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MAXIMUM NORM REGULARITY OF PERIODIC ELLIPTIC DIFFERENCE OPERATORS WITH VARIABLE COEFFICIENTS*

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Abstract. We prove regularity results for divergence form periodic second order elliptic difference operators on the space of functions of mean value zero, valid in maximum norm. The estimates obtained are discrete analogues of the regularity results for continuous operators. The maximum norms of the inverse of such an elliptic operator and of its first spatial differences are uniformly bounded in the grid spacing, and second spatial differences are uniformly bounded except for a logarithmic factor in the grid spacing.

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1. Introduction

We examine the discrete regularity of numerical solutions of the divergence form uniformly elliptic problem

$$\sum_{1\leq j,l\leq d} \nabla \cdot (a_{jl}(x)\nabla)u = f(x) \tag{1.1}$$

for $u : [-\pi,\pi]^d \to \mathbb{R}$ with periodic boundary conditions and mean value zero, with spatially periodic coefficients $a_{jl} : [-\pi,\pi] \to \mathbb{R}$. Under reasonable hypotheses, the solution u of (1.1), satisfies regularity estimates such as $||D^\gamma u||_{L^p} \leq C_p||f||_{L^p}$ for $1 < p < \infty$ and $|\gamma| \leq 2$. We will show that the solution of a suitably discretized problem exhibits an analogous regularity property, when measured in maximum norm.

The problem may be discretized by the finite difference method with divergence form discrete elliptic operators acting on the set of spatially periodic grid functions of mean value zero. By restricting our attention to grid functions with mean value zero, we ensure that every operator in this class is invertible. In Theorem 2.1, the main theorem of this paper, we prove maximum norm estimates for the inverse of an operator of this form and its first spatial differences, uniform in the grid spacing, and maximum norm estimates for second spatial differences, which include a logarithmic factor in the grid spacing. These results generalize those obtained for the discrete Laplacian with periodic boundary conditions obtained by Beale in [1].

Regularity results for elliptic difference operators on bounded domains in various norms other than the maximum norm have been known for some time. In [15], Thomee and Westergren proved interior *a priori* estimates

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in L^2 for elliptic difference operators on bounded domains. In [3,11], these results were extended independently to the case of L^p for $1 < p < \infty$. Each of these papers require the coefficients of the operator to be smooth. In [14], Thomee also established Schauder estimates for discrete elliptic operators with smooth coefficients. In [13], regularity estimates up to the boundary for elliptic systems were obtained, again with the requirement of smooth coefficients. In [4,5], regularity results similar in spirit to those of this paper are derived in Sobolev and Hölder norms for bounded domains with Dirichlet boundary conditions, with less smoothness required of the coefficients but significantly more complicated assumptions on the discrete elliptic operator. Our assumption of periodic boundary conditions on the space of functions of mean value zero simplifies the regularity theory in this paper significantly, allowing us to reduce the regularity of the coefficients without introducing additional hypotheses. Furthermore, maximum norm estimates are particularly well-suited to practical computation, due both to ease of implementation and the fact that estimates in L^p for $1 \leq p < \infty$ do not give any direct pointwise information.

In this paper, we approach the problem of deriving discrete regularity results for periodic boundary conditions from the perspective of analytic semigroup theory. We rely on Proposition 1 from [7] giving maximum norm bounds on the evolution operator $\mathrm{e}^{A_h t}$ and its first and second spatial differences, where A_h a discrete elliptic operator defined on all space. Using this proposition in conjunction with the L^2 bounds on the evolution operator and its spatial differences and H^m Sobolev regularity theory, we derive estimates in maximum norm for A_h^{-1} and its first and second spatial differences, for periodic discrete divergence form elliptic operators A_h acting on the space of periodic grid functions of mean value zero. The results in this paper apply, for instance, to the popular second-order accurate discretization for mixed derivatives found in [9]. By restricting our attention to periodic grid functions of mean value zero, we ensure that the elliptic operators under consideration are in fact invertible. For such an elliptic operator A_h, we show that $(A_h)^{-1}$ and $D_h(A_h)^{-1}$ have maximum norm uniformly bounded in h, and second differences of $(A_h)^{-1}$ are uniformly bounded except for logarithmic factors. Numerical experiments appear to indicate that the blow up factor of $|\ln h|$ in the estimate for second differences is necessary, even for the case where A_h is the constant coefficient discrete Laplacian. The results presented here originally appeared in dissertation form in [6], as a consequence of the maximum norm regularity theory for L-stable difference schemes for parabolic equations.

