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ERROR ESTIMATES FOR A NUMERICAL METHOD FOR THE COMPRESSIBLE NAVIER–STOKES SYSTEM ON SUFFICIENTLY SMOOTH DOMAINS *, **, ***

Eduard Feireisl[1,2], Radim Hošek[1,2], David Maltese[1,2] and Antonín Novotný[1,2]

Abstract. We derive an *a priori* error estimate for the numerical solution obtained by time and space discretization by the finite volume/finite element method of the barotropic Navier–Stokes equations. The numerical solution on a convenient polyhedral domain approximating a sufficiently smooth bounded domain is compared with an exact solution of the barotropic Navier–Stokes equations with a bounded density. The result is unconditional in the sense that there are no assumed bounds on the numerical solution. It is obtained by the combination of discrete relative energy inequality derived in [T. Gallouët, R. Herbin, D. Maltese and A. Novotný, *IMA J. Numer. Anal.* **36** (2016) 543–592.] and several recent results in the theory of compressible Navier–Stokes equations concerning blow up criterion established in [Y. Sun, C. Wang and Z. Zhang, *J. Math. Pures Appl.* **95** (2011) 36–47] and weak strong uniqueness principle established in [E. Feireisl, B.J. Jin and A. Novotný, *J. Math. Fluid Mech.* **14** (2012) 717–730].

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[1] Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic. feireisl@math.cas.cz

[2] Institut Mathématiques de Toulon, EA2134, University of Toulon, BP 20132, 839 57 La Garde, France.

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1. Introduction

We consider the compressible Navier–Stokes equations in the barotropic regime in a space-time cylinder $Q_T = (0,T) \times \Omega$, where $T > 0$ is arbitrarily large and $\Omega \subset R^3$ is a bounded domain:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}). \tag{1.2}$$

In equations (1.1) and (1.2) $\varrho = \varrho(t,x) \geq 0$ and $\mathbf{u} = \mathbf{u}(t,x) \in R^3$, $t \in [0,T)$, $x \in \Omega$ are the unknown density and velocity fields, while $\mathbb{S}$ and p are the viscous stress and pressure characterizing the fluid *via* the constitutive relations

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right), \ \mu > 0, \tag{1.3}$$

$$p \in C^2(0,\infty) \cap C^1[0,\infty), \ p(0) = 0, \ p'(\varrho) > 0 \text{ for all } \varrho \geq 0, \ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma - 1}} = p_\infty > 0, \tag{1.4}$$

where $\gamma \geq 1$.

The assumption $p'(0) > 0$ in (1.4) excludes the constitutive laws for pressure behaving as ϱ^γ as $\varrho \to 0^+$. The error estimates stated in Theorem 3.2 however still hold in the case $\lim_{\varrho \to 0^+} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = 0$, in particular for the isentropic pressure laws $p(\varrho) = \varrho^\gamma$. The proof contains some additional technical difficulties, see also Remark 3.2.

Equations (1.1) and (1.2) are completed with the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.5}$$

and initial conditions

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \ \varrho_0 > 0 \text{ in } \overline{\Omega}. \tag{1.6}$$

We notice that under assumption (1.3), we may write

$$\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = \mu \Delta \mathbf{u} + \frac{\mu}{3} \nabla_x \mathrm{div}_x \mathbf{u}. \tag{1.7}$$

The results on error estimates for numerical schemes for the compressible Navier–Stokes equations are in the mathematical literature on short supply. We refer the reader to papers of Liu [39, 40], Jovanović [28], Gallouet *et al.* [22].

In [22] the authors have developed a methodology of deriving unconditional error estimates for the numerical schemes to the compressible Navier–Stokes equations (1.1)–(1.6) and applied it to the numerical scheme (3.5)–(3.7) discretizing the system on polyhedral domains. They have obtained error estimates for the discrete solution with respect to a *classical solution* of the system on the same (polyhedral) domain. In spite of the fact that [22] provides the first and to the best of our knowledge so far the sole error estimate for discrete solutions of a finite volume/finite element approximation to a model of compressible fluids that does not need any assumed bounds on the numerical solution itself, it has two weak points: 1) The existence of classical solutions on at least a short time interval to the compressible Navier–Stokes equations is known for smooth C^3 domains (see [43] or [4]) but may not be in general true on the polyhedral domains. 2) The numerical solutions are compared with the classical exact solutions (as is usual in any previous existing mathematical literature). In this paper we address both points raised above and to a certain extent remove the limitations of the theory presented in [22].

More precisely, we generalize the result of Gallouet *et al.* ([22], Thm. 3.1) in two directions:

(1) The physical domain Ω filled by the fluid and the numerical domain Ω_h, $h > 0$ approximating the physical domain do not need to coincide.

(2) If the physical domain is sufficiently smooth (at least of class C^3) and the C^3-initial data satisfy natural compatibility conditions, we are able to obtain the unconditional error estimates with respect to any *weak exact solution with bounded density*.

As in [22], and in contrast with any other literature dealing with finite volume or mixed finite volume/finite element methods for compressible fluids [3, 10, 16–19, 23–25, 28, 29, 32–35, 44] and others) this result does not require any assumed bounds on the discrete solution: the sole bounds needed for the result are those provided by the numerical scheme. Moreover, in contrast with [22] and with all above mentioned papers, the exact solution is solely weak solution with bounded density. This seemingly weak hypothesis is compensated by the regularity and compatibility conditions imposed on initial data that make possible a (sophisticated) bootstrapping argument showing that weak solutions with bounded density are in fact strong solutions in the class investigated in [22].

These results are achieved by using the following tools:

(1) The technique introduced in [22] modified in order to accommodate non-zero velocity of the exact sample solution on the boundary of the numerical domain.
(2) Three fundamental recent results from the theory of compressible Navier–Stokes equations, namely
   - Local in time existence of strong solutions in class (2.11) and (2.12) by Cho *et al.* [4].
   - Weak strong uniqueness principle proved in [13] (see also [14]).
   - Blow up criterion for strong solutions in the class (2.11) and (2.12) by Sun *et al.* [41].

   The three above mentioned items allow to show that the weak solution with bounded density emanating from the sufficiently smooth initial data is in fact a strong solution defined on the large time interval $[0, T)$.
(3) Bootstrapping argument using recent results on maximal regularity for parabolic systems by Danchin [8], Denk *et al.* [5] and Krylov [36]. The last item allows to bootstrap the strong solution in the class Cho *et al.* [4] to the class needed for the error estimates in [22], provided a certain compatibility condition for the initial data is satisfied.

2. Preliminaries

2.1. Weak and strong solutions to the Navier–Stokes system

We introduce the notion of the weak solution to system (1.1)–(1.4):

Definition 2.1 (Weak solutions). *Let $\varrho_0 : \Omega \to [0, +\infty)$ and $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^3$ with finite energy $E_0 = \int_\Omega (\frac{1}{2}\varrho_0|\boldsymbol{u}_0|^2 + H(\varrho_0))\,\mathrm{d}x$ and finite mass $0 < M_0 = \int_\Omega \varrho_0\,\mathrm{d}x$. We shall say that the pair $(\varrho, \boldsymbol{u})$ is a weak solution to the problem (1.1)–(1.6) emanating from the initial data $(\varrho_0, \boldsymbol{u}_0)$ if:*

(a) $\varrho \in C_{\mathrm{weak}}([0,T]; L^a(\Omega))$, *for a certain* $a > 1$, $\varrho \geq 0$ *a.e. in* $(0,T)$, *and* $\boldsymbol{u} \in L^2(0,T; W^{1,2}_0(\Omega; \mathbb{R}^3))$.
(b) *the continuity equation* (1.1) *is satisfied in the following weak sense*

$$\int_\Omega \varrho\varphi\,\mathrm{d}x\Big|_0^\tau = \int_0^\tau \int_\Omega (\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t,\ \forall\tau\in[0,T],\ \forall\varphi\in C_c^\infty([0,T]\times\overline{\Omega}). \tag{2.1}$$

(c) $\varrho\boldsymbol{u} \in C_{\mathrm{weak}}([0,T]; L^b(\Omega;\mathbb{R}^3))$, *for a certain* $b > 1$, *and the momentum equation* (1.2) *is satisfied in the weak sense,*

$$\begin{aligned}\int_\Omega \varrho\boldsymbol{u}\cdot\varphi\,\mathrm{d}x\Big|_0^\tau &= \int_0^\tau\int_\Omega (\varrho\boldsymbol{u}\cdot\partial_t\varphi + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\varphi + p(\varrho)\,\mathrm{div}\,\varphi)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \int_0^\tau\int_\Omega (\mu\nabla\boldsymbol{u} : \nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t + (\mu+\lambda)\mathrm{div}\boldsymbol{u}\mathrm{div}\varphi)\,\mathrm{d}x\,\mathrm{d}t,\ \forall\tau\in[0,T],\ \forall\varphi\in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3).\end{aligned} \tag{2.2}$$

(d) *The following energy inequality is satisfied*

$$\int_\Omega \left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + H(\varrho) \right) \mathrm{d}x \Big|_0^\tau + \int_0^\tau \int_\Omega \left(\mu|\nabla \boldsymbol{u}|^2 + (\mu+\lambda)|\operatorname{div} \boldsymbol{u}|^2 \right) \mathrm{d}x\,\mathrm{d}t \leq 0, \ \textit{for a.a. } \tau \in (0,T), \tag{2.3}$$

$$\textit{with } \ H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \mathrm{d}z. \tag{2.4}$$

Here and hereafter the symbol $\int_\Omega g\,\mathrm{d}x\,|_0^\tau$ *is meant for* $\int_\Omega g(\tau,x)\,\mathrm{d}x - \int_\Omega g_0(x)\,\mathrm{d}x$.

In the above definition, we tacitly assume that all the integrals in the formulas (2.1)–(2.3) are defined and we recall that $C_{\rm weak}([0,T];L^a(\Omega))$ is the space of functions of $L^\infty([0,T];L^a(\Omega))$ which are continuous as functions of time in the weak topology of the space $L^a(\Omega)$.

We notice that the function $\varrho \mapsto H(\varrho)$ is a solution of the ordinary differential equation $\varrho H'(\varrho) - H(\varrho) = p(\varrho)$ with the constant of integration fixed such that $H(1) = 0$.

Note that the existence of weak solutions emanating from the finite energy initial data is well-known on bounded Lipschitz domains provided $\gamma > 3/2$, see Lions [38] for 'large' values of γ, Feireisl and coauthors [12] for $\gamma > 3/2$.

Proposition 2.2. *Suppose the* $\Omega \subset R^3$ *is a bounded domain of class* C^3. *Let* r, $\mathbf{V}$ *be a weak solution to problem* (1.1)–(1.6) *in* $(0,T)\times\Omega$, *originating from the initial data*

$$r_0 \in C^3(\overline{\Omega}), \ r_0 > 0 \ \textit{in } \overline{\Omega}, \tag{2.5}$$

$$\mathbf{V}_0 \in C^3(\overline{\Omega}; R^3), \tag{2.6}$$

satisfying the compatibility conditions

$$\mathbf{V}_0|_{\partial\Omega} = 0, \ \nabla_x p(r_0)|_{\partial\Omega} = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{V}_0)|_{\partial\Omega}, \tag{2.7}$$

and such that

$$0 \leq r \leq \overline{r} \ \textit{a.a. in } (0,T)\times\Omega. \tag{2.8}$$

Then r, $\mathbf{V}$ *is a classical solution satisfying the bounds:*

$$\|1/r\|_{C([0,T]\times\overline{\Omega})} + \|r\|_{C^1([0,T]\times\overline{\Omega})} + \|\partial_t \nabla_x r\|_{C([0,T];L^6(\Omega;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(\Omega))} \leq D, \tag{2.9}$$

$$\|\mathbf{V}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{C([0,T];L^6(\Omega;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(\Omega))} \leq D, \tag{2.10}$$

where D *depends on* Ω, T, $\overline{r}$, *and the initial data* r_0, $\mathbf{V}_0$ *(via* $\|(r_0,\mathbf{V}_0)\|_{C^3(\overline{\Omega};R^4)}$ *and* $\min_{x\in\overline{\Omega}} r_0(x)$*).*

Proof. The proof will be carried over in several steps.

Step 1.

According to Cho *et al.* [4], problem (1.1)–(1.6) admits a strong solution unique in the class

$$r \in C([0,T_M); W^{1,6}(\Omega)), \ \partial_t r \in C([0,T_M); L^6(\Omega)), \ 1/r \in L^\infty(Q_T), \tag{2.11}$$

$$\mathbf{V} \in C([0,T_M]; W^{2,2}(\Omega;R^3)) \cap L^2(0,T_M; W^{2,6}(\Omega;R^3)), \ \partial_t \mathbf{V} \in L^2(0,T_M; W^{1,2}_0(\Omega;R^3)). \tag{2.12}$$

defined on a time interval $[0, T_M)$, where $T_M > 0$ is finite or infinite and depends on the initial data. Moreover, for any $T_M^* < T_M$, there is a constant $c = c(T_M^*)$ such that

$$\|r\|_{L^\infty(0,T_M^*;W^{1,6}(\Omega))} + \|\partial_t r\|_{L^\infty(0,T_M^*;L^6(\Omega))} + \|1/r\|_{L^\infty(Q_T)} \tag{2.13}$$

$$+\|\mathbf{V}\|_{L^\infty(0,T_M^*;W^{2,2}(\Omega;R^3))} + \|\mathbf{V}\|_{L^2(0,T_M^*;W^{2,6}(\Omega;R^3))} + \|\partial_t \mathbf{V}\|_{L^2(0,T_M^*;W^{1,2}(\Omega))}$$

$$\leq c\left(\|r_0\|_{W^{1,6}(\Omega)} + \|\mathbf{V}_0\|_{W^{2,2}(\Omega)}\right).$$

Step 2.
By virtue of the weak-strong uniqueness result stated in ([13], Thm. 4.1) (see also [14], Thm. 4.6), the weak solution r, $\mathbf{V}$ coincides on the time interval $[0, T_M)$ with the strong solution, the existence of which is claimed in the previous step. According to Sun *et al.* ([41], Thm. 1.3) if $T_M < \infty$ then

$$\limsup_{t \to T_M -} \|r(t)\|_{L^\infty(\Omega)} = \infty.$$

Since (2.8) holds, we infer that $T_M = T$. At this point we conclude that couple $(r, \mathbf{V})$ possesses regularity (2.11) and (2.12) and that that the bound (2.13) holds with c dependent solely on T.

Step 3.
Since the initial data enjoy the regularity and compatibility conditions stated in (2.5)–(2.7), a straightforward bootstrap argument gives rise to better bounds, specifically, the solution belongs to the Valli–Zajaczkowski (see [43], Thm. 2.5) class

$$r \in C([0,T]; W^{3,2}(\Omega)),\ \partial_t r \in L^2(0,T; W^{2,2}(\Omega)), \tag{2.14}$$

$$\mathbf{V} \in C([0,T]; W^{3,2}(\Omega)) \cap L^2(0,T; W^{4,2}(\Omega; R^3)),\ \partial_t \mathbf{V} \in L^2(0,T; W^{2,2}(\Omega; R^3)), \tag{2.15}$$

where, similarly to the previous step, the norms depend only on the initial data, $\overline{r}$, and T.

Step 4.
We write equation (1.2) in the form

$$\partial_t \mathbf{V} - \frac{1}{r}\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{V}) = -\mathbf{V} \cdot \nabla_x \mathbf{V} + \frac{1}{r}\nabla_x p(r), \tag{2.16}$$

where, by virtue of (2.15) and a simple interpolation argument, $\mathbf{V} \in C^{1+\nu}([0,T] \times \overline{\Omega}; R^{3\times 3})$, and, by the same token $r \in C^{1+\nu}([0,T] \times \overline{\Omega})$ for some $\nu > 0$. Consequently, by means of the standard theory of parabolic equations, see for instance Ladyzhenskaya *et al.* [37], we may infer that r, $\mathbf{V}$ is a classical solution,

$$\partial_t \mathbf{V},\ \nabla_x^2 \mathbf{V} \text{ Hölder continuous in } [0,T] \times \overline{\Omega}. \tag{2.17}$$

and, going back to (1.1),

$$\partial_t r \text{ Hölder continuous in } [0,T] \times \overline{\Omega}. \tag{2.18}$$

Step 5.
We write

$$\nabla_x \partial_t r = -\nabla_x \mathbf{V} \cdot \nabla_x r - \mathbf{V} \cdot \nabla_x^2 r - \nabla_x r \mathrm{div}_x \mathbf{V} - r \nabla_x \mathrm{div}_x \mathbf{V};$$

whence, by virtue (2.14), (2.17), (2.18), and the Sobolev embedding $W^{1,2} \hookrightarrow L^6$,

$$\partial_t r \in C([0,T]; W^{1,6}(\Omega)). \tag{2.19}$$

Next, we differentiate (2.16) with respect to t. Denoting $\mathbf{Z} = \partial_t \mathbf{V}$ we therefore obtain

$$\partial_t \mathbf{Z} - \frac{1}{r}\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{Z}) + \mathbf{V}\cdot\nabla_x \mathbf{Z} = \partial_t\left(\frac{1}{r}\right)\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{V}) - \partial_t \mathbf{V}\cdot\nabla_x \mathbf{V} + \partial_t\left(\frac{1}{r}\nabla_x p(r)\right), \tag{2.20}$$

where, in view of (2.19) and the previously established estimates, the expression on the right-hand side is bounded in $C([0,T];L^6(\Omega;R^3))$. Thus using the L^p-maximal regularity (see Denk *et al.* [5], Krylov [36] or Danchin [8], Thm. 2.2), we deduce that

$$\partial^2_{t,t}\mathbf{V} = \partial_t \mathbf{Z} \in L^2(0,T;L^6(\Omega;R^3)),\ \partial_t \mathbf{V} = \mathbf{Z} \in C([0,T];W^{1,6}(\Omega;R^3)). \tag{2.21}$$

Finally, writing

$$\partial^2_{t,t} r = -\partial_t \mathbf{V}\cdot\nabla_x r - \mathbf{V}\cdot\partial_t\nabla_x r - \partial_t r \mathrm{div}_x \mathbf{V} - r\partial_t \mathrm{div}_x \mathbf{V},$$

and using (2.19), (2.21), we obtain the desired conclusion

$$\partial^2_{t,t} r \in C([0,T];L^6(\Omega)). \qquad \square$$

Here and hereafter, we shall use notation $a \stackrel{<}{\sim} b$ and $a \approx b$. the symbol $a \stackrel{<}{\sim} b$ means that there exists $c = c(\Omega, T, \mu, \gamma) > 0$ such that $a \leq cb$; $a \approx b$ means $a \stackrel{<}{\sim} b$ and $b \stackrel{<}{\sim} a$.

