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COMPARISON RESULTS OF NONSTANDARD P_2 FINITE ELEMENT METHODS FOR THE BIHARMONIC PROBLEM

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Abstract. As modern variant of nonconforming schemes, discontinuous Galerkin finite element methods appear to be highly attractive for fourth-order elliptic PDEs. There exist various modifications and the most prominent versions with first-order convergence properties are the symmetric interior penalty DG method and the C^0 interior penalty method which may compete with the classical Morley nonconforming FEM on triangles. Those schemes differ in their various jump and penalisation terms and also in the norms. This paper proves that the best-approximation errors of all the three schemes are equivalent in the sense that their minimal error in the respective norm and the optimal choice of a discrete approximation can be bounded from below and above by each other. The equivalence constants do only depend on the minimal angle of the triangulation and the penalisation parameter of the schemes; they are independent of any regularity requirement and hold for an arbitrarily coarse mesh.

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1. Introduction

Given the applied volume force $f \in L^2(\Omega)$ on a bounded polygonal Lipschitz domain Ω in the plane with unit normal ν, the biharmonic problem with clamped boundary conditions reads: Seek u such that

$$\Delta^2 u = f \ \text{ in } \ \Omega \qquad \text{and} \qquad u|_{\partial\Omega} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0. \tag{1.1}$$

The weak formulation corresponding to (1.1) seeks $u \in V := H_0^2(\Omega)$ such that

$$\int_\Omega D^2 u : D^2 v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \text{for all } v \in V. \tag{1.2}$$

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This article compares the errors for various nonstandard finite element methods (FEM) of (1.2) based on piecewise quadratic polynomials, namely the nonconforming Morley FEM [10], the discontinuous Galerkin FEM (DGFEM) [12], and the C^0 interior penalty method (C^0 IP) [5,11], with respective solutions u_{M}, u_{dG}, and u_{IP}.

The quasi-optimality results of [15] imply that the errors of these methods are comparable with the best-approximation in the finite element space. However, these methods come along with different discrete scalar products and norms on different finite element spaces and, hence, are not immediately comparable. Given the best-approximation results of [15] in respective norms, the first contribution of this paper is the identification of a common norm $\|\cdot\|_h$ in which the comparison can be verified. Our suggestion is compatible for the three nonstandard FEMs. The second contribution is the equivalence of the best-approximations in the universal norm $\|\cdot\|_h$, which overcomes the additional difficulty that the three finite element spaces are all piecewise quadratic polynomials but involve different constraints. This enables the proof of

$$\min_{v_{\mathrm{dG}}\in P_2(\mathcal{T})}\|v-v_{\mathrm{dG}}\|_h = \min_{v_{\mathrm{M}}\in \mathrm{M}(\mathcal{T})}\|v-v_{\mathrm{M}}\|_h \approx \min_{v_{\mathrm{IP}}\in \mathrm{IP}(\mathcal{T})}\|v-v_{\mathrm{IP}}\|_h$$

(details on the spaces $P_2(\mathcal{T})$, $\mathrm{M}(\mathcal{T})$ and $\mathrm{IP}(\mathcal{T})$ follow in Sect. 2; the norm $\|\cdot\|_h$ is defined in Sect. 3). This paper aims at a comparison in the spirit of [4,9] with the additional difficulty that the finite element spaces based on piecewise quadratic polynomials do not contain a proper conforming subspace. This does not prevent the proof of equivalence of the errors of these methods, up to data oscillations, with the L^2 best-approximation $\Pi_0 D^2u$ of the exact solution's Hessian D^2u onto piecewise constant functions

$$\|u-u_{\mathrm{M}}\|_h \approx \|u-u_{\mathrm{IP}}\|_h \approx \|u-u_{\mathrm{dG}}\|_h \approx \left\|(1-\Pi_0)\, D^2u\right\|_{L^2(\Omega)}. \tag{1.3}$$

The proofs are based on the local equivalence of the discrete discontinuous Galerkin norms to the norm $\|\cdot\|_h$. Hence, the methods of [2,17] have to be excluded from the analysis of this paper.

