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ERROR ESTIMATES OF A STABILIZED LAGRANGE−GALERKIN SCHEME FOR THE NAVIER−STOKES EQUATIONS

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Abstract. Error estimates with optimal convergence orders are proved for a stabilized Lagrange−Galerkin scheme for the Navier−Stokes equations. The scheme is a combination of Lagrange−Galerkin method and Brezzi−Pitkäranta's stabilization method. It maintains the advantages of both methods; (i) It is robust for convection-dominated problems and the system of linear equations to be solved is symmetric. (ii) Since the P1 finite element is employed for both velocity and pressure, the number of degrees of freedom is much smaller than that of other typical elements for the equations, *e.g.*, P2/P1. Therefore, the scheme is efficient especially for three-dimensional problems. The theoretical convergence orders are recognized numerically by two- and three-dimensional computations.

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1. Introduction

The purpose of this paper is to prove the stability and convergence of a stabilized Lagrange−Galerkin scheme for the Navier−Stokes equations. The scheme is a combination of a Lagrange−Galerkin (LG) method and Brezzi−Pitkäranta's stabilization method [8]. It has been proposed by us in [17, 18] and, to the best of our knowledge, it is one of the earliest works which combine the two methods, Lagrange−Galerkin and stabilization. Optimal error estimates are shown for both velocity and pressure.

The LG method is a finite element method embracing the method of characteristics. The LG method has common advantages, robustness for convection-dominated problems and symmetry of the resulting matrix, which are desirable in scientific computation of fluid dynamics. Many authors have studied LG schemes for convection-diffusion problems [5, 10, 12, 22, 24] and for the Navier−Stokes, Oseen and natural convection problems [1, 3, 6, 15, 19–21, 27], see also the bibliography therein. The convergence analysis of LG schemes for the Navier−Stokes equations has been done by Pironneau [21] and improved by Süli [27]. The analysis has been extended to a higher-order time scheme by Boukir *et al.* [6] and to the projection method by Achdou and Guermond [1]. While in these analyses they use a stable element satisfying the conventional inf-sup condition [14],

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we extend the convergence analysis to a stabilized LG scheme. The reason to use the stabilized method is to reduce the number of degrees of freedom (DOF). In fact the cheapest P1 element is employed in our scheme for both velocity and pressure, which is based on Brezzi−Pitkäranta's pressure-stabilization method. Hence, the number of DOF is much smaller than that of typical stable elements, *e.g.*, P2/P1. As a result, the scheme leads to a small-size symmetric resulting matrix, which can be solved by powerful linear solvers for symmetric matrices, *e.g.*, minimal residual method (MINRES) [2, 25]. It is, therefore, efficient especially in three-dimensional computation.

In LG schemes the position at the previous time t^{n-1} of a particle is sought along the trajectory, which is governed by a system of ordinary differential equations. The position at t^{n-1} of a particle at a point at t^n is called upwind point of the point or foot of the trajectory arriving at the point. While the system of ordinary differential equations is assumed to be solved exactly in [1, 27], approximate upwind points are computed explicitly without assuming the exact solvability of the ordinary differential equations in [6, 21]. Therefore, we may say that the latter schemes are fully discrete. Our scheme is also fully discrete since the approximate upwind points are simply obtained by the Euler method. In fully discrete schemes, however, it is not obvious that the approximate upwind points remain in the domain, which should be proved. Such difficulty caused by the nonlinearity of the Navier−Stokes's equations is overcome in the proof by mathematical induction, which has been developed in [6, 27]. Thus, the stability and convergence with optimal error estimates are proved for the velocity in the H^1-norm and for the pressure in the L^2-norm (Thm. 3.3) and for the velocity in the L^2-norm (Thm. 3.6) under the condition $\Delta t = O(h^{d/4})$, where d is the dimension of the space. This condition is caused by the nonlinearity of the problem and it is not required for the Oseen's problem [20]. A stabilized LG scheme with an L^2-type pressure-stabilization for the Navier−Stokes's equations has been proposed in [15], where the exact solvability of the ordinary differential equations is assumed for upwind points. The optimal error estimates are proved under a strong stability condition $\Delta t = O(h^2)$ for $d = 2$.

In the LG method we have to deal with the integration of composite functions that originate from the convection term. It is reported in [16, 23, 28, 29] that instability may occur caused by quadrature error if rough numerical quadrature is employed for the integration. Although several methods have been studied to avoid the instability in [4, 16, 22, 23, 32], here we do not discuss the issue because the integration in our scheme can be computed exactly by a method developed recently in [30, 31]. In our numerical examples we still employ numerical quadrature, but with much care, *cf.* Remark 5.2.

This paper is organized as follows. Our stabilized LG scheme for the Navier−Stokes's equations is presented in Section 2. The main results on the stability and convergence with optimal error estimates are shown in Section 3, and they are proved in Section 4. The theoretical convergence orders are recognized numerically by two- and three-dimensional computations in Section 5. The conclusions are given in Section 6. In the Appendix two lemmas used in Section 4 are proved.

2. A stabilized Lagrange−Galerkin scheme

We prepare the function spaces and the notation to be used throughout the paper. Let Ω be a bounded domain in $\mathbb{R}^d (d = 2, 3)$, $\Gamma \equiv \partial\Omega$ the boundary of Ω, and T a positive constant. For an integer $m \geq 0$ and a real number $p \in [1, \infty]$ we use the Sobolev's spaces $W^{m,p}(\Omega)$, $W_0^{1,\infty}(\Omega)$, $H^m(\Omega)(= W^{m,2}(\Omega))$, $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. For any normed space X with norm $\|\cdot\|_X$, we define function spaces $C([0, T]; X)$ and $H^m(0, T; X)$ consisting of X-valued functions in $C([0, T])$ and $H^m(0, T)$, respectively. We use the same notation $(\cdot, \cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions. The dual pairing between X and the dual space X' is denoted by $\langle\cdot, \cdot\rangle$. The norms on $W^{m,p}(\Omega)^d$ and $H^m(\Omega)^d$ are simply denoted as

$$\|\cdot\|_{m,p} \equiv \|\cdot\|_{W^{m,p}(\Omega)^d}, \quad \|\cdot\|_m \equiv \|\cdot\|_{H^m(\Omega)^d} \ (= \|\cdot\|_{m,2})$$

and the notation $\|\cdot\|_m$ is employed not only for vector-valued functions but also for scalar-valued ones. We also denote the norm on $H^{-1}(\Omega)^d$ by $\|\cdot\|_{-1}$. $L_0^2(\Omega)$ is a subspace of $L^2(\Omega)$ defined by

$$L_0^2(\Omega) \equiv \left\{q \in L^2(\Omega);\ (q, 1) = 0\right\}.$$

We often omit $[0,T]$, Ω and/or d if there is no confusion, *e.g.*, we shall write $C(L^\infty)$ in place of $C([0,T];L^\infty(\Omega)^d)$. For t_0 and $t_1 \in \mathbb{R}$ we introduce the function spaces

$$Z^m(t_0,t_1) \equiv \{v \in H^j(t_0,t_1;H^{m-j}(\Omega)^d);\ j=0,\ldots,m,\ \|v\|_{Z^m(t_0,t_1)} < \infty\}$$

and $Z^m \equiv Z^m(0,T)$, where the norm $\|v\|_{Z^m(t_0,t_1)}$ is defined by

$$\|v\|_{Z^m(t_0,t_1)} \equiv \left\{ \sum_{j=0}^m \|v\|^2_{H^j(t_0,t_1;H^{m-j}(\Omega)^d)} \right\}^{1/2}.$$

We consider the Navier–Stokes's problem; find $(u,p): \Omega\times(0,T) \to \mathbb{R}^d \times \mathbb{R}$ such that

$$\frac{Du}{Dt} - \nabla\cdot[2\nu D(u)] + \nabla p = f \qquad \text{in } \Omega\times(0,T), \tag{2.1a}$$

$$\nabla\cdot u = 0 \qquad \text{in } \Omega\times(0,T), \tag{2.1b}$$

$$u = 0 \qquad \text{on } \Gamma\times(0,T), \tag{2.1c}$$

$$u = u^0 \qquad \text{in } \Omega,\ \text{at } t=0, \tag{2.1d}$$

where u is the velocity, p is the pressure, $f:\ \Omega\times(0,T)\to\mathbb{R}^d$ is a given external force, $u^0:\ \Omega\to\mathbb{R}^d$ is a given initial velocity, $\nu>0$ is a viscosity, $D(u)$ is the strain-rate tensor defined by

$$D_{ij}(u) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j=1,\ldots,d,$$

and D/Dt is the material derivative defined by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\cdot\nabla.$$

Letting $V \equiv H^1_0(\Omega)^d$ and $Q \equiv L^2_0(\Omega)$, we define the bilinear forms a on $V\times V$, b on $V\times Q$ and $\mathcal{A}$ on $(V\times Q)\times(V\times Q)$ by

$$a(u,v) \equiv 2\nu\big(D(u),D(v)\big), \quad b(v,q) \equiv -(\nabla\cdot v,q), \quad \mathcal{A}\big((u,p),(v,q)\big) \equiv a(u,v)+b(v,p)+b(u,q),$$

respectively. Then, we can write the weak formulation of (2.1) as follows: find $(u,p):\ (0,T)\to V\times Q$ such that, for $t\in(0,T)$,

$$\left(\frac{Du}{Dt}(t),v\right) + \mathcal{A}\left((u,p)(t),(v,q)\right) = (f(t),v), \quad \forall (v,q)\in V\times Q, \tag{2.2}$$

with $u(0)=u^0$.

