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A CONVERGENT EXPLICIT FINITE DIFFERENCE SCHEME FOR A MECHANICAL MODEL FOR TUMOR GROWTH

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Abstract. Mechanical models for tumor growth have been used extensively in recent years for the analysis of medical observations and for the prediction of cancer evolution based on image analysis. This work deals with the numerical approximation of a mechanical model for tumor growth and the analysis of its dynamics. The system under investigation is given by a multi-phase flow model: The densities of the different cells are governed by a transport equation for the evolution of tumor cells, whereas the velocity field is given by a Brinkman regularization of the classical Darcy's law. An efficient finite difference scheme is proposed and shown to converge to a weak solution of the system. Our approach relies on convergence and compactness arguments in the spirit of Lions [P.-L. Lions, Mathematical topics in fluid mechanics. Vol. 2. Vol. 10 of *Oxford Lecture Series Math. Appl*. The Clarendon Press, Oxford University Press, New York (1998)].

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1. Introduction

1.1. Motivation

Mechanical models for tumor growth are used extensively in recent years for the prediction of cancer evolution based on imaging analysis. Such models are based on the assumption that the growth of the tumor is mainly limited by the competition for space. Mathematical modeling, analysis and numerical simulations together with experimental and clinical observations are essential components in the effort to enhance our understanding of the cancer development. The goal of this article is to make a further step in the investigation of such models by presenting a convergent explicit finite difference scheme for the numerical approximation of a Hele–Shaw-type model for tumor growth and by providing its detailed mathematical analysis. Even though the main focus in the present work is on the investigation of the evolution of the proliferating cells, it provides a mathematical framework that can potentially accommodate more complex systems that account for the presence of nutrient and drug application. This will be the subject of future investigation [31].

Keywords and phrases. Tumor growth models, cancer progression, mixed models, multi-phase flow, finite difference scheme, existence.

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1.2. Governing equations

In the present context the tissue is considered as a multi-phase fluid and the ability of the tumor to expand into a host tissue is then primarily driven by the cell division rate which depends on the local cell density and the mechanical pressure in the tumor.

1.2.1. Transport equations for the evolution of the cell densities

The dynamics of the cell population density $n(t,x)$ under pressure forces and cell multiplication is described by a transport equation

$$\partial_t n - \operatorname{div}(n\boldsymbol{u}) = n\boldsymbol{G}(p), \quad x \in \Omega,\ t \geq 0 \tag{1.1}$$

where n represents the number density of tumor cells, $\boldsymbol{u}$ the velocity field and p the pressure of the *tumor*. Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$. We assume homogeneous Neumann boundary conditions, that is $\nabla n \cdot \nu = 0$, where ν is the normal vector on $\partial\Omega$ pointing outwards. The pressure law is given by

$$p(n) = an^\gamma, \tag{1.2}$$

where $\gamma \geq 2$ and $a > 0$ is a parameter. In the following, we will set $a = 1$ for simplicity. Following [3, 30], we assume that growth is directly related to the pressure through a function $\boldsymbol{G}(\cdot)$ which satisfies

$$\boldsymbol{G} \in C^1(\mathbb{R}), \quad \boldsymbol{G}'(\cdot) \leq -\beta < 0, \quad \boldsymbol{G}(P_M) = 0 \quad \text{for some } \beta, P_M > 0. \tag{1.3}$$

The pressure P_M is usually called *homeostatic pressure*. Here, and in what follows, for simplicity we let

$$\boldsymbol{G}(p) = \alpha - \beta p^\theta, \tag{1.4}$$

for some $\alpha, \beta, \theta > 0$.

1.2.2. The tumor tissue as a porous medium

The continuous motion of cells within the tumor region, typically due to proliferation, is represented by the velocity field $\boldsymbol{u} := \nabla W$ solving an alternative to Darcy's equation known as *Brinkman's equation*

$$p = W - \mu\Delta W \tag{1.5}$$

where μ is a positive constant describing the viscous-like properties of tumor cells and p is the pressure given by (1.2).

Relation (1.5) consists of two terms. If we consider (1.5) with only W on the right hand side, it is the Darcy law, which in the present setting describes the tendency of cells to move down pressure gradients and results from the friction of the tumor cells with the extracellular matrix. The additional term, $-\mu\Delta W$, is a dissipative force density (analogous to the Laplacian term that appears in the Navier–Stokes equation) and results from the internal cell friction due to cell volume changes. A second interpretation of relation (1.5) is that the tumor tissue may be viewed as "fluid like." In other words, the tumor cells flow through the fixed extracellular matrix like a flow through a porous medium, obeying Brinkman's law.

The resulting model, governed by the transport equation (1.1) for the population density of cells, the elliptic equation (1.5) for the velocity field and a state equation for the pressure law (1.2), now reads

$$\begin{cases} \partial_t n - \operatorname{div}(n\nabla W) = \alpha n - \beta n^{\gamma\theta+1}, & x \in \Omega,\ t \geq 0 \\ -\mu\Delta W + W = n^\gamma, \end{cases} \tag{1.6}$$

where $\alpha, \beta, \gamma, \theta, \mu > 0$. We complete the system (1.6) with an initial data n_0 satisfying (for some constant C)

$$n_0 \geq 0, \quad p(n_0) \leq P_M, \quad \|n_0\|_{L^1(\mathbb{R}^d)} \leq C. \tag{1.7}$$

The objective of this work is to establish global existence of weak solutions to the nonlinear model for tumor growth (1.6) by designing an efficient numerical scheme for its approximation and by showing that this scheme converges when the mesh is refined. The main ingredients of our approach and contribution to the existing theory on Hele–Shaw-type systems for tumor growth include:

- The introduction of a suitable notion of solutions to the nonlinear system (1.6) consisting of the transport equation (1.1) and the Brinkman regularization (1.5).
- The construction of an approximating procedure which relies on an artificial vanishing viscosity approximation and the establishment of the suitable compactness in order to pass into the limit and to conclude convergence to the original system (*cf.* Sect. 3, Lem. 3.7).
- The design of an efficient numerical scheme for the numerical approximation of the nonlinear system (1.1)–(1.5). The numerical approximation introduces numerical viscosity that goes to zero as the mesh is refined, in a similar way that the artificial viscosity vanishes as $\varepsilon \to 0$.
- The proof of the convergence of the numerical scheme. In the center of the analysis lies the proof of the strong convergence of the cell densities. This is achieved by establishing the weak continuity of the *effective viscous pressure* in the spirit of Lions [23] (*cf.* Sect. 4, Lem. 4.8).
- The design of numerical experiments in order to establish that the finite difference scheme is effective in computing approximate solutions to the nonlinear system (1.6) (*cf.* Sect. 4).

For relevant results on the analysis and the numerical approximation of a two-phase flow model in porous media we refer the reader to [6]. Related results on the numerical approximation of compressible fluids employing the weak compactness tools developed by Lions [23] in the discrete setting have been established by Karper *et al.* [16–19] and Gallouët *et al.* [13].

Relevant work on the mathematical analysis of mechanical models of Hele–Shaw-type have been presented by Perthame *et al.* [26–29]. The analysis in [28] establishes the existence of traveling wave solutions of the Hele–Shaw model of tumor growth with nutrient and presents numerical observations in two space dimensions. The present article is according to our knowledge the first article presenting rigorous analytical results on the global existence of general weak solutions to Hele–Shaw-type systems.

A different approach yielding results on the global existence of weak solutions to a nonlinear model for tumor growth in a general moving domain $\Omega_t \subset \mathbb{R}^3$ without any symmetry assumption and for finite large initial data is presented in [8–10]. But in contrast to the present nonlinear system, the transport equation for the evolution of cancerous cells in [9, 10] has a source term which is linear with respect to cell density.

Relevant results on nonlinear models for tumor growth governed by the Darcy's law for the evolution of the velocity field are presented by Zhao [32] based on the framework introduced by Friedman *et al.* [5, 14].

1.3. Outline

The paper is organized as follows: Section 1 presents the motivation, modeling and introduces the necessary preliminary material. Section 2 provides a weak formulation of the problem and states the main result. Section 3 is devoted to the global existence of solutions via a vanishing viscosity approximation. In Section 4 we present an efficient finite difference scheme for the approximation of the weak solution to system (1.6) on rectangular domains and Section 5 is devoted to numerical experiments. A discretized Aubin–Lions lemma and some technical lemmas are presented in Appendices A and B respectively.

2. Weak formulation and main results

Notation 2.1. For $\varphi : (0,T) \times \Omega \to \mathbb{R}$, $\boldsymbol{\varphi} : (0,T) \times \Omega \to \mathbb{R}^d$, we will denote by $\nabla\varphi(t,x) := \nabla_x\varphi(t,x) = (\partial_{x_1}\varphi, \ldots, \partial_{x_d}\varphi)(t,x)$ and $\operatorname{div}\boldsymbol{\varphi}(t,x) := \operatorname{div}_x \boldsymbol{\varphi}(t,x) = \sum_{i=1}^d \partial_{x_i}\boldsymbol{\varphi}^{(i)}(t,x)$ the gradient and divergence in the spatial direction in Ω.

2.1. Weak solutions

Definition 2.2. Let Ω a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, which is either rectangular or has a smooth boundary $\partial\Omega$ and $T > 0$ a finite time horizon. We say that (n, W, p) is a weak solution of problem (1.1)–(1.5) supplemented with initial data (n_0, W_0, p_0) satisfying (1.7) provided that the following hold:

• $(n, W, p) \geq 0$ represents a weak solution of (1.1)–(1.5) on $(0, T) \times \Omega$, *i.e.*, for any test function $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d), T > 0$, the following integral relations hold

$$\int_{\mathbb{R}^d} n(\tau, x)\varphi(\tau, x)\,\mathrm{d}x - \int_{\mathbb{R}^d} n_0\varphi(0, x)\mathrm{d}x = \int_0^\tau \int_{\mathbb{R}^d} (n\partial_t\varphi - n\nabla W \cdot \nabla\varphi + n\boldsymbol{G}(p)\varphi)\,\mathrm{d}x\mathrm{d}t. \tag{2.1}$$

In particular,

$$n \in L^p((0, T) \times \Omega), \text{ for all } p \geq 1.$$

We remark that in the weak formulation, it is convenient that the equations (1.1) hold in the whole space $\mathbb{R}^d$ provided that the densities n are extended to be zero outside the tumor domain.

• Brinkman's equation (1.5) holds in the sense of distributions, *i.e.*, for any test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ satisfying

$$\varphi|_{\partial\Omega} = 0 \text{ for any } t \in [0, T],$$

the following integral relation holds for a.e. $t \in [0, T]$,

$$\int_\Omega n^\gamma \varphi\,\mathrm{d}x = \int_\Omega \Big(\mu\nabla W \cdot \nabla\varphi + W\varphi\Big)\mathrm{d}x. \tag{2.2}$$

and $p = n^\gamma$ almost everywhere. All quantities in (2.2) are required to be integrable, and in particular, $W \in L^\infty([0, T]; H^2(\Omega))$.

The main result of the article now follows.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$, $0 < T < \infty$. Assume that $n_0 \in L^\infty(\Omega)$ with $0 \leq n_0 \leq n_\infty := P_M^{1/\gamma}$ and that $\boldsymbol{G}(\cdot)$ is of the form (1.4). Then the problem (1.1)–(1.5), admits a weak solution in the sense specified in Definition 2.2.*

The following two remarks are now in order.

Remark 2.4. In Section 3, such a solution is obtained as the limit of the vanishing viscosity approximations $(n_\varepsilon, W_\varepsilon, p_\varepsilon)$ of (3.1) to (1.6) as $\varepsilon \to 0$.

Remark 2.5. In Section 4, such a solution is obtained in the case of a rectangular domain, as the limit of the sequence of approximations (n_h, W_h, p_h) computed by the numerical scheme (4.1)–(4.3) as $h \to 0$.

3. Global existence via vanishing viscosity

In this section we prove Theorem 2.3 by constructing an approximating scheme which relies on the addition of an artificial vanishing viscosity approximation

$$\begin{cases} \partial_t n_\varepsilon - \operatorname{div}(n_\varepsilon\nabla W_\varepsilon) = \alpha n_\varepsilon - \beta n_\varepsilon^{\gamma+1} + \varepsilon\Delta n_\varepsilon, \quad x \in \Omega,\ t \geq 0 \\ \mu\Delta W_\varepsilon - W_\varepsilon = n_\varepsilon^\gamma, \\ n_\varepsilon(0, \cdot) = n_0^\varepsilon, \end{cases} \tag{3.1}$$

where n_0^ε is a smoothed version of n_0, that is $n_0^\varepsilon = n_0 * \varphi_\varepsilon$ for a smooth function φ_ε with compact support, and a bounded domain $\Omega \in \mathbb{R}^d$ with smooth boundary or alternatively the d-dimensional torus $\mathbb{T}^d$, and we establish its convergence to the nonlinear system (1.6) at the continuous level. For simplicity, we assume $a = 1$ and homogeneous Neumann boundary conditions for n_ε and W_ε (if the domain is a torus $\mathbb{T}^d$ we can also use periodic boundary conditions).

Theorem 3.1. *For every $\varepsilon > 0$, the parabolic-elliptic system* (3.1) *admits a unique smooth solution* $(n_\varepsilon, W_\varepsilon, p_\varepsilon)$.

Proof. The proof of this result relies on classical arguments (*cf*. Ladyzhenskaya [20]), namely by employing the Contraction Mapping Principle and the regularity of the initial data one can show the existence of a unique solution $(n_\varepsilon, W_\varepsilon, p_\varepsilon)$ defined for a small time $T > 0$. Then one derives *a priori* estimates establishing that the solution does not blow up and in fact is defined for every time. Finally, a bootstrap argument yields the smoothness of the solution. We refer the reader for details to (*cf*. Thm. 5.1.2 in Lunardi [24]) where all the details are presented in the context of a related parabolic partial differential equation. □

The remaining part of this section aims to establish the necessary compactness of the approximate sequence of solutions $(n_\varepsilon, W_\varepsilon, p_\varepsilon)$.