The remaining parts of this paper are organised as follows. Section 2 introduces the three FEMs along with necessary notation on data structures and jumps. Section 3 states equivalence of the best-approximations in those finite element spaces with respect to the norm $\|\cdot\|_h$. In Section 4 this norm is proven to be equivalent to the energy norms of these methods. The numerical examples in Section 5 provide strong empirical evidence of (1.3) on uniform and adaptive meshes. Comments on approximation classes complete the paper in Section 6.

Throughout the paper, standard notation on Lebesgue and Sobolev spaces and their norms is employed. The polynomials of degree $\leq k$ over some domain ω are denoted by $P_k(\omega)$. The integral mean is denoted by $⨍$; the dot denotes the product of two one-dimensional lists of the same length while the colon denotes the scalar product of matrices, *e.g.*, $a\cdot b = a^\top b\in\mathbb{R}$ for $a,b\in\mathbb{R}^2$ and $A:B=\sum_{j,k=1}^2 A_{jk}B_{jk}$ for 2×2 matrices A, B. The notation $a\lesssim b$ abbreviates $a\leq Cb$ for a positive generic constant C that may depend on the domain Ω but not on the mesh-size. The notation $a\approx b$ stands for $a\lesssim b\lesssim a$. The measure $|\cdot|$ is context-sensitive and refers to the number of elements of some finite set or the length of an edge or the area of some domain and not just the modulus of a real number or the Euclidean length of a vector.

2. Finite element methods and their comparison

This section introduces the various finite element formulations for (1.2).

2.1. Morley FEM

Let $\mathcal{T}$ denote a shape-regular triangulation of the polygonal Lipschitz domain $\Omega\subset\mathbb{R}^2$ into closed triangles, that is, $\overline{\Omega}=\cup_{T\in\mathcal{T}}T$, which is regular in the sense that two distinct triangles are either disjoint or share exactly one vertex or one edge. Let $\mathcal{N}$ denote the set of vertices; $\mathcal{N}(\Omega)$ denote the set of interior vertices and $\mathcal{N}(\partial\Omega)$ denote the boundary vertices. Let $\mathcal{E}$ denote the set of edges of $\mathcal{T}$; $\mathcal{E}(\Omega)$ denote the set of interior edges, $\mathcal{E}(\partial\Omega)$

(A) (B)

FIGURE 1. Morley finite element and six-noded Lagrange finite element.

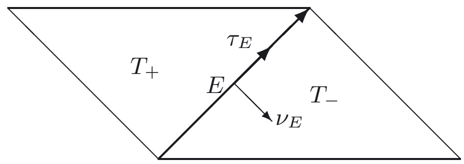

FIGURE 2. Adjacent triangles T_- and T_+ that share the edge $E = \partial T_- \cap \partial T_+$. The unit normal ν_E is the outward normal to T_+.

denote the set of boundary edges. Denote for any edge $E \in \mathcal{E}$ its midpoint by $\mathrm{mid}(E)$. Let

$$H^2(\mathcal{T}) := \left\{ v \in L^2(\Omega) \,\middle|\, \forall T \in \mathcal{T},\ v|_T \in H^2(\mathrm{int}(T)) \right\},$$
$$P_2(\mathcal{T}) := \left\{ v \in L^2(\Omega) \,\middle|\, \forall T \in \mathcal{T},\ v|_T \in P_2(T) \right\}.$$

The nonconforming Morley element space [10] (see Fig. 1a) is defined by

$$\mathrm{M}(\mathcal{T}) := \Bigg\{ v_{\mathrm{M}} \in P_2(\mathcal{T}) \,\Bigg|\, v_{\mathrm{M}} \text{ is continuous at } \mathcal{N}(\Omega), \text{ and vanishes at } \mathcal{N}(\partial\Omega);$$
$$\forall E \in \mathcal{E}(\Omega), \int_E \left[\frac{\partial v_{\mathrm{M}}}{\partial \nu_E} \right]_E \mathrm{d}s = 0; \quad \forall E \in \mathcal{E}(\partial\Omega), \int_E \frac{\partial v_{\mathrm{M}}}{\partial \nu_E} \mathrm{d}s = 0 \Bigg\}.$$