2.2. Extension lemma

Lemma 2.3. *Under the hypotheses of Proposition 2.2, the functions r and $\mathbf{V}$ can be extended outside Ω in such a way that:*

(1) *The extended functions (still denoted by r and $\mathbf{V}$) are such that $\mathbf{V}$ is compactly supported in $[0,T]\times\mathbb{R}^3$ and $r \geq \underline{r} > 0$.*

(2)

$$\|\mathbf{V}\|_{C^1([0,T]\times R^3;R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(R^3;R^3))} + \|\partial_t\nabla_x \mathbf{V}\|_{C([0,T];L^6(R^3;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(R^3))} \tag{2.22}$$

$$\stackrel{<}{\sim} \|\mathbf{V}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t\nabla_x \mathbf{V}\|_{C([0,T];L^6(\Omega;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(\Omega))};$$

(3)

$$\|r\|_{C^1([0,T]\times R^3)} + \|\partial_t\nabla_x r\|_{C([0,T];L^6(R^3;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(R^3))} \tag{2.23}$$

$$\stackrel{<}{\sim} \|r\|_{C^1([0,T]\times\overline{\Omega})} + \|\partial_t\nabla_x r\|_{C([0,T];L^6(\Omega;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(\Omega))} +$$

$$\|\mathbf{V}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(\overline{\Omega};R^3))}\| + \partial_t\nabla_x \mathbf{V}\|_{C([0,T];L^6(\Omega;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(\Omega))};$$

(4)

$$\partial_t r + \mathrm{div}_x(r\mathbf{V}) = 0 \ \textit{in}\ (0,T)\times R^3. \tag{2.24}$$

Proof. We first construct the extension of the vector field $\mathbf{V}$. To this end, we follow the standard construction in the flat domain, see Adams ([1], Chap. 5, Thm. 5.22) and combine it with the standard procedure of 'flattening' of the boundary and the partition of unity technique, we get (2.22) Once this is done, we solve on the whole space the transport equation (2.24). It is easy to show that the unique solution r of this equation possesses regularity and estimates stated in (2.23). $\square$

Remark 2.4. Here and hereafter, we denote $X_T(\mathbb{R}^3)$ a subset of $L^2((0,T)\times\mathbb{R}^3)$ of couples $(r,\mathbf{V})$, $r>0$ with finite norm

$$\|(r,\mathbf{V})\|_{X_T(\mathbb{R}^3)} \equiv \|r\|_{C^1([0,T]\times R^3)} + \|\partial_t\nabla_x r\|_{C([0,T];L^6(R^3;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(R^3))} \tag{2.25}$$

$$\|\mathbf{V}\|_{C^1([0,T]\times R^3;R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(R^3;R^3))} + \|\partial_t\nabla_x\mathbf{V}\|_{C([0,T];L^6(R^3;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(R^3))}$$

We notice that if r, $\mathbf{V}$ are interrelated through (2.7), then the first component of the couple belonging to $X_T(\mathbb{R}^3)$ is always strictly positive on $[0,T]\times R^3$. We set

$$0<\underline{r}=\min_{(t,x)\in[0,T]\times\mathbb{R}^3} r(t,x), \quad \overline{r}=\max_{(t,x)\in[0,T]\times\mathbb{R}^3} r(t,x)<\infty \tag{2.26}$$

2.3. Physical domain, mesh approximation

The physical space is represented by a bounded domain $\Omega\subset R^3$ of class C^3. The numerical domains Ω_h are polyhedral domains,

$$\overline{\Omega}_h = \cup_{K\in\mathcal{T}} K, \tag{2.27}$$

where $\mathcal{T}$ is a set of tetrahedra which have the following property: If $K\cap L\neq\emptyset$, $K\neq L$, then $K\cap L$ is either a common face, or a common edge, or a common vertex. By $\mathcal{E}(K)$, we denote the set of the faces σ of the element $K\in\mathcal{T}$. The set of all faces of the mesh is denoted by $\mathcal{E}$; the set of faces included in the boundary $\partial\Omega_h$ of Ω_h is denoted by $\mathcal{E}_{\rm ext}$ and the set of internal faces (*i.e.* $\mathcal{E}\setminus\mathcal{E}_{\rm ext}$) is denoted by $\mathcal{E}_{\rm int}$.

Further, we ask

$$\mathcal{V}_h\in\partial\Omega_h \text{ a vertex } \Rightarrow\ \mathcal{V}_h\in\partial\Omega. \tag{2.28}$$

Furthermore, we suppose that each K is a tetrahedron such that

$$\xi[K]\approx \mathrm{diam}[K]\approx h, \tag{2.29}$$

where $\xi[K]$ is the radius of the largest ball contained in K.

The properties of this mesh needed in the sequel are formulated in the following lemma, whose proof is left to the reader, see Johnson and Nedelec [27] for the $2D$ case, and [26] for the general $3D$ case.

Lemma 2.5. *There exists a positive constant d_Ω depending solely on the geometric properties of $\partial\Omega$ such that*

$$\mathrm{dist}[x,\partial\Omega]\leq d_\Omega h^2,$$

for any $x\in\partial\Omega_h$. Moreover,

$$|(\Omega_h\setminus\Omega)\cup(\Omega\setminus\Omega_h)|\lesssim h^2.$$

We find important to emphasize that $\Omega_h\not\subset\Omega$, in general.

2.4. Numerical spaces

We denote by $Q_h(\Omega_h)$ the space of piecewise constant functions:

$$Q_h(\Omega_h)=\{q\in L^2(\Omega_h)\,|\ \forall K\in\mathcal{T},\ q_{|K}\in\mathbb{R}\}. \tag{2.30}$$

For a function v in $C(\overline{\Omega}_h)$, we set

$$v_K=\frac{1}{|K|}\int_K v\,\mathrm{d}x \text{ for } K\in\mathcal{T} \text{ and } \Pi_h^Q v(x)=\sum_{K\in\mathcal{T}} v_K 1_K(x),\ x\in\Omega. \tag{2.31}$$

Here and in what follows, 1_K is the characteristic function of K.

We define the Crouzeix–Raviart space with 'zero traces':

$$V_{h,0}(\Omega_h) = \{v \in L^2(\Omega_h),\ \forall K \in \mathcal{T},\ v_{|K} \in \mathbb{P}_1(K), \tag{2.32}$$

$$\forall \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L,\ \int_\sigma v_{|K}\,{\rm d}S = \int_\sigma v_{|L}\,{\rm d}S, \quad \forall \sigma' \in \mathcal{E}_{\rm ext},\ \int_{\sigma'} v\,{\rm d}S = 0\},$$

and 'with general traces'

$$V_h(\Omega_h) = \{v \in L^2(\Omega),\ \forall K \in \mathcal{T},\ v_{|K} \in \mathbb{P}_1(K),\ \forall \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L,\ \int_\sigma v_{|K}\,{\rm d}S = \int_\sigma v_{|L}\,{\rm d}S\}. \tag{2.33}$$

We denote by Π_h^V the standard Crouzeix–Raviart projection, and $\Pi_{h,0}^V$ the Crouzeix–Raviart projection with 'zero trace', specifically,

$$\Pi_h^V : C(\overline{\Omega}_h) \to V_h(\Omega_h),\ \int_\sigma \Pi_h^V[\phi]\ {\rm dS}_x = \int_\sigma \phi\ {\rm dS}_x \text{ for all } \sigma \in \mathcal{E},$$

$$\Pi_{h,0}^V : C(\overline{\Omega}_h) \to V_{h,0}(\Omega_h),\ \int_\sigma \Pi_{h,0}^V[\phi]\ {\rm dS}_x = \int_\sigma \phi\ {\rm dS}_x \text{ for all } \sigma \in \mathcal{E}_{\rm int}, \tag{2.34}$$

$$\int_\sigma \Pi_{h,0}^V[\phi]\ {\rm dS}_x = 0 \text{ whenever } \sigma \in \mathcal{E}_{\rm ext}.$$

If $v \in W^{1,1}(\Omega_h)$, we set

$$v_\sigma = \frac{1}{|\sigma|}\int_\sigma v {\rm d}S \text{ for } \sigma \in \mathcal{E}. \tag{2.35}$$

(See *e.g.* [9], Sect. 4.3) for the definition of traces of functions in $W^{1,1}$.)

Each element $v \in V_h(\Omega_h)$ can be written in the form

$$v(x) = \sum_{\sigma\in\mathcal{E}} v_\sigma \varphi_\sigma(x), \quad x \in \Omega_h, \tag{2.36}$$

where the set $\{\varphi_\sigma\}_{\sigma\in\mathcal{E}} \subset V_h(\Omega_h)$ is the classical Crouzeix–Raviart basis determined by

$$\forall (\sigma,\sigma') \in \mathcal{E}^2,\ \frac{1}{|\sigma'|}\int_{\sigma'} \varphi_\sigma\,{\rm d}S = \delta_{\sigma,\sigma'}. \tag{2.37}$$

Similarly, each element $v \in V_{h,0}(\Omega_h)$ can be written in the form

$$v(x) = \sum_{\sigma\in\mathcal{E}_{\rm int}} v_\sigma \varphi_\sigma(x), \quad x \in \Omega_h. \tag{2.38}$$

We first recall in Lemmas 2.6)–(2.10 the standard properties of the projection Π_h^V. The collection of their proofs in the requested generality can be found in the Appendix of [22] with exception of Lemma 2.11 and its Corollary 2.12. We refer to the monograph of Brezzi and Fortin [2], the Crouzeix's and Raviart's paper [6], Gallouet *et al.* [21] for the original versions of some of these proofs. We present the proof of Lemma 2.11 dealing with the comparison of projections Π_h^V and $\Pi_{h,0}^V$ that we did not find in the literature.

Lemma 2.6. *The following estimates hold true:*

$$\|\Pi_h^V[\phi]\|_{L^\infty(K)} + \|\Pi_{h,0}^V[\phi]\|_{L^\infty(K)} \stackrel{<}{\sim} \|\phi\|_{L^\infty(K)}, \tag{2.39}$$

for all $K \in \mathcal{T}$ *and* $\phi \in C(K)$*;*

$$\|\phi - \Pi_h^V[\phi]||_{L^p(K)} \lesssim h^s \|\nabla^s \phi\|_{L^p(K;\mathbb{R}^{d^s})},\ s = 1, 2,\ 1 \le p \le \infty, \tag{2.40}$$

and

$$||\nabla(\phi - \Pi_h^V[\phi])||_{L^p(K;\mathbb{R}^d)} \le ch^{s-1} \|\nabla^s \phi\|_{L^p(K;\mathbb{R}^{d^s})},\ s = 1, 2,\ 1 \le p \le \infty, \tag{2.41}$$

for all $K \in \mathcal{T}$ *and* $\phi \in C^s(K)$*.*

Lemma 2.7. *Let* $1 \le p < \infty$*. Then*

$$\sum_{\sigma \in \mathcal{E}} |\sigma| h |v_\sigma|^p \approx ||v||^p_{L^p(\Omega_h)}, \tag{2.42}$$

with any $v \in V_h(\Omega_h)$*.*

Lemma 2.8. *The following Sobolev-type inequality holds true:*

$$||v||^2_{L^6(\Omega_h)} \lesssim \sum_{K \in \mathcal{T}} \int_K |\nabla_x v|^2 \mathrm{d}x, \tag{2.43}$$

with any $v \in V_{h,0}(\Omega_h)$*.*

Lemma 2.9. *There holds:*

$$\sum_{K \in \mathcal{T}} \int_K q\ \operatorname{div} \Pi_h^V[\boldsymbol{v}]\, \mathrm{d}x = \int_\Omega q\ \operatorname{div} \boldsymbol{v}\, \mathrm{d}x, \tag{2.44}$$

for all $\boldsymbol{v} \in C^1(\overline{\Omega}_h, \mathbb{R}^d)$ *and all* $q \in Q_h(\Omega_h)$*.*

Lemma 2.10 (Jumps over faces in the Crouzeix–Raviart space)**.** *For all* $v \in V_{h,0}(\Omega_h)$ *there holds*

$$\sum_{\sigma \in \mathcal{E}} \frac{1}{h} \int_\sigma [v]^2_{\sigma, \boldsymbol{n}_\sigma}\, \mathrm{d}S \lesssim \sum_{K \in \mathcal{T}} \int_K |\nabla_x v|^2 \mathrm{d}x, \tag{2.45}$$

where $[v]_{\sigma,\boldsymbol{n}_\sigma}$ *is a jump of* v *with respect to a normal* $\boldsymbol{n}_\sigma$ *to the face* σ*,*

$$\forall x \in \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \quad [v]_{\sigma,\boldsymbol{n}_\sigma}(x) = \begin{cases} v|_K(x) - v|_L(x) \text{ if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,K} \\ v|_L(x) - v|_K(x) \text{ if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,L}, \end{cases}$$

($\boldsymbol{n}_{\sigma,K}$ *is the normal of* σ*, that is outer w.r. to element* K) *and*

$$\forall x \in \sigma \in \mathcal{E}_{\mathrm{ext}}, \quad [v]_{\sigma,\boldsymbol{n}_\sigma}(x) = v(x),\ \textit{with } \boldsymbol{n}_\sigma \textit{ an exterior normal to } \partial\Omega.$$

We will need to compare the projections Π_h^V and $\Pi_{h,0}^V$. Clearly they coincide on 'interior' elements meaning $K \in \mathcal{T}$, $K \cap \partial\Omega_h = \emptyset$. We have the following lemma for the tetrahedra with non void intersection with the boundary.

Lemma 2.11. *We have*

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi]\|_{L^\infty(K)} + h\|\nabla_x(\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi])\|_{L^\infty(K;R^3)} \lesssim \sup_{\sigma \subset K \cap \partial\Omega_h} \|\phi\|_{L^\infty(\sigma)} \textit{ if } K \in \mathcal{T},\ K \cap \partial\Omega_h \ne \emptyset, \tag{2.46}$$

for any $\phi \in C(K)$*.*

Proof. We recall the Crouzeix–Raviart basis (2.37) and the fact that Π_h^V and $\Pi_{h,0}^V$ differ only in basis functions corresponding to $\sigma \in \mathcal{E}_{\rm ext}$. We have

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^v[\phi]\|_{L^\infty(K)} \leq \left\| \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm ext}} \varphi_\sigma \frac{1}{|\sigma|} \int_\sigma \phi \, {\rm d}S \right\|_{L^\infty(K)} \leq c(K) \cdot \sup_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm ext}} \|\phi\|_{L^\infty(\sigma)}, \tag{2.47}$$

and

$$\begin{aligned} h\|\nabla_x(\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi])\|_{L^\infty(K)} &\leq h \left\| \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm ext}} \nabla_x \varphi_\sigma \frac{1}{|\sigma|} \int_\sigma \phi \, {\rm d}S \right\|_{L^\infty(K)} \\ &\leq ch \sup_{\sigma \subseteq K \cap \partial\Omega_h} \|\phi\|_{L^\infty(\sigma)} \left\| \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm ext}} \nabla_x \varphi_\sigma \right\|_{L^\infty(K)}. \end{aligned}$$

The proof is completed by $\| \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm ext}} \nabla_x \varphi_\sigma \|_{L^\infty(K)} \leq c(K) h^{-1}$. □

In fact, in the derivation of the error estimates we will use the consequence of the above observations formulated in the following two corollaries.

Corollary 2.12. *Let $\phi \in C^1(R^3)$ such that $\phi|_{\partial\Omega} = 0$. Then we have,*

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi]\|_{L^\infty(K)} = 0 \text{ if } K \in \mathcal{T},\ K \cap \partial_{\Omega_h} = \emptyset, \tag{2.48}$$

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi]\|_{L^\infty(K)} + h\|\nabla_x(\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi])\|_{L^\infty(K;R^3)} \lesssim h^2 \|\nabla_x \phi\|_{L^\infty(R^3;R^3)}, \tag{2.49}$$

if $K \in \mathcal{T}_h$, $K \cap \partial\Omega_h \neq \emptyset$, $\partial K \not\subset \partial\Omega$.

Proof. Relation (2.48) follows immediately from (2.46), as there is an empty sum on the right hand side for 'interior' elements ($K \cap \partial\Omega_h = \emptyset$).

For any $x \in \partial\Omega_h$ there exists $y \in \partial\Omega$ (and thus $\phi(y) = 0$) such that

$$|\phi(x)| \leq {\rm dist}[x, y] \|\nabla_x \phi\|_{L^\infty(R^3;R^3)} \lesssim h^2 \|\nabla_x \phi\|_{L^\infty(R^3;R^3)}, \tag{2.50}$$

where we used Lemma 2.5 for the latter inequality. The proof is completed by taking supremum over $K \in \mathcal{T}_h$ and combining with (2.50). Note that the mesh regularity property (2.29) supplies a uniform estimate of constants $c(K)$ from the previous lemma, which enables to write the latter inequality in (2.50). □

Corollary 2.13. *For any $\phi \in C(R^3)$,*

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi]\|_{L^p(\Omega_h)} \lesssim h^{1/p} \|\phi\|_{L^\infty(\Omega_h)},\ 1 \leq p < \infty. \tag{2.51}$$

Proof. Apply inverse estimates (see *e.g.* [31], Lem. 2.9) to (2.46). □

We will frequently use the Poincaré, Sobolev and interpolation inequalities on tetrahedra reported in the following lemma.

Lemma 2.14.

(1) *We have,*

$$\|v - v_K\|_{L^p(K)} \lesssim h\|\nabla v\|_{L^p(K)}, \tag{2.52}$$

$$\forall \sigma \in \mathcal{E}(K),\ \|v - v_\sigma\|_{L^p(K)} \lesssim h\|\nabla v\|_{L^p(K)}, \tag{2.53}$$

for any $v \in W^{1,p}(K)$, *where* $1 \le p \le \infty$.

(2) *There holds*

$$\|v - v_K\|_{L^{p^*}(K)} \lesssim \|\nabla v\|_{L^p(K)}, \tag{2.54}$$

$$\forall \sigma \in \mathcal{E}(K),\ \|v - v_\sigma\|_{L^{p^*}(K)} \lesssim \|\nabla v\|_{L^p(K)}, \tag{2.55}$$

for any $v \in W^{1,p}(K)$, $1 \le p < d$, *where* $p^* = \frac{dp}{d-p}$.

(3) *We have,*

$$\|v - v_K\|_{L^q(K)} \le ch^\beta \|\nabla v\|_{L^p(K;\mathbb{R}^d)}, \tag{2.56}$$

$$\|v - v_\sigma\|_{L^q(K)} \le ch^\beta \|\nabla v\|_{L^p(K;\mathbb{R}^d)}, \tag{2.57}$$

for any $v \in W^{1,p}(K)$, $1 \le p < d$, *where* $\frac{1}{q} = \frac{\beta}{p} + \frac{1-\beta}{p^*}$, $p \le q \le p^*$.

We finish the section of preliminaries by recalling two algebraic inequalities: the 'imbedding' inequality

$$\left(\sum_{i=1}^{L} |a_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{L} |a_i|^q\right)^{1/q}, \tag{2.58}$$

for all $a = (a_1, \dots, a_L) \in R^L$, $1 \le q \le p < \infty$ and the discrete Hölder inequality

$$\sum_{i=1}^{L} |a_i||b_i| \le \left(\sum_{i=1}^{L} |a_i|^q\right)^{1/q} \left(\sum_{i=1}^{L} |a_i|^p\right)^{1/p}, \tag{2.59}$$

for all $a = (a_1, \dots, a_L) \in R^L$, $b = (b_1, \dots, b_L) \in R^L$, $\frac{1}{q} + \frac{1}{p} = 1$.

3. Main result

Here and hereafter we systematically use the following abbreviated notation:

$$\hat{\phi} = \Pi_h^Q[\phi],\ \ \phi_h = \Pi_h^V[\phi],\ \ \phi_{h,0} = \Pi_{h,0}^V[\phi], \tag{3.1}$$

where projections Π_h^Q, Π_h^V and $\Pi_{h,0}^V$ are defined in (2.31) and (2.34). For a function $v \in C([0,T], L^1(\Omega))$ we set

$$v^n(x) = v(t_n, x), \tag{3.2}$$

where $t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n < t_{n+1} < \ldots t_N = T$ is a partition of the interval $[0,T]$. Finally, for a function $v \in V_h(\Omega_h)$ we denote

$$\nabla_h v(x) = \sum_{K \in \mathcal{T}} \nabla_x v(x) 1_K(x),\ \ \mathrm{div}_h \mathbf{v}(x) = \sum_{K \in \mathcal{T}} \mathrm{div}_x \mathbf{v}(x) 1_K(x). \tag{3.3}$$

In order to ensure the positivity of the approximate densities, we shall use an upwinding technique for the density in the mass equation. For $q \in Q_h(\Omega_h)$ and $\boldsymbol{u} \in \mathbf{V}_{h,0}(\Omega_h; \mathbb{R}^3)$, the upwinding of q with respect to $\boldsymbol{u}$ is defined, for $\sigma = K|L \in \mathcal{E}_{\rm int}$ by:

$$q_\sigma^{\rm up} = \begin{cases} q_K \text{ if } \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\sigma,K} > 0 \\ q_L \text{ if } \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\sigma,K} \leq 0, \end{cases} \tag{3.4}$$

and we denote

$$\mathrm{Up}_K(q, \mathbf{u}) \equiv \sum_{\sigma \in \mathcal{E}(K) c \cap \mathcal{E}_{\rm int}} q_\sigma^{\rm up} \boldsymbol{u}_\sigma \cdot \mathbf{n}_{\sigma,K} = \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm int}} \left(q_K [\boldsymbol{u}_\sigma \cdot \mathbf{n}_{\sigma,K}]^+ + q_L [\boldsymbol{u}_\sigma \cdot \mathbf{n}_{\sigma,K}]^- \right),$$

where $a^+ = \max(a, 0)$, $a^- = \min(a, 0)$.