3.1. *A priori* estimates

We start by proving that n_ε are uniformly bounded independent of $\varepsilon > 0$ and nonnegative:

Lemma 3.2. *If $0 \le n_\varepsilon(0,\cdot) \le n_\infty := P_M^{1/\gamma} < \infty$ for all $\varepsilon > 0$, then the functions $n_\varepsilon(t,\cdot)$ are uniformly* (*in* $\varepsilon > 0$) *bounded and nonnegative, specifically,*

$$0 \le \min_{(t,x)} n_\varepsilon(t,x) \le \max_{(t,x)} n_\varepsilon(t,x) \le n_\infty.$$

Proof. First we notice that if W_ε has a maximum at an interior point x_0, then $\Delta W_\varepsilon(\cdot, x_0) \le 0$ and therefore $W_\varepsilon = p_\varepsilon + \mu \Delta W_\varepsilon \le p_\varepsilon$. Similarly, if it has a minimum at a point x_0, it will satisfy $\Delta W_\varepsilon(\cdot, x_0) \ge 0$ and therefore $W_\varepsilon \ge p_\varepsilon$. If W_ε attains a strict maximum on the boundary, *i.e.*, there is a point $x_0 \in \partial\Omega$ such that $W_\varepsilon(x_0) > W_\varepsilon(x)$ for any other $x \in \Omega$, we apply Hopf's Lemma (see for example [12], p. 347) to the function $v := W_\varepsilon - \max_{(t,x)} p_\varepsilon(t,x)$ which satisfies

$$-\mu\Delta v + v = p_\varepsilon - \max_{(t,x)} p_\varepsilon(t,x) \le 0,$$

which has a strict maximum at the point x_0. If $v(x_0) \le 0$, then $W_\varepsilon \le W_\varepsilon(x_0) \le \max_{(t,x)} p_\varepsilon(t,x)$ and otherwise Hopf's lemma gives $\nabla W_\varepsilon(x_0)\cdot\nu = \nabla v(x_0)\cdot\nu > 0$ where we have denoted the boundary normal ν, this contradicts the homogeneous boundary conditions. In a similar way we show that $W_\varepsilon \ge \min_{(t,x)} p_\varepsilon(t,x)$ applying Hopf's lemma to $-W_\varepsilon$ and hence

$$\min_{(t,x)} p_\varepsilon(t,x) \le W_\varepsilon \le \max_{(t,x)} p_\varepsilon(t,x). \tag{3.2}$$

We rewrite the evolution equation for n_ε using the equation for the potential W_ε,

$$\partial_t n_\varepsilon - \nabla W_\varepsilon \cdot \nabla n_\varepsilon = n_\varepsilon \boldsymbol{G}(p_\varepsilon) + \frac{1}{\mu} n_\varepsilon (p_\varepsilon - W_\varepsilon) + \varepsilon \Delta n_\varepsilon. \tag{3.3}$$

Now let $t_0 \ge 0$ be the first point in time, where $n_\varepsilon(t_0, x_0) \ge n_\infty$ reaches its maximum for some $x_0 \in \Omega$ (and therefore also $p_\varepsilon(t_0, x_0) \ge P_M$ reaches a maximum). Without loss of generality, we may assume that $t_0 < T$. Then $\nabla n_\varepsilon(t_0, x_0) = 0$ and $\Delta n_\varepsilon(t_0, x_0) \le 0$. Hence

$$\partial_t n_\varepsilon(t_0, x_0) \le n_\varepsilon \boldsymbol{G}(p_\varepsilon) + \frac{1}{\mu} n_\varepsilon (p_\varepsilon - W_\varepsilon).$$

By (3.2), the second term on the right hand side is nonpositive and since $\boldsymbol{G}(p_\varepsilon(t_0, x_0)) \le 0$ for $p_\varepsilon \ge P_M$, we get

$$\partial_t n_\varepsilon(t_0, x_0) \le 0.$$

Hence n_ε will decrease and if initially $n_0 \leq n_\infty$, this implies that $n_\varepsilon(t,\cdot) \leq n_\infty$ for any later time $t \geq 0$. To show the nonnegativity of n_ε, we integrate the evolution equation for n_ε,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon \mathrm{d}x = \int_\Omega n_\varepsilon \boldsymbol{G}(p_\varepsilon)\mathrm{d}x.$$

On the other hand, multiplying the same equation by a regularized version of the sign function, integrating and then passing to the limit in the approximation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |n_\varepsilon| \mathrm{d}x \leq \int_\Omega |n_\varepsilon| \boldsymbol{G}(p_\varepsilon)\mathrm{d}x,$$

Subtracting the two equations from one another, and using that $|n_\varepsilon| - n_\varepsilon \geq 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big||n_\varepsilon| - n_\varepsilon\big| \mathrm{d}x &\leq \int_\Omega \big||n_\varepsilon| - n_\varepsilon\big| \boldsymbol{G}(p_\varepsilon)\mathrm{d}x,\\ &\leq \max_{s\in[0,P_M]} |\boldsymbol{G}(s)| \int_\Omega \big||n_\varepsilon| - n_\varepsilon\big| \mathrm{d}x.\end{aligned}$$

Now using Grönwall's inequality and that $|n_0| - n_0 \equiv 0$ by assumption, we obtain

$$\int_\Omega \big||n_\varepsilon| - n_\varepsilon\big|(t)\mathrm{d}x = 0$$

and thus that $n_\varepsilon(t,x) \geq 0$ almost everywhere. □

Next we prove a simple lemma on the regularity of W_ε.

Lemma 3.3. *We have that*

$$W_\varepsilon \subset L^\infty([0,T];H^2(\Omega)), \quad W_\varepsilon \subset L^\infty([0,T];W^{2,q}(\Omega')),$$

for any $q \in [1,\infty)$, all compact subsets $\Omega' \subset\subset \Omega$, uniformly in $\varepsilon > 0$ and

$$W_\varepsilon, \Delta W_\varepsilon \subset L^\infty((0,T)\times\Omega)),$$

uniformly in $\varepsilon > 0$ as well.

Proof. We square the equation for W_ε and integrate it over the spatial domain and then use integration by parts,

$$\begin{aligned}\int_\Omega |p_\varepsilon|^2 \mathrm{d}x &= \int_\Omega |W_\varepsilon|^2 - 2\mu W_\varepsilon \Delta W_\varepsilon + \mu^2 |\Delta W_\varepsilon|^2 \mathrm{d}x\\ &= \int_\Omega |W_\varepsilon|^2 + 2\mu|\nabla W_\varepsilon|^2 + \mu^2|\nabla^2 W_\varepsilon|^2 \mathrm{d}x.\end{aligned}$$

By the previous Lemma 3.2, we have that p_ε is uniformly bounded in $\varepsilon > 0$ and therefore that the left hand side of the above equation is bounded and that $W_\varepsilon \in L^\infty([0,T];H^2(\Omega))$. Using a Calderon–Zygmund inequality (*e.g.* [15], Thm. 9.11.), we obtain $W_\varepsilon \in L^\infty([0,T];W^{2,q}(\Omega'))$ for all $q \in [1,\infty)$ and compact subsets $\Omega' \subset\subset \Omega$. By the Sobolev embedding theorem, this implies that in particular $\nabla W_\varepsilon \in L^\infty((0,T)\times\Omega')$. The second claim follows from (3.2) and the uniform bound on the pressure proved in Lemma 3.2. □

3.2. Entropy inequalities for n_ε

To prove strong convergence of the approximating sequence $\{(n_\varepsilon, W_\varepsilon, p_\varepsilon)\}_{\varepsilon>0}$, it will be useful to derive entropy inequalities for n_ε. To this end, the following lemma will be useful:

Lemma 3.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth convex, nonnegative function and denote $f_\varepsilon := f(n_\varepsilon)$. Then f_ε satisfies the following identity*

$$\partial_t f_\varepsilon - \operatorname{div}(f_\varepsilon \nabla W_\varepsilon) - \varepsilon \Delta f(n_\varepsilon) = (f'(n_\varepsilon)n_\varepsilon - f_\varepsilon)\Delta W_\varepsilon + f'(n_\varepsilon)n_\varepsilon \boldsymbol{G}(p_\varepsilon) - \varepsilon f''(n_\varepsilon)|\nabla n_\varepsilon|^2 \tag{3.4}$$

where

$$\varepsilon \int_0^T\int_\Omega f''(n_\varepsilon)|\nabla n_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t \le C, \tag{3.5}$$

with $C>0$ a constant independent of $\varepsilon>0$. In particular, this implies that $\partial_t f_\varepsilon = g_\varepsilon + k_\varepsilon$ with $g_\varepsilon \in L^1([0,T]\times\Omega)$ and $k_\varepsilon \in L^1([0,T];W^{-1,2}(\Omega))$.

Proof. The identity (3.4) follows after multiplying the evolution equation for n_ε, (3.3), by $f'(n_\varepsilon)$ and using chain rule. Integrating the inequality in space and time, we obtain

$$\int_\Omega f_\varepsilon(T)\,\mathrm{d}x + \varepsilon\int_0^T\int_\Omega f''(n_\varepsilon)|\nabla n_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t = \int_\Omega f_\varepsilon(0)\,\mathrm{d}x + \int_0^T\int_\Omega (f'(n_\varepsilon)n_\varepsilon - f_\varepsilon)\Delta W_\varepsilon + f'(n_\varepsilon)n_\varepsilon\boldsymbol{G}(p_\varepsilon)\,\mathrm{d}x\mathrm{d}t.$$

The right hand side is bounded by the assumptions on the initial data and the L^∞-bounds proved in Lemmas 3.2 and 3.3. This implies (3.5). Therefore the right hand side of (3.4) is contained in $L^1((0,T)\times\Omega)$. Using (3.5) for the third term on the left hand side, we conclude that it is contained in $L^1([0,T];H^{-1}(\Omega))$. The second term on the left hand side is contained in $L^\infty([0,T];W^{-1,2}(\Omega))$. Hence $\partial_t f_\varepsilon = g_\varepsilon + k_\varepsilon$ with $g_\varepsilon\in L^1([0,T]\times\Omega)$ and $k_\varepsilon \in L^1([0,T];W^{-1,2}(\Omega))$ and in particular, $\partial_t f_\varepsilon \in L^1([0,T];W^{-1,1^*}(\Omega))$ by the Sobolev embedding ($1^* = d/(d-1)$). ☐

Remark 3.5. The preceeding lemma implies that the time derivative of the approximation of the pressure $\partial_t p_\varepsilon = \partial_t|n_h|^\gamma = g_\varepsilon + k_\varepsilon$ where g_ε is uniformly bounded in $L^1([0,T]\times\Omega)$ and k_ε in $L^1([0,T];H^{-1}(\Omega))$. Hence $\partial_t W_\varepsilon = U_\varepsilon + V_\varepsilon$ where $U_\varepsilon\in L^1([0,T];H^1(\Omega))$ solves $-\mu\Delta U_\varepsilon + U_\varepsilon = k_\varepsilon$ and $V_\varepsilon \in L^1([0,T];W^{1,r}(\Omega))$, $1\le r<1^*$ solves $-\mu\Delta V_\varepsilon + V_\varepsilon = g_\varepsilon$ (see [1], Thm. 6.1 for a proof of the second statement). Hence $\partial_t W_\varepsilon \in L^1([0,T];W^{1,r}(\Omega))$ for any $1\le r<1^*$.

3.3. Passing to the limit $\varepsilon \to 0$

The estimates of the previous (sub)sections allow us to pass to the limit $\varepsilon\to 0$ in a subsequence, still denoted ε, and conclude the existence of limit functions

$$\begin{aligned} n_\varepsilon &\rightharpoonup n \ge 0, \quad \text{in } L^q([0,T]\times\Omega),\ 1\le q<\infty,\\ p_\varepsilon &\rightharpoonup \overline{p} \ge 0, \quad \text{in } L^q([0,T]\times\Omega),\ 1\le q<\infty, \end{aligned}$$

where $p_\varepsilon := n_\varepsilon^\gamma$ and $0\le n,\overline{p}\in L^\infty([0,T]\times\Omega)$. Using Aubin–Lions' lemma for W_ε and ∇W_ε, we obtain strong convergence of a subsequence in $L^q([0,T]\times\Omega)$ for any $q\in[0,\infty)$ to limit functions $W,\nabla W\in L^q([0,T]\times\Omega)$. Moreover, from the estimates in Lemma 3.3 we obtain that $W\in L^\infty([0,T]\times\Omega)\cap L^\infty([0,T];W^{2,q}(\Omega))$. Hence we have that $(n,W,\overline{p})$ satisfy for any $\varphi,\psi\in C_0^1([0,T)\times\Omega)$,

$$\begin{aligned} \int_0^T\int_\Omega n\varphi_t - n\nabla W\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t + \int_\Omega n_0\,\varphi(0,x)\mathrm{d}x &= -\int_0^T\int_\Omega \overline{n\boldsymbol{G}(p)}\varphi\,\mathrm{d}x\mathrm{d}t\\ \int_0^T\int_\Omega W\psi + \mu\nabla W\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\Omega \overline{p}\,\psi\,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.6}$$

where $\overline{n\boldsymbol{G}(p)}$ is the weak limit of $n_\varepsilon \boldsymbol{G}(p_\varepsilon)$. To conclude that the limit $(n, W, \overline{p})$ is a weak solution of (1.6), we need to show that n_ε converges strongly and therefore in the limit $\overline{p} = p := n^\gamma$ and $\overline{n\boldsymbol{G}(p)} = n\boldsymbol{G}(p)$. For this purpose, we combine a compensated compactness property (Lem. 3.7) with a monotonicity argument. We will also make use of the following lemma which was proved in a more general version in [7,25]:

Lemma 3.6. *Let $n, f \in L^\infty([0,T]\times\Omega)$ and $\boldsymbol{u} \in L^\infty([0,T]; H^1(\Omega))$ with $\operatorname{div}\boldsymbol{u} \in L^\infty([0,T]\times\Omega)$ satisfy*

$$n_t - \operatorname{div}(\boldsymbol{u}n) = f, \tag{3.7}$$

in the sense of distributions. Then for all $b \in C^1(\mathbb{R})$,

$$b(n)_t - \operatorname{div}(\boldsymbol{u}b(n)) = b'(n)f + [b'(n)n - b(n)]\operatorname{div}\boldsymbol{u}, \tag{3.8}$$

in the sense of distributions.

Proof. We let $0 \le \psi \in C_0^\infty(\mathbb{R}^{d+1})$ be a smooth, radially symmetric mollifier, *i.e.* $\psi(x) = \psi(-x)$ and $\int_{\mathbb{R}^{d+1}} \psi(x)\mathrm{d}x$, with $\operatorname{supp}(\psi) \subset B_1(0)$ and denote for $\delta > 0$, $\psi_\delta(x) := \delta^{-(d+1)}\psi(x/\delta)$. Then we choose as a test function in (3.7) $\psi_\delta(s,y)\varphi(t+s, x+y)$, with φ is compactly supported in $(\delta, T-\delta)\times\Omega^\delta$ where Ω^δ includes all the points x in Ω which have distance $d(x,\partial\Omega) > \delta$ and do a change of variables:

$$\int_0^T\int_\Omega n(t-s,x-y)\psi_\delta(s,y)\partial_t\varphi(t,x) - n(t-s,x-y)\boldsymbol{u}(t,x)\psi_\delta(s,y)\cdot\nabla\varphi(t,x)\,\mathrm{d}x\mathrm{d}t$$
$$= -\int_0^T\int_\Omega f(t-s,x-y)\psi_\delta(s,y)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t.$$

Integrating in (s,y), this becomes

$$\int_0^T\int_\Omega (n*\psi_\delta)(t,x)\partial_t\varphi(t,x) - (n\boldsymbol{u})*\psi_\delta(t,x)\cdot\nabla\varphi(t,x)\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega (f*\psi_\delta)(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t.$$

We define $n_\delta := n*\psi_\delta$ and $f_\delta := f*\psi_\delta$ and choose as a test function $\varphi := b'(n_\delta)\phi$ for a smooth ϕ compactly supported in $(\delta, T-\delta)\times\Omega^\delta$ (which is possible since n_δ is smooth and bounded thanks to the convolution.). Then we can rewrite the last identity using chain rule as

$$\int_0^T\int_\Omega b(n_\delta)\partial_t\phi - b(n_\delta)\boldsymbol{u}\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega (b'(n_\delta)f_\delta + [b'(n_\delta)n_\delta - b(n_\delta)]\operatorname{div}\boldsymbol{u} + b'(n_\delta)r_\delta)\,\phi\,\mathrm{d}x\mathrm{d}t.$$

where $r_\delta := \operatorname{div}((n\boldsymbol{u})*\psi_\delta) - \operatorname{div}(n_\delta\boldsymbol{u})$. By ([22], Lem. 2.3) we have that $r_\delta \to 0$ in $L^2_{\text{loc}}((0,T)\times\Omega)$ and thanks to the properties of the convolution that $b(n_\delta) \to b(n)$ almost everywhere as well as $f_\delta \to f$ a.e. when $\delta \to 0$. Thus we obtain that in the limit $\delta \to 0$, n satisfies

$$\int_0^T\int_\Omega b(n)\partial_t\phi - b(n)\boldsymbol{u}\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega (b'(n)f + [b'(n)n - b(n)]\operatorname{div}\boldsymbol{u})\,\phi\,\mathrm{d}x\mathrm{d}t.$$

which is exactly (3.8) in the sense of distributions. □

Applying Lemma 3.6 for the weak limit n in (3.6) with $b(n) = n^2$, we obtain that n satisfies

$$\int_0^T\int_\Omega n^2\varphi_t - n^2\nabla W\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega (2n\overline{n\boldsymbol{G}(p)} + n^2\Delta W)\varphi\,\mathrm{d}x\mathrm{d}t \tag{3.9}$$

for any test functions $\varphi \in C_0^1((0,T)\times\Omega)$. On the other hand, from (3.4) for $b(n) = n^2$ we obtain after integrating in space and time

$$\int_\Omega n_\varepsilon^2(\tau)\,\mathrm{d}x - \int_\Omega n_\varepsilon^2(0)\,\mathrm{d}x \le \int_0^\tau\int_\Omega n_\varepsilon^2\Delta W_\varepsilon + 2n_\varepsilon^2\boldsymbol{G}(p_\varepsilon)\,\mathrm{d}x\mathrm{d}t.$$

Passing to the limit $\varepsilon \to 0$ in this inequality, we have

$$\int_\Omega \overline{n^2}(\tau)\,\mathrm{d}x - \int_\Omega n_0^2\,\mathrm{d}x \le \int_0^\tau \int_\Omega \overline{n^2 \Delta W} + 2\overline{n^2 \boldsymbol{G}(p)}\,\mathrm{d}x\mathrm{d}t, \tag{3.10}$$

where $\overline{n^2}$ denotes the weak limit of n_ε^2 and $\overline{n^2\Delta W}$ and $\overline{n^2\boldsymbol{G}(p)}$ are the weak limits of $n_\varepsilon^2 \Delta W_\varepsilon$ and $n_\varepsilon^2 \boldsymbol{G}(p_\varepsilon)$ respectively. Letting $\tau \to 0$ in this inequality, we obtain, thanks to the boundedness of the integrand on the right hand side,

$$\int_\Omega \overline{n^2}(0)\,\mathrm{d}x - \int_\Omega n_0^2\,\mathrm{d}x \le 0.$$

On the other hand, since $b(n) = n^2$ is convex, we have $\overline{n^2} \ge n^2$ and hence $\overline{n^2}(0,x) = n_0^2(x)$.