Let Δt be a time increment and $t^n \equiv n\Delta t$ for $n\in\mathbb{N}\cup\{0\}$. For a function g defined in $\Omega\times(0,T)$ we denote generally $g(\cdot,t^n)$ by g^n. Let $X:(0,T)\to\mathbb{R}^d$ be a solution of the system of ordinary differential equations,

$$\frac{\mathrm{d}X}{\mathrm{d}t} = u(X,t). \tag{2.3}$$

Then, it holds that

$$\frac{Du}{Dt}(X(t),t) = \frac{\mathrm{d}}{\mathrm{d}t}u\big(X(t),t\big),$$

when u is smooth. Let $X(\cdot; x, t^n)$ be the solution of (2.3) subject to an initial condition $X(t^n) = x$. For a velocity $w : \Omega \to \mathbb{R}^d$ let $X_1(w, \Delta t) : \Omega \to \mathbb{R}^d$ be a mapping defined by

$$X_1(w, \Delta t)(x) \equiv x - w(x)\Delta t. \tag{2.4}$$

Since the position $X_1(u^{n-1}, \Delta t)(x)$ is an approximation of $X(t^{n-1}; x, t^n)$ for $n \geq 1$, we can consider a first order approximation of the material derivative at (x, t^n),

$$\frac{Du}{Dt}(x, t^n) = \frac{\mathrm{d}}{\mathrm{d}t} u\big(X(t; x, t^n), t\big)\Big|_{t=t^n} = \frac{u^n - u^{n-1} \circ X_1(u^{n-1}, \Delta t)}{\Delta t}(x) + O(\Delta t),$$

where the symbol $\circ$ stands for the composition of functions,

$$(v \circ w)(x) \equiv v\big(w(x)\big),$$

for $v : \Omega \to \mathbb{R}^d$ and $w : \Omega \to \Omega$. $X_1(w, \Delta t)(x)$ is called an upwind point of x with respect to the velocity w. The next proposition gives a sufficient condition to guarantee that all upwind points are in Ω.

Proposition 2.1 [24]**.** *Let $w \in W_0^{1,\infty}(\Omega)^d$ be a given function, and assume that*

$$\Delta t \|w\|_{1,\infty} < 1.$$

Then, it holds that

$$X_1(w, \Delta t)(\Omega) = \Omega.$$

For the sake of simplicity we assume that Ω is a polygonal ($d = 2$) or polyhedral ($d = 3$) domain. Let $\mathcal{T}_h = \{K\}$ be a triangulation of $\bar{\Omega}$ $(= \bigcup_{K \in \mathcal{T}_h} K)$, h_K a diameter of $K \in \mathcal{T}_h$, and $h \equiv \max_{K \in \mathcal{T}_h} h_K$ the maximum element size. Throughout this paper we consider a regular family of triangulations $\{\mathcal{T}_h\}_{h \downarrow 0}$ satisfying the inverse assumption [9], *i.e.*, there exists a positive constant α_0 independent of h such that

$$\frac{h}{h_K} \leq \alpha_0, \quad \forall K \in \mathcal{T}_h, \quad \forall h. \tag{2.5}$$

We define the function spaces X_h, M_h, V_h and Q_h by

$$X_h \equiv \{v_h \in C(\bar{\Omega})^d;\ v_{h|K} \in P_1(K)^d,\ \forall K \in \mathcal{T}_h\}, \quad M_h \equiv \{q_h \in C(\bar{\Omega});\ q_{h|K} \in P_1(K),\ \forall K \in \mathcal{T}_h\},$$

$V_h \equiv X_h \cap V$ and $Q_h \equiv M_h \cap Q$, respectively, where $P_1(K)$ is the space of linear functions on $K \in \mathcal{T}_h$. Let $N_T \equiv \lfloor T/\Delta t \rfloor$ be the total number of time steps, δ_0 a positive constant and $(\cdot, \cdot)_K$ the $L^2(K)^d$ inner product. We define the bilinear forms $\mathcal{C}_h$ on $H^1(\Omega) \times H^1(\Omega)$ and $\mathcal{A}_h$ on $(V \times H^1(\Omega)) \times (V \times H^1(\Omega))$ by

$$\mathcal{C}_h(p, q) \equiv \delta_0 \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K,$$
$$\mathcal{A}_h\big((u, p), (v, q)\big) \equiv a(u, v) + b(v, p) + b(u, q) - \mathcal{C}_h(p, q). \tag{2.6}$$

The bilinear form $\mathcal{C}_h$ has been originally introduced in [8] in order to stabilize the pressure.

Suppose $f \in C([0, T]; L^2(\Omega)^d)$ and $u^0 \in V$. Let an approximate function $u_h^0 \in V_h$ of u^0 be given. Our stabilized LG scheme for (2.1) is to find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that, for $n = 1, \ldots, N_T$,

$$\left(\frac{u_h^n - u_h^{n-1} \circ X_1(u_h^{n-1}, \Delta t)}{\Delta t}, v_h\right) + \mathcal{A}_h\big((u_h^n, p_h^n), (v_h, q_h)\big) = (f^n, v_h), \quad \forall (v_h, q_h) \in V_h \times Q_h. \tag{2.7}$$

Remark 2.2.

(i) By expanding u_h^n and p_h^n in terms of a basis of V_h and Q_h, the scheme (2.7) leads to a symmetric matrix of the form

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix},$$

where A, B and C are sub-matrices derived from $\frac{1}{\Delta t}(u_h^n, v_h) + a(u_h^n, v_h)$, $b(u_h^n, q_h)$ and $\mathcal{C}_h(p_h^n, q_h)$, respectively, and the superscript "T" stands for the transposition.

(ii) The matrix is independent of the time step n and is invertible. The invertibility is derived from the fact that $(u_h^n, p_h^n) = (0, 0)$ when $u_h^{n-1} = f^n = 0$ since we have

$$\frac{1}{\Delta t}\|u_h^n\|_0^2 + 2\nu\|D(u_h^n)\|_0^2 + \delta_0 \sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla p_h^n\|_{L^2(K)^d}^2 = 0$$

by substituting $(u_h^n, -p_h^n) \in V_h \times Q_h$ into (v_h, q_h) in (2.7).

(iii) There exists a unique solution (u_h^n, p_h^n) if $X_1(u_h^{n-1}, \Delta t)$ maps Ω into Ω. The condition is ensured if $\Delta t\|u_h^{n-1}\|_{1,\infty} < 1$, *cf.* Proposition 2.1.

3. Main results

In this section we state the main results, conditional stability and optimal error estimates for the scheme (2.7), which are proved in Section 4.

We use the following norms and a seminorm, $\|\cdot\|_{V_h} \equiv \|\cdot\|_V \equiv \|\cdot\|_1$, $\|\cdot\|_{Q_h} \equiv \|\cdot\|_Q \equiv \|\cdot\|_0$,

$$\|u\|_{l^\infty(X)} \equiv \max_{n=0,\ldots,N_T} \|u^n\|_X, \quad \|u\|_{l^2_m(X)} \equiv \left\{ \Delta t \sum_{n=1}^{m} \|u^n\|_X^2 \right\}^{1/2}, \quad \|u\|_{l^2(X)} \equiv \|u\|_{l^2_{N_T}(X)},$$

$$|p|_h \equiv \left\{ \sum_{K\in\mathcal{T}_h} h_K^2 (\nabla p, \nabla p)_K \right\}^{1/2},$$

for $m \in \{1, \ldots, N_T\}$ and $X = L^\infty(\Omega)$, $L^2(\Omega)$ and $H^1(\Omega)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by

$$\overline{D}_{\Delta t} u^n \equiv \frac{u^n - u^{n-1}}{\Delta t}\cdot$$

Definition 3.1. For $(w, r) \in V \times Q$ we define the Stokes projection $(\hat{w}_h, \hat{r}_h) \in V_h \times Q_h$ of (w, r) by

$$\mathcal{A}_h\big((\hat{w}_h, \hat{r}_h), (v_h, q_h)\big) = \mathcal{A}\big((w, r), (v_h, q_h)\big), \quad \forall (v_h, q_h) \in V_h \times Q_h. \tag{3.1}$$

Hypothesis 3.2. *The solution (u, p) of (2.2) satisfies $u \in C([0,T]; W^{1,\infty}(\Omega)^d) \cap Z^2 \cap H^1(0,T; V \cap H^2(\Omega)^d)$ and $p \in H^1(0,T; Q \cap H^1(\Omega))$.*

Theorem 3.3. *Suppose Hypothesis 3.2 holds. Then, there exist positive constants h_0 and c_0 independent of h and Δt such that, for any pair $(h, \Delta t)$,*

$$h \in (0, h_0], \quad \Delta t \le c_0 h^{d/4}, \tag{3.2}$$

the following hold.

(i) *Scheme* (2.7) *with* u_h^0*, the first component of the Stokes's projection of* $(u^0, 0)$ *by* (3.1)*, has a unique solution* $(u_h, p_h) = \{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$.

(ii) *It holds that*

$$\|u_h\|_{l^\infty(L^\infty)} \leq \|u\|_{C(L^\infty)} + 1. \tag{3.3}$$

(iii) *There exists a positive constant* $\bar{c}$ *independent of* h *and* Δt *such that*

$$\|u_h - u\|_{l^\infty(H^1)}, \quad \left\|\overline{D}_{\Delta t} u_h - \frac{\partial u}{\partial t}\right\|_{l^2(L^2)}, \quad \|p_h - p\|_{l^2(L^2)} \leq \bar{c}(\Delta t + h). \tag{3.4}$$

Remark 3.4. Since the initial pressure p^0 is not given in (2.1), we cannot practice the Stokes's projection of (u^0, p^0). That is the reason why we employ the Stokes projection of $(u^0, 0)$ and set the first component as u_h^0. This choice is sufficient for the error estimates (3.4) and also (3.5) in Theorem 3.6 below.

Hypothesis 3.5. *The Stokes's problem is regular,* i.e.*, for any* $g \in L^2(\Omega)^d$ *the solution* $(w, r) \in V \times Q$ *of the Stokes problem,*

$$\mathcal{A}\big((w, r), (v, q)\big) = (g, v), \quad \forall (v, q) \in V \times Q,$$

belongs to $H^2(\Omega)^d \times H^1(\Omega)$ *and the estimate*

$$\|w\|_2 + \|r\|_1 \leq c_R \|g\|_0$$

holds, where c_R *is a positive constant independent of* g*,* w *and* r.

Theorem 3.6. *Suppose Hypotheses* 3.2 *and* 3.5 *hold. Then, there exists a positive constant* $\tilde{c}$ *independent of* h *and* Δt *such that*

$$\|u_h - u\|_{l^\infty(L^2)} \leq \tilde{c}(\Delta t + h^2), \tag{3.5}$$

where u_h *is the first component of the solution of* (2.7) *stated in Theorem* 3.3(i).

Remark 3.7. Hypothesis 3.5 holds, *e.g.*, if Ω is convex in $\mathbb{R}^2$, *cf.* [14].

4. Proofs of Theorems 3.3 and 3.6

We use c, c_u and $c_{(u,p)}$ to represent the generic positive constants independent of the discretization parameters h and Δt. c_u and $c_{(u,p)}$ are constants depending on u and (u, p), respectively. The symbol "′" (prime) is sometimes used in order to distinguish between two constants, *e.g.*, c_u and c_u'.