The piecewise action of the Hessian D^2 is denoted by D^2_{NC}. Equip $\mathrm{M}(\mathcal{T})$ with the discrete scalar product

$$a_{\mathrm{NC}}(v_{\mathrm{M}}, w_{\mathrm{M}}) := \int_\Omega D^2_{\mathrm{NC}} v_{\mathrm{M}} : D^2_{\mathrm{NC}} w_{\mathrm{M}} \,\mathrm{d}x \quad \text{for all } v_{\mathrm{M}}, w_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$$

and induced norm $|\!|\!|\cdot|\!|\!|_{\mathrm{NC}} = a_{\mathrm{NC}}(\cdot,\cdot)^{1/2}$. The Morley FEM seeks $u_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$ such that

$$a_{\mathrm{NC}}(u_{\mathrm{M}}, v_{\mathrm{M}}) = \int_\Omega f v_{\mathrm{M}} \,\mathrm{d}x \quad \text{for all } v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T}). \tag{2.1}$$

2.2. C^0 IP

The C^0 IP method is based on the continuous Lagrange P_2 finite element space

$$\mathrm{IP}(\mathcal{T}) := P_2(\mathcal{T}) \cap H^1_0(\Omega) \tag{2.2}$$

and penalty terms along edges. For any interior edge $E \in \mathcal{E}(\Omega)$, there will be two adjacent triangles T_+ and T_- such that $E = \partial T_+ \cap \partial T_-$ (see Fig. 2). Let $\nu_E = (\nu_E(1), \nu_E(2))$ denote the fixed normal vector of E that points from T_+ to T_-. For $E \in \mathcal{E}(\partial\Omega)$, let ν_E denote the outward unit normal vector of Ω. The tangential vector of an edge E is denoted by $\tau_E := (-\nu_E(2), \nu_E(1))$. Given any (possibly vector-valued) function v, define the jump and the average of v of across E by

$$[v]_E := v\left|_{T_+} - v\right|_{T_-} \qquad \text{and} \qquad \langle v\rangle_E := \left(v\left|_{T_+} + v\right|_{T_-}\right)/2 \qquad \text{along } E.$$

For a boundary edge $E \in \mathcal{E}(\partial\Omega) \cap \mathcal{E}(T_+)$, the partner $v|_{T_-}$ is set zero.

For any $v \in H^2(\mathcal{T})$ with $v_\pm = v\big|_{T_\pm}$, define the jump and mean of the normal derivative of v on E by

$$\left[\frac{\partial v}{\partial \nu_E}\right]_E = \frac{\partial v_+}{\partial \nu_E}\bigg|_E - \frac{\partial v_-}{\partial \nu_E}\bigg|_E \quad \text{and} \quad \left\langle \frac{\partial v}{\partial \nu_E}\right\rangle_E = \frac{1}{2}\left(\frac{\partial v_+}{\partial \nu_E}\bigg|_E + \frac{\partial v_-}{\partial \nu_E}\bigg|_E\right).$$

For $v \in H^2(\mathcal{T})$ and $z \in E \in \mathcal{E}$, define

$$[v(z)]_E = v_+(z) - v_-(z). \tag{2.3}$$

Given $\sigma_{\mathrm{IP}} > 0$, define the discrete scalar product for $v_{\mathrm{IP}}, w_{\mathrm{IP}} \in \mathrm{IP}(\mathcal{T})$ by

$$\begin{aligned} a_{\mathrm{IP}}\left(v_{\mathrm{IP}}, w_{\mathrm{IP}}\right) := a_{\mathrm{NC}}\left(v_{\mathrm{IP}}, w_{\mathrm{IP}}\right) &+ \sum_{E\in\mathcal{E}} \int_E \left\langle \frac{\partial^2 v_{\mathrm{IP}}}{\partial \nu_E^2}\right\rangle_E \left[\frac{\partial w_{\mathrm{IP}}}{\partial \nu_E}\right]_E \mathrm{d}s \\ &+ \sum_{E\in\mathcal{E}} \int_E \left\langle \frac{\partial^2 w_{\mathrm{IP}}}{\partial \nu_E^2}\right\rangle_E \left[\frac{\partial v_{\mathrm{IP}}}{\partial \nu_E}\right]_E \mathrm{d}s + \sum_{E\in\mathcal{E}} \frac{\sigma_{\mathrm{IP}}}{h_E} \int_E \left[\frac{\partial v_{\mathrm{IP}}}{\partial \nu_E}\right]_E \left[\frac{\partial w_{\mathrm{IP}}}{\partial \nu_E}\right]_E \mathrm{d}s. \end{aligned} \tag{2.4}$$