We now choose smooth test functions φ_ϵ approximating $\varphi(t,x) = \mathbf{1}_{[0,\tau]}(t)$, where $\tau \in (0,T]$, in inequality (3.9) and then pass to the limit in the approximation to obtain the inequality

$$\int_\Omega n^2(\tau)\,\mathrm{d}x - \int_\Omega n_0^2\,\mathrm{d}x = \int_0^\tau \int_\Omega (2n\overline{n\boldsymbol{G}(p)} + n^2 \Delta W)\,\mathrm{d}x\mathrm{d}t \tag{3.11}$$

Subtracting (3.11) from (3.10), we have

$$\int_\Omega \left(\overline{n^2} - n^2\right)(\tau)\mathrm{d}x \le \int_0^\tau\!\!\int_\Omega \left(2\overline{n^2\boldsymbol{G}(p)} - 2n\overline{n\boldsymbol{G}(p)} + \Delta W\left(\overline{n^2} - n^2\right) + \overline{n^2\Delta W} - \overline{n^2}\Delta W\right)\mathrm{d}x\,\mathrm{d}t. \tag{3.12}$$

Now using the explicit expression of $\boldsymbol{G}$, (1.4), the first term on the right hand side can be estimated as follows:

$$\begin{aligned}
\int_0^\tau\!\!\int_\Omega \left(2\overline{n^2\boldsymbol{G}(p)} - 2n\overline{n\boldsymbol{G}(p)}\right)\mathrm{d}x\,\mathrm{d}t &= 2\int_0^\tau\!\!\int_\Omega \alpha\left(\overline{n^2} - n^2\right) - \beta\left(\overline{n^{2+\gamma\theta}} - n\overline{n^{1+\gamma\theta}}\right)\mathrm{d}x\,\mathrm{d}t \\
&\le 2\int_0^\tau\!\!\int_\Omega \alpha\left(\overline{n^2} - n^2\right) - \beta\left(\overline{n^{2+\gamma\theta}} - \overline{n^{2+\gamma\theta}}\right)\mathrm{d}x\,\mathrm{d}t \\
&\le 2\alpha\int_0^\tau\!\!\int_\Omega \left(\overline{n^2} - n^2\right)\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.13}$$

where we have used ([25], Lem. 3.35), which implies $n\overline{n^{1+\gamma\theta}} \le \overline{n^{2+\gamma\theta}}$, for the first inequality. To estimate the second term on the right hand side, we use that ΔW is bounded thanks to Lemma 3.3 and that $\overline{n^2} \ge n^2$ by the convexity of $f(x) = x^2$. Hence

$$\int_0^\tau \int_\Omega \Delta W\left(\overline{n^2} - n^2\right)\mathrm{d}x\mathrm{d}t \le \frac{P_M}{\mu}\int_0^\tau \int_\Omega \left(\overline{n^2} - n^2\right)\mathrm{d}x\mathrm{d}t. \tag{3.14}$$

For the last term, we use the following lemma,

Lemma 3.7. *The weak limits $(n, W, \overline{p})$ of the sequences $\{(n_\varepsilon, W_\varepsilon, p_\varepsilon)\}_{\varepsilon>0}$ satisfy for smooth functions $S : \mathbb{R} \to \mathbb{R}$,*

$$\int_\Omega \left(\overline{S(n)\Delta W} - \overline{S(n)}\Delta W\right)\mathrm{d}x = \frac{1}{\mu}\int_\Omega \left(\overline{p}\;\overline{S(n)} - \overline{pS(n)}\right)\mathrm{d}x \tag{3.15}$$

where $\overline{S(n)\Delta W}$, $\overline{S(n)}$, $\overline{pS(n)}$ are the weak limits of $S(n_\varepsilon)\Delta W_\varepsilon$, $S(n_\varepsilon)$ and $p_\varepsilon S(n_\varepsilon)$ respectively.

Applying this lemma to the second term in (3.12) with $S(n) = n^2$, we can estimate it by

$$\begin{aligned}
\int_0^\tau \int_\Omega \left(\overline{n^2\Delta W} - \overline{n^2}\Delta W\right)\mathrm{d}x &= \frac{1}{\mu}\int_\Omega \left(\overline{p}\;\overline{n^2} - \overline{pn^2}\right)\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{\mu}\int_\Omega \left(\overline{n^\gamma}\;\overline{n^2} - \overline{n^{2+\gamma}}\right)\mathrm{d}x\mathrm{d}t \\
&\le 0,
\end{aligned}$$

using that $\overline{n^\gamma}\,\overline{n^2} \leq \overline{n^{2+\gamma}}$ (*cf.* [25]). Thus,

$$\int_\Omega \left(\overline{n^2} - n^2\right)(\tau)\mathrm{d}x \leq \left(2\alpha + \frac{P_M}{\mu}\right)\int_0^\tau\int_\Omega \left(\overline{n^2} - n^2\right)\mathrm{d}x\,\mathrm{d}t.$$

Hence Grönwall's inequality implies

$$\int_\Omega \left(\overline{n^2} - n^2\right)(\tau)\mathrm{d}x \leq 0.$$

By convexity of the function $f(x) = x^2$ we also have $n^2 \leq \overline{n^2}$ almost everywhere and so

$$\overline{n^2}(t,x) = n^2(t,x)$$

almost everywhere in $(0,T)\times\Omega$. Therefore we conclude that the functions n_ε converge strongly to n almost everywhere and in particular also $\overline{p} = n^\gamma$ which means that the limit $(n, W, \overline{p})$ is a weak solution of equations (1.6).

Proof of Lemma 3.7. We multiply the equation for W_ε by $S(n_\varepsilon)$ and integrate over Ω,

$$\int_\Omega \mu\Delta W_\varepsilon\, S(n_\varepsilon) - W_\varepsilon S(n_\varepsilon)\,\mathrm{d}x = -\int_\Omega p_\varepsilon S(n_\varepsilon)\,\mathrm{d}x.$$

Passing to the limit $\varepsilon \to 0$, we obtain

$$\int_\Omega \mu\overline{\Delta W S(n)} - W\overline{S(n)}\,\mathrm{d}x = -\int_\Omega \overline{pS(n)}\,\mathrm{d}x. \tag{3.16}$$

On the other hand, using the smooth function $S(n_\varepsilon)$ as a test function in the weak formulation of the limit equation

$$-\mu\Delta W + W = \overline{p},$$

and passing to the limit $\varepsilon \to 0$, we obtain

$$\int_\Omega \mu\Delta W\overline{S(n)} - W\overline{S(n)}\,\mathrm{d}x = -\int_\Omega \overline{p}\;\overline{S(n)}\,\mathrm{d}x.$$

Combining the last identity with (3.16), we obtain (3.15). □

4. Global existence *via* a numerical approximation

We consider the problem in two space dimensions in a rectangular domain, for simplicity we use $\Omega = [0,1]^2$, the generalization to other rectangular domains as well as three space dimensions is straightforward but more cumbersome in terms of notation, for this reason we restrict ourself to a square two dimensional domain here. For simplicity, we will also assume $a = 1$ in the Brinkman law in (1.6). We let $h > 0$ the mesh width, and Δt the time step size. We will determine the necessary ratio between h and Δt later on. For $i, j = 1, \ldots, N_x$, where $N_x = 1/h$, h chosen such that N_x is an integer, we denote grid cells $\mathcal{C}_{ij} := ((i-1)h, ih] \times ((j-1)h, jh]$ with cell midpoints $x_{i,j} = ((i-1/2)h, (j-1/2)h)$. In addition, we denote $t^m = m\Delta t$, $m = 0, \ldots N_T$, where $N_T = T/\Delta t$ for some final time $T > 0$. The approximation of a function f at grid point $x_{i,j}$ and time t^m will be denoted $f^m_{i,j}$. We also introduce the finite differences,

$$D_1^\pm f_{ij} = \pm\frac{f_{i\pm1,j} - f_{i,j}}{h}, \quad D_2^\pm f_{ij} = \pm\frac{f_{i,j\pm1} - f_{i,j}}{h}, \quad D_t^\pm f^m = \pm\frac{f^{m\pm1} - f^m}{\Delta t},$$

and define the discrete Laplacian, divergence and gradient operators based on these,

$$\nabla_h^{\pm} := (D_1^{\pm}, D_2^{\pm})^t, \quad \operatorname{div}_h^{\pm} f_{i,j} = D_1^{\pm} f_{i,j}^{(1)} + D_2^{\pm} f_{i,j}^{(2)}, \quad \Delta_h := \operatorname{div}_h^{+} \nabla_h^{-}.$$

Since D_i^+ and D_i^- commute, we have that $\Delta_h = \operatorname{div}_h^+ \nabla_h^- = \operatorname{div}_h^- \nabla_h^+$. For ease of notation, we also let $u_{i+1/2,j}$ and $v_{i,j+1/2}$ denote the discrete velocities in the transport equation, specifically, given $W_{i,j}$, we let

$$u_{i+1/2,j} := D_1^+ W_{i,j}, \quad v_{i,j+1/2} := D_2^+ W_{i,j}. \tag{4.1}$$

4.1. An explicit finite difference scheme

Given $(n_{i,j}^m, W_{i,j}^m)$ at time step m, we define the quantities $(n_{i,j}^{m+1}, W_{i,j}^{m+1})$ at the next time step by

$$-\mu \Delta_h W_{i,j}^m + W_{i,j}^m = p_{i,j}^m, \tag{4.2a}$$

$$p_{i,j}^m := |n_{i,j}^m|^\gamma, \tag{4.2b}$$

$$D_t^+ n_{i,j}^m + D_1^- F_{i+1/2,j}^{(1)}(u^m, n^m) + D_2^- F_{i,j+1/2}^{(2)}(v^m, n^m) = n_{i,j}^m \boldsymbol{G}(p_{i,j}^m), \tag{4.2c}$$

where $p_{i,j} = (n_{i,j})^\gamma$ and the fluxes $F^{(j)}$, $j = 1,2$ are defined by

$$\begin{aligned} F_{i+1/2,j}^{(1)}(u^m, n^m) &= -u_{i+1/2,j}^m \frac{n_{i,j}^m + n_{i+1,j}^m}{2} - \frac{h}{2} |u_{i+1/2,j}| D_1^+ n_{i,j}^m \\ F_{i,j+1/2}^{(2)}(v^m, n^m) &= -v_{i,j+1/2}^m \frac{n_{i,j}^m + n_{i,j+1}^m}{2} - \frac{h}{2} |v_{i,j+1/2}| D_2^+ n_{i,j}^m. \end{aligned} \tag{4.3}$$

We use homogeneous Neumann or periodic boundary conditions for both variables:

$$\begin{aligned} n_{0,j}^m &= n_{1,j}^m, & n_{N_x+1,j}^m &= n_{N_x,j}^m, & j &= 1,\dots,N_x, \\ n_{i,0}^m &= n_{i,1}^m, & n_{i,N_x+1}^m &= n_{i,N_x}^m, & i &= 1,\dots,N_x, \\ W_{0,j}^m &= W_{1,j}^m, & W_{N_x+1,j}^m &= W_{N_x,j}^m, & j &= 1,\dots,N_x, \\ W_{i,0}^m &= W_{i,1}^m, & W_{i,N_x+1}^m &= W_{i,N_x}^m, & i &= 1,\dots,N_x. \end{aligned}$$

The initial condition we approximate taking averages over the cells,

$$n_{i,j}^0 = \frac{1}{|\mathcal{C}_{ij}|} \int_{\mathcal{C}_{ij}} n_0(x)\,\mathrm{d}x, \quad p_{i,j}^0 = |n_{i,j}^0|^\gamma, \quad i,j = 1,\dots,N_x.$$

4.2. Estimates on approximations

In the following, we will prove estimates on the discrete quantities $(n_{i,j}^m, W_{i,j}^m)$ obtained using the scheme (4.1)–(4.3). We therefore define the piecewise constant functions

$$f_h(t,x) = \sum_{m=0}^{N_T} \sum_{i,j=1}^{N_x} f_{i,j}^m \, \mathbf{1}_{\mathcal{C}_{ij}}(x) \mathbf{1}_{[t^m, t^{m+1})}(t), \quad (t,x) \in [0,T] \times \Omega, \tag{4.4}$$

where $f \in \{n, W, p\}$. We first prove that n_h stays nonnegative and uniformly bounded from above.

Lemma 4.1. *If $0 \le n_{i,j}^0 \le n_\infty := P_M^{1/\gamma} < \infty$ uniformly in $h > 0$ and the timestep Δt satisfies the CFL condition*

$$\Delta t \le \min \left\{ \frac{h}{8 \max_{ij} |\nabla_h W_{i,j}^m| + h \boldsymbol{G}^\infty}, \frac{\mu}{4\gamma \overline{n}_\infty^\gamma} \right\} \tag{4.5}$$

(where $\boldsymbol{G}^\infty := \max_{s \in \mathbb{R}^+} \boldsymbol{G}(s) = \boldsymbol{G}(0)$ by the properties of $\boldsymbol{G}$, c.f. (1.3)*), then for any $t > 0$, the functions $n_h(t,\cdot)$ are uniformly (in $h > 0$) bounded and nonnegative, specifically, defining $\overline{n}_\infty = n_\infty + 4\Delta t \sup_{s \ge 0} \left(s^{1/\gamma} \boldsymbol{G}(s) \right)$, we have for all $m \ge 0$,*

$$0 \le \min_{i,j} n_{i,j}^m \le \max_{i,j} n_{i,j}^m \le \overline{n}_\infty.$$

Proof. The proof goes by induction on the timestep m. Clearly, by the assumptions, we have $0 \le n_{i,j}^0 \le \overline{n}_\infty$. For the induction step we therefore assume that this holds for timestep $m > 0$ and show that it implies the nonnegativity and boundedness at timestep $m+1$.