4.1. Preparations

We recall some lemmas and a proposition, which are directly used in our proofs. The next lemma is derived from Korn's inequality [11].

Lemma 4.1. *Let* Ω *be a bounded domain with a Lipschitz-continuous boundary. Then, there exists a positive constant* α_1 *and the following inequalities hold.*

$$\|D(v)\|_0 \leq \|v\|_1 \leq \alpha_1 \|D(v)\|_0, \qquad \forall v \in H_0^1(\Omega)^d. \tag{4.1}$$

We use inverse inequalities and interpolation properties.

Lemma 4.2 [9]**.** *There exist positive constants α_{2i}, $i = 0, \ldots, 4$, independent of h and the following inequalities hold.*

$$|q_h|_h \le \alpha_{20}\|q_h\|_0, \qquad \forall q_h \in Q_h, \tag{4.2a}$$
$$\|v_h\|_{0,\infty} \le \alpha_{21}h^{-d/6}\|v_h\|_1, \qquad \forall v_h \in V_h, \tag{4.2b}$$
$$\|v_h\|_{1,\infty} \le \alpha_{22}h^{-d/2}\|v_h\|_1, \qquad \forall v_h \in V_h, \tag{4.2c}$$
$$\|\Pi_h v\|_{0,\infty} \le \|v\|_{0,\infty}, \qquad \forall v \in C(\bar{\Omega})^d, \tag{4.2d}$$
$$\|\Pi_h v\|_{1,\infty} \le \alpha_{23}\|v\|_{1,\infty}, \qquad \forall v \in W^{1,\infty}(\Omega)^d, \tag{4.2e}$$
$$\|\Pi_h v - v\|_1 \le \alpha_{24}h\|v\|_2, \qquad \forall v \in H^2(\Omega)^d, \tag{4.2f}$$

where $\Pi_h : C(\bar{\Omega})^d \to X_h$ is the Lagrange interpolation operator.

Remark 4.3.

(i) Although the inverse assumption (2.5) is supposed throughout the paper, it is not required for the estimates (4.2a), (4.2d), (4.2e) and (4.2f). The assumption that $\{\mathcal{T}_h\}_{h\downarrow 0}$ is regular is sufficient for them.
(ii) The inverse inequality (4.2b) is sufficient in this paper, while it is not optimal for $d = 2$.
(iii) We note $\alpha_{23} \ge 1$.

Lemma 4.4 [13]**.** *There exists a positive constant α_{30} independent of h such that for any h*

$$\inf_{(w_h,r_h)\in V_h\times Q_h}\ \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\mathcal{A}_h\big((w_h,r_h),(v_h,q_h)\big)}{\|(w_h,r_h)\|_{V\times Q}\|(v_h,q_h)\|_{V\times Q}} \ge \alpha_{30}. \tag{4.3}$$

Remark 4.5. Although the conventional inf-sup condition [14],

$$\inf_{q_h\in Q_h}\ \sup_{v_h\in V_h} \frac{b(v_h,q_h)}{\|v_h\|_1\|q_h\|_0} \ge \beta^* > 0,$$

does not hold true for the pair of V_h and Q_h, the P1/P1 finite element spaces, $\mathcal{A}_h$ satisfies the stability inequality (4.3) for this pair.

Proposition 4.6 [7]**.**

(i) *Suppose $(w,r) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$. Then, there exists a positive constant α_{31} independent of h such that for any h the Stokes projection $(\hat{w}_h, \hat{r}_h)$ of (w,r) by (3.1) satisfies*

$$\|\hat{w}_h - w\|_1, \quad \|\hat{r}_h - r\|_0, \quad |\hat{r}_h - r|_h \le \alpha_{31}h\|(w,r)\|_{H^2\times H^1}. \tag{4.4a}$$

(ii) *Suppose Hypothesis 3.5 additionally holds. Then, there exists a positive constant α_{32} independent of h such that for any h*

$$\|\hat{w}_h - w\|_0 \le \alpha_{32}h^2\|(w,r)\|_{H^2\times H^1}. \tag{4.4b}$$

We recall some results concerning the evaluation of composite functions, which are mainly due to Lemma 4.5 in [1] and Lemma 1 in [10]. In the next lemma a and b are any functions in $W_0^{1,\infty}(\Omega)^d$ satisfying

$$\Delta t\|a\|_{1,\infty},\ \Delta t\|b\|_{1,\infty} \le \delta_1,$$

where δ_1 is a constant stated in (i) of the following lemma. We consider the mappings $X_1(a,\Delta t)$ and $X_1(b,\Delta t)$ defined in (2.4).

Lemma 4.7.

(i) *There exists a constant* $\delta_1 \in (0,1)$ *such that*

$$J(x) \geq 1/2, \quad \forall x \in \Omega, \tag{4.5}$$

where J *is the Jacobian* $\det(\partial X_1(a,\Delta t)/\partial x)$.

(ii) *There exist positive constants* α_{4i}, $i = 0, \ldots, 3$, *independent of* Δt *such that the following inequalities hold.*

$$\|g - g \circ X_1(a,\Delta t)\|_0 \leq \alpha_{40} \Delta t \|a\|_{0,\infty} \|g\|_1, \qquad \forall g \in H^1(\Omega)^d, \tag{4.6a}$$

$$\|g - g \circ X_1(a,\Delta t)\|_{-1} \leq \alpha_{41} \Delta t \|a\|_{1,\infty} \|g\|_0, \qquad \forall g \in L^2(\Omega)^d, \tag{4.6b}$$

$$\|g \circ X_1(b,\Delta t) - g \circ X_1(a,\Delta t)\|_0 \leq \alpha_{42} \Delta t \|b - a\|_0 \|g\|_{1,\infty}, \qquad \forall g \in W^{1,\infty}(\Omega)^d, \tag{4.6c}$$

$$\|g \circ X_1(b,\Delta t) - g \circ X_1(a,\Delta t)\|_{0,1} \leq \alpha_{43} \Delta t \|b - a\|_0 \|g\|_1, \qquad \forall g \in H^1(\Omega)^d. \tag{4.6d}$$

Proof. Since $J_{ij} = \delta_{ij} - \Delta t \partial a_i / \partial x_j$, (4.5) is obvious. It holds that for any $q \in [1,\infty)$, $p \in [1,\infty]$, p' with $1/p + 1/p' = 1$ and $g \in W^{1,qp'}(\Omega)^d$

$$\|g \circ X_1(b,\Delta t) - g \circ X_1(a,\Delta t)\|_{0,q} \leq 2 \|X_1(b,\Delta t) - X_1(a,\Delta t)\|_{0,pq} \|\nabla g\|_{0,qp'}$$

from Lemma 4.5 in [1], which implies (4.6a), (4.6c) and (4.6d). For the proof of (4.6b), refer to Lemma 1 in [10]. □

4.2. An estimate at each time step

Let $(\hat{u}_h, \hat{p}_h)(t) \in V_h \times Q_h$ be the Stokes's projection of $(u,p)(t)$ by (3.1) for $t \in [0,T]$. Letting

$$e_h^n \equiv u_h^n - \hat{u}_h^n, \quad \varepsilon_h^n \equiv p_h^n - \hat{p}_h^n, \quad \eta(t) \equiv (u - \hat{u}_h)(t),$$

we have for $n \geq 1$

$$(\overline{D}_{\Delta t} e_h^n, v_h) + \mathcal{A}_h\big((e_h^n, \varepsilon_h^n), (v_h, q_h)\big) = \langle R_h^n, v_h \rangle, \quad \forall (v_h, q_h) \in V_h \times Q_h, \tag{4.7}$$

where

$$R_h^n \equiv \sum_{i=1}^{4} R_{hi}^n,$$

$$R_{h1}^n \equiv \frac{Du^n}{Dt} - \frac{u^n - u^{n-1} \circ X_1(u^{n-1}, \Delta t)}{\Delta t}, \qquad R_{h2}^n \equiv \frac{1}{\Delta t}\Big\{u^{n-1} \circ X_1(u_h^{n-1}, \Delta t) - u^{n-1} \circ X_1(u^{n-1}, \Delta t)\Big\},$$

$$R_{h3}^n \equiv \frac{1}{\Delta t}\Big\{\eta^n - \eta^{n-1} \circ X_1(u_h^{n-1}, \Delta t)\Big\}, \qquad R_{h4}^n \equiv -\frac{1}{\Delta t}\Big\{e_h^{n-1} - e_h^{n-1} \circ X_1(u_h^{n-1}, \Delta t)\Big\}.$$

(4.7) is derived from (2.7), (3.1) and (2.2). We note $e_h^0 = u_h^0 - \hat{u}_h^0$ and set $\varepsilon_h^0 \equiv p_h^0 - \hat{p}_h^0$, where (u_h^0, p_h^0) is the Stokes projection of $(u^0, 0)$ by (3.1).

Hereafter, let δ_1 be the constant in Lemma 4.7.

Proposition 4.8.

(i) *Let* $(u^0, p^0) \in (H^2(\Omega)^d \cap V) \times (H^1(\Omega) \cap Q)$ *be given and assume that* $\nabla \cdot u^0 = 0$. *Then, there exists a positive constant* c_I *independent of* h *such that for any* h

$$\sqrt{\nu} \|D(e_h^0)\|_0 + \sqrt{\frac{\delta_0}{2}} |\varepsilon_h^0|_h \leq c_I h. \tag{4.8}$$

(ii) *Let $n \in \{1, \ldots, N_T\}$ be a fixed number and let $u_h^{n-1} \in V_h$ be known. Suppose the inequality*

$$\Delta t \|u_h^{n-1}\|_{1,\infty} \le \delta_1 \tag{4.9}$$

holds. Then, there exists a unique solution $(u_h^n, p_h^n) \in V_h \times Q_h$ of (2.7).