The discrete norm reads

$$\|v_{\mathrm{IP}}\|_{\mathrm{IP}}^2 := \|v_{\mathrm{IP}}\|_{\mathrm{NC}}^2 + \sum_{E\in\mathcal{E}} h_E^{-1} \left\|[\partial v_{\mathrm{IP}}/\partial \nu_E]_E\right\|_{L^2(E)}^2.$$

It is known [5,11] that $\|\cdot\|_{\mathrm{IP}}^2 \lesssim a_{\mathrm{IP}}(\cdot,\cdot)$ on $V_{\mathrm{IP}}(\mathcal{T})$ provided σ_{IP} is sufficiently large.

The C^0 IP method [5,11] for (1.2) seeks $u_{\mathrm{IP}} \in \mathrm{IP}(\mathcal{T})$ such that

$$a_{\mathrm{IP}}\left(u_{\mathrm{IP}}, v_{\mathrm{IP}}\right) = \int_\Omega f v_{\mathrm{IP}}\,\mathrm{d}x \quad \text{for all } v_{\mathrm{IP}} \in \mathrm{IP}(\mathcal{T}).$$

2.3. DG method (DGFEM)

This modification of the DG method of [2] was proposed in [12]. The scalar product $a_{\mathrm{dG}}(\cdot,\cdot)$ reads for $v_{\mathrm{dG}}, w_{\mathrm{dG}} \in P_2(\mathcal{T})$ and the penalty parameter $\sigma_{\mathrm{dG}} > 0$

$$\begin{aligned} a_{\mathrm{dG}}\left(v_{\mathrm{dG}}, w_{\mathrm{dG}}\right) &:= a_{\mathrm{NC}}\left(v_{\mathrm{dG}}, w_{\mathrm{dG}}\right) \\ &- \sum_{E\in\mathcal{E}} \left(\int_E \left(\left\langle D_{\mathrm{NC}}^2 v_{\mathrm{dG}}\right\rangle_E \nu_E\right)\cdot \left[\nabla_{\mathrm{NC}} w_{\mathrm{dG}}\right]_E \mathrm{d}s + \int_E \left(\left\langle D_{\mathrm{NC}}^2 w_{\mathrm{dG}}\right\rangle_E \nu_E\right)\cdot \left[\nabla_{\mathrm{NC}} v_{\mathrm{dG}}\right]_E \mathrm{d}s\right) \\ &+ \sum_{E\in\mathcal{E}} \left(\frac{\sigma_{\mathrm{dG}}}{h_E^3}\int_E [v_{\mathrm{dG}}]_E [w_{\mathrm{dG}}]_E \mathrm{d}s + \frac{\sigma_{\mathrm{dG}}}{h_E}\int_E \left[\frac{\partial v_{\mathrm{dG}}}{\partial \nu_E}\right]_E \left[\frac{\partial w_{\mathrm{dG}}}{\partial \nu_E}\right]_E \mathrm{d}s\right). \end{aligned}$$

In general, the two stabilisation terms in the bilinear form may rely on different penalty parameters. The DG norm reads

$$\|v_{\mathrm{dG}}\|_{\mathrm{dG}}^2 := \|v_{\mathrm{dG}}\|_{\mathrm{NC}}^2 + \sum_{E\in\mathcal{E}} \left(h_E^{-3}\left\|[v_{\mathrm{dG}}]_E\right\|_{L^2(E)}^2 + h_E^{-1}\left\|[\partial v_{\mathrm{dG}}/\partial \nu_E]_E\right\|_{L^2(E)}^2\right).$$

Provided the penalty parameter σ_{dG} is sufficiently large, it holds $\|\cdot\|_{\mathrm{dG}}^2 \lesssim a_{\mathrm{dG}}(\cdot,\cdot)$ on $P_2(\mathcal{T})$. The DGFEM seeks $u_{\mathrm{dG}} \in P_2(\mathcal{T})$ such that

$$a_{\mathrm{dG}}\left(u_{\mathrm{dG}}, v_{\mathrm{dG}}\right) = \int_\Omega f v_{\mathrm{dG}}\,\mathrm{d}x \quad \text{for all } v_{\mathrm{dG}} \in P_2(\mathcal{T}).$$