We first show that the $W_{i,j}^m$ are bounded in terms of the $p_{i,j}^m$. To do so, let us assume it has a local maximum $W_{\hat{\imath},\hat{\jmath}}^m$ in a cell $\mathcal{C}_{\hat{\imath}\hat{\jmath}}$, for some $\hat{\imath},\hat{\jmath} \in \{1,\dots,N_x\}$. Then

$$D_k^+ W_{\hat{\imath},\hat{\jmath}}^m \le 0, \quad -D_k^- W_{\hat{\imath},\hat{\jmath}}^m \le 0, \quad k = 1,2,$$

(if $\hat{\imath}$ or $\hat{\jmath} \in \{1, N_x\}$, then because of the Neumann boundary conditions, the forward/backward difference in direction of the boundary is zero and thus the previous inequality is true as well). Hence

$$\Delta_h W_{\hat{\imath},\hat{\jmath}}^m = \frac{1}{h}\sum_{k=1}^{2}\left(D_k^+ W_{\hat{\imath},\hat{\jmath}}^m - D_k^- W_{\hat{\imath},\hat{\jmath}}^m\right) \le 0.$$

Therefore,

$$W_{\hat{\imath},\hat{\jmath}}^m = p_{\hat{\imath},\hat{\jmath}}^m + \mu \Delta_h W_{\hat{\imath},\hat{\jmath}}^m \le p_{\hat{\imath},\hat{\jmath}}^m \le \max_{i,j} |n_{i,j}^m|^\gamma.$$

Similarly, at a local minimum $W_{\hat{\imath},\hat{\jmath}}^m$ of W_h, we have

$$D_k^+ W_{\hat{\imath},\hat{\jmath}}^m \ge 0, \quad -D_k^- W_{\hat{\imath},\hat{\jmath}}^m \ge 0, \quad k = 1,2,$$

and hence

$$\Delta_h W_{\hat{\imath},\hat{\jmath}}^m = \frac{1}{h}\sum_{k=1}^{2}\left(D_k^+ W_{\hat{\imath},\hat{\jmath}}^m - D_k^- W_{\hat{\imath},\hat{\jmath}}^m\right) \ge 0,$$

which implies

$$W_{\hat{\imath},\hat{\jmath}}^m = p_{\hat{\imath},\hat{\jmath}}^m + \mu \Delta_h W_{\hat{\imath},\hat{\jmath}}^m \ge p_{\hat{\imath},\hat{\jmath}}^m \ge \min_{i,j} |n_{i,j}^m|^\gamma \ge 0.$$

Thus,

$$0 \le W_h \le \max_{i,j} |n_{i,j}^m|^\gamma. \tag{4.6}$$

Now we rewrite the scheme (4.2c) as

$$n_{i,j}^{m+1} = \left(\alpha_{i,j}^{(1),m} + \alpha_{i,j}^{(2),m}\right) n_{i,j}^m + \beta_{i,j}^m n_{i+1,j}^m + \zeta_{i,j}^m n_{i-1,j}^m + \eta_{i,j}^m n_{i,j+1}^m + \theta_{i,j}^m n_{i,j-1}^m \tag{4.7}$$

where

$$\begin{aligned}
\alpha_{i,j}^{(1),m} &= 1 - \frac{\Delta t}{2h}\big[(|u_{i+1/2,j}^m| + u_{i+1/2,j}^m) + (|u_{i-1/2,j}^m| - u_{i-1/2,j}^m) \\
&\qquad + (|v_{i,j+1/2}^m| + v_{i,j+1/2}^m) + (|v_{i,j-1/2}^m| - v_{i,j-1/2}^m)\big] \\
\alpha_{i,j}^{(2),m} &= \Delta t\, \boldsymbol{G}(p_{i,j}^m) + \frac{\Delta t}{h}\left[u_{i+1/2,j}^m - u_{i-1/2,j}^m + v_{i,j+1/2}^m - v_{i,j-1/2}^m\right] \\
\beta_{i,j}^m &= \frac{\Delta t}{2h}\left(u_{i+1/2,j}^m + |u_{i+1/2,j}^m|\right) \\
\zeta_{i,j}^m &= \frac{\Delta t}{2h}\left(|u_{i-1/2,j}^m| - u_{i-1/2,j}^m\right) \\
\eta_{i,j}^m &= \frac{\Delta t}{2h}\left(v_{i,j+1/2}^m + |v_{i,j+1/2}^m|\right) \\
\theta_{i,j}^m &= \frac{\Delta t}{2h}\left(|v_{i,j-1/2}^m| - v_{i,j-1/2}^m\right).
\end{aligned}$$

We note that $\beta^m_{i,j}, \zeta^m_{i,j}, \eta^m_{i,j}, \theta^m_{i,j} \ge 0$, and that under the CFL-condition (4.5), also $\alpha^{(1),m}_{i,j} + \alpha^{(2),m}_{i,j} \ge 0$. Hence, assuming that $n^m_{i,j} \ge 0$ for all i, j, we have

$$\begin{aligned} n^{m+1}_{i,j} &\ge \left(\beta^m_{i,j} + \zeta^m_{i,j} + \eta^m_{i,j} + \theta^m_{i,j}\right) \min\{n^m_{i+1,j}, n^m_{i-1,j}, n^m_{i,j+1}, n^m_{i,j-1}\} \\ &\quad + \left(\alpha^{(1),m}_{i,j} + \alpha^{(2),m}_{i,j}\right) n^m_{i,j} \\ &\ge 0. \end{aligned}$$

We proceed to showing the boundedness of n_h. Thanks to the CFL-condition (4.5), we have

$$\alpha^{(1),m}_{i,j} \ge \frac{1}{2}, \quad \beta^m_{i,j}, \zeta^m_{i,j}, \eta^m_{i,j}, \theta^m_{i,j} \le \frac{1}{8}.$$

Moreover, $\alpha^{(1),m}_{i,j} + \beta^m_{i,j} + \zeta^m_{i,j} + \eta^m_{i,j} + \theta^m_{i,j} = 1$. Using the induction hypothesis that $n^m_{i,j} \le \overline{n}_\infty$ for all i, j and the nonnegativity of n_h which we have just proved, we can estimate $n^{m+1}_{i,j}$:

$$\begin{aligned} n^{m+1}_{i,j} &\le \left(\alpha^{(1),m}_{i,j} + \alpha^{(2),m}_{i,j}\right) n^m_{i,j} + \left(\beta^m_{i,j} + \zeta^m_{i,j} + \eta^m_{i,j} + \theta^m_{i,j}\right) \overline{n}_\infty \\ &\le \left(\frac{1}{2} + \alpha^{(2),m}_{i,j}\right) n^m_{i,j} + \frac{1}{2}\overline{n}_\infty \\ &= \overline{n}_\infty - \frac{1}{2}\left(\overline{n}_\infty - n^m_{i,j}\right) + \alpha^{(2),m}_{i,j} n^m_{i,j}. \end{aligned} \tag{4.8}$$

We can rewrite and bound $\alpha^{(2),m}_{i,j}$ using the equation for $W^m_{i,j}$, (4.2a),

$$\begin{aligned} \alpha^{(2),m}_{i,j} &= \Delta t \left(\boldsymbol{G}(p^m_{i,j}) + \Delta_h W^m_{i,j}\right) \\ &= \Delta t \left(\boldsymbol{G}(p^m_{i,j}) + \frac{1}{\mu}\left(W^m_{i,j} - p^m_{i,j}\right)\right) \\ &\le \Delta t \left(\boldsymbol{G}(p^m_{i,j}) + \frac{1}{\mu}\left(\overline{n}^\gamma_\infty - |n^m_{i,j}|^\gamma\right)\right) \\ &\le \Delta t \left(\boldsymbol{G}(p^m_{i,j}) + \frac{\gamma\, \overline{n}^{\gamma-1}_\infty}{\mu}\left(\overline{n}_\infty - n^m_{i,j}\right)\right) \\ &\le \Delta t \boldsymbol{G}(p^m_{i,j}) + \frac{1}{4\overline{n}_\infty}(\overline{n}_\infty - n^m_{i,j}), \end{aligned}$$

where we have used (4.6) for the first inequality, that $f(a) - f(b) = f'(\widetilde{a})(a-b)$ for some intermediate value $\widetilde{a} \in [b, a]$, with $f(a) = a^\gamma$, for the second inequality and the CFL-condition for the last inequality. Now going back to (4.8) and inserting this there, we obtain,

$$\begin{aligned} n^{m+1}_{i,j} &\le \overline{n}_\infty - \frac{1}{2}\left(\overline{n}_\infty - n^m_{i,j}\right) + \left(\Delta t \boldsymbol{G}(p^m_{i,j}) + \frac{1}{4\overline{n}_\infty}(\overline{n}_\infty - n^m_{i,j})\right) n^m_{i,j} \\ &\le \frac{3}{4}\overline{n}_\infty + \frac{1}{4} n^m_{i,j} + \Delta t n^m_{i,j} \boldsymbol{G}(p^m_{i,j}). \end{aligned} \tag{4.9}$$

If $n^m_{i,j} \ge n_\infty$ then $\boldsymbol{G}(p^m_{i,j}) \le 0$ and hence the expression in (4.9) is bounded by $\overline{n}_\infty$. On the other hand, if $n^m_{i,j} \le n_\infty$, we can bound it by

$$\begin{aligned} n^{m+1}_{i,j} &\le \frac{3}{4}\overline{n}_\infty + \frac{1}{4} n^m_{i,j} + \Delta t n^m_{i,j} \boldsymbol{G}(p^m_{i,j}) \\ &\le \frac{3}{4}\overline{n}_\infty + \frac{1}{4}\left(n_\infty + 4\Delta t \sup_{s \ge 0}\left(s^{1/\gamma} \boldsymbol{G}(s)\right)\right) \\ &= \overline{n}_\infty \end{aligned}$$

where we used the definition of $\overline{n}_\infty$ for the last equality. This proves that $n_{i,j}^{m+1} \leq \overline{n}_\infty$ for all i,j if the same holds already for the $n_{i,j}^m$. □

Remark 4.2. The estimates in the proof of the previous lemma are very coarse and therefore one can use a much larger CFL-condition than (4.5) in practice. Also note that $\overline{n}_\infty \to n_\infty$ when $\Delta t \to 0$.

4.2.1. Estimates on the discrete potential $\mathsf{W_h}$

Lemma 4.3. *We have that*

$$W_h, \nabla_h W_h, \nabla_h^2 W_h \subset L^\infty([0,T]; L^2(\Omega)),$$

uniformly in $h > 0$*, where* $\nabla_h = \nabla_h^+$ *or* $\nabla_h = \nabla_h^-$ *(either works) and* $\nabla_h^2 := \nabla_h^\mp \nabla_h^\pm$ *(the discrete version of the Hessian) and*

$$W_h, \Delta_h W_h \subset L^\infty((0,T) \times \Omega)),$$

uniformly in $h > 0$ *as well.*

Proof. To obtain the L^2-estimates, we square the equation for the potential W_h, (4.2a) and sum over all i,j,

$$\mu^2 \sum_{i,j=1}^{N_x} |\Delta_h W_{i,j}^m|^2 - 2\mu \sum_{i,j=1}^{N_x} W_{i,j}^m \Delta_h W_{i,j}^m + \sum_{i,j=1}^{N_x} |W_{i,j}^m|^2 = \sum_{i,j=1}^{N_x} |n_{i,j}^m|^{2\gamma}.$$

Using summation by parts and that W satisfies either periodic or homogeneous Neumann boundary conditions, we obtain

$$\mu^2 \sum_{i,j=1}^{N_x} |\nabla_h^2 W_{i,j}^m|^2 + 2\mu \sum_{i,j=1}^{N_x} |\nabla_h W_{i,j}^m|^2 + \sum_{i,j=1}^{N_x} |W_{i,j}^m|^2 = \sum_{i,j=1}^{N_x} |n_{i,j}^m|^{2\gamma}.$$

From the previous estimates, we know that $n_h \in L^\infty([0,T] \times \Omega)$ uniformly in $h > 0$ and therefore also uniformly bounded in any other L^p-space, which implies together with the above identity, that $W_h, \nabla_h W_h, \nabla_h^2 W_h \in L^2([0,T] \times \Omega)$. That W_h is uniformly bounded follows from (4.6) and the uniform bound on n_h which was proved in the previous Lemma 4.1.

Using this and the uniform boundedness of the pressure, we conclude by (4.2a) that also $\Delta_h W_h$ is uniformly bounded. □

Remark 4.4. Using the discrete Gagliardo–Nirenberg–Sobolev inequality ([2], Thm. 3.4), we obtain that $\nabla_h W_h \in L^\infty([0,T]; L^q(\Omega))$ for $1 \leq q < q^* = 2d/(d-2)$.

4.3. Discrete entropy inequalities for n_h

To prove strong convergence of the approximating sequence $\{(n_h, W_h)\}_{h>0}$, it will be useful to derive entropy inequalities for n_h. To this end, the following lemma will be useful:

Lemma 4.5. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a smooth convex function and assume that* Δt *satisfies the CFL-condition*

$$\Delta t \leq \min\left\{ \frac{h}{16 \max_{i,j} |\nabla_h W_{i,j}^m|}, \frac{h}{8 \max_{ij} |\nabla_h W_{i,j}^m| + h\, \boldsymbol{G}^\infty}, \frac{\mu}{4\gamma \overline{n}_\infty^\gamma} \right\} \tag{4.10}$$

Denote $f_{i,j}^m := f(n_{i,j}^m)$ *and* f_h *a piecewise constant interpolation of it as in* (4.4). *Then* $f_{i,j}^m$ *satisfies the following identity*

$$D_t f_{i,j}^m = \frac{1}{2} D_1^- \left(u_{i+1/2,j}^m \left(f_{i,j}^m + f_{i+1,j}^m \right) \right) + \frac{1}{2} D_2^- \left(v_{i,j+1/2}^m \left(f_{i,j}^m + f_{i,j+1}^m \right) \right) \tag{4.11}$$

$$+ \frac{h}{4} D_1^- \left[f'(n_{i,j}^m) |u_{i+1/2,j}^m| D_1^+ n_{i,j}^m \right] + \frac{h}{4} D_2^- \left[f'(n_{i,j}^m) |v_{i,j+1/2}^m| D_2^+ n_{i,j}^m \right] \tag{4.12}$$

$$+ \frac{h}{4} D_1^+ \left[f'(n_{i,j}^m) |u_{i-1/2,j}^m| D_1^- n_{i,j}^m \right] + \frac{h}{4} D_2^+ \left[f'(n_{i,j}^m) |v_{i,j-1/2}^m| D_2^- n_{i,j}^m \right] \tag{4.13}$$

$$- \frac{h^2}{4} D_1^- \left[f''(\widetilde{n}_{i+1/2,j}^m) u_{i+1/2,j}^m |D_1^+ n_{i,j}^m|^2 \right] \tag{4.14}$$

$$- \frac{h^2}{4} D_2^- \left[f''(\widetilde{n}_{i,j+1/2}^m) v_{i,j+1/2}^m |D_2^+ n_{i,j}^m|^2 \right] \tag{4.15}$$

$$- \frac{h}{4} f''(\widehat{n}_{i-1/2,j}^m) |u_{i-1/2,j}^m| |D_1^- n_{i,j}^m|^2 - \frac{h}{4} f''(\widehat{n}_{i,j-1/2}^m) |v_{i,j-1/2}^m| |D_2^- n_{i,j}^m|^2 \tag{4.16}$$

$$- \frac{h}{4} f''(\widehat{n}_{i+1/2,j}^m) |u_{i+1/2,j}^m| |D_1^+ n_{i,j}^m|^2 - \frac{h}{4} f''(\widehat{n}_{i,j+1/2}^m) |v_{i,j+1/2}^m| |D_2^+ n_{i,j}^m|^2 \tag{4.17}$$

$$+ (f'(n_{i,j}^m) n_{i,j}^m - f_{i,j}^m) \Delta_h W_{i,j}^m + f'(n_{i,j}^m) n_{i,j}^m \boldsymbol{G}(p_{i,j}^m) \tag{4.18}$$

$$+ \frac{\Delta t}{2} f''(\widetilde{n}_{i,j}^{m+1/2}) |D_t^+ n_{i,j}^m|^2, \tag{4.19}$$

where $\widetilde{n}_{i\pm1/2,j}^m,\ \widehat{n}_{i\pm1/2,j}^m \in [\min\{n_{i,j}^m,\ n_{i\pm1,j}^m\},\ \max\{n_{i,j}^m,\ n_{i\pm1,j}^m\}]$, $\widetilde{n}_{i,j\pm1/2}^m,\ \widehat{n}_{i,j\pm1/2}^m \in [\min\{n_{i,j}^m,\ n_{i,j\pm1}^m\}, \max\{n_{i,j}^m, n_{i,j\pm1}^m\}]$ *and* $\widetilde{n}_{i,j}^{m+1/2} \in [\min\{n_{i,j}^m,\ n_{i,j}^{m+1}\},\ \max\{n_{i,j}^m, n_{i,j}^{m+1}\}]$ *and where the term* (4.18) *is uniformly bounded and the terms* (4.16)–(4.17) *and* (4.19) *satisfy*

$$\frac{h^{d+1} \Delta t}{2} \sum_{m=0}^{N_T} \sum_{i,j} f''(\widehat{n}_{i+1/2,j}^m) |u_{i+1/2,j}^m| |D_1^+ n_{i,j}^m|^2 \le C,$$

$$\frac{h^{d+1} \Delta t}{2} \sum_{m=0}^{N_T} \sum_{i,j} f''(\widehat{n}_{i,j+1/2}^m) |v_{i,j+1/2}^m| |D_2^+ n_{i,j}^m|^2 \le C,$$

$$\frac{h^d \Delta t^2}{2} \sum_{m=0}^{N_T} \sum_{i,j} f''(\widetilde{n}_{i,j}^{m+1/2}) |D_t^+ n_{i,j}^m|^2 \le C, \tag{4.20}$$

In particular, this implies that the piecewise constant interpolation $D_t^+ f_h$ *is of the form* $D_t^+ f_h = g_h + k_h$ *where* $g_h \in L^1([0,T] \times \Omega)$ *and* $k_h \in L^\infty([0,T]; W^{-1,q}(\Omega))$ *for any* $1 \le q < \infty$ *if* $d = 2$ *and for* $1 \le q \le q^* = 2d/(d-2)$ *if* $d > 2$*, uniformly in* $h > 0$.