3.1. Numerical scheme

We consider a couple $(\varrho^n, \boldsymbol{u}^n) = (\varrho^{n,(\Delta t,h)}, \boldsymbol{u}^{n,(\Delta t,h)})$ *of* (*numerical*) *solutions of the following algebraic system* (*numerical scheme*)*:*

$$\varrho^n \in Q_h(\Omega_h), \ \varrho^n > 0, \ \mathbf{u}^n \in V_{h,0}(\Omega_h; R^3), \quad n = 0, 1, \dots, N, \tag{3.5}$$

$$\sum_{K \in \mathcal{T}} |K| \frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t} \phi_K + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\rm up} (\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}) \phi_K = 0 \text{ for any } \phi \in Q_h(\Omega_h) \text{ and } n = 1, \dots, N, \tag{3.6}$$

$$\begin{aligned} &\sum_{K \in \mathcal{T}} \frac{|K|}{\Delta t} \left(\varrho_K^n \hat{\boldsymbol{u}}_K^n - \varrho_K^{n-1} \hat{\boldsymbol{u}}_K^{n-1} \right) \cdot \boldsymbol{v}_K + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\rm up} \hat{\boldsymbol{u}}_\sigma^{n,\rm up} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] \cdot \boldsymbol{v}_K \\ &- \sum_{K \in \mathcal{T}} p(\varrho_K^n) \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{v}_\sigma \cdot \mathbf{n}_{\sigma,K} + \mu \sum_{K \in \mathcal{T}} \int_K \nabla \boldsymbol{u}^n : \nabla \boldsymbol{v} \ \mathrm{d}x \\ &+ \frac{\mu}{3} \sum_{K \in \mathcal{T}} \int_K \mathrm{div} \boldsymbol{u}^n \mathrm{div} \boldsymbol{v} \ \mathrm{d}x = 0, \text{ for any } \boldsymbol{v} \in V_{h,0}(\Omega; R^3) \text{ and } n = 1, \dots, N. \end{aligned} \tag{3.7}$$

The numerical solutions depend on the size h of the space discretization and on the time step Δt. For the sake of clarity and in order to simplify notation we will always systematically write in all formulas $(\varrho^n, \boldsymbol{u}^n)$ instead of $(\varrho^{n,(\Delta t,h)}, \boldsymbol{u}^{n,(\Delta t,h)})$.

The numerical method (3.5)–(3.7) has been suggested in ([31], Def. 3.1); it is *strongly nonlinear and implicit.* It is therefore not a trivial question whether this (finite dimensional) problem admits a solution. The problem of the well posedness of this numerical scheme is investigated in Karper ([31], Prop. 3.3). Karper's result states that:

For each fixed $h > 0$, $\Delta t > 0$, *problem* (3.5)–(3.7) *admits a solution* $(\varrho_h^n, \mathbf{u}_h^n)$*:*

$$\varrho_h^n \in Q_h(\Omega_h), \ \mathbf{u}_h^n \in V_{h,0}(\Omega_h; R^3), \ n = 0, 1, \dots, N,$$

and $\varrho_h^n > 0$, $n = 1, \dots, N$, *provided* $\varrho_h^0 > 0$.

The proof uses topological degree theory in the spirit suggested in [20]. All its details are available in Section 11 of [31]. Notice that the above result does not guarantee the *uniqueness* of numerical solutions.

Remark 3.1. Throughout the paper, $q_\sigma^{\rm up}$ is defined in (3.4), where $\mathbf{u}$ is the numerical solution constructed in (3.5)–(3.7).

3.2. Error estimates

The main result of this paper is announced in the following theorem:

Theorem 3.2. *Let $\Omega \subset R^3$ be a bounded domain of class C^3 and let the pressure satisfy (1.4) with $\gamma \geq 3/2$. Let $\{\varrho^n, \mathbf{u}^n\}_{0\leq n\leq N}$ be a family of numerical solutions resulting from the scheme (3.5)–(3.7). Moreover, suppose there are initial data $[r_0, \mathbf{V}_0]$ belonging to the regularity class specified in Proposition 2.2 and giving rise to a weak solution $[r, \mathbf{V}]$ to the initial-boundary value problem (1.1)–(1.6) in $(0,T)\times\Omega$ satisfying*

$$0 \leq r(t,x) \leq \overline{r} \text{ a.a. in } (0,T)\times\Omega.$$

Then $[r, \mathbf{V}]$ is regular and there exists a positive number

$$C = C\Big(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V}, \nabla^2 \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{45})},$$
$$\|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3))}, \|\partial_t^2 \boldsymbol{V}, \partial_t \nabla \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^{12}))},\Big)$$

such that

$$\sup_{1\leq n\leq N} \int_{\Omega\cap\Omega_h} \left[\frac{1}{2}\varrho^n |\hat{\mathbf{u}}^n - \mathbf{V}(t_n,\cdot)|^2 + H(\varrho^n) - H'(r(t_n,\cdot))(\varrho^n - r(t_n,\cdot)) - H(r(t_n))\right] \,\mathrm{d}x \tag{3.8}$$
$$+\Delta t \sum_{1\leq n\leq N} \int_{\Omega\cap\Omega_h} |\nabla_h \mathbf{u}^n - \nabla_x \mathbf{V}(t_n,\cdot)|^2 \,\mathrm{d}x$$
$$\leq C\left(\sqrt{\Delta t} + h^a + \int_{\Omega\cap\Omega_h} \left[\frac{1}{2}\varrho^0 |\hat{\mathbf{u}}^0 - \mathbf{V}_0|^2 + H(\varrho^0) - H'(r_0)(\varrho^0 - r_0) - H(r_0)\right] \,\mathrm{d}x\right),$$

where

$$a = \frac{2\gamma - 3}{\gamma} \text{ if } \tfrac{3}{2} \leq \gamma \leq 2, \quad a = \frac{1}{2} \text{ otherwise.} \tag{3.9}$$

Note that for $\gamma = 3/2$ Theorem 3.2 gives only uniform bounds on the difference of exact and numerical solution, not the convergence.

Remark 3.3. The constitutive assumptions for the pressure (1.4) in Theorem 3.2 require, in particular, $p'(0) > 0$. This condition excludes the isentropic pressure laws

$$p(\varrho) = \varrho^\gamma, \ \gamma > 1. \tag{3.10}$$

Nevertheless, Theorem 3.2 *holds under the same assumptions also for the isentropic pressure laws* (3.10). Here, we have adopted the more restrictive condition (1.4) (in particular $p'(0) > 0$) only for the sake of simplicity and clarity, in order to avoid some unnecessary technical difficulties. It allows to simplify proofs of some estimates: for example estimates (4.7), (4.10) are in this case immediate consequences of the energy inequality (4.2), while in the general case of pressure laws vanishing at 0, the derivation of the same estimates requires more effort (see [22], Cor. 4.1 and Lem. 4.2), where the proofs of these estimates are performed in the general case.

4. Uniform estimates

If we take $\phi = 1$ in formula (3.6) we get immediately the conservation of mass:

$$\forall n = 1, \dots N, \quad \int_{\Omega_h} \varrho^n \,\mathrm{d}x = \int_{\Omega_h} \varrho^0 \,\mathrm{d}x. \tag{4.1}$$

The next Lemma reports the standard energy estimates for the numerical scheme (3.5)–(3.7). The reader can consult Section 4.1 in Gallouet *et al.* ([22], Lem. 4.1) for its laborious but straightforward proof.

Lemma 4.1. *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5)–(3.7) with the pressure p satisfying (1.4). Then there exist*

$$\overline{\varrho}^n_\sigma \in [\min(\varrho^n_K, \varrho^n_L), \max(\varrho^n_K, \varrho^n_L)],\ \sigma = K|L \in \mathcal{E}_{\rm int},\ n = 1, \dots, N,$$
$$\overline{\varrho}^{n-1,n}_K \in [\min(\varrho^{n-1}_K, \varrho^n_K), \max(\varrho^{n-1}_K, \varrho^n_K)],\ K \in \mathcal{T},\ n = 1, \dots, N,$$

such that

$$\sum_{K\in\mathcal{T}} |K| \left(\frac{1}{2} \varrho^m_K |\boldsymbol{u}^m_K|^2 + H(\varrho^m_K) \right) - \sum_{K\in\mathcal{T}} |K| \left(\frac{1}{2} \varrho^0_K |\boldsymbol{u}^0_K|^2 + H(\varrho^0_K) \right)$$
$$+ \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \left(\mu \int_K |\nabla_x \boldsymbol{u}^n|^2 \,{\rm d}x + \frac{\mu}{3} \int_K |{\rm div} \boldsymbol{u}^n|^2 \,{\rm d}x \right)$$
$$+ [D^{m,|\Delta \boldsymbol{u}|}_{\rm time}] + [D^{m,|\Delta \varrho|}_{\rm time}] + [D^{m,|\Delta \boldsymbol{u}|}_{\rm space}] + [D^{m,|\Delta \varrho|}_{\rm space}] = 0, \quad (4.2)$$

for all $m = 1, \dots, N$, where

$$[D^{m,|\Delta \boldsymbol{u}|}_{\rm time}] = \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| \varrho^{n-1}_K \frac{|\boldsymbol{u}^n_K - \boldsymbol{u}^{n-1}_K|^2}{2}, \quad (4.3a)$$

$$[D^{m,|\Delta \varrho|}_{\rm time}] = \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| H''(\overline{\varrho}^{n-1,n}_K) \frac{|\varrho^n_K - \varrho^{n-1}_K|^2}{2}, \quad (4.3b)$$

$$[D^{m,|\Delta \boldsymbol{u}|}_{\rm space}] = \Delta t \sum_{n=1}^{m} \sum_{\sigma = K|L \in \mathcal{E}_{\rm int}} |\sigma| \varrho^{n,{\rm up}}_\sigma \frac{(\boldsymbol{u}^n_K - \boldsymbol{u}^n_L)^2}{2} |\boldsymbol{u}^n_\sigma \cdot \boldsymbol{n}_{\sigma,K}|, \quad (4.3c)$$

$$[D^{m,|\Delta \varrho|}_{\rm space}] = \Delta t \sum_{n=1}^{m} \sum_{\sigma = K|L \in \mathcal{E}_{\rm int}} |\sigma| H''(\overline{\varrho}^n_\sigma) \frac{(\varrho^n_K - \varrho^n_L)^2}{2} |\boldsymbol{u}^n_\sigma \cdot \mathbf{n}_{\sigma,K}|. \quad (4.3d)$$

We have the following corollary of Lemma 4.1.

Corollary 4.2. *Under assumptions of Lemma 4.1, we have:*

(1) *There exists $c = c(M_0, E_0) > 0$ (independent of n, h and Δt) such that*

$$k \sum_{n=1}^{N} \int_K |\nabla_x \boldsymbol{u}^n|^2 \,{\rm d}x \leq c, \quad (4.4)$$

$$k \sum_{n=1}^{N} \|\boldsymbol{u}^n\|^2_{L^6(\Omega_h;\mathbb{R}^3)} \leq c, \quad (4.5)$$

$$\sup_{n=0,\dots N} \|\varrho^n |\hat{\boldsymbol{u}}^n|^2\|_{L^1(\Omega_h)} \leq c. \quad (4.6)$$

(2)

$$\sup_{n=0,\dots N} \|\varrho^n\|_{L^\gamma(\Omega_h)} \leq c, \quad (4.7)$$

(3) *If the pair $(r, \boldsymbol{U})$ belongs to the class (2.25) there is $c = c(M_0, E_0, \underline{r}, \overline{r}, \|\boldsymbol{U}, \nabla \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^{12})}) > 0$ such that for all $n = 1, \dots, N$,*

$$\sup_{n=0,\dots N} \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n | \hat{r}(t_n), \hat{\mathbf{U}}(t_n)) \leq c, \quad (4.8)$$

where

$$\mathcal{E}(\varrho, \boldsymbol{u} | z, \mathbf{v}) = \int_{\Omega_h} \left(\varrho |\boldsymbol{u} - \mathbf{v}|^2 + E(\varrho|z) \right) {\rm d}x, \quad E(\varrho|z) = H(\varrho) - H'(z)(\varrho - z) - H(z). \quad (4.9)$$

(4) *There exists* $c = c(M_0, E_0, \underline{r}, |p'|_{C[\underline{r},\overline{r}]}) > 0$ *such that*

$$\Delta t \sum_{n=1}^{m} \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma|(\varrho_K^n - \varrho_L^n)^2 \Big[\frac{1_{\{\overline{\varrho}_\sigma^n \geq 1\}}}{[\max\{\varrho_K, \varrho_L\}]^{2-\gamma}} + 1_{\{\overline{\varrho}_\sigma^n < 1\}} \Big] |\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \leq c \quad \text{if } \gamma \in [1,2), \tag{4.10}$$

$$\Delta t \sum_{n=1}^{m} \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma|(\varrho_K^n - \varrho_L^n)^2 \, |\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \leq c \quad \text{if } \gamma \geq 2$$

Items (1)–(3) of Corollary 4.2 are direct consequences of Lemma 4.1. Item (4) represents the convenient expression for the numerical dissipation (4.3d). The interested reader can consult Section 4.2 in (Gallouet *et al.* [22], Cor. 4.1, Lem. 4.2) for the detailed proofs of these estimates.

5. Discrete relative energy inequality

The starting point of our error analysis is the discrete relative energy inequality for the numerical scheme (3.5)–(3.7) formulated in the following lemma.

Lemma 5.1. *Let* $(\varrho^n, \boldsymbol{u}^n)$ *be a solution of the discrete problem* (3.5)–(3.7) *with the pressure* p *satisfying* (1.4). *Then there holds for all* $m = 1, \ldots, N$,

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \frac{1}{2}|K| \left(\varrho_K^m |\boldsymbol{u}_K^m - \boldsymbol{U}_K^m|^2 - \varrho_K^0 |\boldsymbol{u}_K^0 - \boldsymbol{U}_K^0|^2 \right) + \sum_{K\in\mathcal{T}} |K| \left(E(\varrho_K^m | r_K^m) - E(\varrho_K^0 | r_K^0) \right) \\ &\qquad + \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \left(\mu \int_K |\nabla_x (\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x + \frac{\mu}{3} \int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x \right) \leq \sum_{i=1}^{6} T_i, \end{aligned} \tag{5.1}$$

for any $0 < r^n \in Q_h(\Omega_h)$, $\boldsymbol{U}^n \in V_{h,0}(\Omega_h; \mathbb{R}^3)$, $n = 1, \ldots, N$, *where*

$$\begin{aligned} T_1 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \left(\mu \int_K \nabla_x \boldsymbol{U}^n : \nabla_x (\boldsymbol{U}^n - \boldsymbol{u}^n) \,\mathrm{d}x + \frac{\mu}{3} \int_K \mathrm{div}\boldsymbol{U}^n \mathrm{div}(\boldsymbol{U}^n - \boldsymbol{u}^n) \,\mathrm{d}x \right), \\ T_2 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{U}_K^n - \boldsymbol{U}_K^{n-1}}{\Delta t} \cdot \left(\frac{\boldsymbol{U}_K^{n-1} + \boldsymbol{U}_K^n}{2} - \boldsymbol{u}_K^{n-1} \right), \\ T_3 &= -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K) \\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\frac{\boldsymbol{U}_K^n + \boldsymbol{U}_L^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \right) \cdot \boldsymbol{U}_K^n [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}], \\ T_4 &= -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K) \\ \sigma=K|L}} |\sigma| p(\varrho_K^n) [\boldsymbol{U}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}], \\ T_5 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} (r_K^n - \varrho_K^n) \left(H'(r_K^n) - H'(r_K^{n-1}) \right), \\ T_6 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K) \\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\mathrm{up}} H'(r_K^{n-1}) [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]. \end{aligned} \tag{5.2}$$

Proof. Lemma 5.1 is proved in Section 5 in (Gallouet *et al.* [22], Thm. 5.1). We provide here the proof for the sake of completeness.

First, noting that the numerical diffusion represented by terms (4.3a)–(4.3d) in the energy identity (4.2) is positive, we infer

$$I_1 + I_2 + I_3 \leq 0, \tag{5.3}$$

with

$$I_1 := \sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{\Delta t}\left(\varrho_K^n |\boldsymbol{u}_K^n|^2 - \varrho_K^{n-1}|\boldsymbol{u}_K^{n-1}|^2\right), \qquad I_2 := \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\left(H(\varrho_K^n) - H(\varrho_K^{n-1})\right),$$
$$I_3 := \sum_{K\in\mathcal{T}} \left(\mu \int_K |\nabla_x \boldsymbol{u}^n|^2 \,\mathrm{d}x + \frac{\mu}{3}\int_K |\mathrm{div}\boldsymbol{u}^n|^2 \,\mathrm{d}x\right).$$

Next, we consider the discrete continuity equation (3.6) with $\phi = \frac{1}{2}|\hat{\mathbf{U}}^n|^2$ as test function in order to obtain

$$I_4 := \sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{\Delta t}(\varrho_K^n - \varrho_K^{n-1})|\boldsymbol{U}_K^n|^2 = -\sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} \frac{1}{2}|\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]|\boldsymbol{U}_K^n|^2 := J_1. \tag{5.4}$$

In the next step, taking $-\boldsymbol{U}^n$ as test function $\mathbf{v}$ in the discrete momentum equation (3.7) one gets

$$I_5 = -\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\left(\varrho_K^n \boldsymbol{u}_K^n - \varrho_K^{n-1}\boldsymbol{u}_K^{n-1}\right)\cdot \boldsymbol{U}_K^n = J_2 + J_3 + J_4,$$

with

$$J_2 = \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\cdot \boldsymbol{U}_K^n\,[\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}],$$
$$J_3 = \mu \sum_{K\in\mathcal{T}} \int_K \nabla\boldsymbol{u}^n : \nabla \boldsymbol{U}^n \,\mathrm{d}x + \frac{\mu}{3}\sum_{K\in\mathcal{T}}\int_K \mathrm{div}\boldsymbol{u}^n \mathrm{div}\boldsymbol{U}^n \,\mathrm{d}x$$

and

$$J_4 = -\sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| p(\varrho_K^n)[\boldsymbol{U}_\sigma^n \cdot \mathbf{n}_{\sigma,K}].$$

We then consider the discrete continuity equation (3.6) with a test function $\phi = H'(r^{n-1})$ and obtain

$$-\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}(\varrho_K^n - \varrho_K^{n-1})H'(r_K^{n-1}) = \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]H'(r_K^{n-1}).$$

Observing that $\varrho_K^n H'(r_K^n) - \varrho_K^{n-1}H'(r_K^{n-1}) = \varrho_K^n\left(H'(r_K^n) - H'(r_K^{n-1})\right) + (\varrho_K^n - \varrho_K^{n-1})H'(r_K^{n-1})$, we rewrite the last identity in the form

$$I_6 := -\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\left(\varrho_K^n H'(r_K^n) - \varrho_K^{n-1}H'(r_K^{n-1})\right) = J_5 + J_6$$
$$\text{with } J_5 = -\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\varrho_K^n\left(H'(r_K^n) - H'(r_K^{n-1})\right) \text{ and } J_6 = \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]H'(r_K^{n-1}). \tag{5.5}$$

Finally, thanks to the convexity of the function H, we have

$$\begin{aligned} I_7 &:= \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\Big[(r_K^n H'(r_K^n) - H(r_K^n)) - \big(r_K^{n-1}H'(r_K^{n-1}) - H(r_K^{n-1})\big)\Big] \\ &= \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} r_K^n \left(H'(r_K^n) - H'(r_K^{n-1})\right) - \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\left(H(r_K^n) - (r_K^n - r_K^{n-1})H'(r_K^{n-1}) - H(r_K^{n-1}\right) \\ &\le \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} r_K^n \left(H'(r_K^n) - H'(r_K^{n-1})\right) := J_7. \end{aligned} \tag{5.6}$$

Now, we gather the expressions (5.3)–(5.6); this is performed in several steps.