Although the regularity theory presented here gives a more complete picture of the regularity properties of discrete elliptic operators by obtaining maximum norm estimates for periodic domains, we are motivated by the possibility for application to interface problems. In future work, we hope to extend the results obtained by Beale and Layton in [2] for elliptic interface problems and by Beale in [1] for parabolic interface problems to the case of interface problems with variable coefficients. In those papers, it was demonstrated that under reasonable hypotheses, for an interface problem with constant coefficients, if the discretization is selected to have $O(h^2)$ truncation error away from the interface but only $O(h)$ truncation error near the interface, the solution may still retain $O(h^2)$ accuracy uniformly throughout the domain, and the gradient of the solution nearly second order accuracy throughout the domain (due to the presence of logarithmic factors). The results in this paper may ultimately be of use for the case when the periodic operator has smooth coefficients across the interface but the solution to the interface problem is required to satisfy a jump condition across the interface. However, we do not resolve this question here.

In Section 2 we introduce preliminaries, define the class of periodic elliptic difference operators A_h under consideration, and state the main result of this paper, which is Theorem 2.1. At the end of this section, we verify that the principal symbol of A_h satisfies a discrete ellipticity condition, which is required to use Proposition 5.2, an adaptation of Proposition 1 in [7]. In Section 3, we introduce the associated semidiscrete parabolic problem. We then prove L^2 bounds for the evolution operator $\mathrm{e}^{A_h t}$ and its spatial differences, and L^2 resolvent estimates for A_h. In Section 4, we adapt the elliptic regularity theory for the continuous problem to obtain Sobolev estimates. To obtain the elliptic resolvent estimates, we make use of the discrete Poincaré's inequality and discrete Sobolev's inequality. The discrete Poincaré's inequality (also known in the literature as Wirtinger's inequality) first appeared in [10]. The discrete Sobolev's inequality was proved originally in [12], and is stated in [11]. In Section 5, we prove the main result of this paper, Theorem 2.1. To prove this theorem, we use the results of Sections 3 and 4 to obtain maximum norm estimates for A_h^{-1} and its first and second spatial differences. The key to discovering these maximum norm estimates is to express the differences of the inverse by

$$\begin{aligned} D_h^\gamma (A_h)^{-1} &= \int_0^\infty D_h^\gamma \mathrm{e}^{A_h t}\,\mathrm{d}t \\ &= \int_0^1 D_h^\gamma \mathrm{e}^{A_h t}\,\mathrm{d}t + \int_1^\infty D_h^\gamma \mathrm{e}^{A_h t}\,\mathrm{d}t \end{aligned}$$

and estimate each term separately. The first integral can be handled by the evolution operator bounds on $D_h^\gamma \mathrm{e}^{A_h t}$ found in Proposition 5.2. The second integral requires more care. From the results of Sections 3 and 4, we have that the L^∞ norm of $D_h^\gamma \mathrm{e}^{A_h t}$ is controlled by the H^m norm of $A_h^{m/2}\mathrm{e}^{A_h t}$, which decays exponentially and guarantees convergence of the integral.

2. Preliminaries and results

For convenience we assume that the domain of interest is $\Omega = [-\pi, \pi]^d$, that the grid spacing is $h = 2\pi/N$ and that the grid functions under consideration are 2π-periodic.

We discretize Ω by the grid I_h^d where $I_h = \left\{x = \frac{2\pi}{N}k : k = 1, \dots, N\right\}$. On I_h^d, we define the space X_h of periodic grid functions of mean value zero by

$$X_h = \left\{ u_h(x) : \sum_{x \in I_h^d} u_h(x) = 0 \right\}. \tag{2.1}$$

By extending each $u_h \in X_h$ periodically, we may regard X_h as a subspace of the space of grid functions defined on $\mathbb{R}_h^d = h\mathbb{Z}^d = \left\{jh : j \in \mathbb{Z}^d\right\}$. Note that we will permit all grid functions to assume complex values.

The shift operator in the positive x_j direction is the operator $S_j^+ : u(x) \to u(x + he_j)$, where e_j is the standard basis vector in the x_j direction. We will also use the shift operator $S_h^\gamma = (S_1^+)^{\gamma_1} \dots (S_d^+)^{\gamma_d}$. For shift operators, the entries in the multi-index γ may assume negative values, and we define $S_j^- = (S_j^+)^{-1}$. The forward difference in the positive x_j direction is the operator $D_j^+ : u(x) \to (u(x + he_j) - u(x))/h$, and the backward difference is $D_j^- = S_j^- D_j^+$. For a multi-index γ with non-negative entries, we define the difference operator $D_h^\gamma = (D_1^+)^{\gamma_1} \dots (D_d^+)^{\gamma_d}$.