2.4. Statement of equivalence

Throughout this paper, the oscillations of a function $f \in L^2(\Omega)$ with respect to a triangulation $\mathcal{T}$ read

$$\operatorname{osc}(f, \mathcal{T}) := \sqrt{\sum_{T \in \mathcal{T}} h_T^4 \left\| f - ⨍_T f \, \mathrm{d}x \right\|_{L^2(T)}^2}.$$

The main result of this paper states that the errors of these methods are equivalent up to oscillations.

Theorem 2.1. *The discrete solutions u_{M}, u_{IP} and u_{dG} of the Morley FEM, C^0 IP and DGFEM satisfy*

$$\|u - u_{\mathrm{M}}\|_{\mathrm{NC}} \approx \|u - u_{\mathrm{IP}}\|_{\mathrm{IP}} \approx \|u - u_{\mathrm{dG}}\|_{\mathrm{dG}} \approx \left\| (1 - \Pi_0) \, D^2 u \right\|_{L^2(\Omega)}$$

up to oscillations $\operatorname{osc}(f, \mathcal{T})$.

The proof follows in the subsequent sections.

3. Equivalence of best-approximations in $\|\cdot\|_h$

Let h_T denote the diameter of T, for all $T \in \mathcal{T}$ and let h_E the length of an edge E. The set of two vertices of an edge E is denoted by $\mathcal{N}(E)$. Define the following seminorm for all $v_h \in H^2(\mathcal{T})$

$$\|v_h\|_h^2 := \|v_h\|_{\mathrm{NC}}^2 + \sum_{E \in \mathcal{E}} \left(⨍_E \left[\frac{\partial v_h}{\partial \nu_E} \right]_E \mathrm{d}s \right)^2 + \sum_{E \in \mathcal{E}} h_E^{-2} \sum_{z \in \mathcal{N}(E)} \left[v_h(z) \right]_E^2 ,$$

which is obviously a norm on $V + P_2(\mathcal{T})$.

The subsequent theorem establishes a comparison between the best-approximations in $\mathrm{M}(\mathcal{T})$, $\mathrm{IP}(\mathcal{T})$ and $P_2(\mathcal{T})$.

Theorem 3.1. *For any $v \in V$, the following distances are equivalent*

$$\min_{v_{\mathrm{dG}} \in P_2(\mathcal{T})} \|v - v_{\mathrm{dG}}\|_h = \min_{v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})} \|v - v_{\mathrm{M}}\|_h \approx \min_{v_{\mathrm{IP}} \in \mathrm{IP}(\mathcal{T})} \|v - v_{\mathrm{IP}}\|_h .$$

The proof of Theorem 3.1 is based on the following two lemmas. Let

$$C(\mathcal{N}) := \left\{ v \in H^2(\mathcal{T}) \;\middle|\; v \text{ is continuous at all } z \in \mathcal{N} \right\} .$$

Lemma 3.2. *Any $v_2 \in P_2(\mathcal{T}) \cap C(\mathcal{N})$ satisfies*

$$\min_{v_{\mathrm{IP}} \in \mathrm{IP}(\mathcal{T})} \|v_2 - v_{\mathrm{IP}}\|_h^2 \lesssim \sum_{E \in \mathcal{E}} h_E^{-2} \left| [v_2]_E \, (\operatorname{mid} E) \right|^2 .$$

Proof. The generalised nodal interpolation operator

$$I_2^* : P_2(\mathcal{T}) \cap C(\mathcal{N}) \to \mathrm{IP}(\mathcal{T})$$

acts on $v_2 \in P_2(\mathcal{T}) \cap C(\mathcal{N})$ as

$$\begin{aligned} \left(I_2^*(v_2) \right)(z) &= v_2(z) && \text{for any } z \in \mathcal{N}(\Omega), \\ \left(I_2^*(v_2) \right)(\operatorname{mid}(E)) &= \langle v_2 \rangle_E \, (\operatorname{mid}(E)) && \text{for any } E \in \mathcal{E}(\Omega) \end{aligned}$$

and provides an upper bound for the minimum

$$\min_{v_{\mathrm{IP}}\in\mathrm{IP}(\mathcal{T})}\|v_2-v_{\mathrm{IP}}\|_h^2 \le \|v_2-I_2^*v_2\|_h^2 = |||v_2-I_2^*v_2|||_{\mathrm{NC}}^2+\sum_{E\in\mathcal{E}}\left(\fint_E\left[\frac{\partial\left(v_2-I_2^*v_2\right)}{\partial\nu_E}\right]_E \mathrm{d}s\right)^2. \tag{3.1}$$