(iii) *Furthermore, suppose Hypothesis 3.2 and the inequality*

$$\Delta t \|u\|_{C(W^{1,\infty})} \le \delta_1 \tag{4.10}$$

hold. Let $p_h^{n-1} \in Q_h$ be known and suppose the equation

$$b(u_h^{n-1}, q_h) - \mathcal{C}_h(p_h^{n-1}, q_h) = 0, \quad \forall q_h \in Q_h, \tag{4.11}$$

holds. Then, it holds that

$$\begin{aligned} &\overline{D}_{\Delta t}\left(\nu\|D(e_h^n)\|_0^2 + \frac{\delta_0}{2}|\varepsilon_h^n|_h^2\right) + \frac{1}{2}\|\overline{D}_{\Delta t} e_h^n\|_0^2 \le A_1(\|u_h^{n-1}\|_{0,\infty})\nu\|D(e_h^{n-1})\|_0^2 \\ &\quad + A_2(\|u_h^{n-1}\|_{0,\infty})\left\{\Delta t\|u\|_{Z^2(t^{n-1},t^n)}^2 + h^2\left(\frac{1}{\Delta t}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}^2 + 1\right)\right\}, \end{aligned} \tag{4.12}$$

where A_i, $i = 1, 2$, are functions defined by

$$A_i(\xi) \equiv c_i(\xi^2 + 1)$$

and c_i, $i = 1, 2$, are positive constants independent of h and Δt. They are defined by (4.19) below.

For the proof we use the next lemma, which is proved in Appendix A.1.

Lemma 4.9. *Suppose Hypothesis 3.2 holds. Let $n \in \{1, \ldots, N_T\}$ be a fixed number and let $u_h^{n-1} \in V_h$ be known. Then, under the conditions (4.9) and (4.10) it holds that*

$$\|R_{h1}^n\|_0 \le c_u \sqrt{\Delta t}\|u\|_{Z^2(t^{n-1},t^n)}, \tag{4.13a}$$

$$\|R_{h2}^n\|_0 \le c_u\big(\|e_h^{n-1}\|_0 + h\|(u,p)^{n-1}\|_{H^2\times H^1}\big), \tag{4.13b}$$

$$\|R_{h3}^n\|_0 \le \frac{ch}{\sqrt{\Delta t}}(\|u_h^{n-1}\|_{0,\infty} + 1)\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}, \tag{4.13c}$$

$$\|R_{h4}^n\|_0 \le c\|u_h^{n-1}\|_{0,\infty}\|e_h^{n-1}\|_1. \tag{4.13d}$$

Proof of Proposition 4.8. We prove (i). Since (u_h^0, p_h^0) and $(\hat{u}_h^0, \hat{p}_h^0)$ are the Stokes's projections of $(u^0, 0)$ and (u^0, p^0) by (3.1), respectively, we have

$$\begin{aligned} \|D(e_h^0)\|_0 &\le \|e_h^0\|_1 = \|u_h^0 - \hat{u}_h^0\|_1 \le \|u_h^0 - u^0\|_1 + \|u^0 - \hat{u}_h^0\|_1 \le 2\alpha_{31} h\|(u^0,p^0)\|_{H^2\times H^1}, \\ |\varepsilon_h^0|_h &= |p_h^0 - \hat{p}_h^0|_h \le |p_h^0 - 0|_h + |\hat{p}_h^0 - p^0|_h + |p^0|_h \le \alpha_{20}\big(\|p_h^0 - 0\|_0 + \|\hat{p}_h^0 - p^0\|_0\big) + h\|p^0\|_1 \\ &\le (2\alpha_{20}\alpha_{31} + 1)h\|(u^0,p^0)\|_{H^2\times H^1}, \end{aligned}$$

which imply (4.8) for $c_I \equiv \{2\sqrt{\nu}\alpha_{31} + \sqrt{\delta_0/2}(2\alpha_{20}\alpha_{31} + 1)\}\|(u^0,p^0)\|_{H^2\times H^1}$.

(ii) is obtained from (4.9) and Remark 2.2-(iii).

We prove (iii). Substituting $(\overline{D}_{\Delta t} e_h^n, 0)$ into (v_h, q_h) in (4.7) and using the inequality $(a^2 - b^2)/2 \le a(a-b)$, we have

$$\|\overline{D}_{\Delta t} e_h^n\|_0^2 + \overline{D}_{\Delta t}\big(\nu\|D(e_h^n)\|_0^2\big) + b(\overline{D}_{\Delta t} e_h^n, \varepsilon_h^n) \le \sum_{i=1}^4 \langle R_{hi}^n, \overline{D}_{\Delta t} e_h^n\rangle, \tag{4.14}$$

where we have noted that $X_1(u^{n-1}, \Delta t)$ in R^n_{hi} $(i = 1, 2)$ maps Ω onto Ω by (4.10). From (4.11) and (2.7) with $v_h = 0 \in V_h$ we have that

$$b(u^k_h, q_h) - \mathcal{C}_h(p^k_h, q_h) = 0, \quad \forall q_h \in Q_h, \tag{4.15}$$

for $k = n-1$ and n. Since $(\hat{u}^n_h, \hat{p}^n_h)$ is the Stokes's projection of (u^n, p^n) by (3.1), we have

$$b(\hat{u}^k_h, q_h) - \mathcal{C}_h(\hat{p}^k_h, q_h) = b(u^k, q_h) = 0, \quad \forall q_h \in Q_h, \tag{4.16}$$

for $k = n-1$ and n. The equalities (4.15) and (4.16) imply that

$$b(\overline{D}_{\Delta t} e^n_h, q_h) - \mathcal{C}_h(\overline{D}_{\Delta t} \varepsilon^n_h, q_h) = 0, \quad \forall q_h \in Q_h,$$

which leads to

$$-b(\overline{D}_{\Delta t} e^n_h, \varepsilon^n_h) + \mathcal{C}_h(\overline{D}_{\Delta t} \varepsilon^n_h, \varepsilon^n_h) = 0 \tag{4.17}$$

by putting $q_h = -\varepsilon^n_h \in Q_h$. Adding (4.17) to (4.14) and using Lemma 4.9 and the inequality $ab \le \beta a^2/2 + b^2/(2\beta)$ $(\beta > 0)$, we have

$$\begin{aligned}
&\|\overline{D}_{\Delta t} e^n_h\|_0^2 + \overline{D}_{\Delta t}\Big(\nu \|D(e^n_h)\|_0^2 + \frac{\delta_0}{2}|\varepsilon^n_h|_h^2\Big) \le \sum_{i=1}^4 \langle R^n_{hi}, \overline{D}_{\Delta t} e^n_h\rangle \\
&\quad \le \left(\sum_{i=1}^4 \beta_i\right) \|\overline{D}_{\Delta t} e^n_h\|_0^2 + \frac{c_u \alpha_1^2}{\nu}\Big(\frac{1}{\beta_2} + \frac{\|u^{n-1}_h\|_{0,\infty}^2}{\beta_4}\Big)\nu \|D(e^{n-1}_h)\|_0^2 \\
&\qquad + c'_u \Big\{ \frac{\Delta t}{\beta_1} \|u\|^2_{Z^2(t^{n-1},t^n)} + h^2 \Big(\frac{1}{\beta_2} \|(u,p)\|^2_{C(H^2\times H^1)} + \frac{\|u^{n-1}_h\|_{0,\infty}^2 + 1}{\beta_3 \Delta t} \|(u,p)\|^2_{H^1(t^{n-1},t^n;H^2\times H^1)} \Big) \Big\}
\end{aligned} \tag{4.18}$$

for any positive numbers β_i $(i = 1, \ldots, 4)$, where the inequality $\|e^{n-1}_h\|_0 \le \|e^{n-1}_h\|_1$ has been used. By setting $\beta_i = 1/8$ for $i = 1, \ldots, 4$ in (4.18) we have that

$$\begin{aligned}
&\overline{D}_{\Delta t}\left(\nu \|D(e^n_h)\|_0^2 + \frac{\delta_0}{2}|\varepsilon^n_h|_h^2\right) + \frac{1}{2}\|\overline{D}_{\Delta t} e^n_h\|_0^2 \le \frac{c_u}{\nu}\left(\|u^{n-1}_h\|_{0,\infty}^2 + 1\right)\nu \|D(e^{n-1}_h)\|_0^2 \\
&\qquad + c_{(u,p)} \left\{ \Delta t \|u\|^2_{Z^2(t^{n-1},t^n)} + h^2\left(\|u^{n-1}_h\|_{0,\infty}^2 + 1\right)\left(\frac{1}{\Delta t}\|(u,p)\|^2_{H^1(t^{n-1},t^n;H^2\times H^1)} + 1\right)\right\}.
\end{aligned}$$

Putting

$$c_1 \equiv c_u/\nu, \qquad c_2 \equiv c_{(u,p)}, \tag{4.19}$$

we obtain (4.12). □

4.3. Proof of Theorem 3.3

The proof is performed by induction through three steps.

Step 1. (Setting c_0 and h_0): Let c_I and A_i $(i = 1, 2)$ be the constant and the functions in Proposition 4.8, respectively. Let a_1, a_2 and c_* be constants defined by

$$\begin{aligned}
&a_1 \equiv A_1(\|u\|_{C(L^\infty)} + 1), \quad a_2 \equiv A_2(\|u\|_{C(L^\infty)} + 1), \\
&c_* \equiv \frac{\alpha_1}{\sqrt{\nu}} \exp(a_1 T/2) \max\Big\{ a_2^{1/2} \|u\|_{Z^2}, a_2^{1/2}\big(\|(u,p)\|_{H^1(H^2\times H^1)} + T^{1/2}\big) + c_I \Big\}.
\end{aligned}$$

We can choose sufficiently small positive constants c_0 and h_0 such that

$$\alpha_{21}\Big\{c_*(c_0 h_0^{d/12}+h_0^{1-d/6})+(\alpha_{24}+\alpha_{31})h_0^{1-d/6}\|(u,p)\|_{C(H^2\times H^1)}\Big\}\le 1, \tag{4.20a}$$

$$c_0\Big[\alpha_{22}\Big\{c_*(c_0+h_0^{1-d/4})+(\alpha_{24}+\alpha_{31})h_0^{1-d/4}\|(u,p)\|_{C(H^2\times H^1)}\Big\}+\alpha_{23}h_0^{d/4}\|u\|_{C(W^{1,\infty})}\Big]\le \delta_1, \tag{4.20b}$$

since all the powers of h_0 are positive.