For each non-negative integer m, we define the discrete Sobolev space $H^m(I_h^d)$ using the norm

$$||u_h||^2_{H^m(I_h^d)} = \sum_{0 \le |\gamma| \le m} ||D_h^\gamma u_h||^2_{L^2(I_h^d)}. \tag{2.2}$$

The $L^2(I_h^d)$ norm is induced by the inner product

$$\langle u_h, v_h \rangle = \sum_{x_0 \in I_h^d} u_h(x_0)\overline{v_h(x_0)} h^d, \tag{2.3}$$

so that $L^2(I_h^d)$ can be regarded as a complex Hilbert space. The maximum norm of a grid function is given by

$$||u_h||_{L^\infty(I_h^d)} = \sup_{x_0 \in I_h^d} |u_h(x_0)|. \tag{2.4}$$

We will use these subscripts also to denote the associated operator norms, though we often omit the space I_h^d from the subscript. For example, for an operator A_h defined on $L^\infty(I_h^d)$, we have

$$||A_h||_{L^\infty} = \sup \left\{ ||A_h u_h||_{L^\infty(I_h^d)} : u_h \in L^\infty(I_h^d),\ ||u_h||_{L^\infty(I_h^d)} = 1 \right\}.$$

For simplicity we will consider difference operators of the form

$$A_h u_h = \sum_{jl} D_j^+ \left(a_{jl}(x) D_l^- u_h\right) \tag{2.5}$$

where the $a_{jl} = a_{lj}$ are required to be 2π-periodic $C^{\lfloor d/2 \rfloor + 3}(\mathbb{R}^d)$ real-valued functions that satisfy the ellipticity condition

$$\sum_{jl} a_{jl}(x)\xi_j \xi_l \geq c|\xi|^2, \tag{2.6}$$

for all vectors $\xi \in \mathbb{R}^d$, with c independent of x. We note that although the operator A_h in (2.5) is only first-order accurate, all of the results in this paper also hold for the more commonly used second-order accurate operator

$$\widetilde{A_h} u_h = \sum_{jl} \left[\frac{1}{2} D_j^+ \left(a_{jl}(x) D_l^- u_h\right) + \frac{1}{2} D_j^- \left(a_{jl}(x) D_l^+ u_h\right)\right] \tag{2.7}$$

replacing A_h. The proofs require only minor modification. (For more on this discretization, see *e.g.* [9].)

The main result of this paper is the following theorem, which states that the inverse and first differences of the inverse of A_h are uniformly bounded in h, and that second differences of the inverse are nearly uniformly bounded in h. The proof appears at the end of Section 5.

Theorem 2.1. *For the operator A_h in* (2.5) (*or for the operator $\widetilde{A_h}$ in* (2.7) *replacing A_h*) *with coefficients $a_{jl} \in C^{\lfloor d/2 \rfloor + 3}(\mathbb{R}^d)$ acting on the space X_h of periodic grid functions of mean value zero, for all multi-indices γ with $|\gamma| \leq 2$, there exist constants C_1 and C_2 for which*

$$\left|\left|D_h^\gamma (A_h)^{-1}\right|\right|_{L^\infty(I_h^d)} \leq C_1, \qquad |\gamma| = 0, 1 \tag{2.8}$$

$$\left|\left|D_h^\gamma (A_h)^{-1}\right|\right|_{L^\infty(I_h^d)} \leq C_2(1 + |\log h|), \qquad |\gamma| = 2. \tag{2.9}$$

In Section 5, we will make use of Proposition 1 from [7]. This result requires us to verify that the symbol of the difference operator A_h satisfies a discrete ellipticity condition. As this is unrelated to the other parts of Section 5, we present this verification now.

The *symbol* of a difference operator is obtained by replacing the shift operator $S_h^\gamma : u(x) \to u(x + \gamma h)$ by $\mathrm{e}^{i\langle \gamma, \xi \rangle}$. For example, the symbol of D_j^+ is $(\mathrm{e}^{i\xi_j} - 1)/h$. For each fixed y, we define the *principal symbol* $p_h(y, \xi)$ to be the symbol associated with the principal part of $A_h(y)$, which is the difference operator $\sum_{jl} a_{jl}(y,h) D_j^+ D_l^-$.

Proposition 2.2. *The principal symbol $p_h(y, \xi)$ of A_h satisfies the uniform ellipticity condition*

$$h^2 p_h(y, \xi) \leq -c|\xi|^2, \qquad \xi \in [-\pi, \pi]^d, \tag{2.10}$$

for some constant c independent of h and y.

Proof. The principal symbol of A_h is

$$p_h(y, \xi) = h^{-2} \sum_{jl} a_{jl}(y) \left(\mathrm{e}^{i\xi_j} - 1\right)\left(1 - \mathrm{e}^{-i\xi_l}\right).$$

Algebraic manipulations exploiting the symmetry of the functions a_{jl} and the representations of the trigonometric functions in terms of the exponential function permit us to rewrite

$$h^2 p_h(y, \xi) = -\left[\sum_{jl} a_{jl}(y) \sin \xi_j \sin \xi_l\right] - 4\left[\sum_{jl} a_{jl}(y) \sin^2 \frac{\xi_j}{2} \sin^2 \frac{\xi_l}{2}\right].$$