Proof. We first rewrite the scheme for $n_{i,j}^m$ as

$$\begin{aligned} D_t^+ n_{i,j}^m = &\frac{1}{2} u_{i+1/2,j}^m D_1^+ n_{i,j}^m + \frac{1}{2} u_{i-1/2,j}^m D_1^- n_{i,j}^m \\ &+ \frac{1}{2} v_{i,j+1/2}^m D_2^+ n_{i,j}^m + \frac{1}{2} v_{i,j-1/2}^m D_2^- n_{i,j}^m \\ &+ \frac{h}{2} D_1^- \left[|u_{i+1/2,j}^m| D_1^+ n_{i,j}^m \right] + \frac{h}{2} D_2^- \left[|v_{i,j+1/2}^m| D_2^+ n_{i,j}^m \right] \\ &+ n_{i,j}^m \Delta_h W_{i,j}^m + n_{i,j}^m \boldsymbol{G}(p_{i,j}^m). \end{aligned} \tag{4.21}$$

Then, using the Taylor expansion,

$$f(b) - f(a) = f'(a)(b-a) + f''(\widetilde{a}) \frac{(a-b)^2}{2},$$

where $\widetilde{a} \in [\min\{a,b\}, \max\{a,b\}]$, we can write

$$\begin{aligned}
D_t^+ f_{i,j}^m &= f'(n_{i,j}^m) D_t^+ n_{i,j}^m + \frac{\Delta t}{2} f''(\widetilde{n}_{i,j}^{m+1/2}) |D_t^+ n_{i,j}^m|^2 \\
D_1^\pm f_{i,j}^m &= f'(n_{i,j}^m) D_1^\pm n_{i,j}^m \pm \frac{h}{2} f''(\widetilde{n}_{i\pm1/2,j}^m) |D_1^\pm n_{i,j}^m|^2 \\
D_2^\pm f_{i,j}^m &= f'(n_{i,j}^m) D_2^\pm n_{i,j}^m \pm \frac{h}{2} f''(\widetilde{n}_{i,j\pm1/2}^m) |D_2^\pm n_{i,j}^m|^2 \\
D_1^\pm f'(n_{i,j}^m) &= f''(\widehat{n}_{i\pm1/2,j}^m) D_1^\pm n_{i,j}^m \\
D_2^\pm f'(n_{i,j}^m) &= f''(\widehat{n}_{i,j\pm1/2}^m) D_2^\pm n_{i,j}^m,
\end{aligned}$$

where $\widetilde{n}_{i,j}^{m+1/2}, \widetilde{n}_{i\pm1/2,j}^m, \widetilde{n}_{i,j\pm1/2}^m, \widehat{n}_{i\pm1/2,j}^m$ and $\widehat{n}_{i,j\pm1/2}^m$ are intermediate values. Hence, multiplying equation (4.21) by $f'(n_{i,j}^m)$, it becomes

$$\begin{aligned}
D_t^+ f_{i,j}^m &= \frac{\Delta t}{2} f''(\widetilde{n}_{i,j}^{m+1/2}) |D_t^+ n_{i,j}^m|^2 \\
&\quad + \frac{1}{2} u_{i+1/2,j}^m D_1^+ f_{i,j}^m - \frac{h}{4} f''(\widetilde{n}_{i+1/2,j}^m) u_{i+1/2,j}^m |D_1^+ n_{i,j}^m|^2 \\
&\quad + \frac{1}{2} u_{i-1/2,j}^m D_1^- f_{i,j}^m + \frac{h}{4} f''(\widetilde{n}_{i-1/2,j}^m) u_{i-1/2,j}^m |D_1^- n_{i,j}^m|^2 \\
&\quad + \frac{1}{2} v_{i,j+1/2}^m D_2^+ f_{i,j}^m - \frac{h}{4} f''(\widetilde{n}_{i,j+1/2}^m) v_{i,j+1/2}^m |D_2^+ n_{i,j}^m|^2 \\
&\quad + \frac{1}{2} v_{i,j-1/2}^m D_2^- f_{i,j}^m + \frac{h}{4} f''(\widetilde{n}_{i,j-1/2}^m) v_{i,j-1/2}^m |D_2^- n_{i,j}^m|^2 \\
&\quad + \frac{h}{4} D_1^- \left[f'(n_{i,j}^m) |u_{i+1/2,j}^m| D_1^+ n_{i,j}^m \right] - \frac{h}{4} f''(\widehat{n}_{i-1/2,j}^m) |u_{i-1/2,j}^m| |D_1^- n_{i,j}^m|^2 \\
&\quad + \frac{h}{4} D_2^- \left[f'(n_{i,j}^m) |v_{i,j+1/2}^m| D_2^+ n_{i,j}^m \right] - \frac{h}{4} f''(\widehat{n}_{i,j-1/2}^m) |v_{i,j-1/2}^m| |D_2^- n_{i,j}^m|^2 \\
&\quad + \frac{h}{4} D_1^+ \left[f'(n_{i,j}^m) |u_{i-1/2,j}^m| D_1^- n_{i,j}^m \right] - \frac{h}{4} f''(\widehat{n}_{i+1/2,j}^m) |u_{i+1/2,j}^m| |D_1^+ n_{i,j}^m|^2 \\
&\quad + \frac{h}{4} D_2^+ \left[f'(n_{i,j}^m) |v_{i,j-1/2}^m| D_2^- n_{i,j}^m \right] - \frac{h}{4} f''(\widehat{n}_{i,j+1/2}^m) |v_{i,j+1/2}^m| |D_2^+ n_{i,j}^m|^2 \\
&\quad + f'(n_{i,j}^m) n_{i,j}^m \Delta_h W_{i,j}^m + f'(n_{i,j}^m) n_{i,j}^m \boldsymbol{G}(p_{i,j}^m) \\
&= \frac{\Delta t}{2} f''(\widetilde{n}_{i,j}^{m+1/2}) |D_t^+ n_{i,j}^m|^2 \\
&\quad + \frac{1}{2} D_1^- \left(u_{i+1/2,j}^m \left(f_{i,j}^m + f_{i+1,j}^m \right) \right) + \frac{1}{2} D_2^- \left(v_{i,j+1/2}^m \left(f_{i,j}^m + f_{i,j+1}^m \right) \right) \\
&\quad + \frac{h}{4} D_1^- \left[f'(n_{i,j}^m) |u_{i+1/2,j}^m| D_1^+ n_{i,j}^m \right] + \frac{h}{4} D_2^- \left[f'(n_{i,j}^m) |v_{i,j+1/2}^m| D_2^+ n_{i,j}^m \right] \\
&\quad + \frac{h}{4} D_1^+ \left[f'(n_{i,j}^m) |u_{i-1/2,j}^m| D_1^- n_{i,j}^m \right] + \frac{h}{4} D_2^+ \left[f'(n_{i,j}^m) |v_{i,j-1/2}^m| D_2^- n_{i,j}^m \right] \\
&\quad - \frac{h^2}{4} D_1^- \left[f''(\widetilde{n}_{i+1/2,j}^m) u_{i+1/2,j}^m |D_1^+ n_{i,j}^m|^2 \right] \\
&\quad - \frac{h^2}{4} D_2^- \left[f''(\widetilde{n}_{i,j+1/2}^m) v_{i,j+1/2}^m |D_2^+ n_{i,j}^m|^2 \right] \\
&\quad - \frac{h}{4} f''(\widehat{n}_{i-1/2,j}^m) |u_{i-1/2,j}^m| |D_1^- n_{i,j}^m|^2 - \frac{h}{4} f''(\widehat{n}_{i,j-1/2}^m) |v_{i,j-1/2}^m| |D_2^- n_{i,j}^m|^2 \\
&\quad - \frac{h}{4} f''(\widehat{n}_{i+1/2,j}^m) |u_{i+1/2,j}^m| |D_1^+ n_{i,j}^m|^2 - \frac{h}{4} f''(\widehat{n}_{i,j+1/2}^m) |v_{i,j+1/2}^m| |D_2^+ n_{i,j}^m|^2 \\
&\quad + (f'(n_{i,j}^m) n_{i,j}^m - f_{i,j}^m) \Delta_h W_{i,j}^m + f'(n_{i,j}^m) n_{i,j}^m \boldsymbol{G}(p_{i,j}^m).
\end{aligned}$$

which implies (4.11)–(4.19). In particular, for $f(x) = x^2$, this becomes

$$
\begin{aligned}
D_t^+ f_{i,j}^m = {} & \Delta t |D_t^+ n_{i,j}^m|^2 \\
& + \frac{1}{2} D_1^+ \left(u_{i-1/2,j}^m \left(f_{i,j}^m + f_{i-1,j} \right) \right) + \frac{1}{2} D_2^+ \left(v_{i,j-1/2}^m \left(f_{i,j}^m + f_{i,j-1} \right) \right) \\
& - \frac{h^2}{2} D_1^- \left[u_{i+1/2,j}^m |D_1^+ n_{i,j}^m|^2 \right] - \frac{h^2}{2} D_2^- \left[v_{i,j+1/2}^m |D_2^+ n_{i,j}^m|^2 \right] \\
& + \frac{h}{2} D_1^- \left[n_{i,j}^m |u_{i+1/2,j}^m| D_1^+ n_{i,j}^m \right] + \frac{h}{2} D_2^- \left[n_{i,j}^m |v_{i,j+1/2}^m| D_2^+ n_{i,j}^m \right] \\
& + \frac{h}{2} D_1^+ \left[n_{i,j}^m |u_{i-1/2,j}^m| D_1^- n_{i,j}^m \right] + \frac{h}{2} D_2^+ \left[n_{i,j}^m |v_{i,j-1/2}^m| D_2^- n_{i,j}^m \right] \\
& - \frac{h}{2} |u_{i-1/2,j}^m| |D_1^- n_{i,j}^m|^2 - \frac{h}{2} |v_{i,j-1/2}^m| |D_2^- n_{i,j}^m|^2 \\
& - \frac{h}{2} |u_{i+1/2,j}^m| |D_1^+ n_{i,j}^m|^2 - \frac{h}{2} |v_{i,j+1/2}^m| |D_2^+ n_{i,j}^m|^2 \\
& + f_{i,j}^m \Delta_h W_{i,j}^m + 2 f_{i,j}^m \boldsymbol{G}(p_{i,j}^m).
\end{aligned}
\tag{4.22}
$$

We estimate the first term on the right hand side of the inequality inserting (4.21),

$$
\begin{aligned}
|D_t^+ n_{i,j}^m|^2 \le {} & 2 \Big| \frac{1}{2} u_{i+1/2,j}^m D_1^+ n_{i,j}^m + \frac{1}{2} u_{i-1/2,j}^m D_1^- n_{i,j}^m + \frac{1}{2} v_{i,j+1/2}^m D_2^+ n_{i,j}^m \\
& + \frac{1}{2} v_{i,j-1/2}^m D_2^- n_{i,j}^m + \frac{h}{2} D_1^- \left[|u_{i+1/2,j}^m| D_1^+ n_{i,j}^m \right] + \frac{h}{2} D_2^- \left[|v_{i,j+1/2}^m| D_2^+ n_{i,j}^m \right] \Big|^2 \\
& + 2 \big| n_{i,j}^m \Delta_h W_{i,j}^m + n_{i,j}^m \boldsymbol{G}(p_{i,j}^m) \big|^2 \\
\le {} & 4 \Big| \frac{1}{2} u_{i+1/2,j}^m D_1^+ n_{i,j}^m + \frac{1}{2} u_{i-1/2,j}^m D_1^- n_{i,j}^m + \frac{h}{2} D_1^- \left[|u_{i+1/2,j}^m| D_1^+ n_{i,j}^m \right] \Big|^2 \\
& + 4 \Big| \frac{1}{2} v_{i,j+1/2}^m D_2^+ n_{i,j}^m + \frac{1}{2} v_{i,j-1/2}^m D_2^- n_{i,j}^m + \frac{h}{2} D_2^- \left[|v_{i,j+1/2}^m| D_2^+ n_{i,j}^m \right] \Big|^2 \\
& + 2 \big| n_{i,j}^m \Delta_h W_{i,j}^m + n_{i,j}^m \boldsymbol{G}(p_{i,j}^m) \big|^2 \\
\le {} & 8 \big| u_{i+1/2,j}^m D_1^+ n_{i,j}^m \big|^2 + 8 \big| u_{i-1/2,j}^m D_1^- n_{i,j}^m \big|^2 + 8 \big| v_{i,j+1/2}^m D_2^+ n_{i,j}^m \big|^2 \\
& + 8 \big| u_{i,j-1/2}^m D_2^- n_{i,j}^m \big|^2 + 2 \big| n_{i,j}^m \Delta_h W_{i,j}^m + n_{i,j}^m \boldsymbol{G}(p_{i,j}^m) \big|^2 \\
\le {} & 8 \max_{i,j} |\nabla_h W_{i,j}^m| \big\{ |u_{i+1/2,j}^m| \, |D_1^+ n_{i,j}^m \big|^2 + |u_{i-1/2,j}^m| \, |D_1^- n_{i,j}^m \big|^2 \\
& + |v_{i,j+1/2}^m| \, |D_2^+ n_{i,j}^m \big|^2 + |v_{i,j-1/2}^m| \, |D_2^- n_{i,j}^m \big|^2 \big\} \\
& + 2 \big| n_{i,j}^m \Delta_h W_{i,j}^m + n_{i,j}^m \boldsymbol{G}(p_{i,j}^m) \big|^2
\end{aligned}
$$

Thus if we assume that Δt satisfies the CFL-condition (4.10), we have

$$
\begin{aligned}
\Delta t \sum_{i,j} |D_t^+ n_{i,j}^m|^2 \le h \sum_{i,j} & \big\{ |u_{i+1/2,j}^m| |D_1^+ n_{i,j}^m|^2 + |v_{i,j+1/2}^m| |D_2^+ n_{i,j}^m|^2 \big\} \\
& + h \sum_{i,j} \big| n_{i,j}^m \Delta_h W_{i,j}^m + n_{i,j}^m \boldsymbol{G}(p_{i,j}^m) \big|^2
\end{aligned}
$$