Step 2. (Induction): For $n\in\{0,\ldots,N_T\}$ we define property $\mathrm{P}(n)$ as follows:

$$\mathrm{P}(n):\begin{cases}\text{(a) } \nu\|D(e_h^n)\|_0^2+\dfrac{\delta_0}{2}|\varepsilon_h^n|_h^2+\dfrac{1}{2}\|\overline{D}_{\Delta t}e_h\|^2_{l^2_n(L^2)}\\ \qquad\le \exp(a_1 n\Delta t)\Big[\nu\|D(e_h^0)\|_0^2+\dfrac{\delta_0}{2}|\varepsilon_h^0|_h^2+a_2\Big\{\Delta t^2\|u\|^2_{Z^2(0,t^n)}+h^2\big(\|(u,p)\|^2_{H^1(0,t^n;H^2\times H^1)}+n\Delta t\big)\Big\}\Big],\\ \text{(b) } \|u_h^n\|_{0,\infty}\le\|u\|_{C(L^\infty)}+1,\\ \text{(c) } \Delta t\|u_h^n\|_{1,\infty}\le\delta_1,\end{cases}$$

where $\|\overline{D}_{\Delta t}e_h\|_{l^2_n(L^2)}$ vanishes for $n=0$. $\mathrm{P}(n)$-(a) can be rewritten as

$$x_n+\Delta t\sum_{i=1}^{n}y_i\le\exp(a_1 n\Delta t)\left(x_0+\Delta t\sum_{i=1}^{n}b_i\right), \tag{4.21}$$

where

$$x_n\equiv\nu\|D(e_h^n)\|_0^2+\frac{\delta_0}{2}|\varepsilon_h^n|_h^2,\qquad y_i\equiv\frac12\|\overline{D}_{\Delta t}e_h^i\|_0^2,$$
$$b_i\equiv a_2\left\{\Delta t\|u\|^2_{Z^2(t^{i-1},t^i)}+h^2\left(\frac{1}{\Delta t}\|(u,p)\|^2_{H^1(t^{i-1},t^i;H^2\times H^1)}+1\right)\right\}.$$

We firstly prove the general step in the induction. Supposing that $\mathrm{P}(n-1)$ holds true for an integer $n\in\{1,\ldots,N_T\}$, we prove that $\mathrm{P}(n)$ also holds. Since $\mathrm{P}(n-1)$-(c) is nothing but (4.9), there exists a unique solution $(u_h^n,p_h^n)\in V_h\times Q_h$ of equation (2.7) from Proposition 4.8(ii). We prove $\mathrm{P}(n)$-(a). (4.10) holds thanks to the estimate,

$$\Delta t\|u\|_{C(W^{1,\infty})}\le c_0 h_0^{d/4}\|u\|_{C(W^{1,\infty})}\le c_0\alpha_{23}h_0^{d/4}\|u\|_{C(W^{1,\infty})}\le\delta_1,$$

from condition (3.2), Remark 4.3(iii) and (4.20b). (4.11) is obtained from (2.7) for $n\ge2$ and from the definition of (u_h^0,p_h^0), *i.e.*, the Stokes's projection of $(u^0,0)$ by (3.1), for $n=1$. Hence (4.12) holds from Proposition 4.8(iii). Since the inequalities $A_i(\|u_h^{n-1}\|_{0,\infty})\le a_i$ $(i=1,2)$ hold from $\mathrm{P}(n-1)$-(b), (4.12) implies

$$\overline{D}_{\Delta t}x_n+y_n\le a_1x_{n-1}+b_n,$$

which leads to

$$x_n+\Delta t y_n\le\exp(a_1\Delta t)(x_{n-1}+\Delta t b_n) \tag{4.22}$$

by $1\le 1+a_1\Delta t\le\exp(a_1\Delta t)$. From $\mathrm{P}(n-1)$-(a), *i.e.*,

$$x_{n-1}+\Delta t\sum_{i=1}^{n-1}y_i\le\exp\{a_1(n-1)\Delta t\}\left(x_0+\Delta t\sum_{i=1}^{n-1}b_i\right), \tag{4.23}$$

we have that

$$
\begin{aligned}
x_n + \Delta t \sum_{i=1}^{n} y_i &\le \exp(a_1 \Delta t)(x_{n-1} + \Delta t b_n) + \Delta t \sum_{i=1}^{n-1} y_i \quad (\text{by } (4.22))\\
&\le \exp(a_1 \Delta t)\left(x_{n-1} + \Delta t \sum_{i=1}^{n-1} y_i + \Delta t b_n \right)\\
&\le \exp(a_1 \Delta t)\left[\exp\{a_1(n-1)\Delta t\} \left(x_0 + \Delta t \sum_{i=1}^{n-1} b_i \right) + \Delta t b_n \right] \quad (\text{by } (4.23))\\
&\le \exp(a_1 n \Delta t)\left(x_0 + \Delta t \sum_{i=1}^{n} b_i \right),
\end{aligned}
$$

which is (4.21), *i.e.*, P(n)-(a).

For the proofs of P(n)-(b) and (c) we prepare the estimate of $\|e_h^n\|_1$. From P(n)-(a) and (4.8) we have that

$$
\begin{aligned}
&\nu \|D(e_h^n)\|_0^2 + \frac{\delta_0}{2} |\varepsilon_h^n|_h^2 + \frac{1}{2} \|\overline{D}_{\Delta t} e_h\|_{l_n^2(L^2)}^2 \le \exp(a_1 T)\Big[c_I^2 h^2 + a_2 \Big\{ \Delta t^2 \|u\|_{Z^2}^2 + h^2 \big(\|(u,p)\|_{H^1(H^2\times H^1)}^2 + T \big) \Big\} \Big]\\
&\quad \le \exp(a_1 T)\Big[a_2 \Delta t^2 \|u\|_{Z^2}^2 + h^2 \Big\{ a_2 \big(\|(u,p)\|_{H^1(H^2\times H^1)}^2 + T \big) + c_I^2 \Big\} \Big] \le \big\{ c_3(\Delta t + h) \big\}^2,
\end{aligned} \tag{4.24}
$$

where

$$
c_3 \equiv \exp(a_1 T/2) \max\Big\{ a_2^{1/2} \|u\|_{Z^2},\ a_2^{1/2} \big(\|(u,p)\|_{H^1(H^2\times H^1)} + T^{1/2} \big) + c_I \Big\}.
$$

(4.24) implies

$$
\|e_h^n\|_1 \le \alpha_1 \|D(e_h^n)\|_0 \le \frac{\alpha_1}{\sqrt{\nu}} c_3 (\Delta t + h) = c_*(\Delta t + h). \tag{4.25}
$$

We prove P(n)-(b) and (c). Let Π_h be the Lagrange interpolation operator stated in Lemma 4.2. We have that

$$
\begin{aligned}
\|u_h^n\|_{0,\infty} &\le \|u_h^n - \Pi_h u^n\|_{0,\infty} + \|\Pi_h u^n\|_{0,\infty} \le \alpha_{21} h^{-d/6} \|u_h^n - \Pi_h u^n\|_1 + \|\Pi_h u^n\|_{0,\infty}\\
&\le \alpha_{21} h^{-d/6} (\|u_h^n - \hat{u}_h^n\|_1 + \|\hat{u}_h^n - u^n\|_1 + \|u^n - \Pi_h u^n\|_1) + \|\Pi_h u^n\|_{0,\infty}\\
&\le \alpha_{21} h^{-d/6} \{ c_*(\Delta t + h) + \alpha_{31} h \|(u^n, p^n)\|_{H^2\times H^1} + \alpha_{24} h \|u^n\|_2 \} + \|u^n\|_{0,\infty} \quad (\text{by } (4.25))\\
&\le \alpha_{21} \{ c_*(c_0 h_0^{d/12} + h_0^{1-d/6}) + (\alpha_{24} + \alpha_{31}) h_0^{1-d/6} \|(u,p)\|_{C(H^2\times H^1)} \} + \|u\|_{C(L^\infty)} \quad (\text{by } (3.2))\\
&\le 1 + \|u\|_{C(L^\infty)} \quad (\text{by } (4.20\text{a})),\\
\Delta t \|u_h^n\|_{1,\infty} &\le c_0 h^{d/4} (\|u_h^n - \Pi_h u^n\|_{1,\infty} + \|\Pi_h u^n\|_{1,\infty}) \le c_0 h^{d/4} (\alpha_{22} h^{-d/2} \|u_h^n - \Pi_h u^n\|_1 + \|\Pi_h u^n\|_{1,\infty})\\
&\le c_0 \{ \alpha_{22} h^{-d/4} (\|u_h^n - \hat{u}_h^n\|_1 + \|\hat{u}_h^n - u^n\|_1 + \|u^n - \Pi_h u^n\|_1) + h^{d/4} \|\Pi_h u^n\|_{1,\infty} \}\\
&\le c_0 [\alpha_{22} h^{-d/4} \{ c_*(\Delta t + h) + \alpha_{31} h \|(u^n, p^n)\|_{H^2\times H^1} + \alpha_{24} h \|u^n\|_2 \} + \alpha_{23} h^{d/4} \|u^n\|_{1,\infty}]\\
&\le c_0 [\alpha_{22} h^{-d/4} \{ c_*(c_0 h^{d/4} + h) + (\alpha_{24} + \alpha_{31}) h \|(u^n, p^n)\|_{H^2\times H^1} \} + \alpha_{23} h^{d/4} \|u^n\|_{1,\infty}]\\
&\le c_0 [\alpha_{22} \{ c_*(c_0 + h_0^{1-d/4}) + (\alpha_{24} + \alpha_{31}) h_0^{1-d/4} \|(u,p)\|_{C(H^2\times H^1)} \} + \alpha_{23} h_0^{d/4} \|u\|_{C(W^{1,\infty})}]\\
&\le \delta_1 \quad (\text{by } (4.20\text{b})).
\end{aligned}
$$

Therefore, P(n) holds true.

The proof of P(0) is easier than that of the general step. P(0)-(a) obviously holds with equality. P(0)-(b) and (c) are obtained as follows.

$$\begin{aligned}\|u_h^0\|_{0,\infty} &\le \|u_h^0 - \Pi_h u^0\|_{0,\infty} + \|\Pi_h u^0\|_{0,\infty} \le \alpha_{21} h^{-d/6}(\|u_h^0 - u^0\|_1 + \|u^0 - \Pi_h u^0\|_1) + \|\Pi_h u^0\|_{0,\infty}\\ &\le \alpha_{21}(\alpha_{31}+\alpha_{24})h^{1-d/6}\|u^0\|_2 + \|u^0\|_{0,\infty} \le 1 + \|u\|_{C(L^\infty)} \quad (\text{by (4.20a)}),\\ \Delta t\|u_h^0\|_{1,\infty} &\le c_0 h^{d/4}(\|u_h^0 - \Pi_h u^0\|_{1,\infty} + \|\Pi_h u^0\|_{1,\infty}) \le c_0 h^{d/4}(\alpha_{22}h^{-d/2}\|u_h^0 - \Pi_h u^0\|_1 + \|\Pi_h u^0\|_{1,\infty})\\ &\le c_0\{\alpha_{22}h^{-d/4}(\|u_h^0 - u^0\|_1 + \|u^0 - \Pi_h u^0\|_1) + h^{d/4}\|\Pi_h u^0\|_{1,\infty}\}\\ &\le c_0\{\alpha_{22}(\alpha_{31}+\alpha_{24})h^{1-d/4}\|u^0\|_2 + \alpha_{23}h^{d/4}\|u^0\|_{1,\infty}\} \le \delta_1 \quad (\text{by (4.20b)}).\end{aligned}$$

Thus, the induction is completed.