Step 1: *Term* $I_1 + I_4 + I_5$. We obtain by direct calculation,

$$\begin{aligned} I_1 + I_4 + I_5 = &\sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{\Delta t}\left(\varrho_K^n |\boldsymbol{u}_K^n - \boldsymbol{U}_K^n|^2 - \varrho_K^{n-1}|\boldsymbol{u}_K^{n-1} - \boldsymbol{U}_K^{n-1}|^2\right) \\ &- \sum_{K\in\mathcal{T}} |K|\varrho_K^{n-1}\frac{\boldsymbol{U}_K^n - \boldsymbol{U}_K^{n-1}}{\Delta t}\cdot\left(\frac{\boldsymbol{U}_K^{n-1} + \boldsymbol{U}_K^n}{2} - \boldsymbol{u}_K^{n-1}\right). \end{aligned} \tag{5.7}$$

Step 2: *Term* $J_1 + J_2$. Employing the definition (3.4) of the upwinding, one gets

$$J_1 + J_2 = -\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\left(\frac{\boldsymbol{U}_K^n + \boldsymbol{U}_L^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\right)\cdot\boldsymbol{U}_K^n[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]. \tag{5.8}$$

Step 3: *Term* $I_3 - J_3$. This term can be written in the form

$$\begin{aligned} I_3 - J_3 = &\sum_{K\in\mathcal{T}}\left(\mu\int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2\,\mathrm{d}x + \frac{\mu}{3}\int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2\,\mathrm{d}x\right) \\ &- \sum_{K\in\mathcal{T}}\mu\int_K\left(\nabla\boldsymbol{U}^n : \nabla(\boldsymbol{U}^n - \boldsymbol{u}^n) + \frac{\mu}{3}\int_K \mathrm{div}\boldsymbol{U}^n\mathrm{div}(\boldsymbol{U}^n - \boldsymbol{u}^n)\right). \end{aligned} \tag{5.9}$$

Step 4: *Term* $I_2 + I_6 + I_7$. By virtue of (5.3), (5.5) and (5.6), we easily find that

$$I_2 + I_6 + I_7 = \sum_{K\in\mathcal{T}}\frac{|K|}{\Delta t}\left(E(\varrho_K^n\,|\,r_K^n) - E(\varrho_K^{n-1}\,|\,r_K^{n-1})\right), \tag{5.10}$$

where the function E is defined in (4.9).

Step 5: *Term* $J_5 + J_6 + J_7$. Coming back to (5.5) and (5.6), we deduce that

$$J_5 + J_6 + J_7 = \sum_{K\in\mathcal{T}}\frac{|K|}{\Delta t}(r_K^n - \varrho_K^n)\left(H'(r_K^n) - H'(r_K^{n-1})\right) + \sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]H'(r_K^{n-1}). \tag{5.11}$$

Step 6: *Conclusion*

According to (5.3)–(5.6), we have

$$\sum_{i=1}^7 I_i \le \sum_{i=1}^7 J_i;$$

whence, writing this inequality by using expressions (5.7)–(5.11) calculated in steps 1-5, we get

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{\Delta t}\left(\varrho_K^n|\boldsymbol{u}_K^n-\boldsymbol{U}_K^n|^2-\varrho_K^{n-1}|\boldsymbol{u}_K^{n-1}-\boldsymbol{U}_K^{n-1}|^2\right)+\sum_{K\in\mathcal{T}}\frac{|K|}{\Delta t}\left(E(\varrho_K^n|r_K^n)-E(\varrho_K^{n-1}|r_K^{n-1})\right)\\
&\qquad+\sum_{K\in\mathcal{T}}\left(\mu\int_K|\nabla_x(\boldsymbol{u}^n-\boldsymbol{U}^n)|^2\,\mathrm{d}x+\frac{\mu}{3}\int_K|\mathrm{div}(\boldsymbol{u}^n-\boldsymbol{U}^n)|^2\,\mathrm{d}x\right)\\
&\le\sum_{K\in\mathcal{T}}\left(\mu\int_K\nabla_x\boldsymbol{U}_h^n:\nabla_x(\boldsymbol{U}^n-\boldsymbol{u}^n)\,\mathrm{d}x+\frac{\mu}{3}\int_K\mathrm{div}\boldsymbol{U}^n\mathrm{div}(\boldsymbol{U}^n-\boldsymbol{u}^n)\,\mathrm{d}x\right)\\
&\qquad+\sum_{K\in\mathcal{T}}|K|\varrho_K^{n-1}\frac{\boldsymbol{U}_K^n-\boldsymbol{U}_K^{n-1}}{\Delta t}\cdot\left(\frac{\boldsymbol{U}_K^{n-1}+\boldsymbol{U}_K^n}{2}-\boldsymbol{u}_K^{n-1}\right)\\
&\qquad-\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|\varrho_\sigma^{n,\mathrm{up}}\left(\frac{\boldsymbol{U}_K^n+\boldsymbol{U}_L^n}{2}-\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\right)\cdot\boldsymbol{U}_K^n[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]\\
&\qquad-\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|p(\varrho_K^n)[\boldsymbol{U}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]+\sum_{K\in\mathcal{T}}\frac{|K|}{\Delta t}(r_K^n-\varrho_K^n)\left(H'(r_K^n)-H'(r_K^{n-1})\right)\\
&\qquad+\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}_K}|\sigma|\varrho_\sigma^{n,\mathrm{up}}H'(r_K^{n-1})[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}].
\end{aligned}
\tag{5.12}
$$

We obtain formula (5.1) by summing $(5.12)^n$ from $n=1$ to $n=m$ and multiplying the resulting inequality by Δt. □

6. Approximate discrete relative energy inequality

In this section, we transform the right hand side of the relative energy inequality (5.1) to a form that is more convenient for the comparison with the strong solution. This transformation is given in the following lemma.

Lemma 6.1 (Approximate relative energy inequality)**.** *Let $(\varrho^n,\boldsymbol{u}^n)$ be a solution of the discrete problem (3.5)–(3.7), where the pressure satisfies (1.4) with $\gamma\ge 3/2$. Then there exists*

$$
\begin{aligned}
c=c\Big(&M_0,E_0,\underline{r},\overline{r},|p'|_{C^1[\underline{r},\overline{r}]},\|(\partial_t r,\nabla r,\boldsymbol{V},\partial_t\boldsymbol{V},\nabla\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{18})},\\
&\|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))},\|\partial_t\nabla r\|_{L^2(0,T;L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3))}\Big)>0,
\end{aligned}
$$

such that for all $m=1,\dots,N$, we have:

$$
\begin{aligned}
&\int_{\Omega_h}\left(\varrho^m|\hat{\boldsymbol{u}}^m-\hat{\boldsymbol{V}}_{h,0}^m|^2+E(\varrho^m|\hat{r}^m)\right)\mathrm{d}x-\int_{\Omega_h}\left(\varrho^0|\hat{\boldsymbol{u}}^0-\hat{\boldsymbol{V}}_{h,0}^0|^2+E(\varrho^0|\hat{r}^0)\right)\mathrm{d}x\\
&+\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\left(\mu\int_K|\nabla_x(\boldsymbol{u}^n-\boldsymbol{V}_{h,0}^n)|^2\,\mathrm{d}x+\frac{\mu}{3}\int_K|\mathrm{div}(\boldsymbol{u}^n-\boldsymbol{V}_{h,0}^n)|^2\,\mathrm{d}x\right)\le\sum_{i=1}^6 S_i+R_{h,\Delta t}^m+G^m,
\end{aligned}
\tag{6.1}
$$

for any couple $(r, \mathbf{V})$ *belonging to the class* (2.25), *where*

$$
\begin{aligned}
S_1 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \left(\mu \int_K \nabla_x \boldsymbol{V}_{h,0}^n : \nabla_x (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x + \frac{\mu}{3} \int_K \mathrm{div}\boldsymbol{V}_{h,0}^n \mathrm{div}(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x \right), \\
S_2 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t} \cdot \left(\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n \right), \\
S_3 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \right) \cdot \left(\boldsymbol{V}_{h,0,\sigma}^n - \boldsymbol{V}_{h,0,K}^n \right) \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K}, \\
S_4 &= -\Delta t \sum_{n=1}^{m} \int_{\Omega_h} p(\varrho^n)\,\mathrm{div}\,\boldsymbol{V}^n\,\mathrm{d}x, \\
S_5 &= \Delta t \sum_{n=1}^{m} \int_{\Omega_h} (\hat{r}^n - \varrho^n) \frac{p'(\hat{r}^n)}{\hat{r}^n} [\partial_t r]^n\,\mathrm{d}x, \\
S_6 &= -\Delta t \sum_{n=1}^{m} \int_{\Omega_h} \frac{\varrho^n}{\hat{r}^n} p'(\hat{r}^n) \boldsymbol{u}^n \cdot \nabla r^n\,\mathrm{d}x,
\end{aligned}
\tag{6.2}
$$

and

$$
|G^m| \le c\,\Delta t \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n \Big| \hat{r}^n, \hat{\boldsymbol{V}}^n), \ |R_{h,\Delta t}^m| \le c(\sqrt{\Delta t} + h^a), \tag{6.3}
$$

with the power a *defined in* (3.9) *and with the functional* $\mathcal{E}$ *introduced in* (4.9). (*Recall that in agreement with the notation* (2.35), (3.1)–(3.3), $\mathbf{V}_{h,0}^n = \Pi_{h,0}^V[\mathbf{V}(t_n)]$, $\mathbf{V}_{h,0,K}^n = \Pi_h^Q \Pi_{h,0}^V \mathbf{V}(t_n)|_K$, $\mathbf{V}_{h,0,\sigma}^n = \frac{1}{|\sigma|}\int_\sigma \mathbf{V}_{h,0}^n$, $\hat{r}^n = \Pi_h^Q[r(t_n)]$, *where the projections* Π^Q, Π^V *are defined in* (2.31)) *and* (2.34).)

Proof. We take as test functions $\mathbf{U}^n = \mathbf{V}_{h,0}^n$ and $r^n = \hat{r}^n$ in the discrete relative energy inequality (5.1). We keep the left hand side and the first term (term T_1) at the right hand side as they stay. The transformation of the remaining terms at the right hand side (terms $T_2 - T_6$) is performed in the following steps:

Step 1: *Term* T_2. We have

$$
T_2 = T_{2,1} + R_{2,1} + R_{2,2}, \ \text{with } T_{2,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t} \cdot \left(\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n \right), \tag{6.4}
$$

and

$$
R_{2,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} R_{2,1}^{n,K}, \quad R_{2,2} = \Delta t \sum_{n=1}^{m} R_{2,2}^n,
$$

where

$$
R_{2,1}^{n,K} = -\frac{|K|}{2} \varrho_K^{n-1} \frac{(\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1})^2}{\Delta t} = -\frac{|K|}{2} \varrho_K^{n-1} \frac{([\boldsymbol{V}^n - \boldsymbol{V}^{n-1}]_{h,0,K})^2}{\Delta t},
$$

and

$$
R_{2,2}^n = -\sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t} \cdot \left(\boldsymbol{u}_K^{n-1} - \boldsymbol{u}_K^n \right).
$$

We may write by virtue of the first order Taylor formula applied to function $t \mapsto \mathbf{V}(t,x)$,

$$\left|\frac{[\boldsymbol{V}^n - \boldsymbol{V}^{n-1}]_{h,0,K}}{\Delta t}\right| = \left|\frac{1}{|K|}\int_K \left[\frac{1}{\Delta t}\left[\int_{t_{n-1}}^{t_n} \partial_t \mathbf{V}(z,x)\mathrm{d}z\right]_{h,0}\right]\mathrm{d}x\right|$$
$$= \left|\frac{1}{|K|}\int_K \left[\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} [\partial_t \mathbf{V}(z)]_{h,0}(x)\mathrm{d}z\right]\mathrm{d}x\right| \leq \|[\partial_t \mathbf{V}]_{h,0}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} \leq \left\|\partial_t \mathbf{V}\right\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))},$$

where we have used the property (2.39) of the projection $\Pi^V_{h,0}$ on the space $V_{h,0}(\Omega_h)$. Therefore, thanks to the mass conservation (4.1), we get

$$|R^{n,K}_{2,1}| \leq \frac{M_0}{2}|K|\Delta t \|\partial_t \boldsymbol{V}\|^2_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))}. \tag{6.5}$$

To treat term $R^n_{2,2}$ we use the discrete Hölder inequality and identity (4.1) in order to get

$$|R^n_{2,2}| \leq \Delta t\ cM_0 \|\partial_t \boldsymbol{V}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))} + cM_0^{1/2}\left(\sum_{K\in\mathcal{T}} |K|\varrho^{n-1}_K |\boldsymbol{u}^{n-1}_K - \boldsymbol{u}^n_K|^2\right)^{1/2} \|\partial_t \boldsymbol{V}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))};$$

whence, by virtue of estimate (4.2) for the upwind dissipation term (4.3a), one obtains

$$|R_{2,2}| \leq \sqrt{\Delta t}\, c(M_0, E_0, \|\partial_t \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)}). \tag{6.6}$$

Step 2: *Term* T_3. Employing the definition (3.4) of upwind quantities, we easily establish that

$$T_3 = T_{3,1} + R_{3,1},$$
$$\text{with } T_{3,1} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho^{n,\mathrm{up}}_\sigma \left(\hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma - \hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}\right)\cdot \boldsymbol{V}^n_{h,0,K}\boldsymbol{u}^n_\sigma \cdot \boldsymbol{n}_{\sigma,K}, \quad R_{3,1} = \Delta t \sum_{n=1}^m \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} R^{n,\sigma}_{3,1},$$
$$\text{and } R^{n,\sigma}_{3,1} = |\sigma|\varrho^n_K \frac{|\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^n_{h,0,L}|^2}{2}[\boldsymbol{u}^n_\sigma \cdot \boldsymbol{n}_{\sigma,K}]^+ + |\sigma|\varrho^n_L \frac{|\boldsymbol{V}^n_{h,0,L} - \boldsymbol{V}^n_{h,0,K}|^2}{2}[\boldsymbol{u}^n_\sigma \cdot \boldsymbol{n}_{\sigma,L}]^+,\ \forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}.$$

Writing

$$\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^n_{h,0,L} = [\boldsymbol{V}^n_{h,0} - \boldsymbol{V}^n_h]_K + \boldsymbol{V}^n_{h,K} - \boldsymbol{V}^n_h + \boldsymbol{V}^n_h - \boldsymbol{V}^n_{h,\sigma}$$
$$+\boldsymbol{V}^n_{h,\sigma} - \boldsymbol{V}^n_h + \boldsymbol{V}^n_h - \boldsymbol{V}^n_{h,L} + [\boldsymbol{V}^n_h - \boldsymbol{V}^n_{h,0}]_L,\ \sigma = K|L \in \mathcal{E}_{\mathrm{int}},$$

and employing estimates (2.48) (if $K\cap\partial\Omega_h = \emptyset$), (2.49) (if $K\cap\partial\Omega_h \neq \emptyset$) to evaluate the L^∞-norm of the first term, (2.52) then $(2.41)_{s=1}$ and (2.53) after $(2.41)_{s=1}$ to evaluate the L^∞-norm of the second and third terms, and performing the same tasks at the second line, we get

$$\|\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^n_{h,0,L}\|_{L^\infty(K\cup L;\mathbb{R}^3)} \leq ch\|\nabla \boldsymbol{V}\|_{L^\infty(K\cup L;\mathbb{R}^9)}; \tag{6.7}$$

consequently

$$|R^{n,\sigma}_{3,1}| \leq h^2\ c\|\nabla \boldsymbol{V}\|^2_{L^\infty((0,T)\times\Omega;\mathbb{R}^9)}|\sigma|(\varrho^n_K + \varrho^n_L)|\boldsymbol{u}^n_\sigma|,\ \forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}},$$

whence

$$|R_{3,1}| \leq h\ c\|\nabla \boldsymbol{V}\|^2_{L^\infty((0,T)\times\Omega;\mathbb{R}^9)}\left(\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)} h|\sigma|(\varrho^n_K + \varrho^n_L)^{6/5}\right)^{5/6}$$
$$\times\left[\Delta t\sum_{n=1}^m\left(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}^n_\sigma|^6\right)^{1/3}\right]^{1/2} \leq h\ c(M_0, E_0, \|\nabla \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}), \tag{6.8}$$

provided $\gamma \geq 6/5$, thanks to the discrete Hölder inequality, the equivalence relation (2.29), the equivalence of norms (2.42) and energy bounds listed in Corollary 4.2.

Clearly, for each face $\sigma = K|L \in \mathcal{E}_{\rm int}$, $\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K} + \boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L} = 0$; whence, finally

$$T_{3,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\rm up} \left(\hat{\boldsymbol{u}}_\sigma^{n,\rm up} - \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\rm up} \right) \cdot \left(\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,\sigma}^n \right) \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}. \tag{6.9}$$

Before the next transformation of term $T_{3,1}$, we realize that

$$\mathbf{V}_{h,0,K}^n - \mathbf{V}_{h,0,\sigma}^n = [\mathbf{V}_{h,0}^n - \mathbf{V}_h^n]_K + \mathbf{V}_{h,K}^n - \mathbf{V}_h^n + \mathbf{V}_h^n - \mathbf{V}_{h,\sigma}^n + [\mathbf{V}_h^n - \mathbf{V}_{h,0}^n]_\sigma;$$

whence by virtue of (2.48) and (2.49), (2.52) and (2.53) and $(2.41)_{s=1}$, similarly as in (6.7),

$$\| \boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,\sigma}^n \|_{L^\infty(K;\mathbb{R}^3)} \leq ch \| \nabla_x \boldsymbol{V} \|_{L^\infty(0,T;L^\infty(\Omega:R^3))}, \quad \sigma \subset K. \tag{6.10}$$

Let us now decompose the term $T_{3,1}$ as

$$T_{3,1} = T_{3,2} + R_{3,2}, \text{ with } R_{3,2} = \Delta t \sum_{n=1}^{m} R_{3,2}^n,$$

$$T_{3,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\rm up} \left(\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\rm up} - \hat{\boldsymbol{u}}_\sigma^{n,\rm up} \right) \cdot \left(\boldsymbol{V}_{h,0,\sigma}^n - \boldsymbol{V}_{h,0,K}^n \right) \hat{\boldsymbol{u}}_\sigma^{n,\rm up} \cdot \boldsymbol{n}_{\sigma,K}, \text{ and}$$

$$R_{3,2}^n = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\rm up} \left(\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\rm up} - \hat{\boldsymbol{u}}_\sigma^{n,\rm up} \right) \cdot \left(\boldsymbol{V}_{h,0,\sigma}^n - \boldsymbol{V}_{h,0,K}^n \right) \left(\boldsymbol{u}_\sigma^n - \hat{\boldsymbol{u}}_\sigma^{n,\rm up} \right) \cdot \boldsymbol{n}_{\sigma,K}.$$

By virtue of discrete Hölder's inequality and estimate (6.10), we get

$$|R_{3,2}^n| \leq c \|\nabla \mathbf{V}\|_{L^\infty(Q_T;\mathbb{R}^9)} \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| \varrho_\sigma^{n,\rm up} \left| \hat{\boldsymbol{u}}_\sigma^{n,\rm up} - \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\rm up} \right|^2 \right)^{1/2}$$
$$\times \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| |\varrho_\sigma^{n,\rm up}|^{\gamma_0} \right)^{1/(2\gamma_0)} \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| \left| \boldsymbol{u}_\sigma^n - \hat{\boldsymbol{u}}_\sigma^{n,\rm up} \right|^q \right)^{1/q},$$

where $\frac{1}{2} + \frac{1}{2\gamma_0} + \frac{1}{q} = 1$, $\gamma_0 = \min\{\gamma, 2\}$ and $\gamma \geq 3/2$. For the sum in the last term of the above product, we have

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| \left| \boldsymbol{u}_\sigma^n - \hat{\boldsymbol{u}}_\sigma^{n,\rm up} \right|^q \leq c \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| |\boldsymbol{u}_\sigma^n - \boldsymbol{u}_K^n|^q$$

$$\leq c \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \left(\|\boldsymbol{u}_\sigma^n - \boldsymbol{u}^n\|_{L^q(K;\mathbb{R}^3)}^q + \sum_{K\in\mathcal{T}} \|\boldsymbol{u}^n - \boldsymbol{u}_K^n\|_{L^q(K;\mathbb{R}^3)}^q \right) \right) \leq ch^{\frac{2\gamma_0-3}{2\gamma_0}q} \left(\sum_{K\in\mathcal{T}} \|\nabla_x \boldsymbol{u}^n\|_{L^2(K;\mathbb{R}^9}^2 \right)^{q/2},$$

where we have used the definition (3.4), the discrete Minkowski inequality, interpolation inequalities (2.56) and (2.57) and the discrete 'imbedding' inequality (2.58). Now we can go back to the estimate of $R_{3,2}^n$ taking into account the upper bounds (4.4), (4.7) and (4.8), in order to get

$$|R_{3,2}| \leq h^a \, c \left(M_0, E_0, \|\nabla \mathbf{V}\|_{L^\infty(Q_T;\mathbb{R}^9)} \right), \tag{6.11}$$

provided $\gamma \geq 3/2$, where a is given in (6.3).