Now summing (4.22) over all i,j, multiplying with h^d and using the latter inequality, we obtain

$$\begin{aligned} h^d D_t^+ \sum_{i,j} f_{i,j}^m &= -h^{d+1} \sum_{i,j} \left(|u_{i-1/2,j}^m||D_1^- n_{i,j}^m|^2 + |v_{i,j-1}^m||D_2^- n_{i,j}^m|^2 \right) \\ &\quad + h^d \Delta t \sum_{i,j} |D_t^+ n_{i,j}^m|^2 + h^d \sum_{i,j} f_{i,j}^m \left(\Delta_h W_{i,j}^m + 2\boldsymbol{G}(p_{i,j}^m)\right) \\ &\leq h^d \sum_{i,j} f_{i,j}^m \left(\Delta_h W_{i,j}^m + 2\boldsymbol{G}(p_{i,j}^m)\right) \\ &\quad + h^{d+1} \sum_{i,j} |n_{i,j}^m \Delta_h W_{i,j}^m + n_{i,j}^m \boldsymbol{G}(p_{i,j}^m)|^2 \\ &\leq C, \end{aligned}$$

where $C > 0$ is a constant independent of h, thanks to the L^∞-bounds on n_h and $\Delta_h W_h$ obtained in Lemmas 4.1 and 4.3. This implies that

$$\begin{aligned} &h^{d+1} \Delta t \sum_{m=0}^{N_T} \sum_{i,j} \left(|u_{i-1/2,j}^m||D_1^- n_{i,j}^m|^2 + |v_{i,j-1/2}^m||D_2^- n_{i,j}^m|^2 \right) \leq C \\ &h^d \Delta t^2 \sum_{m=0}^{N_T} \sum_{i,j} |D_t^+ n_{i,j}^m|^2 \leq C. \end{aligned}$$

and therefore using Hölder's inequality and the uniform L^∞-bounds on n_h, (4.20). Using summation by parts, we realize that the other terms, (4.11)–(4.15) are in $L^\infty([0,T]; W^{-1,q}(\Omega))$ for $q \in [1, 2^*)$ where $2^* = 2d/(d-2)$ if $d \geq 3$ and any finite number greater than one if $d = 2$. □

Remark 4.6. The preceeding lemma implies that the forward time difference of the approximation of the pressure $D_t^+ p_h = D_t^+ |n_h|^\gamma$ is of the form $D_t^+ p_h = g_h + k_h$ where $g_h \in L^1([0,T] \times \Omega)$ and $k_h \in L^\infty([0,T]; W^{-1,q}(\Omega))$ for any $1 \leq q < \infty$ if $d = 2$ and for $1 \leq q \leq q^* = 2d/(d-2)$ if $d > 2$, uniformly in $h > 0$. Using this, we have that $D_t^+ W_h = U_h + V_h$ where U_h and V_h solve

$$-\mu \Delta_h U_h + U_h = g_h, \quad \text{and} \quad -\mu \Delta_h V_h + V_h = k_h.$$

By Lemma B.1, we have $U_h, \nabla_h U_h \in L^1([0,T]; L^q(\Omega))$ for $1 \leq q \leq d/(d-1)$ and by standard results, V_h, $\nabla_h V_h \in L^\infty([0,T]; L^2(\Omega))$. Hence $D_t W_h, D_t \nabla_h W_h \in L^1([0,T]; L^q(\Omega)) + L^\infty([0,T]; L^2(\Omega))$.

Remark 4.7 (CFL-condition)**.** The estimates from Lemma 4.3 imply that the velocity $\boldsymbol{u}_h := \nabla_h W_h \in L^\infty([0,T]; L^{2^*}(\Omega))$ uniformly in $h > 0$, $2^* = 2d/(d-2)$ or any number in $[1,\infty)$ if $d = 2$, using the Sobolev embedding theorem. Using an inverse inequality, we can bound it in the $L^\infty((0,T) \times \Omega)$-norm as follows:

$$\max_{(x,t) \in (0,T) \times \Omega} |\boldsymbol{u}_h| \leq C h^{-\frac{d}{2^*}} \left(\int_\Omega |\boldsymbol{u}_h|^{2^*} \mathrm{d}x \right)^{\frac{1}{2^*}} \leq C h^{-\frac{d}{2^*}}$$

Thus the time step size Δt is of order $\mathcal{O}(h^{1+d/2^*})$. In practice a linear CFL-condition seems to work well though.

4.4. Passing to the limit $h \to 0$

The estimates of the previous (sub)sections allow us to pass to the limit $h \to 0$ in a subsequence still denoted h,

$$\begin{aligned} n_h &\rightharpoonup n \geq 0, \quad \text{in } L^q([0,T] \times \Omega),\ 1 \leq q < \infty, \\ p_h &\rightharpoonup \overline{p} \geq 0, \quad \text{in } L^q([0,T] \times \Omega),\ 1 \leq q < \infty, \end{aligned}$$

where $p_h := n_h^\gamma$ and $0 \le n, \overline{p} \in L^\infty([0,T]\times\Omega)$. Using the "discretized" Aubin–Lions Lemma A.1 for W_h and $\nabla_h W_h$, we obtain strong convergence of a subsequence in $L^q([0,T]\times\Omega)$ for any $q\in[0,\infty)$ in the case of W_h and $1\le q\le 2^*$ in the case of $\nabla_h W_h$ ($2^* = 2d/(d-2)$ if $d\ge 3$ and any finite number greater than or equal to one if $d=2$), to limit functions $W, \nabla W \in L^q([0,T]\times\Omega)$. Moreover, from the estimates in Lemma 4.3 we obtain that $W\in L^\infty([0,T]\times\Omega)\cap L^\infty([0,T];H^2(\Omega))$. Hence we have that $(n,W,\overline{p})$ satisfy for any $\varphi,\psi\in C^1([0,T]\times\Omega)$,

$$\begin{aligned}\int_0^T\int_\Omega n\varphi_t - n\nabla W\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t &= -\int_0^T\int_\Omega \overline{n\boldsymbol{G}(p)}\varphi\,\mathrm{d}x\mathrm{d}t\\ \int_0^T\int_\Omega W\psi + \mu\nabla W\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\Omega \overline{p}\,\psi\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

where $\overline{n\boldsymbol{G}(p)}$ is the weak limit of $n_h\boldsymbol{G}(p_h)$. To conclude that the limit $(n,W,\overline{p})$ is a weak solution of (1.6), we proceed as in the previous Section 3 and show that n_h in fact converges strongly: First, we recall that the limit n satisfies (3.9).

On the other hand, from (4.22), we obtain (under the CFL-condtion (4.10))

$$\begin{aligned}D_t^+|n_{i,j}^m|^2 \le\; &\frac{1}{2}D_1^+\left(u_{i-1/2,j}^m\left(|n_{i,j}^m|^2+|n_{i-1,j}^m|^2\right)\right)\\ &+\frac{1}{2}D_2^+\left(v_{i,j-1/2}^m\left(|n_{i,j}^m|^2+|n_{i,j-1}^m|^2\right)\right)\\ &-\frac{h^2}{2}D_1^-\left[u_{i+1/2,j}^m|D_1^+n_{i,j}^m|^2\right]-\frac{h^2}{2}D_2^-\left[v_{i,j+1/2}^m|D_2^+n_{i,j}^m|^2\right]\\ &+\frac{h}{2}D_1^-\left[n_{i,j}^m|u_{i+1/2,j}|D_1^+n_{i,j}^m\right]+\frac{h}{2}D_2^-\left[n_{i,j}^m|v_{i,j+1/2}|D_2^+n_{i,j}^m\right]\\ &+\frac{h}{2}D_1^+\left[n_{i,j}^m|u_{i-1/2,j}|D_1^-n_{i,j}^m\right]+\frac{h}{2}D_2^+\left[n_{i,j}^m|v_{i,j-1/2}|D_2^-n_{i,j}^m\right]\\ &+|n_{i,j}^m|^2\Delta_h W_{i,j}^m+2|n_{i,j}^m|^2\boldsymbol{G}(p_{i,j}^m),\end{aligned}\tag{4.23}$$

Considering this inequality in terms of the piecewise constant functions n_h, W_h and p_h, multiplying it with a nonnegative C^1-test function φ, integrating and then passing to the limit $h\to 0$, we obtain (using the bounds (4.20), the weak convergence of n_h and p_h and the strong convergence of W_h and $\nabla_h W_h$),

$$-\int_0^T\int_\Omega \overline{n^2}\varphi_t - \overline{n^2}\nabla W\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \le \int_0^T\int_\Omega\left(\overline{n^2\Delta W}+2\overline{n^2\boldsymbol{G}(p)}\right)\varphi\,\mathrm{d}x\mathrm{d}t,\tag{4.24}$$

where $\overline{n^2}$ denotes the weak limit of n_h^2 and $\overline{n^2\Delta W}$ and $\overline{n^2\boldsymbol{G}(p)}$ are the weak limits of $n_h^2\Delta_h W_h$ and $n_h^2\boldsymbol{G}(p_h)$ respectively.

Adding (3.9) and (4.24), we have

$$-\int_0^T\int_\Omega\left(\overline{n^2}-n^2\right)\varphi_t-\left(\overline{n^2}-n^2\right)\nabla W\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t\le\int_0^T\int_\Omega\left(2\overline{n^2\boldsymbol{G}(p)}-2n\overline{n\boldsymbol{G}(p)}+\overline{n^2\Delta W}-n^2\Delta W\right)\varphi\,\mathrm{d}x\mathrm{d}t.$$

We now choose smooth test functions φ_ϵ approximating $\varphi(t,x)=\mathbf{1}_{[0,\tau]}(t)$, where $\tau\in(0,T]$, in this inequality and then pass to the limit $\epsilon\to 0$ to obtain

$$\begin{aligned}&\int_\Omega\left(\overline{n^2}-n^2\right)(\tau)\mathrm{d}x-\int_\Omega\left(\overline{n^2}(0,x)-n^2(0,x)\right)\mathrm{d}x\\ &\qquad\le\int_0^\tau\int_\Omega\left(2\overline{n^2\boldsymbol{G}(p)}-2n\overline{n\boldsymbol{G}(p)}+\Delta W\left(\overline{n^2}-n^2\right)+\overline{n^2\Delta W}-\overline{n^2}\Delta W\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{4.25}$$

By convexity of $f(x)=x^2$, we have $\overline{n^2}\geq n^2$, on the other hand, the discrete L^2-entropy inequality, (4.23), implies

$$\int_\Omega |n_h(\tau,x)|^2\,\mathrm{d}x \leq \int_\Omega |n_h^0|^2 \mathrm{d}x + \int_0^\tau\int_\Omega \left(|n_h|^2\Delta_h W_h + 2|n_h|^2\boldsymbol{G}(p_h)\right)\mathrm{d}x\mathrm{d}t,$$

which gives, passing to the limit $h\to 0$,

$$\int_\Omega \overline{|n|^2}(\tau,x)\,\mathrm{d}x \leq \int_\Omega |n_0|^2 \mathrm{d}x + \int_0^\tau\int_\Omega \left(\overline{|n|^2\Delta W} + 2\overline{|n|^2\boldsymbol{G}(p)}\right)\mathrm{d}x\mathrm{d}t.$$

Letting $\tau\to 0$, the second term on the right hand side vanishes (as the integrand is bounded), and we obtain

$$\int_\Omega \overline{|n|^2}(0,x)\,\mathrm{d}x \leq \int_\Omega |n_0|^2 \mathrm{d}x.$$

We deduce that $\overline{|n|^2}(0,\cdot)=|n_0|^2$ almost everywhere and that therefore the second term on the left hand side of (4.25) is zero. We have already estimated the first two terms on the right hand side of (4.25) in (3.13) and (3.14). To bound the other term, we use a discretized version of Lemma 3.7.

Lemma 4.8. *The weak limits $(n,W,\overline{p})$ of the sequences $\{(n_h,W_h,p_h)\}_{h>0}$ satisfy for any smooth function $S:\mathbb{R}\to\mathbb{R}$,*

$$\int_\Omega \left(\overline{S(n)\Delta W} - \overline{S(n)}\Delta W\right)\mathrm{d}x = \frac{1}{\mu}\int_\Omega \left(\overline{p}\;\overline{S(n)} - \overline{pS(n)}\right)\mathrm{d}x \tag{4.26}$$

where $\overline{S(n)\Delta W}$, $\overline{S(n)}$, $\overline{pS(n)}$ are the weak limits of $S(n_h)\Delta_h W_h$, $S(n_h)$ and $p_h S(n_h)$ respectively.

Applying this lemma to the last term in (3.12) with $S(n)=n^2$, we can estimate it by

$$\begin{aligned}\int_0^\tau\int_\Omega \left(\overline{n^2\Delta W} - \overline{n^2}\Delta W\right)\mathrm{d}x &= \frac{1}{\mu}\int_\Omega \left(\overline{p}\;\overline{n^2} - \overline{pn^2}\right)\mathrm{d}x\mathrm{d}t\\ &= \frac{1}{\mu}\int_\Omega \left(\overline{n^\gamma}\;\overline{n^2} - \overline{n^{2+\gamma}}\right)\mathrm{d}x\mathrm{d}t\\ &\leq 0,\end{aligned}$$

using again that by Exercise 3.37 in [25], $\overline{n^\gamma}\;\overline{n^2}\leq\overline{n^{2+\gamma}}$. Thus,

$$\int_\Omega \left(\overline{n^2}-n^2\right)(\tau)\mathrm{d}x \leq \left(2\alpha + \frac{P_M}{\mu}\right)\int_0^\tau\int_\Omega \left(\overline{n^2}-n^2\right)\mathrm{d}x\,\mathrm{d}t.$$

Grönwall's inequality thus implies

$$\int_\Omega \left(\overline{n^2}-n^2\right)(\tau)\mathrm{d}x \leq 0$$

By convexity of the function $f(x)=x^2$ we also have $n^2\leq\overline{n^2}$ almost everywhere and hence

$$\overline{n^2}=n^2$$

almost everywhere in $(0,T)\times\Omega$. Therefore we conclude that the functions n_h converge strongly to n almost everywhere, thus also $\overline{p}=n^\gamma$ and so the limit $(n,W,\overline{p})$ is a weak solution of the equations (1.6).

Proof of Lemma 4.8. We multiply the equation for W_h by $S(n_h)$ and integrate it over the spatial domain Ω,

$$\int_\Omega \mu\Delta_h W_h\,S(n_h) - W_h S(n_h)\,\mathrm{d}x = -\int_\Omega p_h S(n_h)\,\mathrm{d}x.$$

Passing to the limit $h \to 0$ in the last equation, we obtain

$$\int_\Omega \mu \overline{\Delta W S(n)} - W\overline{S(n)}\,\mathrm{d}x = -\int_\Omega \overline{pS(n)}\,\mathrm{d}x. \tag{4.27}$$

On the other hand, using $[S(n_h) * \psi_\delta](x)$, where ψ_δ is a smooth mollifier converging to a Dirac measure at zero when δ is sent to zero, as a test function in the weak formulation of the limit equation

$$-\mu\Delta W + W = \overline{p},$$

and passing first to the limit $\delta \to 0$ and then $h \to 0$, we obtain

$$\int_\Omega \mu\Delta W\overline{S(n)} - W\overline{S(n)}\,\mathrm{d}x = -\int_\Omega \overline{p}\;\overline{S(n)}\,\mathrm{d}x$$

Combining the last identity with (4.27), we obtain (4.26). □

5. Numerical examples

To test the scheme in practice, we compute approximations for the following two examples.

5.1. Gaussian initial data

As a first example, we consider the initial data

$$n_0(x) = \frac{1}{2}\exp\left(-10\left(x_1^2 + x_2^2\right)\right), \tag{5.1}$$

on the domain $\Omega = [-2.5, 2.5]^2$ and $h = 1/64$ with pressure law $p = n^3$ and $\boldsymbol{G}(p) = 1 - p$ and $\mu = 1$. Strictly speaking, these are not homogeneous Neumann boundary conditions, but since the gradient of n_0 near the boundary is very small, this works well in practice.