Step 3. Finally we derive the results (i), (ii) and (iii) of the theorem. The induction completed in the previous step implies that P(N_T) holds true. Hence we have (i) and (ii). The first inequality of (3.4) in (iii) is obtained from (4.25) and the estimate

$$\|u_h - u\|_{l^\infty(H^1)} \le \|e_h\|_{l^\infty(H^1)} + \|\eta\|_{l^\infty(H^1)} \le \|e_h\|_{l^\infty(H^1)} + \alpha_{31}h\|(u,p)\|_{C(H^2\times H^1)}.$$

Combining the estimate

$$\begin{aligned}\left\|\overline{D}_{\Delta t}u_h^n - \frac{\partial u^n}{\partial t}\right\|_0 &\le \|\overline{D}_{\Delta t}e_h^n\|_0 + \|\overline{D}_{\Delta t}\eta^n\|_0 + \left\|\overline{D}_{\Delta t}u^n - \frac{\partial u^n}{\partial t}\right\|_0\\ &\le \|\overline{D}_{\Delta t}e_h^n\|_0 + \frac{\alpha_{31}h}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} + \sqrt{\frac{\Delta t}{3}}\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(t^{n-1},t^n;L^2)}\end{aligned}$$

with (4.24), we get the second inequality of (3.4). Here, for the estimates of the last two terms, we have used the equalities

$$\left(\overline{D}_{\Delta t}\eta^n\right)(x) = \int_0^1 \frac{\partial \eta}{\partial t}(x, t^{n-1}+s\Delta t)\mathrm{d}s, \qquad \left(\overline{D}_{\Delta t}u^n - \frac{\partial u^n}{\partial t}\right)(x) = -\Delta t\int_0^1 s\frac{\partial^2 u}{\partial t^2}(x, t^{n-1}+s\Delta t)\mathrm{d}s.$$

We prove the third inequality of (3.4). We have that

$$\begin{aligned}\|\varepsilon_h^n\|_0 &\le \|(e_h^n,\varepsilon_h^n)\|_{V\times Q} \le \frac{1}{\alpha_{30}}\sup_{(v_h,q_h)\in V_h\times Q_h}\frac{\mathcal{A}_h\big((e_h^n,\varepsilon_h^n),(v_h,q_h)\big)}{\|(v_h,q_h)\|_{V\times Q}} = \frac{1}{\alpha_{30}}\sup_{(v_h,q_h)\in V_h\times Q_h}\frac{\langle R_h^n, v_h\rangle - (\overline{D}_{\Delta t}e_h^n, v_h)}{\|(v_h,q_h)\|_{V\times Q}}\\ &\le c_{(u,p)}\left\{\sqrt{\Delta t}\|u\|_{Z^2(t^{n-1},t^n)} + h\Big(\frac{1}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} + 1\Big) + \|e_h^{n-1}\|_1 + \|\overline{D}_{\Delta t}e_h^n\|_0\right\} \qquad (4.26)\end{aligned}$$

for $n = 1,\ldots,N_T$. Here we have used Lemmas 4.4 and 4.9, the inequality $\|e_h^{n-1}\|_0 \le \|e_h^{n-1}\|_1$ and (3.3). We obtain the result by combining (4.26), (4.24) and the estimate

$$\|p_h - p\|_{l^2(L^2)} \le \|\varepsilon_h\|_{l^2(L^2)} + \|\hat{p}_h - p\|_{l^2(L^2)} \le \|\varepsilon_h\|_{l^2(L^2)} + \sqrt{T}\alpha_{31}h\|(u,p)\|_{C(H^2\times H^1)}.$$

4.4. Proof of Theorem 3.6

We use the next lemma, which is proved in Appendix A.2.

Lemma 4.10. *Suppose Hypotheses 3.2 and 3.5 hold. Let $n \in \{1,\ldots,N_T\}$ be a fixed number and $u_h^{n-1} \in V_h$ be known. Then, under the conditions (4.9) and (4.10) we have that*

$$\|R_{h2}^n\|_0 \le c_u\left(\|e_h^{n-1}\|_0 + h^2\|(u,p)^{n-1}\|_{H^2\times H^1}\right), \qquad (4.27a)$$

$$\|R_{h3}^n\|_{V_h'} \le c_u\left(\|(u,p)^{n-1}\|_{H^2\times H^1}\|e_h^{n-1}\|_0 + \frac{h^2}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} + h^2\sum_{k=1}^{2}\|(u,p)^{n-1}\|_{H^2\times H^1}^k\right), \qquad (4.27b)$$

$$\|R_{h4}^n\|_{V_h'} \le c_u\left(1 + h^{-d/6}\|e_h^{n-1}\|_1\right)\left(\|e_h^{n-1}\|_0 + h^2\|(u,p)^{n-1}\|_{H^2\times H^1}\right). \qquad (4.27c)$$

Proof of Theorem 3.6. Since we have $\|e_h\|_{l^\infty(H^1)} \le c_*(\Delta t + h) \le c_*(c_0 + h_0^{1-d/4})h^{d/4}$ from (4.25) and (3.2), (4.27c) implies

$$\|R_{h4}^n\|_{V_h'} \le c_u c_* \big(\|e_h^{n-1}\|_0 + h^2 \|(u,p)^{n-1}\|_{H^2\times H^1}\big). \tag{4.28}$$

Substituting $(e_h^n, -\varepsilon_h^n)$ into (v_h, q_h) in (4.7) and using Lemma 4.1, (4.13a), (4.27a), (4.27b), (4.28) and the inequality $ab \le \beta a^2/2 + b^2/(2\beta)$ $(\beta > 0)$, we have

$$\begin{aligned}
&\overline{D}_{\Delta t}\left(\frac{1}{2}\|e_h^n\|_0^2\right) + \frac{2\nu}{\alpha_1^2}\|e_h^n\|_1^2 + \delta_0|\varepsilon_h^n|_h^2 \le \sum_{i=1}^4 \langle R_{hi}^n, e_h^n\rangle \\
&\le c_u\left(\frac{1}{\beta_2} + \frac{\|(u,p)\|^2_{C(H^2\times H^1)}}{\beta_3} + \frac{c_*^2}{\beta_4}\right)\|e_h^{n-1}\|_0^2 + \left(\sum_{i=1}^4 \beta_i\right)\|e_h^n\|_1^2 + c_u'\left[\frac{\Delta t}{\beta_1}\|u\|^2_{Z^2(t^{n-1},t^n)}\right. \\
&\quad \left. + \frac{h^4}{\beta_3 \Delta t}\|(u,p)\|^2_{H^1(t^{n-1},t^n;H^2\times H^1)} + h^4\left\{\left(\frac{1}{\beta_2} + \frac{c_*^2}{\beta_4}\right)\|(u,p)\|^2_{C(H^2\times H^1)} + \frac{1}{\beta_3}\sum_{k=1}^2 \|(u,p)\|^{2k}_{C(H^2\times H^1)}\right\}\right]
\end{aligned}$$

for any $\beta_i > 0$ $(i = 1,\ldots,4)$, where the inequality $\|e_h^n\|_0 \le \|e_h^n\|_1$ has been employed. Hence, we have that

$$\overline{D}_{\Delta t}\left(\frac{1}{2}\|e_h^n\|_0^2\right) + \frac{\nu}{\alpha_1^2}\|e_h^n\|_1^2 \le c_{(u,p)}\|e_h^{n-1}\|_0^2 + c'_{(u,p)}\left(\Delta t\|u\|^2_{Z^2(t^{n-1},t^n)} + \frac{h^4}{\Delta t}\|(u,p)\|^2_{H^1(t^{n-1},t^n;H^2\times H^1)} + h^4\right)$$

by setting $\beta_i = \nu/(4\alpha_1^2)$ $(i = 1,\ldots,4)$. From the discrete Gronwall's inequality there exists a positive constant c_4 independent of h and Δt such that

$$\|e_h\|_{l^\infty(L^2)} \le c_4(\|e_h^0\|_0 + \Delta t + h^2).$$

Using (4.4b), we have

$$\begin{aligned}
\|e_h^0\|_0 &\le \|u_h^0 - u^0\|_0 + \|u^0 - \hat{u}_h^0\|_0 \le 2\alpha_{32}h^2\|(u^0,p^0)\|_{H^2\times H^1}, \\
\|u_h - u\|_{l^\infty(L^2)} &\le \|e_h\|_{l^\infty(L^2)} + \|\eta\|_{l^\infty(L^2)} \le \|e_h\|_{l^\infty(L^2)} + \alpha_{32}h^2\|(u,p)\|_{C(H^2\times H^1)}.
\end{aligned}$$

Combining these three inequalities together, we get (3.5). □

5. Numerical results

In this section two- and three-dimensional test problems are computed by scheme (2.7) in order to recognize the theoretical convergence orders numerically.

For the computation of the integral

$$\int_K u_h^{n-1} \circ X_1(u_h^{n-1}, \Delta t)(x) v_h(x)\, \mathrm{d}x \tag{5.1}$$

appearing in scheme (2.7) we employ numerical quadrature formulae [26] of degree five for $d = 2$ (seven points) and 3 (fifteen points). The results obtained in Theorems 3.3 and 3.6 hold for any fixed δ_0. Here we set $\delta_0 = 1$. The system of linear equations is solved by MINRES [2, 25].