For any $E\in\mathcal{E}$, the edge-patch ω_E is defined as the union of the triangles that share E (see Fig. 2), that is $\omega_E=T_+\cup T_-$ for interior edges with $E=\partial T_+\cap\partial T_-$ and $\omega_E=T_+$ for boundary edges with $E=\partial T_+\cap\partial\Omega$. Furthermore denote the set of triangles sharing E by $\mathcal{T}(\omega_E):=\{T\in\mathcal{T}\,|\,T\subset\overline{\omega}_E\}$. Let $\varphi_E\in P_2(\mathcal{T}(\omega_E))\cap C_0(\omega_E)$ denote the piecewise quadratic edge-bubble function with $\varphi(\mathrm{mid}(E))=1$ which vanishes at $\mathcal{N}$ and the midpoints of edges $F\in E$ with $F\neq E$. The triangle inequality and the scaling $\left\|D^2_{\mathrm{NC}}\varphi_E\right\|_{\omega_E}\lesssim h_E^{-1}$ prove for the first contribution

$$\begin{aligned}|||v_2-I_2^*v_2|||_{\mathrm{NC}}^2 &= \sum_{T\in\mathcal{T}}\left\|\sum_{E\in\mathcal{E}(T)}\left(v_2(\mathrm{mid}\,E)-\langle v_2\rangle_E(\mathrm{mid}\,E)\right)D^2_{\mathrm{NC}}\varphi_E\right\|_{L^2(T)}^2\\ &\lesssim \sum_{E\in\mathcal{E}}\left|[v_2]_E(\mathrm{mid}\,E)\right|^2\left\|D^2_{\mathrm{NC}}\varphi_E\right\|_{L^2(\omega_E)}^2\\ &\lesssim \sum_{E\in\mathcal{E}}h_E^{-2}\left|[v_2]_E(\mathrm{mid}\,E)\right|^2.\end{aligned}$$

Similarly, the second contribution of (3.1) satisfies

$$\begin{aligned}\sum_{E\in\mathcal{E}}\left|\fint_E\left[\frac{\partial\left(v_2-I_2^*v_2\right)}{\partial\nu_E}\right]_E\mathrm{d}s\right|^2 &\lesssim \sum_{E\in\mathcal{E}}\left|[v_2]_E(\mathrm{mid}\,E)\right|^2\left|\fint_E\frac{\partial\varphi_E}{\partial\nu_E}\,\mathrm{d}s\right|^2\\ &\lesssim \sum_{E\in\mathcal{E}}h_E^{-2}\left|[v_2]_E(\mathrm{mid}\,E)\right|^2.\end{aligned}$$

□

Lemma 3.3. *Any* $v_{\mathrm{M}}\in\mathrm{M}(\mathcal{T})$ *satisfies*

$$\min_{v_{\mathrm{IP}}\in\mathrm{IP}(\mathcal{T})}\|v_{\mathrm{M}}-v_{\mathrm{IP}}\|_h\lesssim\min_{v\in H_0^2(\Omega)}\|v_{\mathrm{M}}-v\|_h.$$

Proof. Lemma 3.2 plus some equivalence of norms prove

$$\min_{v_{\mathrm{IP}}\in\mathrm{IP}(\mathcal{T})}\|v_{\mathrm{M}}-v_{\mathrm{IP}}\|_h\lesssim\sum_{E\in\mathcal{E}}h_E^{-2}\left[v_{\mathrm{M}}(\mathrm{mid}(E))\right]_E^2\lesssim\sum_{E\in\mathcal{E}}h_E^{-3}\left\|[v_{\mathrm{M}}]_E\right\|_{L^2(E)}^2.$$