Finally, we rewrite term $T_{3,2}$ as

$$
\begin{aligned}
&T_{3,2} = T_{3,3} + R_{3,3}, \text{ with } R_{3,3} = \Delta t \sum_{n=1}^{m} R_{3,3}^n,\\
&T_{3,3} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \right) \cdot \left(\boldsymbol{V}_{h,0,\sigma}^n - \boldsymbol{V}_{h,0,K}^n \right) \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K}, \text{ and}\\
&R_{3,3}^n = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \right) \cdot \left(\boldsymbol{V}_{h,0,\sigma}^n - \boldsymbol{V}_{h,0,K}^n \right) \left(\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} \right) \cdot \boldsymbol{n}_{\sigma,K};
\end{aligned}
\tag{6.12}
$$

whence

$$
|R_{3,3}| \leq c(\|\nabla \boldsymbol{V}\|_{L^\infty(Q_T,\mathbb{R}^9)})\ \Delta t \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \hat{\mathbf{u}}^n \,|\, \hat{r}^n, \hat{\boldsymbol{V}}_{h,0}^n). \tag{6.13}
$$

Step 3: *Term* T_4. Integration by parts over each $K \in \mathcal{T}$ gives

$$
T_4 = -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_{\mathrm{K}} p(\varrho_K^n) \mathrm{div}_x \boldsymbol{V}_{h,0}^n \,\mathrm{d}x.
$$

We may write

$$
\|\mathrm{div}_x(\boldsymbol{V}_{0,h}^n - \boldsymbol{V}_h^n)\|_{L^\infty(K)} \leq ch \|\nabla_x \boldsymbol{V}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))}, \tag{6.14}
$$

where we have used (2.48)–(2.49). Therefore, employing identity (2.44) we obtain

$$
T_4 = T_{4,1} + R_{4,1}, \quad T_{4,1} = -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_{\mathrm{K}} p(\varrho_K^n) \mathrm{div}_x \boldsymbol{V}^n \,\mathrm{d}x, \tag{6.15}
$$

$$
R_{4,1} = -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_{\mathrm{K}} p(\varrho_K^n) \mathrm{div}_x (\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_h^n) \,\mathrm{d}x.
$$

Due to (1.4) and (4.7), $p(\varrho^n)$ is bounded uniformly in $L^\infty(L^1(\Omega))$; employing this fact and (6.14) we immediately get

$$
|R_{4,1}| \leq h\ c(E_0, M_0, \|\nabla \mathbf{V}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))}). \tag{6.16}
$$

Step 4: *Term* T_5. Using the Taylor formula, we get

$$
H'(r_K^n) - H'(r_K^{n-1}) = H''(r_K^n)(r_K^n - r_K^{n-1}) - \frac{1}{2} H'''(\overline{r}_K^n)(r_K^n - r_K^{n-1})^2,
$$

where $\overline{r}_K^n \in [\min(r_K^{n-1}, r_K^n), \max(r_K^{n-1}, r_K^n)]$. We infer

$$
\begin{aligned}
&T_5 = T_{5,1} + R_{5,1}, \text{ with } T_{5,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} \frac{r_K^n - r_K^{n-1}}{\Delta t},\ R_{5,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} R_{5,1}^{n,K}, \text{ and}\\
&R_{5,1}^{n,K} = \frac{1}{2} |K| H'''(\overline{r}_K^n) \frac{(r_K^n - r_K^{n-1})^2}{\Delta t} (\varrho_K^n - r_K^n).
\end{aligned}
$$

Consequently, by the first order Taylor formula applied to function $t \mapsto r(t,x)$ on the interval (t_{n-1}, t_n) and thanks to the mass conservation (4.1)

$$
|R_{5,1}| \leq \Delta t\ c(M_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}])}, \|\partial_t r\|_{L^\infty(Q_T)}). \tag{6.17}
$$

Let us now decompose $T_{5,1}$ as follows:

$$T_{5,1} = T_{5,2} + R_{5,2}, \text{ with } T_{5,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} [\partial_t r]^n \mathrm{d}x,\ R_{5,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} R_{5,2}^{n,K}, \text{ and}$$
$$R_{5,2}^{n,K} = \int_K (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} \left(\frac{r_K^n - r_K^{n-1}}{\Delta t} - [\partial_t r]^n \right) \mathrm{d}x. \tag{6.18}$$

In accordance with (3.2), here and in the sequel, $[\partial_t r]^n(x) = \partial_t r(t_n, x)$. We write using twice the Taylor formula in the integral form and the Fubini theorem,

$$|R_{5,2}^{n,K}| = \frac{1}{\Delta t} \Big| p'(r_K^n) r_K^n (\varrho_K^n - r_K^n) \int_K \int_{t_{n-1}}^{t_n} \int_s^{t_n} \partial_t^2 r(z) \mathrm{d}z \mathrm{d}s \mathrm{d}x \Big|$$

$$\leq \frac{p'(r_K^n)}{r_K^n} \int_{t_{n-1}}^{t_n} \int_K |\varrho_K^n - r_K^n| \Big| \partial_t^2 r(z) \Big| \mathrm{d}x \mathrm{d}z \mathrm{d}s$$

$$\leq \frac{p'(r_K^n)}{r_K^n} \|\varrho^n - \hat{r}^n\|_{L^\gamma(K)} \int_{t_{n-1}}^{t_n} \|\partial_t^2 r(z)\|_{L^{\gamma'}(K)} \mathrm{d}z \mathrm{d}s.$$

Therefore, by virtue of Corollary 4.2, we have estimate

$$|R_{5,2}| \leq \Delta t\ c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}]}, \|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}). \tag{6.19}$$

Step 5: *Term* T_6. We decompose this term as follows:

$$T_6 = T_{6,1} + R_{6,1}, \quad R_{6,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} R_{6,1}^{n,\sigma,K}, \text{ with}$$
$$T_{6,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| \varrho_K^n \left(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}) \right) \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}, \text{ and}$$
$$R_{6,1}^{n,\sigma,K} = |\sigma| \left(\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n \right) \left(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}) \right) \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}, \text{ for } \sigma = K|L \in \mathcal{E}_{\mathrm{int}}.$$

We will now estimate the term $R_{6,1}^{n,\sigma,K}$. We shall treat separately the cases $\gamma < 2$ and $\gamma \geq 2$. The 'simple' case $\gamma \geq 2$ is left to the reader. The more complicated case $\gamma < 2$ will be treated as follows: We first write

$$|R_{6,1}^{n,\sigma,K}| \leq \sqrt{h}\, \|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)} |\sigma| |\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n| \left[\frac{1_{\{\overline{\varrho}_\sigma^n \geq 1\}}}{[\max\{\varrho_K, \varrho_L\}]^{(2-\gamma)/2}} + 1_{\{\overline{\varrho}_\sigma^n < 1\}} \right] \sqrt{|\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|}$$

$$\times \left[1_{\{\overline{\varrho}_\sigma^n \geq 1\}} [\max\{\varrho_K, \varrho_L\}]^{(2-\gamma)/2} + 1_{\{\overline{\varrho}_\sigma^n < 1\}} \right] \sqrt{h} \sqrt{|\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|},$$

where we have employed the first order Taylor formula applied to function $x \mapsto H'(r(t_{n-1}, x)$. Consequently, the application of the discrete Hölder and Young inequalities yield

$$
\begin{aligned}
|R_{6,1}| &\leq \sqrt{h}\, c\|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)} \\
&\quad \times \Delta t \sum_{n=1}^{m} \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h \left[1_{\{\overline{\varrho}_\sigma^n \geq 1\}} [\max\{\varrho_K, \varrho_L\}]^{2-\gamma} + 1_{\{\overline{\varrho}_\sigma^n < 1\}} \right] |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}| \right)^{1/2} \\
&\quad \times \left(\sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| h (\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n)^2 \left[\frac{1_{\{\overline{\varrho}_\sigma^n \geq 1\}}}{[\max\{\varrho_K, \varrho_L\}]^{2-\gamma}} + 1_{\{\overline{\varrho}_\sigma^n < 1\}} \right] |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}| \right)^{1/2} \\
&\leq \sqrt{h}\, c\|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)} \\
&\quad \times \Delta t \sum_{n=1}^{m} \left\{ \left[|\Omega_h|^{\frac{5}{6}} + \left(\sum_{K\in\mathcal{T}} |\sigma| h (\varrho_K^n)^{\frac{6}{5}(2-\gamma)} \right)^{\frac{5}{6}} \right] \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|^6 \right)^{\frac{1}{6}} \right. \\
&\quad \left. \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| h (\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n)^2 \left[\frac{1_{\{\overline{\varrho}_\sigma^n \geq 1\}}}{[\max\{\varrho_K, \varrho_L\}]^{2-\gamma}} + 1_{\{\overline{\varrho}_\sigma^n < 1\}} \right] |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K} \right\}^{1/2} \\
&\leq \sqrt{h}\, c\|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)} \left\{ \Delta t \sum_{n=1}^{m} \left[|\Omega_h|^{\frac{5}{6}} + \left(\sum_{K\in\mathcal{T}} |\sigma| h (\varrho_K^n)^{\frac{6}{5}(2-\gamma)} \right)^{\frac{5}{6}} \right] \left(\sum_{\sigma\in\mathcal{E}} |\sigma| h |\boldsymbol{u}_\sigma^n|^6 \right)^{1/6} \right. \\
&\quad \left. + \Delta t \sum_{n=1}^{m} \left[\sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| h (\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n)^2 \left[\frac{1_{\{\overline{\varrho}_\sigma^n \geq 1\}}}{[\max\{\varrho_K, \varrho_L\}]^{2-\gamma}} + 1_{\{\overline{\varrho}_\sigma^n < 1\}} \right] |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}| \right] \right\} \\
&\leq \sqrt{h}\, c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C([\underline{r},\overline{r}])}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}),
\end{aligned}
$$

where, in order to get the last line, we have used the estimate (4.10) of the numerical dissipation to evaluate the second term, and finally equivalence of norms $(2.42)_{p=6}$ together with (4.5) and (4.7), under assumption $\gamma \geq 12/11$, to evaluate the first term.

Let us now decompose the term $T_{6,1}$ as

$$
\begin{aligned}
&T_{6,1} = T_{6,2} + R_{6,2}, \text{ with } T_{6,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| \varrho_K^n H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1})[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}], \\
&R_{6,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{K}} \sum_{\sigma\in\mathcal{E}(K)} R_{6,2}^{n,\sigma,K}, \text{ and} \\
&R_{6,2}^{n,\sigma,K} = |\sigma| \varrho_K^n \left(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}) - H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1}) \right) [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}].
\end{aligned}
$$

Therefore, by virtue of the second order Taylor formula applied to function H', the Hölder inequality, (2.42), and (4.5), (4.7) in Corollary 4.2, we have, provided $\gamma \geq 6/5$,

$$
\begin{aligned}
|R_{6,2}| &\leq hc \left(|H''|_{C([\underline{r},\overline{r}])} + |H'''|_{C([\underline{r},\overline{r}])} \right) \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)} \Delta t \sum_{n=1}^{m} \|\varrho^n\|_{L^\gamma(\Omega_h)} \|\boldsymbol{u}^n\|_{L^6(\Omega_h;\mathbb{R}^3)} \\
&\leq h\, c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}])}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}).
\end{aligned} \tag{6.20}
$$

Let us now deal with the term $T_{6,2}$. Noting that $\displaystyle\int_K \nabla r^{n-1}\,\mathrm{d}x = \sum_{\sigma\in\mathcal{E}(K)} |\sigma|(r_\sigma^{n-1} - r_K^{n-1})\boldsymbol{n}_{\sigma,K}$, we may write $T_{6,2} = T_{6,3} + R_{6,3}$, with

$$
\begin{aligned}
T_{6,3} &= -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \varrho_K^n H''(r_K^{n-1})\boldsymbol{u}^n \cdot \nabla r^{n-1}\,\mathrm{d}x,\\
R_{6,3} &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \varrho_K^n H''(r_K^{n-1})(\boldsymbol{u}^n - \boldsymbol{u}_K^n)\cdot \nabla r^{n-1}\,\mathrm{d}x\\
&\quad + \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho_K^n H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1})(\boldsymbol{u}_\sigma^n - \boldsymbol{u}_K^n)\cdot \boldsymbol{n}_{\sigma,K}.
\end{aligned}
$$

Consequently, by virtue of Hölder's inequality, interpolation inequality (2.56) (to estimate $\|\mathbf{u}^n - \mathbf{u}_K^n\|_{L^{\gamma_0'}(K;\mathbb{R}^3)}$ by $h^{(5\gamma_0-6)/(2\gamma_0)}\|\nabla_x \mathbf{u}^n\|_{L^2(K;\mathbb{R}^9)}$, $\gamma_0 = \min\{\gamma, 2\}$) in the first term, and by the Taylor formula applied to function $x \mapsto r(t_{n-1}, x)$, then Hölder's inequality and (2.56)–(2.57) (to estimate $\|\mathbf{u}_\sigma^n - \mathbf{u}_K^n\|_{L^{\gamma_0'}(K;\mathbb{R}^3)}$ by $h^{(5\gamma_0-6)/(2\gamma_0)}\|\nabla_x \mathbf{u}^n\|_{L^2(K;\mathbb{R}^9)}$), we get

$$
|R_{6,3}| \le h^b\, c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}])}\|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}),\quad b = \frac{5\gamma_0 - 6}{2\gamma_0}, \tag{6.21}
$$

provided $\gamma \ge 6/5$, where we have used at the end the discrete imbedding and Hölder inequalities (2.58) and (2.59) and finally estimates (4.4) and (4.7).

Finally we write $T_{6,3} = T_{6,4} + R_{6,4}$, with

$$
\begin{aligned}
T_{6,4} &= -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \varrho_K^n \frac{p'(r_K^n)}{r_K^n}\boldsymbol{u}^n \cdot \nabla r^n\,\mathrm{d}x,\\
R_{6,4} &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \varrho_K^n \left(H''(r_K^n)\nabla r^n - H''(r_K^{n-1})\nabla r^{n-1}\right)\cdot \boldsymbol{u}^n\,\mathrm{d}x,
\end{aligned} \tag{6.22}
$$

where by the same token as in (6.19),

$$
|R_{6,4}| \le \Delta t\, c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}])}, \|\nabla r, \partial_t r\|_{L^\infty(Q_T;\mathbb{R}^4)}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/(5\gamma-6)}(\Omega;\mathbb{R}^3))}), \tag{6.23}
$$

provided $\gamma \ge 6/5$.

We are now in position to conclude the proof of Lemma 6.1: we obtain the inequality (6.1) by gathering the principal terms (6.4), (6.12), (6.15), (6.18), (6.22) and the residual terms estimated in (6.5), (6.6), (6.8), (6.11), (6.13), (6.17), (6.19), (6.20), (6.21), (6.23) at the right hand side $\sum_{i=1}^{6} T_i$ of the discrete relative energy inequality (5.1). □

7. A discrete identity satisfied by the strong solution

This section is devoted to the proof of a discrete identity satisfied by any strong solution of problem (1.1)–(1.6) in the class (2.9)–(2.10) extended eventually to $\mathbb{R}^3$ according to Lemma 2.3. This identity is stated in Lemma 7.1 below. It will be used in combination with the approximate relative energy inequality stated in Lemma 6.1 to deduce the convenient form of the relative energy inequality verified by any function being a strong solution to the compressible Navier–Stokes system. This last step is performed in the next section.

Lemma 7.1 (A discrete identity for strong solutions)**.** *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5)–(3.7) with the pressure satisfying (1.4), where $\gamma \geq 3/2$. There exists*

$$c = c\Big(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V}, \nabla^2 \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{45})},$$

$$\|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3))}, \|\partial_t^2 \boldsymbol{V}, \partial_t \nabla \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^{12}))}\Big) > 0,$$

such that for all $m = 1, \dots, N$, we have:

$$\sum_{i=1}^{6} \mathcal{S}_i + \mathcal{R}^m_{h,\Delta t} = 0, \tag{7.1}$$

where

$$\mathcal{S}_1 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \left(\mu \int_K \nabla_x \boldsymbol{V}^n_{h,0} : \nabla_x (\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\, \mathrm{d}x + \frac{\mu}{3} \int_K \mathrm{div} \boldsymbol{V}^n_{h,0} \mathrm{div} (\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\, \mathrm{d}x \right),$$

$$\mathcal{S}_2 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| r_K^{n-1} \frac{\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^{n-1}_{h,0,K}}{\Delta t} \cdot \left(\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K\right),$$

$$\mathcal{S}_3 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r^{n,\mathrm{up}}_\sigma \left(\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma \right) \cdot \left(\boldsymbol{V}^n_{h,0,\sigma} - \boldsymbol{V}^n_{h,0,K} \right) \hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} \cdot \boldsymbol{n}_{\sigma,K}$$

$$\mathcal{S}_4 = -\Delta t \sum_{n=1}^{m} \int_{\Omega_h} p(\hat{r}^n)\, \mathrm{div}\, \boldsymbol{V}^n\, \mathrm{d}x,$$

$$\mathcal{S}_5 = 0,$$

$$\mathcal{S}_6 = -\Delta t \sum_{n=1}^{m} \int_{\Omega_h} p'(\hat{r}^n) \boldsymbol{u}^n \cdot \nabla r^n\, \mathrm{d}x,$$

and

$$|\mathcal{R}^m_{h,\Delta t}| \leq c\left(h^{5/6} + \Delta t\right),$$

for any couple $(r, \mathbf{V})$ belonging to (2.25) and satisfying the continuity equation (1.1) on $(0,T)\times\mathbb{R}^3$ and momentum equation (1.2) with boundary conditions (1.5) on $(0,T)\times\Omega$ in the classical sense. (Recall that in agreement with notation (2.35), (3.1)–(3.3), $\mathbf{V}^n_{h,0} = \Pi^V_{h,0}[\mathbf{V}(t_n)]$, $\mathbf{V}^n_{h,0,K} = [\mathbf{V}^n_{h,0}]_K$, $\mathbf{V}^n_{h,0,\sigma} = [\mathbf{V}^n_{h,0}]_\sigma$, $\hat{r}^n = \Pi^Q_h[r(t_n)]$, where projections Π^Q, Π^V are defined in (2.31) and (2.34)).

Before starting the proof we recall an auxiliary algebraic inequality whose straightforward proof is left to the reader, and introduce some notations.

Lemma 7.2. *Let p satisfies assumptions (1.4). Let $0 < a < b < \infty$. Then there exists $c = c(a,b) > 0$ such that for all $\varrho \in [0,\infty)$ and $r \in [a,b]$ there holds*

$$E(\varrho|r) \geq c(a,b) \left(1_{R_+\setminus[a/2,2b]}(\varrho) + \varrho^\gamma 1_{R_+\setminus[a/2,2b]}(\varrho) + (\varrho - r)^2 1_{[a/2,2b]}(\varrho) \right),$$

where $E(\varrho|r)$ is defined in (4.9).