Example 5.1. In problem (2.1) we set $\Omega = (0,1)^d$, $T = 1$ and we consider four values of ν,

$$\nu = 10^{-k}, \quad k = 1,\ldots,4.$$

The functions f and u^0 are given so that the exact solution is as follows:

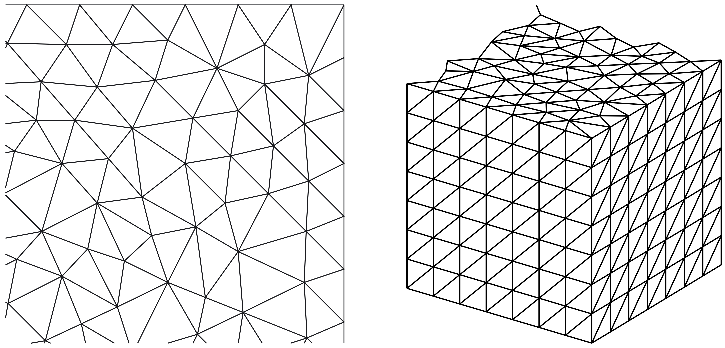

FIGURE 1. Portions of the meshes for $d = 2$ (*left*, $N = 64$, in $[0.9, 1]^2$) and for $d = 3$ (*right*, $N = 64$, in $[0.9, 1]^3$).

for $d = 2$:

$$u(x,t) = \left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right)(x,t), \quad p(x,t) = \sin\{\pi(x_1 + 2x_2 + t)\},$$
$$\psi(x,t) \equiv \frac{\sqrt{3}}{2\pi}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1 + x_2 + t)\},$$

for $d = 3$:

$$u(x,t) = \operatorname{rot}\Psi(x,t), \quad p(x,t) = \sin\{\pi(x_1 + 2x_2 + x_3 + t)\},$$
$$\Psi_1(x,t) \equiv \frac{8\sqrt{3}}{27\pi}\sin(\pi x_1)\sin^2(\pi x_2)\sin^2(\pi x_3)\sin\{\pi(x_2 + x_3 + t)\},$$
$$\Psi_2(x,t) \equiv \frac{8\sqrt{3}}{27\pi}\sin^2(\pi x_1)\sin(\pi x_2)\sin^2(\pi x_3)\sin\{\pi(x_3 + x_1 + t)\},$$
$$\Psi_3(x,t) \equiv \frac{8\sqrt{3}}{27\pi}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin(\pi x_3)\sin\{\pi(x_1 + x_2 + t)\}.$$

These solutions are normalized so that $\|u\|_{C(L^\infty)} = \|p\|_{C(L^\infty)} = 1$.

Let N be the division number of each side of the domain. We set $N = 64, 128, 256$ and 512 for $d = 2$ and $N = 64$ and 128 for $d = 3$, and (re)define $h \equiv 1/N$. Portions of the meshes are shown in Figure 1 for $d = 2$ (*left*, $N = 64$, in $[0.9, 1]^2$) and 3 (*right*, $N = 64$, in $[0.9, 1]^3$). Setting $\Delta t = \gamma_1 h$ and $\gamma_2 h^2$ ($\gamma_1 = 4,\ \gamma_2 = 256$), we solve Example 5.1 by scheme (2.7) with u_h^0, the first component of the Stokes's projection of $(u^0, 0)$ by (3.1). Two relations between Δt and h, *i.e.*, $\Delta t = \gamma_1 h$ and $\gamma_2 h^2$, are employed in order to recognize the convergence orders of (3.4) and (3.5), respectively and we have $(\Delta t =)\gamma_1 h = \gamma_2 h^2$ for $h = 1/64$. For the solution (u_h, p_h) of scheme (2.7) we define the relative errors $Er1$ and $Er2$ by

$$Er1 \equiv \frac{\|u_h - \Pi_h u\|_{l^2(H^1)} + \|p_h - \Pi_h p\|_{l^2(L^2)}}{\|\Pi_h u\|_{l^2(H^1)} + \|\Pi_h p\|_{l^2(L^2)}}, \qquad Er2 \equiv \frac{\|u_h - \Pi_h u\|_{l^\infty(L^2)}}{\|\Pi_h u\|_{l^\infty(L^2)}},$$

where Π_h is the Lagrange interpolation operator to the corresponding space X_h or M_h. Figure 2 shows the graphs of $Er1$ *versus* h for $d = 2$ and 3 (*left*, $\Delta t = \gamma_1 h$) and $Er2$ *versus* h for $d = 2$ (*right*, $\Delta t = \gamma_2 h^2$) in a logarithmic scale, where the symbols are summarized in Table 1. The values of $Er1$, $Er2$ and the slopes are presented in Table 2. We can see that $Er1$ is almost of first order in h for both $d = 2$ and 3 and that $Er2$ is almost of second order in h. These results are consistent with Theorems 3.3 and 3.6.

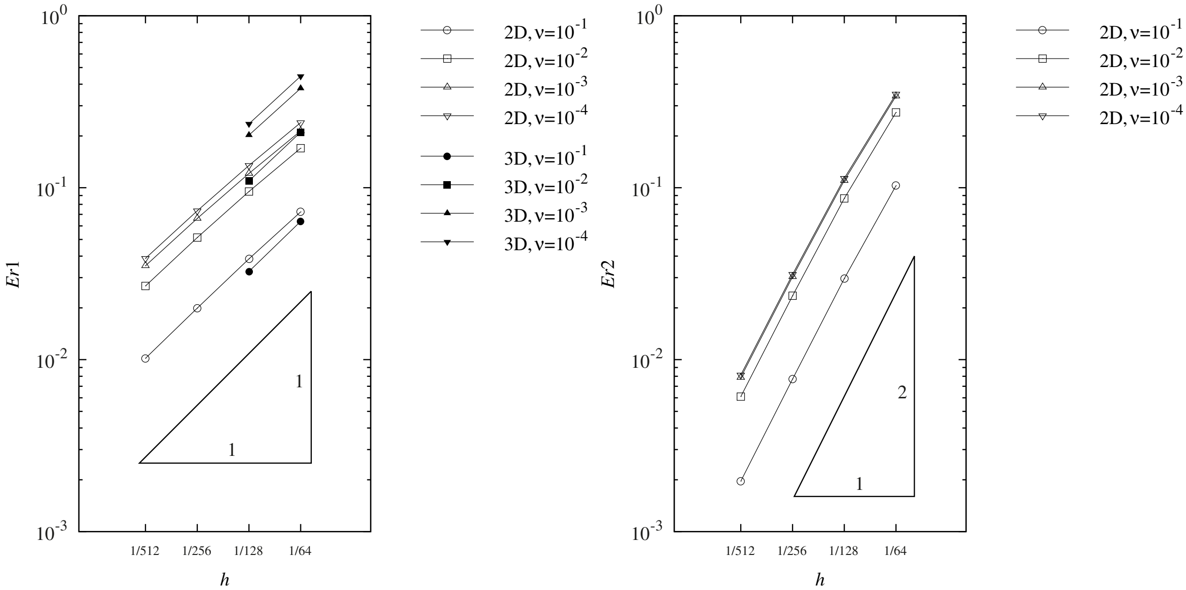

FIGURE 2. $Er1$ *vs.* h for $d = 2$ and 3 (*left*, $\Delta t = \gamma_1 h$, $\gamma_1 = 4$) and $Er2$ *vs.* h for $d = 2$ (*right*, $\Delta t = \gamma_2 h^2$, $\gamma_2 = 256$).

TABLE 1. Symbols used in Figure 2.

	ν			
d	10^{-1}	10^{-2}	10^{-3}	10^{-4}
2	○	□	△	▽
3	●	■	▲	▼

Remark 5.2. In order to examine the influence on the results of numerical quadrature we have also solved Example 5.1 using quadrature formulae of degree two with three points for $d = 2$ and four points for $d = 3$. The differences of the results have been too small to distinguish them on the graphs.

6. CONCLUSIONS

A combined finite element scheme with a Lagrange−Galerkin's method and Brezzi−Pitkäranta's stabilization method for the Navier−Stokes's equations proposed in [17,18] has been theoretically analyzed. Convergence with the optimal error estimates of order $O(\Delta t + h)$ for the velocity in the H^1-norm and the pressure in the L^2-norm (Thm. 3.3) and of order $O(\Delta t + h^2)$ for the velocity in the L^2-norm (Thm. 3.6) have been proved. The scheme has the advantages of both methods: robustness for convection-dominated problems, symmetry of the resulting matrix and a small number of DOF. We note that it is a fully discrete stabilized LG scheme in the sense that the exact solvability of ordinary differential equations describing the particle path is not required. In order to provide the initial approximate velocity we have introduced a stabilized Stokes projection, which works well in the analysis without any loss of convergence order. The theoretical convergence orders have been recognized numerically by two- and three-dimensional computations in Example 5.1. It is not difficult to consider a fully discrete stabilized LG scheme of second order in time based on the ideas of [6, 12], and its convergence with optimal error estimates will be proved by extending the argument of this paper.

TABLE 2. Values of $Er1$, $Er2$ and slopes of the graphs in Figure 2.

		$Er1$				$Er2$	
	N	$d=2$	Slope	$d=3$	Slope	$d=2$	Slope
$\nu = 10^{-1}$:	64	7.24×10^{-2}	–	6.37×10^{-2}	–	1.03×10^{-1}	–
	128	3.85×10^{-2}	0.91	3.25×10^{-2}	0.97	2.96×10^{-2}	1.80
	256	1.99×10^{-2}	0.95	–	–	7.71×10^{-3}	1.94
	512	1.01×10^{-2}	0.97	–	–	1.96×10^{-3}	1.97
$\nu = 10^{-2}$:	64	1.70×10^{-1}	–	2.10×10^{-1}	–	2.74×10^{-1}	–
	128	9.51×10^{-2}	0.84	1.10×10^{-1}	0.94	8.66×10^{-2}	1.66
	256	5.13×10^{-2}	0.89	–	–	2.35×10^{-2}	1.88
	512	2.68×10^{-2}	0.93	–	–	6.09×10^{-3}	1.95
$\nu = 10^{-3}$:	64	2.14×10^{-1}	–	3.78×10^{-1}	–	3.41×10^{-1}	–
	128	1.21×10^{-1}	0.82	2.02×10^{-1}	0.90	1.10×10^{-1}	1.63
	256	6.63×10^{-2}	0.87	–	–	3.03×10^{-2}	1.86
	512	3.51×10^{-2}	0.92	–	–	7.88×10^{-3}	1.95
$\nu = 10^{-4}$:	64	2.39×10^{-1}	–	4.45×10^{-1}	–	3.50×10^{-1}	–
	128	1.35×10^{-1}	0.83	2.35×10^{-1}	0.92	1.13×10^{-1}	1.63
	256	7.34×10^{-2}	0.88	–	–	3.13×10^{-2}	1.85
	512	3.88×10^{-2}	0.92	–	–	8.14×10^{-3}	1.94

APPENDIX A.