For any $E\in\mathcal{E}$, the Friedrichs and Poincaré inequalities along E yield

$$h_E^{-3}\left\|[v_{\mathrm{M}}]_E\right\|_{L^2(E)}^2\lesssim h_E\left\|\partial^2[v_{\mathrm{M}}]_E/\partial\tau_E^2\right\|_{L^2(E)}^2\le h_E\left\|\left[D^2_{\mathrm{NC}}v_{\mathrm{M}}\right]_E\tau_E\right\|_{L^2(E)}^2.$$

Denote the endpoints of E by z_1,z_2 and let, for $j\in\{1,2\}$, $\phi_j\in P_1(\mathcal{T})$ be the piecewise affine function with $\phi_j(z_j)=1$ and $\phi_j(y)=0$ for all $y\in\mathcal{N}\setminus\{z_j\}$. Let $\flat_E:=6\phi_1\phi_2\in H_0^1(\omega_E)$. This piecewise quadratic edge-bubble function satisfies

$$\|\flat_E\|_{L^\infty(\Omega)}=3/2\quad\text{and}\quad\int_E\flat_E\,\mathrm{d}s=|E|.$$

Define $\psi_E := (\flat_E \left[D^2_{\mathrm{NC}} v_{\mathrm{M}}\right]_E \tau_E) \in H^1_0(\omega_E;\mathbb{R}^2)$. Since $\left[D^2_{\mathrm{NC}} v_{\mathrm{M}}\right]_E$ is constant along E, it follows that

$$\left\| \left[D^2_{\mathrm{NC}} v_{\mathrm{M}}\right]_E \tau_E \right\|^2_{L^2(E)} = \left\| \flat_E^{1/2} \left[D^2_{\mathrm{NC}} v_{\mathrm{M}}\right]_E \tau_E \right\|^2_{L^2(E)}.$$

The Curl of a vector field $\beta \in H^1(\Omega;\mathbb{R}^2)$ is defined as

$$\mathrm{Curl}\,\beta := \begin{pmatrix} -\partial\beta_1/\partial x_2 & \partial\beta_1/\partial x_1 \\ -\partial\beta_2/\partial x_2 & \partial\beta_2/\partial x_1 \end{pmatrix}.$$

For any $v \in H^2_0(\Omega)$, an integration by parts reveals that

$$\left\| \flat_E^{1/2} \left[D^2_{\mathrm{NC}} v_{\mathrm{M}}\right]_E \tau_E \right\|^2_{L^2(E)} = \int_E \left(\left[D^2_{\mathrm{NC}} v_{\mathrm{M}}\right]_E \tau_E\right)\cdot \psi_E \,\mathrm{d}s = \int_{\omega_E} D^2_{\mathrm{NC}}\left(v_{\mathrm{M}} - v\right) : \mathrm{Curl}\,\psi_E \,\mathrm{d}x$$

(here, the L^2-orthogonality of $\mathrm{Curl}\,\psi_E$ on $D^2 v$ has been used). The Cauchy and inverse inequalities prove that this is bounded by

$$\left\|D^2_{\mathrm{NC}}\left(v_{\mathrm{M}} - v\right)\right\|_{L^2(\omega_E)} \|\mathrm{Curl}\,\psi_E\|_{L^2(\omega_E)} \lesssim \left\|D^2_{\mathrm{NC}}\left(v_{\mathrm{M}} - v\right)\right\|_{L^2(\omega_E)} \left|\left[D^2_{\mathrm{NC}} v_{\mathrm{M}}\right]_E \tau_E\right|.$$

This implies

$$h_E \left\| \left[D^2_{\mathrm{NC}} v_{\mathrm{M}}\right]_E \tau_E \right\|^2_{L^2(E)} \lesssim \min_{v\in H^2_0(\Omega)} \left\|D^2_{\mathrm{NC}}(v_{\mathrm{M}} - v)\right\|_{L^2(\omega_E)}.$$