If we consider Lemma 7.2 with $\varrho = \varrho^n(x)$, $r = \hat{r}^n(x)$, $a = \underline{r}$, $b = \overline{r}$ (where r is a function belonging to class (2.25) and $\underline{r}$, $\overline{r}$ are its lower and upper bounds, respectively), we obtain

$$E(\varrho^n(x)|\hat{r}^n(x)) \geq c(\underline{r},\overline{r}) \left(1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho^n(x)) + (\varrho^n)^\gamma(x) 1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho^n(x)) + (\varrho^n(x) - \hat{r}^n(x))^2 1_{[\underline{r}/2,2\overline{r}]}(\varrho^n(x)) \right). \tag{7.2}$$

Now, for fixed numbers $\underline{r}$ and $\overline{r}$ and fixed functions ϱ^n, $n = 0, \dots, N$, we introduce the residual and essential subsets of Ω (relative to ϱ^n) as follows:

$$N^n_{\rm ess} = \{x \in \Omega \,\Big|\, \frac{1}{2}\underline{r} \leq \varrho^n(x) \leq 2\overline{r}\},\ N^n_{\rm res} = \Omega \setminus N^n_{\rm ess}, \tag{7.3}$$

and we set

$$[g]_{\rm ess}(x) = g(x)1_{N^n_{\rm ess}}(x),\ [g]_{\rm res}(x) = g(x)1_{N^n_{\rm res}}(x),\ \ x \in \Omega,\ \ g \in L^1(\Omega).$$

Integrating inequality (7.2) we deduce

$$c(\underline{r}, \overline{r}) \sum_{K\in\mathcal{T}} \int_K \left(\Big[1\Big]_{\rm res} + \Big[(\varrho^n)^\gamma\Big]_{\rm res} + \Big[\varrho^n - \hat{r}^n\Big]^2_{\rm ess} \right) {\rm d}x \leq \mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| \hat{r}^n, \boldsymbol{V}^n), \tag{7.4}$$

for any pair $(r, \boldsymbol{V})$ belonging to the class (2.25) and any $\varrho^n \in Q_h(\Omega_h)$, $\varrho^n \geq 0$.

We are now ready to proceed to the proof of Lemma 7.1.

Proof. Since $(r, \boldsymbol{V})$ satisfies (1.1) on $(0,T)\times\Omega$ and belongs to the class (2.25), equation (1.2) can be rewritten in the form

$$r\partial_t \boldsymbol{V} + r\boldsymbol{V}\cdot\nabla\boldsymbol{V} + \nabla p(r) - \mu\Delta\boldsymbol{V} - \mu/3\nabla\,{\rm div}\,\boldsymbol{V} = 0 \ \text{ in } (0,T)\times\Omega.$$

From this fact, we deduce the identity

$$\sum_{i=1}^5 \mathcal{T}_i = \mathcal{R}_0, \tag{7.5}$$

where

$$\mathcal{R}_0 = -\Delta t \sum_{n=1}^m \int_{\Omega_h\setminus\Omega} \Big(r^n[\partial_t \boldsymbol{V}]^n + r\boldsymbol{V}^n\cdot\nabla\boldsymbol{V}^n + \nabla p(r^n) - \mu\Delta\boldsymbol{V}^n - \frac{\mu}{3}\nabla\,{\rm div}\,\boldsymbol{V}^n\Big)\cdot(\boldsymbol{V}^n_{h,0} - \mathbf{u}^n){\rm d}x,$$

$$\mathcal{T}_1 = -\Delta t \sum_{n=1}^m \int_{\Omega_h} \Big(\mu\Delta\boldsymbol{V}^n + \frac{\mu}{3}\nabla\,{\rm div}\,\boldsymbol{V}^n\Big)\cdot(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,{\rm d}x, \quad \mathcal{T}_2 = \Delta t \sum_{n=1}^m \int_{\Omega_h} r^n[\partial_t\boldsymbol{V}]^n\cdot(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,{\rm d}x,$$

$$\mathcal{T}_3 = \Delta t \sum_{n=1}^m \int_{\Omega_h} r^n\boldsymbol{V}^n\cdot\nabla\boldsymbol{V}^n\cdot(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,{\rm d}x, \quad \mathcal{T}_4 = \Delta t \sum_{n=1}^m \int_{\Omega_h} \nabla p(r^n)\cdot\boldsymbol{V}^n_{h,0}\,{\rm d}x,$$

$$\mathcal{T}_5 = 0, \qquad \mathcal{T}_6 = -\Delta t \sum_{n=1}^m \int_{\Omega_h} \nabla p(r^n)\cdot\boldsymbol{u}^n\,{\rm d}x.$$

In the steps below, we deal with each of the terms $\mathcal{R}_0$ and $\mathcal{T}_i$.

Step 0: *Term* $\mathcal{R}_0$. By the Hölder inequality

$$|\mathcal{R}_0| \leq |\Omega_h\setminus\Omega|^{5/6}\, c(\overline{r}, |p'|_{C[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \mathbf{V}, \nabla\mathbf{V}, \nabla^2\mathbf{V})\|_{L^\infty(Q_T;\mathbb{R}^{43})} \Delta t \sum_{n=1}^m (\|\boldsymbol{u}^n\|_{L^6(\Omega_h)} + \|\boldsymbol{V}^n_{h,0}\|_{L^6(\Omega_h)})$$

$$\leq h^{5/3}\, c(M_0, E_0, \overline{r}, |p'|_{C[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \mathbf{V}, \nabla\mathbf{V}, \nabla^2\mathbf{V})\|_{L^\infty(Q_T;\mathbb{R}^{43})}, \tag{7.6}$$

where we have used (4.5), (2.48), (2.49) and (2.39).

Step 1: *Term* $\mathcal{T}_1$. Integrating by parts, we get:

$$\mathcal{T}_1 = \mathcal{T}_{1,1} + \mathcal{R}_{1,1},$$

$$\text{with } \mathcal{T}_{1,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \left(\mu \nabla \boldsymbol{V}_{h,0}^n : \nabla(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) + \frac{\mu}{3} \operatorname{div} \boldsymbol{V}_{h,0}^n \operatorname{div}(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \right) \mathrm{d}x,$$

$$\text{and } \mathcal{R}_{1,1} = I_1 + I_2, \text{ with}$$

$$I_1 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \left(\mu \nabla(\boldsymbol{V}^n - \boldsymbol{V}_{h,0}^n) : \nabla(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) + \frac{\mu}{3} \operatorname{div}(\boldsymbol{V}^n - \boldsymbol{V}_{h,0}^n) \operatorname{div}(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \right) \mathrm{d}x, \tag{7.7}$$

$$I_2 = -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \left(\mu \boldsymbol{n}_{\sigma,K} \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) + \frac{\mu}{3} \operatorname{div} \boldsymbol{V}^n (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \cdot \boldsymbol{n}_{\sigma,K} \right) \mathrm{d}S$$

$$= -\Delta t \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}} \int_\sigma \left(\mu \boldsymbol{n}_\sigma \cdot \nabla \boldsymbol{V}^n \cdot \left[\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n \right]_{\sigma,\mathbf{n}_\sigma} + \frac{\mu}{3} \operatorname{div} \boldsymbol{V}^n \left[\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n \right]_{\sigma,\mathbf{n}_\sigma} \cdot \boldsymbol{n}_\sigma \right) \mathrm{d}S,$$

where in the last line $\mathbf{n}_\sigma$ is the unit normal to the face σ and $[\cdot]_{\sigma,\mathbf{n}_\sigma}$ is the jump over sigma (with respect to $\mathbf{n}_\sigma$) defined in Lemma 2.10.

To estimate I_1, we use the Cauchy–Schwartz inequality, decompose $\mathbf{V}^n - \mathbf{V}_{h,0}^n = \mathbf{V}^n - \mathbf{V}_h^n + \mathbf{V}_h^n - \mathbf{V}_{h,0}^n$ and employ estimates $(2.41)_{s=2}$, (2.48)–(2.49) to evaluate the norms involving $\nabla(\mathbf{V}^n - \mathbf{V}_{h,0}^n)$, and decompose $\mathbf{V}_{h,0}^n = \mathbf{V}_{h,0}^n - \mathbf{V}_h^n + \mathbf{V}_h^n$ use (2.48)–(2.49), $(2.40)_{s=1}$, (4.4), the Minkowski inequality to estimate the norms involving $\nabla(\mathbf{V}_{h,0}^n - \mathbf{u}^n)$. We get

$$|I_1| \leq h\, c(M_0, E_0, \|\nabla \mathbf{V}, \nabla^2 \boldsymbol{V}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{36}))}).$$

Since the integral over any face $\sigma \in \mathcal{E}_{\rm int}$ of the jump of a function from $V_{h,0}(\Omega_h)$ is zero, we may write

$$I_2 = \Delta t \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}_{\rm int}} \int_\sigma \left(\mu \boldsymbol{n}_\sigma \cdot (\nabla \boldsymbol{V}^n - (\nabla \boldsymbol{V}^n)_\sigma) \cdot \left[\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n \right]_{\sigma,\mathbf{n}_\sigma} \right.$$

$$\left. + \frac{\mu}{3} (\operatorname{div} \boldsymbol{V}^n - (\operatorname{div} \boldsymbol{V}^n)_\sigma) \left[\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n \right]_{\sigma,\mathbf{n}_\sigma} \cdot \boldsymbol{n}_\sigma \right) \mathrm{d}S;$$

whence by using the first order Taylor formula applied to functions $x \mapsto \nabla \mathbf{V}^n(x)$ to evaluate the differences $\nabla \boldsymbol{V}^n - (\nabla \boldsymbol{V}^n)_\sigma$, $\operatorname{div} \boldsymbol{V}^n - [\operatorname{div} \boldsymbol{V}^n]_\sigma$, and Hölder's inequality,

$$|I_2| \leq \Delta t\, h\, c \|\nabla^2 \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{27})} \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}_{\rm int}} \sqrt{|\sigma|}\sqrt{h} \left(\frac{1}{\sqrt{h}} \left\| \left[\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n \right]_{\sigma,\mathbf{n}_\sigma} \right\|_{L^2(\sigma;\mathbb{R}^3)} \right)$$

$$\leq \Delta t\, h\, c \|\nabla^2 \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{27})} \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}_{\rm int}} \left(|\sigma| h + \frac{1}{h} \left\| \left[\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n \right]_{\sigma,\mathbf{n}_\sigma} \right\|^2_{L^2(\sigma;\mathbb{R}^3)} \right).$$

Therefore,

$$|\mathcal{R}_{1,1}| \leq h\, c(M_0, E_0, \|\mathbf{V}, \nabla \boldsymbol{V}, \nabla^2 \boldsymbol{V}\|_{L^\infty(Q_T,\mathbb{R}^{39})}), \tag{7.8}$$

where we have employed Lemma 2.10, (4.4) and (2.48)–(2.49), (2.40).

Step 2: *Term* $\mathcal{T}_2$. Let us now decompose the term $\mathcal{T}_2$ as

$$\mathcal{T}_2 = \mathcal{T}_{2,1} + \mathcal{R}_{2,1},$$
$$\text{with } \mathcal{T}_{2,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K r^{n-1} \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x, \quad \mathcal{R}_{2,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \mathcal{R}_{2,1}^{n,K},$$
$$\text{and } \mathcal{R}_{2,1}^{n,K} = \int_K (r^n - r^{n-1})[\partial_t \mathbf{V}]^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x + \int_K r^{n-1} \left([\partial_t \boldsymbol{V}]^n - \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \right) \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x.$$

The remainder $\mathcal{R}_{2,1}^{n,K}$ can be rewritten as follows

$$\mathcal{R}_{2,1}^{n,K} = \int_K \Big[\int_{t_{n-1}}^{t_n} \partial_t r(t,\cdot)\mathrm{d}t \Big] [\partial_t \mathbf{V}]^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x + \frac{1}{\Delta t} \int_K r^{n-1} \Big[\int_{t_{n-1}}^{t_n} \int_s^{t_n} \partial_t^2 \boldsymbol{V}(z,\cdot)\mathrm{d}z\mathrm{d}s \Big] \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x;$$

whence, by the Hölder inequality,

$$|\mathcal{R}_{2,1}^{n,K}| \le \Delta t \Big[(\|r\|_{L^\infty(Q_T)} + \|\partial_t r\|_{L^\infty(Q_T)})(\|\partial_t \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)} |K|^{5/6} (\|\boldsymbol{u}^n\|_{L^6(K)} + \|\boldsymbol{V}_{h,0}^n\|_{L^6(K)})$$
$$+ \|\partial_t^2 \boldsymbol{V}^n\|_{L^{6/5}(\Omega;\mathbb{R}^3))} (\|\boldsymbol{u}^n\|_{L^6(K)} + \|\boldsymbol{V}_{h,0}^n\|_{L^6(K)}) \Big].$$

Consequently, by the same token as in (6.19) or (6.23),

$$|\mathcal{R}_{2,1}| \le \Delta t\, c \left(M_0, E_0, \overline{r}, \|(\partial_t r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{16})}, \|\partial_t^2 \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^3))} \right), \tag{7.9}$$

where we have used the discrete Hölder and Young inequalities, the estimates (2.39), (2.48) and (2.49) and the energy bound (4.4) from Corollary 4.2.

Step 2a: *Term* $\mathcal{T}_{2,1}$. We decompose the term $\mathcal{T}_{2,1}$ as

$$\mathcal{T}_{2,1} = \mathcal{T}_{2,2} + \mathcal{R}_{2,2},$$
$$\text{with } \mathcal{T}_{2,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K r_K^{n-1} \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x,\ \mathcal{R}_{2,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \mathcal{R}_{2,2}^{n,K},$$
$$\text{and } \mathcal{R}_{2,2}^{n,K} = \int_K (r^{n-1} - r_K^{n-1}) \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x;$$

therefore,

$$|\mathcal{R}_{2,2}^n| = |\sum_{K\in\mathcal{T}} \mathcal{R}_{2,2}^{n,K}| \le h\, c \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)} \|\partial_t \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)} \|\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\|_{L^6(\Omega;\mathbb{R}^3)}.$$

Consequently, by virtue of formula (4.5) for $\mathbf{u}^n$ and estimates (2.39), (2.48) and (2.49),

$$|\mathcal{R}_{2,2}| \le h\, c(M_0, E_0, \|(\nabla r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{18})}). \tag{7.10}$$

Step 2b: *Term* $\mathcal{T}_{2,2}$. We decompose the term $\mathcal{T}_{2,2}$ as

$$\mathcal{T}_{2,2} = \mathcal{T}_{2,3} + \mathcal{R}_{2,3},$$

$$\text{with } \mathcal{T}_{2,3} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K r_K^{n-1} \frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t} \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x,\ \mathcal{R}_{2,3} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \mathcal{R}_{2,3}^{n,K},$$

$$\text{and } \mathcal{R}_{2,3}^{n,K} = \int_K r_K^{n-1} \left(\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} - \left[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \right]_h \right) \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x$$

$$+ \int_K r_K^{n-1} \left(\left[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \right]_h - \left[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \right]_{h,K} \right) \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x$$

$$+ \int_K r_K^{n-1} \left(\left[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \right]_{h,K} - \left[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \right]_{h,0,K} \right) \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}_n)\,\mathrm{d}x = I_1^K + I_2^K + I_3^K.$$

We calculate carefully

$$|I_3^K| = \frac{1}{\Delta t} r_K^{n-1} \int_K \left\{ \int_{t_{n-1}}^{t_n} \Big[[\partial_t \boldsymbol{V}(z)]_h - [\partial_t \boldsymbol{V}(z)]_{h,0} \Big]_K \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \mathrm{d}z \right\} \mathrm{d}x$$

$$\leq \frac{1}{\Delta t} r_K^{n-1} \int_{t_{n-1}}^{t_n} \Big\| \Big[[\partial_t \boldsymbol{V}(z)]_h - [\partial_t \boldsymbol{V}(z)]_{h,0} \Big]_K \Big\|_{L^{6/5}(K;\mathbb{R}^3)} \|\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\|_{L^6(K;\mathbb{R}^3)} \mathrm{d}z.$$

Summing over polyhedra $K \in \mathcal{T}$ we get simply by using the discrete Sobolev inequality

$$\sum_{K\in\mathcal{T}} |I_3^K| \leq \frac{1}{\Delta t} r_K^{n-1} \int_{t_{n-1}}^{t_n} \left\{ \left(\sum_{K\in\mathcal{T}} \|\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\|_{L^6(K;\mathbb{R}^3)}^6 \right)^{1/6} \left(\sum_{K\in\mathcal{T}} \Big\| [\partial_t \boldsymbol{V}(z)]_h - [\partial_t \boldsymbol{V}(z)]_{h,0} \Big\|_{L^{6/5}(K;\mathbb{R}^3)}^{6/5} \right)^{5/6} \right\} \mathrm{d}z$$

$$\leq \frac{1}{\Delta t} r_K^{n-1} \int_{t_{n-1}}^{t_n} \|\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\|_{L^6(\Omega_h;\mathbb{R}^3)} \Big\| [\partial_t \boldsymbol{V}(z)]_h - [\partial_t \boldsymbol{V}(z)]_{h,0} \Big\|_{L^{6/5}(\Omega_h;\mathbb{R}^3)} \mathrm{d}z$$

$$\leq \frac{h^{5/6}}{\Delta t} \int_{t_{n-1}}^{t_n} \|\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\|_{L^6(\Omega_h;\mathbb{R}^3)} \|\partial_t \boldsymbol{V}(z)\|_{L^\infty(\Omega_h;\mathbb{R}^3)} \mathrm{d}z,$$

where we have used estimate (2.51) to obtain the last line.

As far as the term I_2^K is concerned, we write

$$|I_2^K| = \frac{1}{\Delta t} r_K^{n-1} \Big| \int_K \left(\Big[\int_{t_{n-1}}^{t_n} \partial_t \boldsymbol{V}(z) \mathrm{d}z \Big]_h - \Big[\int_{t_{n-1}}^{t_n} \partial_t \boldsymbol{V}(z) \mathrm{d}z \Big]_{h,K} \right) \cdot (\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n)\,\mathrm{d}x \Big|$$

$$\leq \frac{h}{\Delta t} r_K^{n-1} \int_{t_{n-1}}^{t_n} \Big\| \nabla_x \Big[\partial_t \boldsymbol{V}(z) \Big]_h \Big\|_{L^{6/5}(K;\mathbb{R}^3)} \|\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\|_{L^6(K;\mathbb{R}^3)},$$

where we have used the Fubini theorem, Hölder's inequality and (2.52), $(2.41)_{s=1}$. Further, employing the Sobolev inequality on the Crouzeix–Raviart space $V_{h,0}(\Omega_h)$ (2.43), the Hölder inequality and estimate $(2.41)_{s=1}$, we get

$$\sum_{K\in\mathcal{T}} |I_2^K| \leq \frac{h}{\Delta t} r_K^{n-1} \|\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\|_{L^6(\Omega_h;\mathbb{R}^3)} \int_{t_{n-1}}^{t_n} \Big\| \nabla_x \partial_t \boldsymbol{V}(z) \Big\|_{L^{6/5}(\Omega_h;\mathbb{R}^3)} \mathrm{d}z.$$

We reserve the similar treatment to the term I_1^K. Resuming these calculations and summing over n from 1 to m we get by using Corollary 4.2 and estimates (2.48)–(2.49), (2.39),

$$|\mathcal{R}_{2,3}| \leq h^{5/6}\, c(M_0, E_0, \|(r, \boldsymbol{V}, \nabla \boldsymbol{V}, \partial_t \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{16})}, \|\partial_t \nabla \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^9))}). \tag{7.11}$$

Step 2c: *Term* $\mathcal{T}_{2,3}$. We rewrite this term in the form

$$\begin{aligned}
&\mathcal{T}_{2,3} = \mathcal{T}_{2,4} + \mathcal{R}_{2,4},\ \mathcal{R}_{2,4} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \mathcal{R}_{2,4}^{n,K},\\
&\text{with } \mathcal{T}_{2,4} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K r_K^{n-1} \frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t} \cdot (\boldsymbol{u}_K^n - \boldsymbol{V}_{h,0,K}^n)\,\mathrm{d}x, \qquad (7.12)\\
&\text{and } \mathcal{R}_{2,4}^{n,K} = \int_K r_K^{n-1} \frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t} \cdot \left((\boldsymbol{u}^n - \boldsymbol{u}_K^n) - (\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n)\right)\,\mathrm{d}x.
\end{aligned}$$

First, we estimate the L^∞ norm of $\frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t}$ as in (6.5). Next, we decompose

$$\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n = \boldsymbol{V}_{h,0}^n - \boldsymbol{V}_h^n + \boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n + [\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n]_K,$$

and use $(2.52)_{p=2}$ to estimate $\boldsymbol{u}^n - \boldsymbol{u}_K^n$, $(2.52)_{p=\infty}$, $(2.41)_{s=1}$ to estimate $\boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n$ and (2.48)–(2.49) to evaluate $\|[\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n]_K\|_{L^\infty(K;\mathbb{R}^3)} \le \|\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n\|_{L^\infty(K;\mathbb{R}^3)}$. Thanks to the Hölder inequality and (4.4) we finally deduce

$$|\mathcal{R}_{2,4}| \le h\, c\left(M_0, E_0, \overline{r}, \|(\boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{15})}\right). \qquad (7.13)$$

