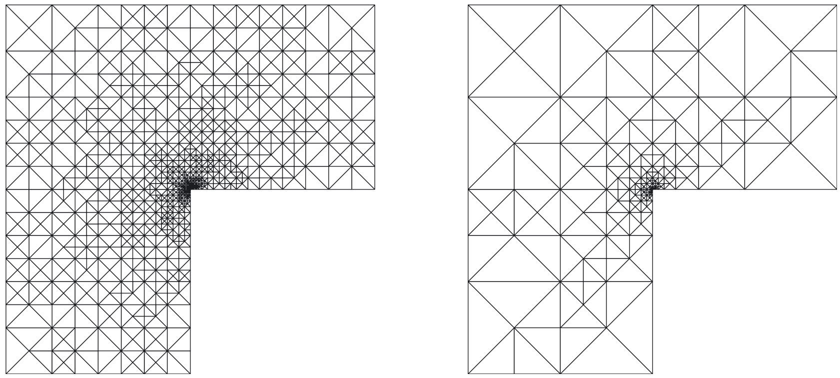

FIGURE 7. Adaptively refined triangulations for $k = 0$ and for $k = 1$ for the numerical experiment on the L-shaped domain from Section 9.3.

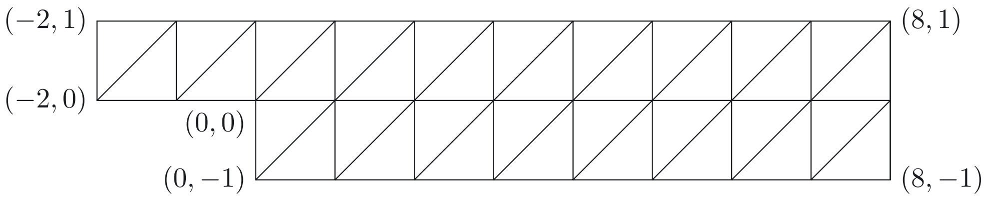

FIGURE 8. Initial mesh of the backward-facing step from Section 9.4.

The singularity of u leads to a strong refinement towards the re-entrant corner. The marking with respect to the data-approximation ($\mu_\ell^2 > \kappa\lambda_\ell^2$ in Algorithm 8.1) is only applied at the first three refinements for $k = 0$. All other marking steps for $k = 0, 1$ used the Dörfler marking ($\mu_\ell^2 \leq \kappa\lambda_\ell^2$).

9.4. Backward-facing step

This benchmark example considers the domain $\Omega = ((-2, 8) \times (-1, 1)) \setminus ([-2, 0] \times [-1, 0])$ with initial mesh from Figure 8 with volume force $f = 0$ and Dirichlet data u_D defined by

$$u_D|_{\partial\Omega}(x, y) = \begin{cases} (0, 0) & \text{if } -2 < x < 8, \\ (-y(y-1)/10, 0) & \text{if } x = -2, \\ (-(y+1)(y-1)/80, 0) & \text{if } x = 8. \end{cases}$$

The extension

$$u_D(x, y) = \begin{cases} (-x^2 y(y-1)/40, 0) & \text{if } x < 0, \\ (-x^2(y+1)(y-1)/5120, 0) & \text{if } x > 0 \end{cases}$$

of u_D to Ω and $\varphi = 0$ are chosen for the numerical computations.

The error estimators for $k = 0, 1$ are plotted in Figure 9 against the number of degrees of freedom. For uniform refinement and $k = 0$, the error estimator η shows a convergence rate of $h^{4/5} \approx \texttt{ndof}^{2/5}$. The error estimator for $k = 1$ shows a convergence rate of $h^{2/3} \approx \texttt{ndof}^{1/3}$ for uniform refinements. Since this is the expected convergence rate for the interior angle of $3\pi/2$ at the re-entrant corner for $k = 0, 1$, the better convergence rate for $k = 0$ is possibly a preasymptotic effect. This convergence rate for a first-order method was also observed