In Figure 1 we show the approximations at times $t = 0, 1, 2, 4$. We observe that the cell density in the middle first reaches the maximum possible and then starts spreading with a relatively narrow transition region between zero density and maximum density.

5.2. Two Gaussians

As a second example, we use the inital data consisting of two Gaussian pulses with centers at $x = (0.7, 0)$ and $x = (-0.6, 0.2)$,

$$n_0(x) = \frac{1}{2}\exp\left(-10\left((x_1 - 0.7)^2 + x_2^2\right)\right) + \frac{1}{2}\exp\left(-20\left((x_1 + 0.6)^2 + (x_2 - 0.2)^2\right)\right) \tag{5.2}$$

on the same domain, $\Omega = [-2.5, 2.5]^2$, with $\mu = 1$, pressure law $p = n^{10}$ and $\boldsymbol{G}(p) = 1 - p$ and mesh width $h = 1/64$. The approximations computed at times $t = 0, 2, 4, 6$ are shown in Figure 2. The interface between the area with maximum cell density and zero cell density seems to be sharper than in the previous example, this appears to be caused by the pressure law with the higher exponent γ. Further tests with higher and lower exponents confirmed that assertion.

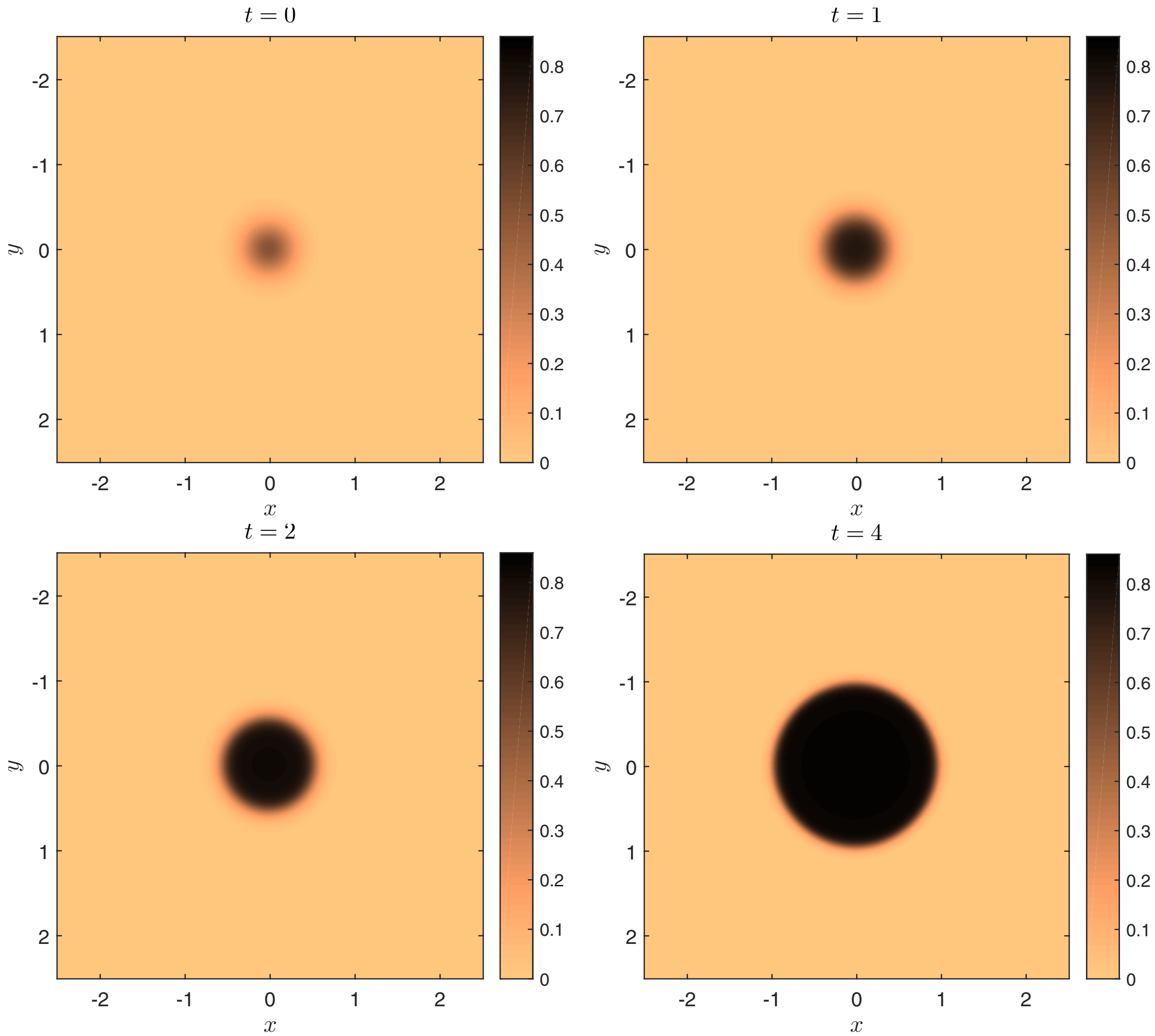

FIGURE 1. The approximations of the cell density n for initial data (5.1) on $\Omega = [-2.5, 2.5]^2$ with mesh width $h = 1/64$.

APPENDIX A. DISCRETIZED AUBIN–LIONS LEMMA

Lemma A.1. *Let* $u_h : [0,T) \times \Omega \to \mathbb{R}^k$ *be a piecewise constant function defined on a grid on* $[0,T) \times \Omega$, Ω *a bounded rectangular domain, satisfying*

$$\int_0^T \int_\Omega |u_h|^q + |\nabla_h u_h|^q \, \mathrm{d}x\mathrm{d}t \leq C \tag{A.1}$$

for some $\infty > q > 1$, *uniformly with respect to* $h > 0$ *and*

$$D_t u_h = A_h f_h + g_h + k_h, \tag{A.2}$$

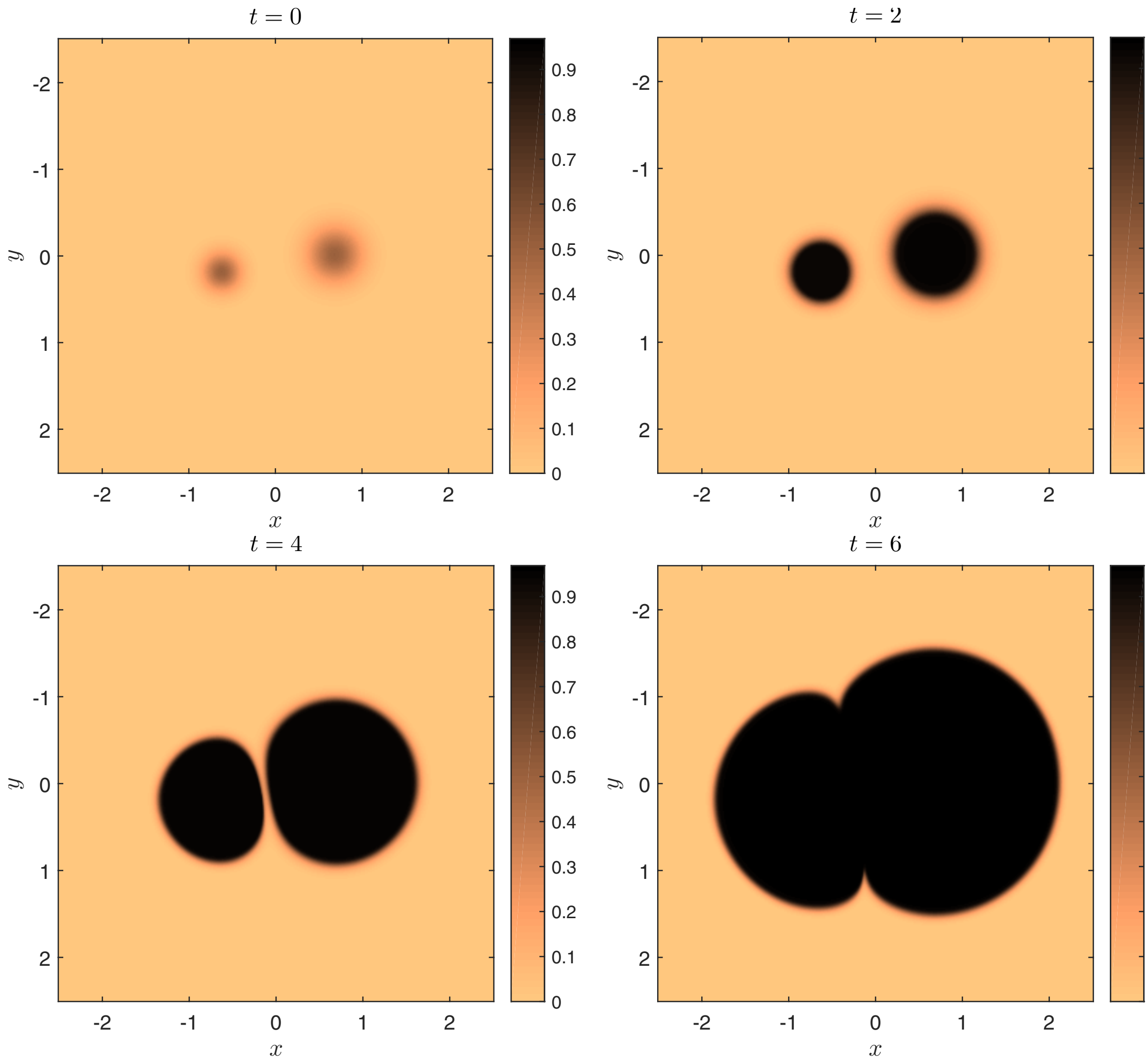

FIGURE 2. The approximations of the cell density n for initial data (5.2) on $\Omega = [-2.5, 2.5]^2$ with mesh width $h = 1/64$.

where A_h is a first order linear finite difference operator, and $f_h, g_h, k_h : \Omega \to \mathbb{R}^{d\times k}$ are piecewise constant functions, satisfying uniformly in $h > 0$,

$$\int_0^T \int_\Omega |f_h|^{r_1} + |g_h|^{r_2} + |k_h| \,\mathrm{d}x\mathrm{d}t \leq C, \tag{A.3}$$

for some $\infty > r_1, r_2 > 1$. Then $u_h \to u$ in $L^q([0,T)\times\Omega)$.

Proof. Denote $\widehat{u}_h$ a piecewise linear interpolation of u_h in space piecewise constant in time and similarly, let $\widehat{g}_h$, $\widehat{f}_h$ and $\widehat{k}_h$ piecewise linear interpolations of g_h, f_h and k_h respectively in space and piecewise constant in time such that

$$D_t \widehat{u}_h = A_h \widehat{f}_h + \widehat{g}_h + \widehat{k}_h. \tag{A.4}$$

By Ladyshenskaya's norm equivalences ([21], p. 230 ff]), we have

$$\int_0^T \|\widehat{u}_h\|^q_{W^{1,q}(\Omega)}\,\mathrm{d}t \le C\int_0^T\int_\Omega |u_h|^q + |\nabla_h u_h|^q\,\mathrm{d}x\mathrm{d}t$$
$$\int_0^T \|\widehat{f}_h\|^{r_1}_{L^{r_1}(\Omega)} + \|\widehat{g}_h\|^{r_2}_{L^{r_2}(\Omega)} + \|\widehat{k}_h\|_{L^1(\Omega)}\mathrm{d}t \le C\int_0^T\int_\Omega |f_h|^{r_1} + |g_h|^{r_2} + |k_h|\,\mathrm{d}x\mathrm{d}t$$

where the right hand sides are bounded by assumptions (A.1) and (A.3). Since $L^1(\Omega)\subset W^{-1,s}(\Omega)$ for $1\le s\le 1^* = d/(d-1)$, we have that $\widehat{k}_h \in L^1([0,T];W^{-1,s}(\Omega))$ for $1\le s\le 1^* = d/(d-1)$ and hence thanks to this and (A.4), we obtain

$$\widehat{u}_h \in L^q([0,T);W^{1,q}(\Omega)),\quad D_t\widehat{u}_h \in L^1([0,T);W^{-1,\min\{r_1,1^*\}}(\Omega)),$$

uniformly with respect to the discretization parameter $h>0$. Thus we can apply the version ([11], Thm. 1) of the Aubin–Lions lemma to find that up to a subsequence $\widehat{u}_h \to u$ in $L^q([0,T)\times\Omega)$ and the limit $u\in L^q([0,T);W^{1,q}(\Omega))$. By ([21], Lem. 3.2, p. 226) this implies that also $u_h\to u$ in $L^q([0,T)\times\Omega)$ (and $\nabla_h u_h \rightharpoonup \nabla u$). □

Remark A.2 (Derivatives)**.** If the u_h in Lemma A.1 are of the form $\nabla_h v_h$ for some v_h piecewise constant function, this lemma implies that $\nabla_h v_h \to \nabla v$ in L^q, again applying ([21], Lem. 3.2, p. 226).

Appendix B. Technical lemmas

In this section, we prove the following lemma:

Lemma B.1. *For each $h>0$, let u_h solve the difference equation*

$$-\operatorname{div}_h(A_h\nabla_h u_h) + c_h u_h = f_h,\quad \text{on } \Omega, \tag{B.1}$$

on a rectangular domain $\Omega\in\mathbb{R}^d$, with homogeneous Neumann boundary conditions, where A_h is a diagonal positive definite $d\times d$-matrix with entries $a_h^{(ii)}\ge\eta>0$ and $c_h\ge\nu>0$ for some η and ν not depending on $h>0$ and $x\in\Omega$, and

$$\|f_h\|_{L^1(\Omega)}\le M,$$

uniformly in $h>0$. We have denoted $\nabla_h := \nabla_h^-$ and $\operatorname{div}_h := \operatorname{div}_h^+$ (or alternatively $\nabla_h := \nabla_h^+$ and $\operatorname{div}_h := \operatorname{div}_h^-$). Then for all $h>0$,

$$\|u_h\|_{L^q(\Omega)} + \|\nabla_h u_h\|_{L^q(\Omega)} \le C,$$

where $1\le q<d/(d-1)$, for a constant $C>0$ independent of $h>0$.

The proof of this lemma will be a (simplified) finite difference version of the proof of Theorem 2.1 in [4]. But before proving the lemma, we need to introduce some notation.

Notation B.2. For any $r\in(1,\infty)$, we denote by $L^{r,\infty}(\Omega)$ the Marcinkiewicz space with norm defined by

$$\|u\|_{L^{r,\infty}(\Omega)} = \sup_{\lambda>0}\lambda|\{x\in\Omega : |u(x)|\ge\lambda\}|^{1/r}.$$

The Marcinkiewicz spaces are continuously embedded in $L^q(\Omega)$ for any $1\le q<r$, [15]:

$$\|u\|_{L^q(\Omega)} \le C(q,r,|\Omega|)\|u\|_{L^{r,\infty}(\Omega)},\quad q\in[1,r). \tag{B.2}$$

Moreover, we need the trunctation operator S_k defined as follows:

Notation B.3. Let $k > 0$ be a real number. Then we define the truncation operator $S_k : \mathbb{R} \to \mathbb{R}$ by

$$S_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k\frac{s}{|s|}, & \text{if } |s| \ge k. \end{cases}$$

It will be convenient in the proof to use the following tuple notation for the finite difference approximations:

Notation B.4. We denote $\underline{i} := (i_1, \dots, i_d)$, $i_\ell = 1, \dots, N_\ell$, N_ℓ the number of cells in the ℓth spatial direction, a d-dimensional tuple and and $u_{\underline{i}}$ the approximation in cell $\mathcal{C}_{\underline{i}} := ((i_1 - 1)h, i_1 h] \times \cdots \times ((i_d - 1)h, i_d h]$. The piecewise constant function u_h can be written as

$$u_h(x) := \sum_{\underline{i}} u_{\underline{i}} \mathbf{1}_{\mathcal{C}_{\underline{i}}}(x), \quad x \in \Omega.$$

We also need the following auxilary result:

Lemma B.5. *Let u_h solve the difference equation (B.1) under the assumptions of Lemma B.1. Then*

$$\int_\Omega |\nabla_h S_k(u_h)|^2 + |S_k(u_h)|^2 \mathrm{d}x \le CMk, \quad \forall\, k > 0, \tag{B.3}$$

for some constant $C > 0$ independent of $h > 0$.