Step 3: *Term* $\mathcal{T}_3$. Let us first decompose $\mathcal{T}_3$ as

$$\begin{aligned}
&\mathcal{T}_3 = \mathcal{T}_{3,1} + \mathcal{R}_{3,1},\\
&\text{with } \mathcal{T}_{3,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K r_K^n \boldsymbol{V}_{h,0,K}^n \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n)\,\mathrm{d}x, \quad \mathcal{R}_{3,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \mathcal{R}_{3,1}^{n,K},\\
&\text{and } \mathcal{R}_{3,1}^{n,K} = \int_K (r^n - r_K^n)\boldsymbol{V}^n \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x + \int_K r_K^n (\boldsymbol{V}^n - \boldsymbol{V}_{h,0}^n) \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x\\
&\qquad + \int_K r_K^n (\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n) \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x\\
&\qquad + \int_K r_K^n \boldsymbol{V}_{h,0,K}^n \cdot \nabla \boldsymbol{V}^n \cdot \left(\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n - (\boldsymbol{u}^n - \boldsymbol{u}_K^n)\right)\,\mathrm{d}x.
\end{aligned}$$

We have

$$\|r^n - r_K^n\|_{L^\infty(K)} \lesssim h \|\nabla r^n\|_{L^\infty(K)},$$

by the Taylor formula,

$$\|\boldsymbol{V}^n - \boldsymbol{V}_{h,0}^n\|_{L^\infty(K;\mathbb{R}^3)} \lesssim h \|\nabla \boldsymbol{V}^n\|_{L^\infty(K;\mathbb{R}^9)},$$

by virtue of $(2.40)_{s=1}$ and (2.48) and (2.49),

$$\begin{aligned}
\|\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n\|_{L^\infty(K;\mathbb{R}^3)} &\le \|\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,}^n\|_{L^\infty(K;\mathbb{R}^3)} + \|\boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n\|_{L^\infty(K;\mathbb{R}^3)}\\
&+ \|[\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n]_K\|_{L^\infty(K;\mathbb{R}^3)} \lesssim h \|\nabla \boldsymbol{V}^n\|_{L^\infty(K;\mathbb{R}^9)}
\end{aligned}$$

by virtue of (2.52), $(2.40)_{s=1}$ $(2.41)_{s=1}$ and (2.48)–(2.49),

$$\|\boldsymbol{u}^n - \boldsymbol{u}_K^n\|_{L^\infty(K;\mathbb{R}^3)} \lesssim h \|\nabla \boldsymbol{u}^n\|_{L^\infty(K;\mathbb{R}^9)}.$$

Consequently by employing several times the Hölder inequality (for integrals over K) and the discrete Hölder inequality (for the sums over $K \in \mathcal{T}$), and using estimate (4.4), we arrive at

$$|\mathcal{R}_{3,1}| \leq h\, c(M_0, E_0, \overline{r}, \|(\nabla r, \boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{15})}). \tag{7.14}$$

Now we shall deal wit term $\mathcal{T}_{3,1}$. Integrating by parts, we get:

$$\begin{aligned}\int_K r_K^n \boldsymbol{V}_{h,0,K}^n \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n)\,\mathrm{d}x &= \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r_K^n [\boldsymbol{V}_{h,0,K}^n \cdot \mathbf{n}_{\sigma,K}] \boldsymbol{V}_\sigma^n \cdot (\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n) \\ &= \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r_K^n [\boldsymbol{V}_{h,0,K}^n \cdot \mathbf{n}_{\sigma,K}] (\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n) \cdot (\boldsymbol{V}_{h,K}^n - \boldsymbol{u}_K^n),\end{aligned}$$

thanks to the the fact that $\sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \boldsymbol{V}_{h,K}^n \cdot \mathbf{n}_{\sigma,K}\mathrm{d}S = 0$.

Next we write

$$\mathcal{T}_{3,1} = \mathcal{T}_{3,2} + \mathcal{R}_{3,2}, \quad \mathcal{R}_{3,2} = \Delta t \sum_{n=1}^m \mathcal{R}_{3,2}^n,$$

$$\mathcal{T}_{3,2} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \hat{r}_\sigma^{n,\mathrm{up}} [\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}] (\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n) \cdot (\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}), \tag{7.15}$$

$$\begin{aligned}\text{and } \mathcal{R}_{3,2}^n &= \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| (r_K^n - \hat{r}_\sigma^{n,\mathrm{up}}) [\boldsymbol{V}_{h,0,K}^n \cdot \mathbf{n}_{\sigma,K}] (\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n) \cdot (\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n) \\ &+ \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \hat{r}_\sigma^{n,\mathrm{up}} \Big[\Big(\boldsymbol{V}_{h,0,K}^n - \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} \Big) \cdot \mathbf{n}_{\sigma,K} \Big] (\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n) \cdot (\boldsymbol{V}_{h,K}^n - \boldsymbol{u}_K^n) \\ &+ \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \hat{r}_\sigma^{n,\mathrm{up}} [\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}] (\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n) \cdot \Big((\boldsymbol{V}_{h,0,K}^n - \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}}) - (\boldsymbol{u}_K^n - \hat{\boldsymbol{u}}_{h,\sigma}^{n,\mathrm{up}}) \Big).\end{aligned}$$

We may write

$$\mathbf{V}_\sigma^n - \mathbf{V}_{h,0,K}^n = \mathbf{V}_\sigma^n - \mathbf{V}^n + \mathbf{V}^n - \mathbf{V}_h^n + \mathbf{V}_h^n - \mathbf{V}_{h,K}^n + [\mathbf{V}_h^n - \mathbf{V}_{h,0}^n]_K,$$

and use several times the Taylor formula along with $(2.40)_{s=1}$, (2.52), $(2.41)_{s=1}$, (2.48)–(2.49) (in order to estimate $r_K^n - \hat{r}_\sigma^{n,\mathrm{up}}$, $\mathbf{V}_\sigma^n - \mathbf{V}_{h,0,K}^n$, $\mathbf{V}_{h,K}^n - \hat{\mathbf{V}}_{h,\sigma}^{n,\mathrm{up}}$) to get the bound

$$|\mathcal{R}_{3,2}^n| \leq h\, c\|r\|_{W^{1,\infty}(\Omega)} \Big(1 + \|\boldsymbol{V}\|_{W^{1,\infty}(Q_T;\mathbb{R}^3)}\Big)^3 \sum_{K\in\mathcal{T}} h|\sigma||\boldsymbol{u}_K^n|$$

$$+ c\|r\|_{W^{1,\infty}(\Omega)} \Big(1 + \|\boldsymbol{V}\|_{W^{1,\infty}(Q_T;\mathbb{R}^3)}\Big)^2 \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}_K^n - \boldsymbol{u}_\sigma^n|.$$

We have by the Hölder inequality

$$\sum_{K\in\mathcal{T}} h|\sigma||\boldsymbol{u}_K^n| \le c\left(\sum_{\sigma\in\mathcal{T}} h|\sigma||\boldsymbol{u}_K^n|^6\right)^{1/6} \le c\left[\left(\sum_{K\in\mathcal{T}} \|\boldsymbol{u}^n-\boldsymbol{u}_K^n\|^6_{L^6(K;\mathbb{R}^3)}\right)^{1/6} + \left(\sum_{K\in\mathcal{T}} \|\boldsymbol{u}^n\|^6_{L^6(K;\mathbb{R}^3)}\right)^{1/6}\right] \le c\left(\sum_{K\in\mathcal{T}} \|\nabla\boldsymbol{u}_n\|^2_{L^2(K;\mathbb{R}^9)}\right)^{1/2},$$

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}_K^n-\boldsymbol{u}_\sigma^n| \le c\left[\left(\sum_{K\in\mathcal{T}} \|\boldsymbol{u}^n-\boldsymbol{u}_K^n\|^2_{L^2(K;\mathbb{R}^3)}\right)^{1/2} + \left(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} \|\boldsymbol{u}^n-\boldsymbol{u}_\sigma^n\|^2_{L^2(K;\mathbb{R}^3)}\right)^{1/2}\right] \le h\,c\left(\sum_{K\in\mathcal{T}} \|\nabla\boldsymbol{u}_n\|^2_{L^2(K;\mathbb{R}^9)}\right)^{1/2},$$

where we have used $(2.54)_{p=2}$, (2.52)–$(2.53)_{p=2}$. Consequently, we may use (4.4) to conclude

$$|\mathcal{R}_{3,2}| \le h\,c\left(M_0, E_0, \overline{r}, \|\nabla r, \boldsymbol{V}, \nabla\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{15})}\right). \tag{7.16}$$

Finally, we replace in $\mathcal{T}_{3,2}$ $\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n$ by $\boldsymbol{V}_{h,0,\sigma}^n - \boldsymbol{V}_{h,0,K}^n$. We get

$$\mathcal{T}_{3,2} = \mathcal{T}_{3,3} + \mathcal{R}_{3,3}, \quad \mathcal{R}_{3,3} = \Delta t\sum_{n=1}^m \mathcal{R}_{3,3}^n,$$

$$\mathcal{T}_{3,3} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\hat{r}_\sigma^{n,\mathrm{up}}[\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}}\cdot\mathbf{n}_{\sigma,K}](\boldsymbol{V}_{h,0,\sigma}^n - \boldsymbol{V}_{h,0,K}^n)\cdot(\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}), \tag{7.17}$$

and

$$\mathcal{R}_{3,3}^n = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\hat{r}_\sigma^{n,\mathrm{up}}\boldsymbol{V}_{h,0,K}^n\cdot\mathbf{n}_{\sigma,K}\left([\boldsymbol{V}^n - \boldsymbol{V}_{h,0}]_\sigma^n - [\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n]_K\right)\cdot(\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}),$$

committing error

$$|\mathcal{R}_{3,3}^n| \le h\,c\left(M_0, E_0, \overline{r}, \|\nabla r, \boldsymbol{V}, \nabla\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{15})}\right), \tag{7.18}$$

as in the previous step.

Step 4: *Terms* $\mathcal{T}_4$ We write

$$\mathcal{T}_4 = \mathcal{T}_{4,1} + \mathcal{R}_{4,1}, \quad \mathcal{T}_{4,1} = -\int_{\Omega_h} \nabla p(r^n)\cdot\mathbf{V}^n\,\mathrm{d}x,$$

$$\mathcal{R}_{4,1} = \int_{\Omega_h} \nabla p(r^n)\cdot(\mathbf{V}^n - \boldsymbol{V}_{h,0}^n)\,\mathrm{d}x;$$

whence

$$|\mathcal{R}_{4,1}| \le hc\left(\overline{r}, |p'|_{C[\underline{r},\overline{r}]}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}\right), \tag{7.19}$$

by virtue of $(2.40)_{s=1}$, (2.48)–(2.49).

Next, employing the integration by parts

$$\mathcal{T}_{4,2} = \mathcal{T}_{4,2} + \mathcal{R}_{4,2}, \quad \mathcal{T}_{4,2} = \int_{\Omega_h} p(r^n) \,\mathrm{div}\, \mathbf{V}^n \,\mathrm{d}x,$$

$$\mathcal{R}_{4,2} = -\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K),\sigma\in\partial\Omega_h} \int_\sigma p(r^n)\mathbf{V}^n \cdot \mathbf{n}_{\sigma,K} \mathrm{d}S = -\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K),\sigma\in\partial\Omega_h} \int_\sigma p(r^n) \left(\mathbf{V}^n - \boldsymbol{V}^n_{h,0,\sigma}\right) \cdot \mathbf{n}_{\sigma,K} \mathrm{d}S.$$

Writing

$$\mathbf{V}^n - \boldsymbol{V}^n_{h,0,\sigma} = \mathbf{V}^n - \boldsymbol{V}^n_h + \mathbf{V}^n_h - \boldsymbol{V}^n_{h,\sigma} + [\boldsymbol{V}^n_h - \boldsymbol{V}^n_{h,0}]_\sigma,$$

we deduce by using $(2.40)_{s=1}$, $(2.41)_{s=1}$, $(2.53)_{p=\infty}$, (2.48), (2.49),

$$\|\mathbf{V}^n - \boldsymbol{V}^n_{h,0,\sigma}\|_{L^\infty(K;\mathbb{R}^3)} \stackrel{<}{\sim} h \|\nabla \boldsymbol{V}^n\|_{L^\infty(K;\mathbb{R}^3)}, \ \sigma \in K.$$

Now, we employ the fact that

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K),\sigma\in\partial\Omega_h} \int_\sigma \mathrm{d}S \approx 1;$$

whence

$$|\mathcal{R}_{4,2}| \leq hc(\overline{r}, |p|_{C[\underline{r},\overline{r}]}, \|\nabla \mathbf{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}) \tag{7.20}$$

Finally,

$$\mathcal{T}_{4,2} = \mathcal{T}_{4,3} + \mathcal{R}_{4,3}, \quad \mathcal{T}_{4,3} = \int_{\Omega_h} p(\hat{r}^n) \,\mathrm{div}\, \mathbf{V}^n \,\mathrm{d}x, \quad \mathcal{R}_{4,3} = \int_{\Omega_h} (p(r^n) - p(\hat{r}^n)) \,\mathrm{div}\, \mathbf{V}^n \,\mathrm{d}x; \tag{7.21}$$

whence

$$|\mathcal{R}_{4,3}| \leq hc(|p'|_{C[\underline{r},\overline{r}]}, \|(\nabla r, \nabla \mathbf{V})\|_{L^\infty(Q_T;\mathbb{R}^{12})}). \tag{7.22}$$

Step 5: *Term* $\mathcal{T}_6$ We decompose $\mathcal{T}_6$ as

$$\begin{aligned} \mathcal{T}_6 &= \mathcal{T}_{6,1} + \mathcal{R}_{6,1}, \text{with } \mathcal{T}_{6,1} = -\Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K p'(\hat{r}^n) \boldsymbol{u}^n \cdot \nabla r^n \,\mathrm{d}x, \\ \mathcal{R}_{6,1} &= \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K (p'(\hat{r}^n) - p'(r^n)) \cdot \boldsymbol{u}^n \cdot \nabla r^n \,\mathrm{d}x. \end{aligned} \tag{7.23}$$

Consequently, by the Taylor formula, Hölder inequality and estimate (4.5),

$$|\mathcal{R}_{6,1}| \leq h\, c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}])}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}). \tag{7.24}$$

Gathering the formulae (7.7), (7.12), (7.17), (7.21), (7.23) and estimates for the residual terms (7.8), (7.9)–(7.13), (7.14)–(7.18), (7.19), (7.20), (7.22), (7.24) concludes the proof of Lemma 7.1. □

8. A Gronwall inequality

In this section we put together the relative energy inequality (6.1) and the identity (7.1) derived in the previous section. The final inequality resulting from this manipulation is formulated in the following lemma.

Lemma 8.1. *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5)–(3.7) with the pressure satisfying (1.4), where $\gamma \geq 3/2$. Then there exists a positive number*

$$c = c\Big(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V}, \nabla^2 \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{45})},$$
$$\|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3))}, \|\partial_t^2 \boldsymbol{V}, \partial_t \nabla \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^{12}))}\Big),$$

such that for all $m = 1, \dots, N$, there holds:

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m | \hat{r}^m, \hat{\boldsymbol{V}}^m_{h,0}) + \Delta t \frac{\mu}{2} \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K |\nabla_x (u^n - \mathbf{V}^n_{h,0})|^2 \mathrm{d}x$$
$$\leq c\Big[h^a + \sqrt{\Delta t} + \mathcal{E}(\varrho^0, \boldsymbol{u}^0 | \hat{r}(0), \hat{\boldsymbol{V}}_{h,0}(0))\Big] + c\,\Delta t \sum_{n=1}^m \mathcal{E}(\varrho^n, \boldsymbol{u}^n | \hat{r}^n, \hat{\boldsymbol{V}}^n_{h,0}),$$

with any couple $(r, \mathbf{V})$ belonging to (2.25) and satisfying the continuity equation (1.1) on $(0,T) \times \mathbb{R}^3$ and momentum equation (1.2) with boundary conditions (1.5) on $(0,T) \times \Omega$ in the classical sense, where a is defined in (3.9) and $\mathcal{E}$ is given in (4.9).

Proof. We observe that

$$S_6 - \mathcal{S}_6 = \Delta t \sum_{n=1}^m \int_{\Omega_h} p'(\hat{r}^n) \frac{\hat{r}^n - \varrho^n}{\hat{r}^n} \mathbf{V}^n \cdot \nabla r^n \,\mathrm{d}x + \Delta t \sum_{n=1}^m \int_{\Omega_h} p'(\hat{r}^n) \frac{\hat{r}^n - \varrho^n}{\hat{r}^n} (\mathbf{u}^n - \mathbf{V}^n) \cdot \nabla r^n \,\mathrm{d}x.$$

Gathering the formulae (6.1) and (6.2), one gets

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m \Big| \hat{r}^m, \hat{\boldsymbol{V}}^m_{h,0}) - \mathcal{E}(\varrho^0, \boldsymbol{u}^0 \Big| \hat{r}(0), \hat{\boldsymbol{V}}_{h,0}(0)) + \mu \Delta t \sum_{n=1}^m \sum_{K \in \mathcal{T}} \Big| \nabla (\boldsymbol{u}^n - \boldsymbol{V}^n_{0,h}) \Big|^2_{L^2(K;\mathbb{R}^3)} \leq \sum_{i=1}^4 \mathcal{P}_i + \mathcal{Q}, \tag{8.1}$$

where

$$\mathcal{P}_1 = \Delta t \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| (\varrho_K^{n-1} - r_K^{n-1}) \frac{\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^{n-1}_{h,0,K}}{\Delta t} \cdot \left(\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K\right),$$
$$\mathcal{P}_2 = \Delta t \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma = K|L \in \mathcal{E}_K} |\sigma| \left(\varrho^{n,\mathrm{up}}_\sigma - \hat{r}^{n,\mathrm{up}}_\sigma\right) \left(\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma\right) \cdot \left(\boldsymbol{V}^n_{h,0,\sigma} - \boldsymbol{V}^n_{h,0,K}\right) \boldsymbol{V}^{n,\mathrm{up}}_{h,0,\sigma} \cdot \boldsymbol{n}_{\sigma,K},$$
$$\mathcal{P}_3 = -\Delta t \sum_{n=1}^m \int_{\Omega_h} \left(p(\varrho^n) - p'(\hat{r}^n)(\varrho^n - \hat{r}^n) - p(\hat{r}^n)\right) \mathrm{div} \mathbf{V}^n,$$
$$\mathcal{P}_4 = \Delta t \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K p'(\hat{r}^n) \frac{\hat{r}^n - \varrho^n}{\hat{r}^n} (\mathbf{u}^n - \mathbf{V}^n) \cdot \nabla r^n \,\mathrm{d}x,$$
$$\mathcal{Q} = \mathcal{R}^m_{h,\Delta t} + R^m_{h,\Delta t} + G^m.$$

Now, we estimate conveniently the terms $\mathcal{P}_i$, $i = 1, \dots, 4$ in four steps.

Step 1: *Term* $\mathcal{P}_1$. We estimate the L^∞ norm of $\frac{\boldsymbol{V}^n_{h,0,K}-\boldsymbol{V}^{n-1}_{h,0,K}}{\Delta t}$ by L^∞ norm of $\partial_t \mathbf{V}$ in the same manner as in (6.5). According to Lemma 7.2, $|\varrho - r|^\gamma 1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho) \leq c(p)E^p(\varrho|r)$, with any $p \geq 1$; in particular,

$$|\varrho - r|^{6/5} 1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho) \leq cE(\varrho|r) \tag{8.2}$$

provided $\gamma \geq 6/5$.