By the uniform positivity of the matrix of $a_{jl}(y)$, we have

$$h^2 p_h(y,\xi) \le -c\,|\sin\xi|^2 - 4c\left|\sin^2\frac{\xi}{2}\right|^2$$

for some c independent of y. The function on the right of the preceding inequality is strictly negative for all $\xi \in [-\pi,\pi]^d$ with $\xi \neq 0$. Furthermore, on the neighborhood $[-\pi/2,\pi/2]^d$ of the origin, $\lambda_1|\xi|^2 \le |\sin\xi|^2 \le \lambda_2|\xi|^2$, for some constants $\lambda_1,\lambda_2 > 0$, so that we have

$$h^2 p_h(y,\xi) \le -c'|\xi|^2$$

for some constant c' independent of y. □

3. The semidiscrete problem

To enable us to apply semigroup theory, we must introduce the semidiscrete problem

$$\begin{aligned} u_{h,t} &= A_h u_h \\ A_h u_h &= \sum_{jl} D_j^+\left(a_{jl}(x) D_l^- u_h\right) \\ u_h(x,0) &= u_0(x) \in X_h \end{aligned} \tag{3.1}$$

for real-valued initial data u_0.

For any grid function $v_h \in X_h$, it is readily verified that $A_h v_h \in X_h$ using summation by parts. (Note that the first difference of an arbitrary periodic grid function is in X_h, and $A_h v_h$ is the sum of first differences of periodic grid functions.) This ensures that the solution $u_h(x,t)$ to (3.1) satisfies $u_h(x,\cdot) \in X_h$, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{x_0} u_h(x_0,t)\right) = \sum_{x_0} u_{h,t}(x_0,t) = \sum_{x_0} A_h u_h(x_0,t) = 0.$$

For $u_h(x,t)$ solving (3.1), we calculate for $t > 0$:

$$\begin{aligned} \langle A_h u_h, u_h\rangle &= h^d \sum_{x_0}\sum_{jl} D_j^+\left(a_{jl}(x_0) D_l^- u_h(x_0)\right) u_h(x_0) \\ &= -h^d \sum_{x_0}\sum_{jl} a_{jl}(x_0)(D_l^- u_h(x_0))(D_j^- u_h(x_0)), \end{aligned}$$

after summing by parts. By the uniform ellipticity condition on the matrix $a_{jl}(x)$, we find

$$\langle A_h u_h, u_h\rangle \le -Ch^d \sum_{x_0}\sum_j (D_j^- u_h(x_0))^2$$

for some constant C independent of h. We recognize the right hand side is a sum of norms, so that

$$\langle A_h u_h, u_h\rangle \le -C\sum_j \left|\left|D_j^- u_h\right|\right|_{L^2}^2.$$

By the translation invariance of the L^2 norm, we have

$$\langle A_h u_h, u_h\rangle \le -C'\sum_j \left|\left|D_j^+ u_h\right|\right|_{L^2}^2. \tag{3.2}$$

The discrete Poincaré inequality then gives us that

$$\langle A_h u_h, u_h \rangle \leq -C \, ||u_h||_{L^2}^2 \, . \tag{3.3}$$

However,

$$\langle A_h u_h, u_h \rangle = \langle u_{h,t}, u_h \rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \langle u_h, u_h \rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \, ||u_h||_{L^2}^2 \, .$$

Using this in (3.3), we find that there exists a positive constant c, independent of h, for which

$$||u_h||_{L^2}^2 \leq \mathrm{e}^{-ct} ||u_0||_{L^2}^2.$$

Thus, the solution operator $\mathrm{e}^{A_h t}$ satisfies

$$\left|\left|\mathrm{e}^{A_h t}\right|\right|_{L^2} \leq \mathrm{e}^{-ct}, \qquad t > 0$$

for $c > 0$ independent of h.

Because A_h is a bounded operator (with bound depending on h), for each $t \in \mathbb{C}$, $\mathrm{e}^{A_h t}$ is a bounded operator (with bound also depending on h). Then for any $M > 0$, and for all t in the wedge $\mathbb{T}_M$ defined by

$$\mathbb{T}_M = \{t = t_1 + it_2 : |t_2| \leq M \, |t_1| \, , \, t_1 > 0\}, \tag{3.4}$$

we have

$$\left|\left|\mathrm{e}^{A_h t}\right|\right|_{L^2} = \left|\left|\mathrm{e}^{A_h t_1 + iA_h t_2}\right|\right|_{L^2} = \left|\left|\mathrm{e}^{iA_h t_2}\mathrm{e}^{A_h t_1}\right|\right|_{L^2} .$$

Because A_h is self-adjoint, for all s real $\mathrm{e}^{iA_h s}$ is a unitary group by Stone's theorem, and therefore we may use that $\left|\left|\mathrm{e}^{iA_h t_2}\right|\right|_{L^2} = 1$. We thus conclude

$$\left|\left|\mathrm{e}^{A_h t}\right|\right|_{L^2} \leq \left|\left|\mathrm{e}^{A_h t_1}\right|\right|_{L^2} \leq \mathrm{e}^{-ct_1} \leq \mathrm{e}^{-c'|t|}, \qquad t \in \mathbb{T}_M \tag{3.5}$$

for some c' depending on M but not on h.