A.1. Proof of Lemma 4.9

Let $t(s) \equiv t^{n-1} + s\Delta t$ ($s \in [0,1]$). We prove (4.13a). Let $y(x,s) \equiv x - (1-s)u^{n-1}(x)\Delta t$. We have that

$$
\begin{aligned}
R_{h1}^n(x) &= \left\{ \left(\frac{\partial}{\partial t} + u^n(x)\cdot\nabla \right) u \right\}(x,t^n) - \frac{1}{\Delta t}\Big[u\,(y(x,s),t(s)) \Big]_{s=0}^{1} \\
&= \left\{ \left(\frac{\partial}{\partial t} + u^{n-1}(x)\cdot\nabla \right) u \right\}(x,t^n) + \left\{ \left((u^n - u^{n-1})(x)\cdot\nabla \right) u^n \right\}(x) \\
&\quad - \int_0^1 \left\{ \left(\frac{\partial}{\partial t} + u^{n-1}(x)\cdot\nabla \right) u \right\}(y(x,s),t(s))\,\mathrm{d}s \\
&= \Delta t \int_0^1 \mathrm{d}s \int_s^1 \left\{ \left(\frac{\partial}{\partial t} + u^{n-1}(x)\cdot\nabla \right)^2 u \right\}(y(x,s_1),t(s_1))\,\mathrm{d}s_1 + \Delta t \int_0^1 \left\{ \left(\frac{\partial u}{\partial t}(x,t(s))\cdot\nabla \right) u^n \right\}(x)\mathrm{d}s \\
&= \Delta t \int_0^1 s_1 \left\{ \left(\frac{\partial}{\partial t} + u^{n-1}(x)\cdot\nabla \right)^2 u \right\}(y(x,s_1),t(s_1))\,\mathrm{d}s_1 + \Delta t \int_0^1 \left\{ \left(\frac{\partial u}{\partial t}(x,t(s))\cdot\nabla \right) u^n \right\}(x)\mathrm{d}s \\
&\equiv R_{h11}^n(x) + R_{h12}^n(x).
\end{aligned}
$$

Each term R_{h1i}^n is estimated as follows:

$$
\|R_{h11}^n\|_0 \le \Delta t \int_0^1 s_1 \left\| \left\{ \left(\frac{\partial}{\partial t} + u^{n-1}(\cdot)\cdot\nabla \right)^2 u \right\}(y(\cdot,s_1),t(s_1)) \right\|_0 \mathrm{d}s_1 \le c_u\sqrt{\Delta t}\|u\|_{Z^2(t^{n-1},t^n)}, \tag{A.1a}
$$

$$
\|R_{h12}^n\|_0 \le c_u \Delta t \int_0^1 \left\| \frac{\partial u}{\partial t}(\cdot,t(s)) \right\|_0 \mathrm{d}s \le c_u\sqrt{\Delta t} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(t^{n-1},t^n;L^2)}, \tag{A.1b}
$$

where for the last inequality of (A.1a) we have changed the variable from x to y and used the evaluation $\det(\partial y(x,s_1)/\partial x) \ge 1/2$ ($\forall s_1 \in [0,1]$) from Lemma 4.7-(i). From (A.1) we get (4.13a).

(4.13b) is obtained as follows:

$$\|R_{h2}^n\|_0 \le \alpha_{42}\|u_h^{n-1} - u^{n-1}\|_0\|u^{n-1}\|_{1,\infty} \le \alpha_{42}\|u^{n-1}\|_{1,\infty}(\|\eta^{n-1}\|_0 + \|e_h^{n-1}\|_0) \\ \le \alpha_{42}\|u^{n-1}\|_{1,\infty}(\alpha_{31}h\|(u,p)^{n-1}\|_{H^2\times H^1} + \|e_h^{n-1}\|_0). \tag{A.2}$$

We prove (4.13c). Let $y(x,s) \equiv x - (1-s)u_h^{n-1}(x)\Delta t$. Since we have that

$$R_{h3}^n = \frac{1}{\Delta t}\left[\eta\left(y(\cdot,s),t(s)\right)\right]_{s=0}^1 = \int_0^1 \left\{\left(\frac{\partial}{\partial t} + u_h^{n-1}(\cdot)\cdot\nabla\right)\eta\right\}(y(\cdot,s),t(s))\,\mathrm{d}s,$$

we also have

$$\begin{aligned}
\|R_{h3}^n\|_0 &\le \int_0^1 \left\|\left\{\left(\frac{\partial}{\partial t} + u_h^{n-1}(\cdot)\cdot\nabla\right)\eta\right\}(y(\cdot,s),t(s))\right\|_0 \mathrm{d}s \\
&\le \int_0^1 \left(\left\|\frac{\partial\eta}{\partial t}\left(y(\cdot,s),t(s)\right)\right\|_0 + \|u_h^{n-1}\|_{0,\infty}\left\|\nabla\eta\left(y(\cdot,s),t(s)\right)\right\|_0\right)\mathrm{d}s \\
&\le \sqrt{2}\int_0^1 \left\{\left\|\frac{\partial\eta}{\partial t}\left(\cdot,t(s)\right)\right\|_0 + \|u_h^{n-1}\|_{0,\infty}\left\|\nabla\eta\left(\cdot,t(s)\right)\right\|_0\right\}\mathrm{d}s \quad \text{(by Lem. 4.7-(i))} \\
&\le \sqrt{\frac{2}{\Delta t}}\left(\left\|\frac{\partial\eta}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^2)} + \|u_h^{n-1}\|_{0,\infty}\|\nabla\eta\|_{L^2(t^{n-1},t^n;L^2)}\right) \\
&\le \sqrt{\frac{2}{\Delta t}}\alpha_{31}h(\|u_h^{n-1}\|_{0,\infty}+1)\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)},
\end{aligned}$$

which implies (4.13c).

(4.13d) is obtained as follows:

$$\|R_{h4}^n\|_0 = \frac{1}{\Delta t}\left\|e_h^{n-1} - e_h^{n-1}\circ X_1(u_h^{n-1},\Delta t)\right\|_0 \le \alpha_{40}\|u_h^{n-1}\|_{0,\infty}\|e_h^{n-1}\|_1.$$

A.2. Proof of Lemma 4.10

(4.27a) is obtained by combining (4.4b) with (A.2). For (4.27b) we divide R_{h3}^n into three terms,

$$\begin{aligned}
R_{h3}^n &= \overline{D}_{\Delta t}\eta^n + \frac{1}{\Delta t}\left\{\eta^{n-1} - \eta^{n-1}\circ X_1(u^{n-1},\Delta t)\right\} + \frac{1}{\Delta t}\left\{\eta^{n-1}\circ X_1(u^{n-1},\Delta t) - \eta^{n-1}\circ X_1(u_h^{n-1},\Delta t)\right\} \\
&\equiv R_{h31}^n + R_{h32}^n + R_{h33}^n.
\end{aligned}$$

We have that, by virtue of (4.4b),

$$\|R_{h31}^n\|_{V_h'} \le \|\overline{D}_{\Delta t}\eta^n\|_0 \le \frac{1}{\sqrt{\Delta t}}\left\|\frac{\partial\eta^n}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^2)} \le \frac{\alpha_{32}h^2}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}, \tag{A.3a}$$

$$\|R_{h32}^n\|_{V_h'} \le \alpha_{41}\|u^{n-1}\|_{1,\infty}\|\eta^{n-1}\|_0 \le \alpha_{41}\|u^{n-1}\|_{1,\infty}\ \alpha_{32}h^2\|(u,p)^{n-1}\|_{H^2\times H^1}, \tag{A.3b}$$

$$\begin{aligned}
\|R_{h33}^n\|_{V_h'} &= \sup_{v_h\in V_h}\frac{1}{\|v_h\|_1}\frac{1}{\Delta t}\left(\eta^{n-1}\circ X_1(u_h^{n-1},\Delta t) - \eta^{n-1}\circ X_1(u^{n-1},\Delta t), v_h\right) \\
&\le \sup_{v_h\in V_h}\frac{1}{\|v_h\|_1}\frac{1}{\Delta t}\left\|\eta^{n-1}\circ X_1(u_h^{n-1},\Delta t) - \eta^{n-1}\circ X_1(u^{n-1},\Delta t)\right\|_{0,1}\|v_h\|_{0,\infty} \\
&\le \alpha_{43}\left\|u_h^{n-1} - u^{n-1}\right\|_0\|\eta^{n-1}\|_1\alpha_{21}h^{-d/6} &\text{(A.3c)} \\
&\le \alpha_{21}\alpha_{43}h^{-d/6}\|\eta^{n-1}\|_1(\|e_h^{n-1}\|_0 + \|\eta^{n-1}\|_0) \\
&\le \alpha_{21}\alpha_{43}\alpha_{32}h^{1-d/6}\|(u,p)^{n-1}\|_{H^2\times H^1}\left(\|e_h^{n-1}\|_0 + \alpha_{32}h^2\|(u,p)^{n-1}\|_{H^2\times H^1}\right) \\
&\le c\|(u,p)^{n-1}\|_{H^2\times H^1}\left(\|e_h^{n-1}\|_0 + h^2\|(u,p)^{n-1}\|_{H^2\times H^1}\right). &\text{(A.3d)}
\end{aligned}$$

From (A.3a), (A.3b) and (A.3d) we obtain (4.27b).

For (4.27c) we use the bound on R^n_{h3}. R^n_{h4} is obtained by replacing η^{n-1} with $-e_h^{n-1}$ in $R^n_{h32}+R^n_{h33}$. Hence, from (A.3b) and (A.3c) we have

$$\begin{aligned}\|R^n_{h4}\|_{V_h'} &\le \alpha_{41}\|u^{n-1}\|_{1,\infty}\|e_h^{n-1}\|_0 + \alpha_{21}\alpha_{43}h^{-d/6}\|e_h^{n-1}\|_1\big\|u_h^{n-1}-u^{n-1}\big\|_0\\ &\le \alpha_{41}\|u^{n-1}\|_{1,\infty}\|e_h^{n-1}\|_0 + \alpha_{21}\alpha_{43}h^{-d/6}\|e_h^{n-1}\|_1\big(\|e_h^{n-1}\|_0+\alpha_{32}h^2\|(u,p)^{n-1}\|_{H^2\times H^1}\big)\\ &\le c_u(1+h^{-d/6}\|e_h^{n-1}\|_1)\left(\|e_h^{n-1}\|_0 + h^2\|(u,p)^{n-1}\|_{H^2\times H^1}\right),\end{aligned}$$

which implies (4.27c).

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