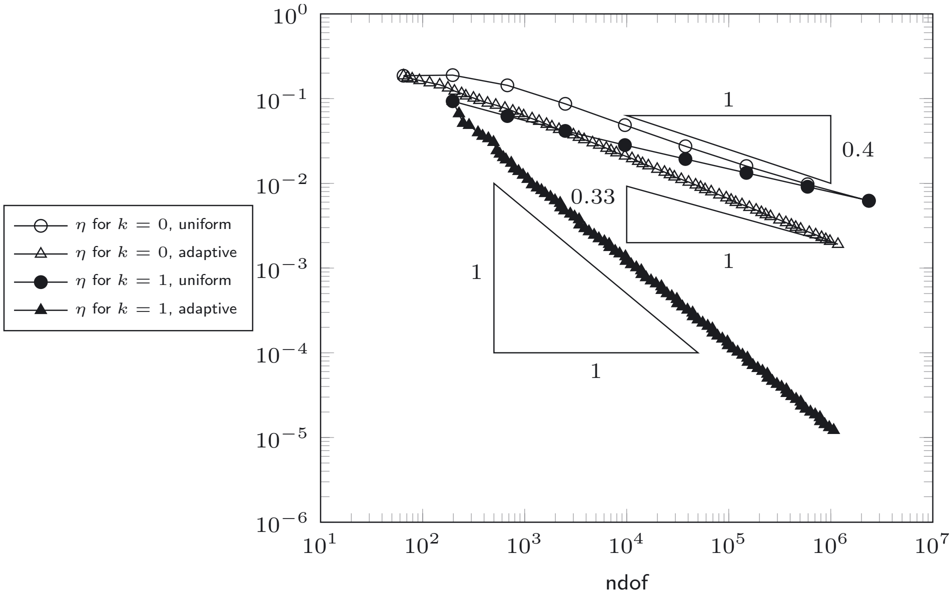

FIGURE 9. Error estimators for the backward facing step experiment from Section 9.4.

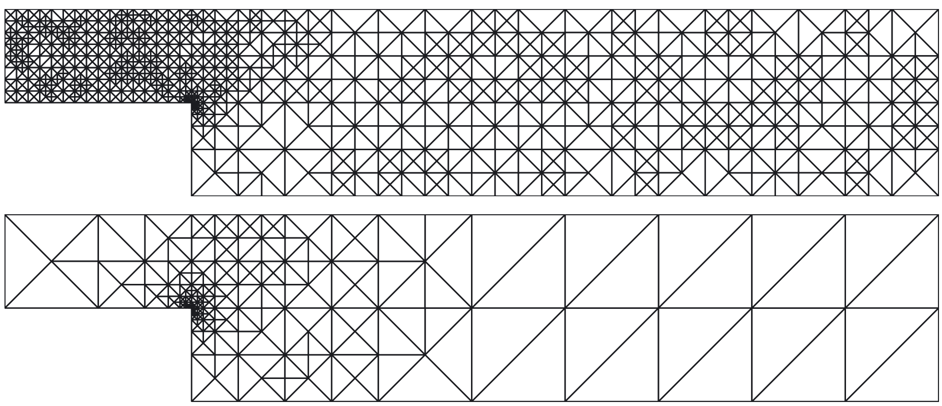

FIGURE 10. Adaptively refined triangulations for $k = 0$ and $k = 1$ for the numerical experiment for the backward-facing step from Section 9.4.

for a pseudostress approach in [19]. The adaptive refinement leads to optimal convergence rates of $\texttt{ndof}^{(k+1)/2}$. Figure 10 depicts triangulations with approximately 1500 degrees of freedom created by the adaptive algorithm for $k = 0, 1$. The singularity of u leads to a strong refinement towards the re-entrant corner. The marking with respect to the data-approximation ($\mu_\ell^2 > \kappa\lambda_\ell^2$ in Algorithm 8.1) is applied at the levels 25, 31, 37, 41, 47, 52, 58, 64, 69, 75, 80, and 85 for $k = 1$. All other marking steps for $k = 0, 1$ used the Dörfler marking ($\mu_\ell^2 \leq \kappa\lambda_\ell^2$).

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