We now modify a standard theorem from analytic semigroup theory, found, for instance, in [8].

Proposition 3.1. *For n a positive integer or half-integer, there exist positive constants ω and C_n, independent of h, for which*

$$\left|\left|A_h^n \mathrm{e}^{A_h t}\right|\right|_{L^2} \leq C_n t^{-n} \mathrm{e}^{-\omega t}, \qquad t > 0. \tag{3.6}$$

Proof. By (3.5), we have

$$\left|\left|\mathrm{e}^{A_h t}\right|\right|_{L^2} \leq \mathrm{e}^{-c|t|}, \qquad t \in \mathbb{T}_M$$

for some $c > 0$ independent of h.

If $\mu = c - \epsilon$, then the modified operator $A_h + \mu$ obeys

$$\left|\left|\mathrm{e}^{(A_h+\mu)t}\right|\right|_{L^2} \leq \mathrm{e}^{-\epsilon|t|}, \qquad t \in \mathbb{T}_M.$$

We define $\delta = 2/M$. Then for $z \in F_1 = \{z = z_1 + iz_2 : z_1 \geq 0, |z_2| \leq \delta z_1\}$, expressing the resolvent as an integral involving the evolution operator, we have

$$\begin{aligned} \left|\left|((A_h+\mu) - z)^{-1}\right|\right|_{L^2} &= \left|\left|\int_0^\infty \mathrm{e}^{-zt}\mathrm{e}^{(A_h+\mu)t} \, \mathrm{d}t\right|\right|_{L^2} \\ &\leq \int_0^\infty \mathrm{e}^{-c'|z|t}\mathrm{e}^{-\epsilon t} \, \mathrm{d}t \end{aligned}$$

for some c' depending on δ. Thus,

$$\left|\left|((A_h+\mu)-z)^{-1}\right|\right|_{L^2} \le \frac{C}{1+|z|} \tag{3.7}$$

for all $z \in F_1$.

We next suppose that $z \in F_2 = \{z = z_1 + iz_2 : z_2 \ge 0, |z_1| \le z_2/\delta\}$. We deform the contour of integration to the ray $R_2 = t_1 - iMt_1$. On R_2, we have that $\operatorname{Re} zt = t_1(z_1 + Mz_2) = t_1(z_1 + 2/\delta z_2) \ge t_1 z_2/\delta$. This implies $|\mathrm{e}^{-zt}| \le \mathrm{e}^{-c''|z||t|}$ for $z \in F_2$ and $t \in R_2$ with c'' depending on δ. We therefore obtain an estimate of the same form as (3.7), and a similar argument extends the estimate to $F_3 = \{z = z_1 + iz_2 : z_2 \le 0, |z_1| \le |z_2|/\delta\}$ and hence $F_\delta = F_1 \cup F_2 \cup F_3$.

We now consider $z' + \mu \in F_\delta$ (*i.e.* $z' \in F_\delta - \mu$), so that we can write $z' = z - \mu$ for $z \in F_\delta$. Then we have:

$$\begin{aligned}\left|\left|(A_h - z')^{-1}\right|\right|_{L^2} &= \left|\left|(A_h - (z-\mu))^{-1}\right|\right|_{L^2}\\ &= \left|\left|(A_h + \mu - z)^{-1}\right|\right|_{L^2}.\end{aligned}$$

We use the resolvent estimate for $z \in F_\delta$ in (3.7), so that

$$\left|\left|(A_h - z')^{-1}\right|\right|_{L^2} \le \frac{C}{1+|z|} \le \frac{C}{1+|z'+\mu|},$$

from which we conclude

$$\left|\left|(A_h - z')^{-1}\right|\right|_{L^2} \le \frac{C}{1+|z'|}, \qquad z' \in F_\delta - \mu. \tag{3.8}$$

This improved resolvent estimate on $F_\delta - \mu$ now allows us to write, for $t > 0$,

$$A_h^n \mathrm{e}^{A_h t} = \int_\Gamma z^n \mathrm{e}^{zt}(z - A_h)^{-1}\,\mathrm{d}z$$

for the contour $\Gamma = R_+ \cup R_- \subset F_\delta - \mu$, where

$$R_\pm = \{-\eta + (-1 \pm i\delta)s,\ s \ge 0\}$$

for $0 < \eta < \mu$ (for more detail on this contour representation, see, for instance, Thm. 12.31 of [8]). Because $-A_h$ is positive, $-A_h$ has a well-defined unique positive self-adjoint square root. As A_h is negative definite and $(-A_h)^{1/2}$ is well-defined, we take $A_h^{1/2} = i(-A_h)^{1/2}$.