We get by using the Hölder inequality,

$$\left| \sum_{K\in\mathcal{T}} |K|(\varrho_K^{n-1} - r_K^{n-1}) \frac{\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^{n-1}_{h,0,K}}{\Delta t} \cdot \left(\boldsymbol{V}^n_{h,K} - \boldsymbol{u}^n_K\right) \right| \leq c\|\partial_t \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)}$$
$$\times \left[\left(\sum_{K\in\mathcal{T}} |K||\varrho_K^{n-1} - r_K^{n-1}|^2 1_{[\underline{r}/2,2\overline{r}]}(\varrho_K) \right)^{1/2} + \left(\sum_{K\in\mathcal{T}} |K||\varrho_K^{n-1} - r_K^{n-1}|^{6/5} 1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho_K) \right)^{5/6} \right]$$
$$\times \left(\sum_{K\in\mathcal{T}} |K| \left|\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K\right|^6 \right)^{1/6} \leq c\left(\|(\partial_t \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^3)} \right) \left(\mathcal{E}^{1/2}\left(\varrho^{n-1}, \hat{\boldsymbol{u}}^{n-1}|\hat{r}^{n-1}, \hat{\boldsymbol{V}}^{n-1}_{h,0}\right) \right.$$
$$\left. + \mathcal{E}^{5/6}\left(\varrho^{n-1}, \hat{\boldsymbol{u}}^{n-1}|\hat{r}^{n-1}, \hat{\boldsymbol{V}}^{n-1}_{h,0}\right) \right) \left(\sum_{K\in\mathcal{T}} \|\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K\|^6_{L^6(K;\mathbb{R}^3)} \right)^{1/6},$$

where we have used (8.2) and estimate (4.8) to obtain the last line. Now, we write $\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K = ([\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n]_K - (\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)) + (\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)$ and use the Minkowski inequality together with formulas (2.54), (2.43) to get

$$\left(\sum_{K\in\mathcal{T}} \|\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K\|^6_{L^6(K;\mathbb{R}^3)} \right)^{1/6} \leq \left(\sum_{K\in\mathcal{T}} \|\nabla(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\|^2_{L^2(K;\mathbb{R}^3)} \right)^{1/2}.$$

Finally, employing Young's inequality, and estimate (4.8), we arrive at

$$|\mathcal{P}_1| \leq c\left(\delta, M_0, E_0, \underline{r}, \overline{r}, \|(\boldsymbol{V}, \nabla\boldsymbol{V}, \partial_t \boldsymbol{V})\|_{L^\infty(Q_T,\mathbb{R}^{15})}\right)$$
$$\times \left(\Delta t \mathcal{E}(\varrho^0, \hat{\boldsymbol{u}}^0|\hat{r}^0, \hat{\boldsymbol{V}}^0_{h,0}) + \Delta t \sum_{n=1}^m \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n|\hat{r}^n, \hat{\boldsymbol{V}}^n_{h,0}) \right) + \delta \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \|\nabla(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\|^2_{L^2(K;\mathbb{R}^3)}, \tag{8.3}$$

with any $\delta > 0$.

Step 2: *Term* $\mathcal{P}_2$. We rewrite $\boldsymbol{V}^n_{h_0,\sigma} - \boldsymbol{V}^n_{h_0,K} = \boldsymbol{V}^n_{h,\sigma} - \boldsymbol{V}^n_{h,K} + [\boldsymbol{V}^n_{h,0} - \boldsymbol{V}^n_h]_\sigma + [\boldsymbol{V}^n_{h,0} - \boldsymbol{V}^n_h]_K$ and estimate the L^∞ norm of this expression by $h\|\nabla\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}$ by virtue of (2.48)–(2.49), (2.52)–(2.53), $(2.41)_{s=1}$. Now we write $\mathcal{P}_2 = \Delta t \sum_{n=1}^m \mathcal{P}_2^n$ where Lemma 7.2 and the Hölder inequality yield, similarly as in the previous step,

$$|\mathcal{P}_2^n| \leq c(\underline{r}, \overline{r}, \|\nabla\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^9)})$$
$$\times \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h \left(E^{1/2}(\varrho^{n,\mathrm{up}}_\sigma|\hat{r}^{n,\mathrm{up}}_\sigma) + E^{2/3}(\varrho^{n,\mathrm{up}}_\sigma|\hat{r}^{n,\mathrm{up}}_\sigma \right) |\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}| \, |\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma|$$
$$\leq c(\underline{r}, \overline{r}, \|(\boldsymbol{V}, \nabla\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{12})}) \left[\left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h \left(E(\varrho^{n,\mathrm{up}}_\sigma|\hat{r}^{n,\mathrm{up}}_\sigma) \right) \right)^{1/2} \right.$$
$$\left. + \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h E(\varrho^{n,\mathrm{up}}_\sigma|\hat{r}^{n,\mathrm{up}}_\sigma) \right)^{2/3} \right] \times \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h \left| \hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma \right|^6 \right)^{1/6},$$

provided $\gamma \geq 3/2$. Next, we observe that the contribution of the face $\sigma = K|L$ to the sums $\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}$ $|\sigma|hE(\varrho_\sigma^{n,\mathrm{up}}|\hat{r}_\sigma^{n,\mathrm{up}})$ and $\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|h|\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}}-\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}|^6$ is less or equal than $2|\sigma|h(E(\varrho_K^n|\hat{r}_K^n)+E(\varrho_L^n|\hat{r}_L^n))$, and than $2|\sigma|h(|\boldsymbol{V}_{h,0,K}^n-\boldsymbol{u}_K^n|^6+|\boldsymbol{V}_{h,0,L}^n-\boldsymbol{u}_L^n|^6)$, respectively. Consequently, we get by the same reasoning as in the previous step, under assumption $\gamma \geq 3/2$,

$$|\mathcal{P}_2| \leq c(\delta, M_0, E_0, \underline{r}, \overline{r}, \|(\boldsymbol{V}, \nabla\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{12})})\,\Delta t\sum_{n=1}^m \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n|\hat{r}^n, \hat{\boldsymbol{V}}_{h,0}^n) + \delta\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\|\nabla(\boldsymbol{V}_{h,0}^n-\boldsymbol{u}^n)\|^2_{L^2(K;\mathbb{R}^3)}. \tag{8.4}$$

Step 3: *Term* $\mathcal{P}_3$. We realize that

$$p(\varrho_K^n) - p'(r_K^n)(\varrho_K^n - r_K^n) - p(r_K^n) \leq c(\underline{r}, \overline{r})E(\varrho_K|r_K),$$

by virtue of Lemma 7.2 in combination with assumption (1.4). Consequently,

$$|\mathcal{P}_3| \leq c\|\operatorname{div}\boldsymbol{V}\|_{L^\infty(Q_T)}\Delta t\sum_{n=1}^m \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n|\hat{r}^n, \hat{\boldsymbol{V}}_{h,0}^n). \tag{8.5}$$

Step 4: *Term* $\mathcal{P}_4$. We write $\boldsymbol{u}^n - \boldsymbol{V}^n$ as the sum $(\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n) + (\boldsymbol{V}_{h,0}^n - \boldsymbol{V}^n)$ accordingly splitting $\mathcal{P}_4$ into two terms

$$\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K p'(\hat{r}^n)\frac{\hat{r}^n-\varrho^n}{\hat{r}^n}(\mathbf{u}^n-\boldsymbol{V}_{h,0}^n)\cdot\nabla r^n\,\mathrm{d}x \quad\text{and}\quad \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K p'(\hat{r}^n)\frac{\hat{r}^n-\varrho^n}{\hat{r}^n}(\boldsymbol{V}_{h,0}^n-\mathbf{V}^n)\cdot\nabla r^n\,\mathrm{d}x.$$

Reasoning similarly as in Step 2, we get

$$\begin{aligned}|\mathcal{P}_4| &\leq h^2\, c(\delta, M_0, E_0, \underline{r}, \overline{r}, |p'|_{C([\underline{r},\overline{r}])}\|(\nabla r, \nabla\boldsymbol{V})\|_{L^\infty(\Omega;\mathbb{R}^9)})\\ &\quad + c(\delta, \|\underline{r}, \overline{r}, |p'|_{C([\underline{r},\overline{r}])}\|\nabla r\|_{L^\infty(\Omega;\mathbb{R}^3)})\,\Delta t\sum_{n=1}^m \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n|\hat{r}^n, \hat{\boldsymbol{V}}_{h,0}^n) + \delta\,\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\|\nabla(\boldsymbol{V}_{h,0}^n-\boldsymbol{u}^n)\|^2_{L^2(K;\mathbb{R}^3)}.\end{aligned} \tag{8.6}$$

Gathering the formulae (8.1) and (8.3)–(8.6) with δ sufficiently small (with respect to μ), we conclude the proof of Lemma 8.1. □

9. End of the proof of the error estimate (Thm. 3.2)

Finally, Lemma 8.1 in combination with the bound (4.8) yields

$$\mathcal{E}(\varrho^m, \hat{\boldsymbol{u}}^m|\hat{r}^m, \hat{\boldsymbol{V}}_{h,0}^m) \leq c\Big[h^A + \sqrt{\Delta t} + \Delta t + \mathcal{E}(\varrho^0, \hat{\boldsymbol{u}}^0|\hat{r}(0), \hat{\boldsymbol{V}}_{h,0}(0))\Big] + c\,\Delta t\sum_{n=1}^{m-1}\mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n|\hat{r}^n, \hat{\boldsymbol{V}}_{h,0}^n);$$

whence by the discrete standard version of the Gronwall lemma one gets at the first step

$$\mathcal{E}(\varrho^m, \hat{\boldsymbol{u}}^m|\hat{r}^m, \hat{\boldsymbol{V}}_{h,0}^m) \leq c\Big[h^a + \sqrt{\Delta t} + \mathcal{E}(\varrho^0, \hat{\boldsymbol{u}}^0|\hat{r}(0), \hat{\boldsymbol{V}}_{h,0}(0))\Big].$$

Going with this information back to Lemma 8.1, one gets finally

$$\mathcal{E}(\varrho^m, \hat{\boldsymbol{u}}^m|\hat{r}^m, \hat{\boldsymbol{V}}_{h,0}^m) + \Delta t\frac{\mu}{2}\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K|\nabla_x(\mathbf{u}^n-\mathbf{V}_{h,0}^n)|^2\mathrm{dx} \leq c\Big[h^a + \sqrt{\Delta t} + \mathcal{E}(\varrho^0, \hat{\boldsymbol{u}}^0|\hat{r}(0), \hat{\boldsymbol{V}}_{h,0}(0))\Big]. \tag{9.1}$$

Now, we write

$$\varrho_K^n(\boldsymbol{u}_K^n - \mathbf{V}_{h,0,K}^n)^2 = \varrho_K^n(\boldsymbol{u}_K^n - \mathbf{V}^n)^2 + 2\varrho_K^n\mathbf{V}^n(\boldsymbol{u}_K^n - \mathbf{V}_{h,0,K}^n) + \varrho_K^n(\mathbf{V}^n - \mathbf{V}_{h,0,K}^n)^2,$$

where

$$\|\mathbf{V}^n - \mathbf{V}_{h,0,K}^n\|_{L^\infty(K;\mathbb{R}^3)} \stackrel{<}{\sim} \|\mathbf{V}^n - \mathbf{V}_h^n\|_{L^\infty(K;\mathbb{R}^3)} + \|\mathbf{V}_h^n - \mathbf{V}_{h,K}^n\|_{L^\infty(K;\mathbb{R}^3)} + \|[\mathbf{V}_h^n - \mathbf{V}_{h,0}^n]_K\|_{L^\infty(K;\mathbb{R}^3)}$$

$$\stackrel{<}{\sim} h\left(\|\nabla_x\mathbf{V}^n\|_{L^\infty(K;\mathbb{R}^9)} + \|\nabla_x\mathbf{V}_h^n\|_{L^\infty(K;\mathbb{R}^9)} + \|\mathbf{V}_h^n - \mathbf{V}_{h,0}^n\|_{L^\infty(K;\mathbb{R}^3)}\right) \stackrel{<}{\sim} h\|\nabla\mathbf{V}^n\|_{L^\infty(K;\mathbb{R}^9)}.$$

In the above calculation we have employed formula (2.40) to estimate the first term, estimates $(2.52)_{s=1}$, $(2.41)_{s=1}$ to estimate the second term, and formulas (2.48) and (2.49) for $K \cap \partial\Omega_h = \emptyset$ and $K \cap \partial\Omega_h \neq \emptyset$, respectively, to evaluate the last term. We conclude that

$$\sum_{K\in\mathcal{T}} \frac{1}{2}|K|\left(\varrho_K^m|\boldsymbol{u}_K^m - \boldsymbol{V}_{h,0,K}^m|^2 - \varrho_K^0|\boldsymbol{u}_K^0 - \boldsymbol{V}_{h,0,K}^0|^2\right) \geq \int_{\Omega\cap\Omega_h} \varrho^m(\hat{\boldsymbol{u}}^m - \mathbf{V}^m)^2\mathrm{d}x - \int_{\Omega\cap\Omega_h} \varrho^0(\hat{\boldsymbol{u}}^0 - \mathbf{V}^0)^2\mathrm{d}x + L_1, \tag{9.2}$$

where

$$|L_1| \stackrel{<}{\sim} h\, M_0\|\nabla_x\mathbf{V}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^9)}.$$

Similarly, we find with help of (4.8),

$$\|E(\varrho_K^n|\hat{r}^n) - E(\varrho_K^n, r^n)\|_{L^\infty(K)} \leq h\, c(M_0, \underline{r}, \overline{r}, |p|_{C^1[\underline{r},\overline{r}]}\|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)});$$

whence

$$\sum_{K\in\mathcal{T}} |K|\left(E(\varrho_K^n|\hat{r}^n) - E(\varrho_K^0|\hat{r}^0)\right) \geq \int_{\Omega\cap\Omega_h} E(\varrho^m|r^m)\mathrm{d}x - \int_{\Omega\cap\Omega_h} E(\varrho^0|r^0)\mathrm{d}x + L_2, \tag{9.3}$$

where

$$|L_2| \leq h\, c(M_0, \underline{r}, \overline{r}, |p|_{C^1[\underline{r},\overline{r}]}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}).$$

Finally, by virtue of (2.48)–(2.49) and $(2.41)_{s=2}$

$$\|\nabla(\boldsymbol{V}_{h,0}^n - \boldsymbol{V}^n)\|_{L^2(K;\mathbb{R}^3)} \stackrel{<}{\sim} h\|(\nabla\boldsymbol{V}^n, \nabla^2\boldsymbol{V}^n)\|_{L^\infty(K;\mathbb{R}^{12})};$$

whence

$$\Delta t\sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K |\nabla_x(\mathbf{u}^n - \mathbf{V}_{h,0}^n)|^2\mathrm{dx} \geq \Delta t\sum_{n=1}^m \int_{\Omega\cap\Omega_h} |(\nabla_h\mathbf{u}^n - \nabla_x\mathbf{V}^n)|^2\mathrm{dx} + L_3, \tag{9.4}$$

where

$$|L_3| \leq h^2 c(\|(\nabla\boldsymbol{V}^n, \nabla^2\boldsymbol{V}^n)\|_{L^\infty(K;\mathbb{R}^{12})}).$$

Theorem 3.2 is a direct consequence of estimate (9.1) and identities (9.2)–(9.4). Theorem 3.2 is thus proved.

10. Concluding remarks

In the convergence proofs one usually needs to complete the numerical scheme by stabilizing terms, so that the new numerical scheme reads

$$\sum_{K\in\mathcal{T}_h} |K|\frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t}\phi_K + \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}}(\mathbf{u}_\sigma^n\cdot\mathbf{n}_{\sigma,K})\phi_K + T_c(\phi) = 0, \tag{10.1}$$

for any $\phi \in Q_h(\Omega_h)$ and $n = 1, \dots, N$,

$$\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\left(\varrho_K^n \boldsymbol{u}_K^n - \varrho_K^{n-1}\boldsymbol{u}_K^{n-1}\right)\cdot \boldsymbol{v}_K + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]\cdot\boldsymbol{v}_K \tag{10.2}$$

$$- \sum_{K\in\mathcal{T}} p(\varrho_K^n)\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\boldsymbol{v}_\sigma\cdot\mathbf{n}_{\sigma,K} + \mu\sum_{K\in\mathcal{T}}\int_K \nabla\boldsymbol{u}^n : \nabla\boldsymbol{v}\ \mathrm{d}x$$

$$+\frac{\mu}{3}\sum_{K\in\mathcal{T}}\int_K \mathrm{div}\boldsymbol{u}^n\mathrm{div}\boldsymbol{v}\ \mathrm{d}x + T_m(\phi) = 0,\ \text{for any } \boldsymbol{v}\in V_{h,0}(\Omega;R^3) \text{ and } n = 1,\dots,N,$$

where

$$T_c(\phi) = h^{1-\varepsilon}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}}|\sigma|[\varrho^n]_{\sigma,\boldsymbol{n}_\sigma}[\phi]_{\sigma\boldsymbol{n}_\sigma},\quad T_m(\phi) = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}}|\sigma|[\varrho^n]_{\sigma,\boldsymbol{n}_\sigma}\{\hat{\boldsymbol{u}}^n\}_\sigma[\hat{\phi}]_{\sigma,\boldsymbol{n}_\sigma},\ \varepsilon\in[0,1),$$

(see [20, 30]). These terms are designed to provide the supplementary positive term

$$h^{1-\varepsilon}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}}|\sigma|[\varrho^n]^2_{\sigma,\boldsymbol{n}_\sigma},$$

to the left hand side of the discrete energy identity (4.2). They contribute to the right hand side of the discrete relative energy (5.1) by supplementary terms whose absolute value is bounded from above by

$$h^{(1-\varepsilon)/2}\,c\left(M_0, E_0, \sup_{n=0,\dots,N}\|r^n, \mathbf{U}^n, \nabla\mathbf{U}^n\|_{L^\infty(\Omega_h;\mathbb{R}^{13})}, \sup_{n=0,\dots,N}\sup_{\sigma\in\mathcal{E}\mathrm{int}}[r^n]_{\sigma,\boldsymbol{n}_\sigma}/h\right).$$

Consequently, they give rise to the contributions at the right hand side of the approximate relative energy inequality (6.1) whose bound is

$$h^{(1-\varepsilon)/2}\,c\left(M_0, E_0, \|r, \nabla r, \mathbf{U}, \nabla\mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^{16})}\right).$$

Similar estimates are true, if we replace in the numerical scheme everywhere classical upwind formula (3.4)

$$\mathrm{Up}_K(q,\mathbf{u}) = \sum_{\sigma\in\mathcal{E}(K)} q_\sigma^{\mathrm{up}}\boldsymbol{u}_\sigma\cdot\mathbf{n}_{\sigma,K} = \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}\left(q_K[\boldsymbol{u}_\sigma\cdot\mathbf{n}_{\sigma,K}]^+ + q_L[\boldsymbol{u}_\sigma\cdot\mathbf{n}_{\sigma,K}]^-\right),$$

by the modified upwind suggested in [15]:

$$\mathrm{Up}_K(q,\mathbf{u}) = \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}\frac{q_K}{2}\left([\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K} + h^{1-\varepsilon}]^+ + [\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K} - h^{1-\varepsilon}]^+\right)$$

$$+\frac{q_L}{2}\left([\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K} + h^{1-\varepsilon}]^- + [\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K} - h^{1-\varepsilon}]^-\right), \tag{10.3}$$

where $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$. We will finish by formulating the error estimate for the numerical problem (3.5), (10.1), (10.2) or for (3.5), (3.6), (3.7) with modified upwind (10.3).

Theorem 10.1. *Let Ω, p, $[r_0, \mathbf{V}^0]$, $[r, V]$ satisfy assumptions of Theorem 3.2. Let $(\varrho^n, \mathbf{u}^n)_{n=0,\dots,N}$ be a family of numerical solutions to the scheme (3.5), (10.1), (10.2) or to the scheme (3.5), (3.6), (3.7) with modified upwind (10.3), where $\varepsilon\in[0,1)$. Then error estimate (3.8)* holds true with the exponent

$$a = \min\left\{\frac{2\gamma-3}{\gamma}, \frac{1-\varepsilon}{2}\right\} \text{ if } \tfrac{3}{2}\le\gamma<2,\quad a = \frac{1-\varepsilon}{2} \text{ if } \gamma\ge 2.$$

Finally, a natural question arises as top what extent the obtained error estimates are optimal. In the light of the results obtained in [28, 29], it may seem we loose, in particular in terms of the spatial discretization parameter h for $\gamma\to 3/2$. On the other hand, however, it is worth noting we do not make any extra assumption concerning boundedness of the numerical solutions in contrast with [28].

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