Proof. Given $k > 0$, we multiply equation (B.1) by $S_k(u_h)$ and integrate over the domain Ω. After changing variables in the integrals, we obtain

$$\int_\Omega (A_h \nabla_h u_h) \cdot \nabla_h S_k(u_h) + c_h u_h S_k(u_h)\,\mathrm{d}x = \int_\Omega f_h S_k(u_h)\,\mathrm{d}x. \tag{B.4}$$

The right hand side can be bounded by Mk using Hölder's inequality. The left hand side, we can rewrite and estimate as follows

$$\begin{aligned}
&\int_\Omega (A_h \nabla_h u_h) \cdot \nabla_h S_k(u_h) + c_h u_h S_k(u_h)\,\mathrm{d}x \\
&\quad= \int_\Omega (A_h \nabla_h S_k(u_h)) \cdot \nabla_h S_k(u_h) + c_h |S_k(u_h)|^2\,\mathrm{d}x \\
&\qquad+ \int_\Omega \left(A_h \left(\nabla_h \left[u_h - S_k(u_h)\right]\right)\right) \cdot \nabla_h S_k(u_h) + c_h \left(u_h - S_k(u_h)\right) S_k(u_h)\,\mathrm{d}x \\
&\quad\ge \eta \|\nabla_h S_k(u_h)\|^2_{L^2(\Omega)} + \nu \|S_k(u_h)\|^2_{L^2(\Omega)} \\
&\qquad+ \int_\Omega \left(A_h \left(\nabla_h \left[u_h - S_k(u_h)\right]\right)\right) \cdot \nabla_h S_k(u_h) + c_h \left(u_h - S_k(u_h)\right) S_k(u_h)\,\mathrm{d}x.
\end{aligned}$$

$(u_h - S_k(u_h))$ is either zero or has the same sign as $S_k(u_h)$. Therefore $(u_h - S_k(u_h))S_k(u_h) \ge 0$ and

$$\int_\Omega c_h \left(u_h - S_k(u_h)\right) S_k(u_h)\,\mathrm{d}x \ge 0.$$

In order to prove that the other term is positive as well, we will show that

$$D_\ell^- S_k(u_{\underline{i}}) D_\ell^- \left(u_{\underline{i}} - S_k(u_{\underline{i}})\right) \ge 0, \quad \forall\, \underline{i},\ \ell = 1, \dots, d.$$

The proof of this fact consists of boring case distinctions and is exactly analoguous for $\ell = 1, 2, (3)$, therefore we will do it only for $\ell = 1$ and omit writing the tuple index $\underline{i}$. Then we have

$$D_1^-(u_i - S_k(u_i))D_1^- S_k(u_i) = \begin{cases} (u_i - k)(k - u_{i-1}), & u_i > k,\, |u_{i-1}| \le k, \\ (u_i + k)(-k - u_{i-1}), & u_i < -k,\, |u_{i-1}| \le k, \\ 0, & |u_i| \le k,\, |u_{i-1}| \le k, \\ (-u_{i-1} + k)(u_i - k), & |u_i| \le k,\, u_{i-1} > k, \\ (-u_{i-1} - k)(u_i + k), & |u_i| \le k,\, u_{i-1} < -k, \\ 0, & u_i > k,\, u_{i-1} > k, \\ 0, & u_i < -k,\, u_{i-1} < -k, \\ (u_i - u_{i-1} - 2k)2k, & u_i > k,\, u_{i-1} < -k, \\ -(u_i - u_{i-1} + 2k)2k, & u_i < -k,\, u_{i-1} > k. \end{cases}$$

The potential reader is welcome to check that these are all the possible cases and that each of the terms on the right hand side is nonnegative. Thus we have that

$$\int_\Omega (A_h \nabla_h u_h) \cdot \nabla_h S_k(u_h) + c_h u_h S_k(u_h)\, \mathrm{d}x \ge \eta \|\nabla_h S_k(u_h)\|^2_{L^2(\Omega)} + \nu \|S_k(u_h)\|^2_{L^2(\Omega)}$$

which implies (B.3) together with the estimate on the right hand side of (B.4) □

Proof of Lemma B.1. First, we note that by the discrete Gagliardo–Nirenberg–Sobolev inequality ([2], Thm. 3.4),

$$\int_\Omega |S_k(u_h)|^{2^*} \mathrm{d}x \le C^{2^*} \left(\int_\Omega |\nabla_h S_k(u_h)|^2 + |S_k(u_h)|^2 \mathrm{d}x \right)^{\frac{2^*}{2}},$$

where $2^* = 2d/(d-2)$ if $d \ge 3$ and any number with $1 \le 2^* < \infty$ if $d = 2$, and where C is a constant depending on $|\Omega|$ but not on $h > 0$. By Lemma B.5, we can bound the right hand side and obtain therefore

$$\int_\Omega |S_k(u_h)|^{2^*} \mathrm{d}x \le C(kM)^{\frac{2^*}{2}}. \tag{B.5}$$

Now we define the set $\mathcal{B}(k)$ by

$$\mathcal{B}(k) = \{\mathcal{C}_{\underline{i}} \subset \Omega : |u_{\underline{i}}| \ge k\}.$$

We have

$$\int_{\mathcal{B}(k)} |S_k(u_h)|^{2^*} \mathrm{d}x \ge k^{2^*} |\mathcal{B}(k)|,$$

and therefore, using (B.5),

$$|\mathcal{B}(k)| \le \frac{1}{k^{2^*}} \int_{\mathcal{B}(k)} |S_k(u_h)|^{2^*} \mathrm{d}x \le \frac{1}{k^{2^*}} \int_\Omega |S_k(u_h)|^{2^*} \mathrm{d}x \le \frac{CM^{\frac{2^*}{2}}}{k^{\frac{2^*}{2}}} \tag{B.6}$$

which implies that $u_h \in L^{r,\infty}(\Omega)$ for $r = 2^*/2$ (which is $d/(d-2)$ if $d \ge 3$) since the choice of $k > 0$ was arbitrary. Now denote

$$\begin{aligned} \partial\mathcal{B}(k) &:= \{\mathcal{C}_{\underline{i}} \subset \Omega : \exists\, \underline{j}, |\underline{i} - \underline{j}| = 1,\, |u_{\underline{j}}| \ge k\} \\ \overline{\mathcal{B}(k)} &:= \mathcal{B}(k) \cup \partial\mathcal{B}(k), \\ \mathcal{B}(k)^c &:= \Omega \backslash \overline{B}(k), \end{aligned}$$

where $|\underline{i}-\underline{j}| = \max_{1\le \ell\le d} |i_\ell - j_\ell|$. Informally speaking, the cells in $\partial\mathcal{B}(k)$ have a neighbor cell which is contained in $\mathcal{B}(k)$. We have

$$|\partial\mathcal{B}(k)| \le (3^d-1)|\mathcal{B}(k)| \le \frac{CM^{\frac{2^*}{2}}}{k^{\frac{2^*}{2}}},$$

by (B.6). Now let $\lambda > 0$, $k > 0$ and decompose

$$\{x\in\Omega : |\nabla_h u_h(x)| \ge \lambda\} = \{x\in\Omega : |\nabla_h u_h(x)| \ge \lambda \,\text{and}\, x\in\overline{\mathcal{B}(k)}\} \cup \{x\in\Omega : |\nabla_h u_h(x)| \ge \lambda \,\text{and}\, x\in\mathcal{B}(k)^c\}.$$

Hence

$$|\{x\in\Omega : |\nabla_h u_h(x)| \ge \lambda\}| \le |\overline{\mathcal{B}(k)}| + |\{x\in\Omega : |\nabla_h u_h(x)| \ge \lambda \,\text{and}\, x\in\mathcal{B}(k)^c\}|.$$

On $\mathcal{B}(k)^c$ and the cells bordering the set, we have $|u_h| \le k$ and therefore $u_h = |S_k(u_h)|$. Hence we can estimate the size of the second set in the above inequality,

$$\begin{aligned}&|\{x\in\Omega : |\nabla_h u_h(x)| \ge \lambda \,\text{and}\, x\in\mathcal{B}(k)^c\}|\\ &\quad= |\{x\in\Omega : |\nabla_h S_k(u_h)(x)| \ge \lambda \,\text{and}\, x\in\mathcal{B}(k)^c\}|\\ &\quad\le |\{x\in\Omega : |\nabla_h S_k(u_h)(x)| \ge \lambda\,\}|\\ &\quad\le \frac{1}{\lambda^2}\int_\Omega |\nabla_h S_k(u_h)|^2 \mathrm{d}x,\end{aligned}$$

where we have used Chebyshev inequality for the last step. Now we can estimate the size of the set $\{x\in\Omega : |\nabla_h u_h(x)| \ge \lambda\}$ using (B.3) once more,

$$|\{x\in\Omega : |\nabla_h u_h(x)| \ge \lambda\}| \le \frac{CM^{\frac{2^*}{2}}}{k^{\frac{2^*}{2}}} + \frac{CkM}{\lambda^2}\cdot$$

Choosing $k = \lambda^{\frac{4}{2^*+2}}$, we obtain

$$\lambda^{\frac{22^*}{2^*+2}}|\{x\in\Omega : |\nabla_h u_h(x)| \ge \lambda\}| \le C(d, M, |\Omega|).$$

If $d \ge 3$, we have $\frac{22^*}{2^*+2} = \frac{\mathrm{d}}{d-1}$ and so $u_h, \nabla_h u_h \in L^{r,\infty}(\Omega)$ for $1 \le r \le d/(d-1)$. For $d = 2$, since 2^* is an arbitrary finite positive number, we can achieve the same. Using the embedding of the Marcinkiewicz spaces, (B.2), we obtain the claim of the lemma. □

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References

[1] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J.L. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **22** (1995) 241–273.

[2] M. Bessemoulin-Chatard, C. Chainais-Hillairet and F. Filbet, On discrete functional inequalities for some finite volume schemes. *IMA J. Numer. Anal.* (2014).

[3] H. Byrne and D. Drasdo, Individual-based and continuum models of growing cell populations: a comparison. *J. Math. Biol.* **58** (2009) 657–687.

[4] J. Casado-Díaz, T. Chacón Rebollo, V. Girault, M. Gómez Marmol and F. Murat, Finite elements approximation of second order linear elliptic equations in divergence form with right-hand side in L^1. *Numer. Math.* **105** (2007) 337–374.

[5] D. Chen and A. Friedman, A two-phase free boundary problem with discontinuous velocity: application to tumor model. *J. Math. Anal. Appl.* **399** (2013) 378–393.

[6] G.M. Coclite, S. Mishra, N.H. Risebro and F. Weber, Analysis and numerical approximation of Brinkman regularization of two-phase flows in porous media. *Comput. Geosci.* **18** (2014) 637–659.

[7] R.J. DiPerna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** (1989) 511–547.

[8] D. Donatelli and K. Trivisa, On a nonlinear model for the evolution of tumor growth with a variable total density of cancerous cells. Preprint (2015).

[9] D. Donatelli and K. Trivisa, On a nonlinear model for the evolution of tumor growth with drug application. *Nonlinearity* **28** (2015) 1463.

[10] D. Donatelli and K. Trivisa, On a nonlinear model for tumor growth: global in time weak solutions. *J. Math. Fluid Mech.* **16** (2014) 787–803.

[11] M. Dreher and A. Jüngel, Compact families of piecewise constant functions in $L^p(0,T;B)$. *Nonlin. Anal.* **75** (2012) 3072–3077.

[12] L.C. Evans, Partial differential equations. Vol. 19 of *Graduate Studies in Mathematics*, 2nd edition. American Mathematical Society, Providence, RI (2010).

[13] R. Eymard, T. Gallouët, R. Herbin and J.C. Latché, A convergent finite element-finite volume scheme for the compressible Stokes problem. II. The isentropic case. *Math. Comp.* **79** (2010) 649–675.

[14] A. Friedman, A hierarchy of cancer models and their mathematical challenges. Mathematical models in cancer (Nashville, TN, 2002). *Discrete Contin. Dyn. Syst. Ser. B* **4** (2004) 147–159.

[15] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Reprint of the 1998 edition. *Classics in Mathematics*. Springer-Verlag, Berlin (2001).

[16] K.H. Karlsen and T.K. Karper, A convergent nonconforming finite element method for compressible Stokes flow. *SIAM J. Numer. Anal.* **48** (2010) 1846–1876.

[17] K.H. Karlsen and T.K. Karper, Convergence of a mixed method for a semi-stationary compressible Stokes system. *Math. Comp.* **80** (2011) 1459–1498.

[18] K.H. Karlsen and T.K. Karper, A convergent mixed method for the Stokes approximation of viscous compressible flow. *IMA J. Numer. Anal.* **32** (2012) 725–764.

[19] T.K. Karper, A convergent FEM-DG method for the compressible Navier–Stokes equations. *Numer. Math.* **125** (2013) 441–510.

[20] O.A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Vol. 2 of *Mathematics and its Applications*. Gordon and Breach, Science Publishers, New York-London-Paris (1969).

[21] O.A. Ladyzhenskaya, The boundary value problems of mathematical physics, Translated from the Russian by Jack Lohwater [Arthur J. Lohwater]. Vol. 49 of *Appl. Math. Sci.* Springer-Verlag, New York (1985).

[22] P.-L. Lions, Mathematical topics in fluid mechanics. 1-Incompressible models. Vol. 3 of *Oxford Lecture Series Math. Appl.* The Clarendon Press, Oxford University Press, New York (1996).

[23] P.-L. Lions, Mathematical topics in fluid mechanics. 2-Compressible models. Vol. 10 of *Oxford Lecture Series Math. Appl.* The Clarendon Press, Oxford University Press, New York (1998).

[24] A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems. Vol. 16 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel (1995).

[25] A. Novotný and I. Straškraba, *Introduction to the mathematical theory of compressible flow*. Vol. 27 of *Oxford Lecture Series Math. Appl.* Oxford University Press, Oxford (2004).

[26] B. Perthame, F. Quirós, M. Tang and N. Vauchelet, Derivation of a Hele–Shaw type system from a cell model with active motion. *Interfaces Free Bound.* **16** (2014) 489–508.

[27] B. Perthame, F. Quirós and J.L. Vázquez, The Hele-Shaw asymptotics for mechanical models of tumor growth. *Arch. Ration. Mech. Anal.* **212** (2014) 93–127.

[28] B. Perthame, M. Tang and N. Vauchelet, Traveling wave solution of the Hele-Shaw model of tumor growth with nutrient. *Math. Models Methods Appl. Sci.* **24** (2014) 2601–2626.

[29] B. Perthame and N. Vauchelet, Incompressible limit of mechanical model of tumor growth with viscosity. *Phil. Trans. R. Sci. A* **373** (2015) 20140283.

[30] J. Ranft, M. Basan, J. Elgeti, J.-F. Joanny, J. Prost and F. Jülicher, Fluidization of tissues by cell division and apoptosis. *Proc. Natl. Acad. Sci.* **107** (2010) 20863–20868.

[31] K. Trivisa and F. Weber, Analysis and Simulation on a Model for the Evolution of Tumors under the Influence of Nutrient and Drug Application. Preprint arXiv:1602.06476 (2016).

[32] J.-H. Zhao, A parabolic-hyperbolic free boundary problem modeling tumor growth with drug application. *Electron. J. Differ. Eq.* **18** (2010).