To bound the contour integral, we use $z = \eta + (-1 \pm i\delta)r$ for $r \ge 0$ on each ray, and noting that $|z|$ is roughly a constant multiple of r for r large, we have $|z^n| \le C(1 + r^n)$, with the constant C depending on n and δ. We also have $|\mathrm{e}^{zt}| = \mathrm{e}^{-\eta t - rt}$, and, by the resolvent estimate in (3.8), we have $||(z - A_h)^{-1}||_{L^2} \le C/(1 + r)$. Thus, using a similar bound for each of the two rays, we have

$$\left|\left|A_h^n \mathrm{e}^{A_h t}\right|\right|_{L^2} \le C\int_0^\infty (1+r^n)\mathrm{e}^{-\eta t - rt}\frac{1}{1+r}\,\mathrm{d}r$$

for C depending on n and δ.

Estimating the right hand side, for $n \ge 1$,

$$\begin{aligned}\left|\left|A_h^n \mathrm{e}^{A_h t}\right|\right|_{L^2} &\le C\mathrm{e}^{-\eta t}\int_0^\infty (1+r)^{n-1}\mathrm{e}^{-rt}\,\mathrm{d}r\\ &\le C\mathrm{e}^{-\eta t}\left(\int_0^\infty \mathrm{e}^{-rt}\,\mathrm{d}r + \int_0^\infty r^{n-1}\mathrm{e}^{-rt}\,\mathrm{d}r\right)\\ &\le C\mathrm{e}^{-\eta t}\left(\frac{1}{t} + \frac{1}{t^n}\right)\\ &\le Ct^{-n}\mathrm{e}^{-\omega t},\end{aligned}$$

where the exponent has changed from $\eta > 0$ to any smaller value $\omega > 0$ in the last step to absorb the negative power of t into the exponential and constant. For $n = \frac{1}{2}$, we have

$$\begin{aligned}\left|\left|A_h^{1/2}\mathrm{e}^{A_h t}\right|\right|_{L^2} &\leq C\mathrm{e}^{-\eta t}\int_0^\infty (1+r^{1/2})\mathrm{e}^{-rt}\frac{1}{1+r}\,\mathrm{d}r\\ &\leq C\mathrm{e}^{-\eta t}\int_0^\infty \frac{1}{r^{1/2}}\mathrm{e}^{-rt}\,\mathrm{d}r\\ &\leq Ct^{-1/2}\mathrm{e}^{-\eta t},\end{aligned}$$

and the result holds in this case for any $0 < \omega \leq \eta$ in the statement of the theorem. □

4. Elliptic regularity in H^m

We require some elliptic regularity results for H^m. The proof of the following result is standard, but we include it for completeness.

Proposition 4.1. *Suppose $a_{jl} \in C^m$ for a positive integer m. Then there exists a constant C_m, depending on the a_{jl} but independent of h, such that if $u_h \in X_h$ solves $A_h u_h = f_h$ for $f_h \in X_h$, we have*

$$||u_h||_{H^{m+1}} \leq C_m\, ||f_h||_{H^{m-1}}. \tag{4.1}$$

Proof. As $A_h u_h = f_h$, for an arbitrary grid function $v_h \in X_h$ we have

$$\langle A_h u_h, v_h\rangle = \langle f_h, v_h\rangle\,. \tag{4.2}$$

Let $v_h = (-1)^{m+1} D_h^{\gamma -} D_h^\gamma u_h$ for γ a multi-index with $|\gamma| = m$, where $D_h^{\gamma-} = (D_1^-)^{\gamma_1}\ldots(D_d^-)^{\gamma_d}$. Then we have

$$\sum_{jl}\left\langle D_j^+(a_{jl}(D_l^- u_h)), (-1)^{m+1}D_h^{\gamma-}D_h^\gamma u_h\right\rangle = \left\langle f_h, (-1)^{m+1}D_h^{\gamma-}D_h^\gamma u_h\right\rangle. \tag{4.3}$$

We denote the left side of (4.3) by B and the right side by E.

We first examine B. Summing by parts $m-1$ times, we have

$$B = \sum_{jl}\left\langle D_h^\gamma(a_{jl}(D_l^- u_h)), D_j^- D_h^\gamma u_h\right\rangle.$$

This can be expressed as

$$\begin{aligned}B &= \sum_{jl}\left\langle \sum_{\beta\leq\gamma}\binom{\gamma}{\beta}(S_h^\beta D_h^{\gamma-\beta}a_{jl})D_h^\beta D_l^- u_h, D_h^\gamma D_j^- u_h\right\rangle\\ &= \sum_{jl}\left\langle (S_h^\gamma a_{jl})D_l^- D_h^\gamma u_h, D_j^- D_h^\gamma u_h\right\rangle\\ &\quad + \sum_{jl}\left\langle \sum_{\beta<\gamma}\binom{\gamma}{\beta}(S_h^\beta D_h^{\gamma-\beta}a_{jl})D_h^\beta D_l^- u_h, D_j^- D_h^\gamma u_h\right\rangle\\ &= B_1 + B_2.\end{aligned}$$

By the ellipticity hypothesis and translation invariance of the L^2 norm, we find

$$B_1 \geq C\sum_j \left|\left|D_j^- D_h^\gamma u_h\right|\right|_{L^2}^2 = C\sum_j\left|\left|D_j^+ D_h^\gamma u_h\right|\right|_{L^2}^2. \tag{4.4}$$

Turning our attention to B_2, by the Cauchy−Schwarz's inequality and the hypothesis that the $a_{jl} \in C^m$, so that the differences $D_h^{\gamma-\beta} a_{jl}$ are uniformly bounded, we have

$$|B_2| \leq C_m \sum_{jl} \sum_{\beta<\gamma} \left|\left|D_h^\beta D_l^- u_h\right|\right|_{L^2} \left|\left|D_j^- D_h^\gamma\right|\right|_{L^2}.$$

Using the Cauchy−Schwarz's inequality with ϵ, we find

$$|B_2| \leq C_\epsilon \left|\left|u_h\right|\right|_{H^m}^2 + \epsilon \sum_j \left|\left|D_j^- D_h^\gamma u_h\right|\right|_{L^2}^2. \tag{4.5}$$

Combining (4.4) and (4.5), we find

$$-C_1 \left|\left|u_h\right|\right|_{H^m}^2 + C_2 \sum_j \left|\left|D_j^+ D_h^\gamma u_h\right|\right|_{L^2}^2 \leq B,$$

with constants depending on m but not on h.

We now turn our focus to the right hand side of (4.3) and bound E. Summing by parts $m-1$ times and using the Cauchy−Schwarz's inequality with ϵ gives us

$$|E| \leq C_\epsilon \left|\left|f_h\right|\right|_{H^{m-1}}^2 + \epsilon \left|\left|u_h\right|\right|_{H^{m+1}}^2.$$

As $B = E$, we have

$$-C_1 \left|\left|u_h\right|\right|_{H^m}^2 + C_2 \sum_j \left|\left|D_j^+ D_h^\gamma u_h\right|\right|_{L^2}^2 \leq C_3 \left|\left|f_h\right|\right|_{H^{m-1}}^2 + \epsilon \left|\left|u_h\right|\right|_{H^{m+1}}^2.$$

Summing over all $|\gamma| = m$, we find that

$$\left|\left|u_h\right|\right|_{H^{m+1}}^2 \leq C_m \left(\left|\left|f_h\right|\right|_{H^{m-1}}^2 + \left|\left|u_h\right|\right|_{H^m}^2\right). \tag{4.6}$$

Applying this estimate repeatedly, we see that

$$\left|\left|u_h\right|\right|_{H^{m+1}}^2 \leq C_m' \left(\left|\left|f_h\right|\right|_{H^{m-1}}^2 + \left|\left|u_h\right|\right|_{H^1}^2)\right). \tag{4.7}$$

To remove the $\left|\left|u_h\right|\right|_{H^1}$ term on the right, we take $v_h = -u_h$ in (4.2), so that

$$\langle A_h u_h, -u_h\rangle = \langle f_h, -u_h\rangle. \tag{4.8}$$

By (3.3), the left side is bounded below by $c \left|\left|u_h\right|\right|_{H^1}^2$. For the right side, we find

$$|\langle f_h, -u_h\rangle| \leq \epsilon \left|\left|u_h\right|\right|_{L^2}^2 + C_\epsilon \left|\left|f_h\right|\right|_{L^2}^2 \leq \epsilon \left|\left|u_h\right|\right|_{H^1}^2 + C_\epsilon \left|\left|f_h\right|\right|_{L^2}^2.$$

Because the left and right sides of (4.8) are equal, we have

$$c \left|\left|u_h\right|\right|_{H^1}^2 \leq \epsilon \left|\left|u_h\right|\right|_{H^1}^2 + C_\epsilon \left|\left|f_h\right|\right|_{L^2}^2,$$

and thus

$$\left|\left|u_h\right|\right|_{H^1}^2 \leq C' \left|\left|f_h\right|\right|_{L^2}^2 \leq C' \left|\left|f_h\right|\right|_{H^{m-1}}^2. \tag{4.9}$$

Substituting (4.9) into (4.7) and taking square roots yields the result. □

By a simple inductive argument, as a consequence of Proposition 4.1, we can bound differences by powers of A_h. For m odd, we use that $\left|\left|u_h\right|\right|_{H^1} \leq C \left|\left|A_h^{1/2} u_h\right|\right|_{L^2}$, which is an immediate consequence of (3.2).

Corollary 4.2. *Suppose $a_{jl} \in C^m$. Then there exists a constant C_m, independent of h, for which*

$$\left|\left|u_h\right|\right|_{H^m} \leq C_m \left|\left|A_h^{m/2} u_h\right|\right|_{L^2} \tag{4.10}$$

for all $u_h \in X_h$.

5. Maximum norm estimates for $D_h^\gamma A_h^{-1}$

With the aid of Sobolev space theory, we can now obtain L^∞ estimates for large time exhibiting exponential decay. Throughout, we assume that the coefficients a_{jl} of the operator A_h are 2π-periodic $C^{\lfloor d/2\rfloor+3}(\mathbb{R}^d)$ real-valued functions satisfying the ellipticity condition (2.6).

Proposition 5.1. *There exist constants C and c, independent of h, for which*

$$\left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty} \le C\mathrm{e}^{-ct}, \qquad t\ge 1, \qquad |\gamma|\le 2. \tag{5.1}$$

Proof. Suppose v_h is an arbitrary vector in X_h. For $m=\lfloor d/2\rfloor+1$, the discrete Sobolev lemma gives us

$$\begin{aligned}\left|\left|D_h^\gamma \mathrm{e}^{A_h t}v_h\right|\right|_{L^\infty} &\le C_m\left|\left|D_h^\gamma \mathrm{e}^{A_h t}v_h\right|\right|_{H^m}\\ &\le C\left|\left|\mathrm{e}^{A_h t}v_h\right|\right|_{H^{m+2}}.\end{aligned}$$

Because all of the a_{jl} are C^{m+2}, using Corollary 4.2 we find that

$$\left|\left|D_h^\gamma \mathrm{e}^{A_h t}v_h\right|\right|_{L^\infty} \le C\left|\left|A_h^{m/2+1}\mathrm{e}^{A_h t}v_h\right|\right|_{L^2}.$$

For $t\ge 1$, using Proposition 3.1 and the fact that for all $u_h\in X_h$ we have $||u_h||_{L^2}\le C\,||u_h||_{L^\infty}$ for some C independent of h (as we are on a bounded domain), we have

$$\left|\left|D_h^\gamma \mathrm{e}^{A_h t}v_h\right|\right|_{L^\infty} \le C\mathrm{e}^{-ct}\,||v_h||_{L^2}\le C\mathrm{e}^{-ct}\,||v_h||_{L^\infty},$$

which establishes the proposition. □

We will require Proposition 1 from [7] to address the small time case, which we restate in the setting of this problem for the sake of completeness. Proposition 2.2 enables us to apply this result.

Proposition 5.2. *There exist constants C and μ, independent of h, so that for all $t>0$ we have*

$$\left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty} \le Ct^{-|\gamma|/2}\mathrm{e}^{\mu t}, \quad |\gamma|\le 2. \tag{5.2}$$

We can now prove Theorem 2.1, improving the L^∞ resolvent estimates for A_h.

Proof of Theorem 2.1. For any multi-index γ with $|\gamma|\le 2$, we can write

$$D_h^\gamma (A_h)^{-1} = \int_0^\infty D_h^\gamma \mathrm{e}^{A_h t}\,\mathrm{d}t,$$

provided the integral converges. We estimate in L^∞:

$$\left|\left|D_h^\gamma (A_h)^{-1}\right|\right|_{L^\infty} \le \int_0^\infty \left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty}\,\mathrm{d}t.$$

We split the interval of integration, so that

$$\left|\left|D_h^\gamma (A_h)^{-1}\right|\right|_{L^\infty} \le \int_0^1 \left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty}\,\mathrm{d}t + \int_1^\infty \left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty}\,\mathrm{d}t. \tag{5.3}$$

We first examine the integral on the left. For $|\gamma|=0,1$, we use the bound in Proposition 5.2 to find that

$$\int_0^1 \left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty}\,\mathrm{d}t \le C\int_0^1 |t|^{-|\gamma|/2}\,\mathrm{d}t \le C. \tag{5.4}$$

For $|\gamma| = 2$, we further divide the interval of integration:

$$\int_0^1 \left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty} \,\mathrm{d}t = \int_0^{h^2} \left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty} \,\mathrm{d}t + \int_{h^2}^1 \left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty} \,\mathrm{d}t.$$

We again use Proposition 5.2 for the interval $[h^2, 1]$. However, on the interval $[0, h^2]$, we bound the second difference by C/h^2, so that

$$\begin{aligned}\int_0^1 \left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty} \,\mathrm{d}t &\le \int_0^{h^2} Ch^{-2}\,\mathrm{d}t + \int_{h^2}^1 Ct^{-1}\,\mathrm{d}t \\ &\le C\left(1 + |\log h|\right). \end{aligned} \tag{5.5}$$

For the integral on the right in (5.3), by Proposition 5.1 we have

$$\int_1^\infty \left|\left|D_h^\gamma \mathrm{e}^{A_h t}\right|\right|_{L^\infty} \,\mathrm{d}t \le C \int_1^\infty \mathrm{e}^{-ct}\,\mathrm{d}t \le C'.$$

Combining this with (5.4) for $|\gamma| = 0, 1$ or (5.5) for $|\gamma| = 2$ yields the result. □

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