The combination of the preceding estimates concludes the proof. $\square$

Proof of Theorem 3.1. The Morley interpolation operator $I_{\mathrm{M}} : H^2(\Omega) \to \mathrm{M}(\mathcal{T})$ is defined for all $v \in H^2(\Omega)$ by

$$(I_{\mathrm{M}} v)(z) = v(z) \qquad \text{for each vertex } z \in \mathcal{N},$$
$$\frac{\partial I_{\mathrm{M}} v}{\partial \nu_E}(\mathrm{mid}(E)) = \fint_E \nabla v\cdot \nu_E \,\mathrm{d}s \quad \text{for each edge } E \in \mathcal{E},$$

and enjoys the integral mean property of the Hessians [7]

$$D^2_{\mathrm{NC}} I_{\mathrm{M}} = \Pi_0 D^2. \tag{3.2}$$

The inclusions $\mathrm{M}(\mathcal{T}) \subset P_2(\mathcal{T})$ and $\mathrm{IP}(\mathcal{T}) \subset P_2(\mathcal{T})$ plus (3.2) imply

$$\begin{aligned}\min_{v_{\mathrm{dG}}\in P_2(\mathcal{T})} \|u - v_{\mathrm{dG}}\|_h &\le \min_{v_{\mathrm{M}}\in \mathrm{M}(\mathcal{T})} \|u - v_{\mathrm{M}}\|_h \le \|u - I_{\mathrm{M}} u\|_h \\ &= \left\|(1-\Pi_0)D^2 u\right\|_{L^2(\Omega)} = \min_{v_{\mathrm{dG}}\in P_2(\mathcal{T})} \|u - v_{\mathrm{dG}}\|_h \le \min_{v_{\mathrm{IP}}\in \mathrm{IP}(\mathcal{T})} \|u - v_{\mathrm{IP}}\|_h.\end{aligned}$$

Therefore, it remains to prove

$$\min_{v_{\mathrm{IP}}\in \mathrm{IP}(\mathcal{T})} \|u - v_{\mathrm{IP}}\|_h \lesssim \min_{v_{\mathrm{M}}\in \mathrm{M}(\mathcal{T})} \|u - v_{\mathrm{M}}\|_h.$$

The triangle inequality and $\|u - I_{\mathrm{M}} u\|_h = \min_{v_{\mathrm{M}}\in \mathrm{M}(\mathcal{T})} \|u - v_{\mathrm{M}}\|_h$ from the first part of the proof reveal

$$\begin{aligned}\min_{v_{\mathrm{IP}}\in \mathrm{IP}(\mathcal{T})} \|u - v_{\mathrm{IP}}\|_h &\le \|u - I_{\mathrm{M}} u\|_h + \min_{v_{\mathrm{IP}}\in \mathrm{IP}(\mathcal{T})} \|I_{\mathrm{M}} u - v_{\mathrm{IP}}\|_h \\ &= \min_{v_{\mathrm{M}}\in \mathrm{M}(\mathcal{T})} \|u - v_{\mathrm{M}}\|_h + \min_{v_{\mathrm{IP}}\in \mathrm{IP}(\mathcal{T})} \|I_{\mathrm{M}} u - v_{\mathrm{IP}}\|_h.\end{aligned}$$

Lemma 3.3 and the projection property (3.2) imply

$$\min_{v_{\mathrm{IP}}\in \mathrm{IP}(\mathcal{T})}\|I_{\mathrm{M}}u-v_{\mathrm{IP}}\|_h \lesssim \min_{v\in H_0^2(\Omega)}\|I_{\mathrm{M}}u-v\|_h \le \|u-I_{\mathrm{M}}u\|_h = \min_{v_{\mathrm{M}}\in \mathrm{M}(\mathcal{T})}\|u-v_{\mathrm{M}}\|_h. \qquad \square$$

4. Equivalence of norms

One key argument in comparison results is the equivalence of various norms.

Theorem 4.1 (Discrete dG norm equivalence)**.** *The norm $\|\cdot\|_h$ satisfies*

$$\|\cdot\|_h = \|\!|\cdot|\!\|_{\mathrm{NC}} \quad \text{on } V+\mathrm{M}(\mathcal{T}), \tag{4.1}$$

$$\|\cdot\|_h \approx \|\cdot\|_{\mathrm{dG}} \quad \text{on } V+P_2(\mathcal{T}), \tag{4.2}$$

$$\|\cdot\|_h \approx \|\cdot\|_{\mathrm{IP}} \quad \text{on } V+\mathrm{IP}(\mathcal{T}). \tag{4.3}$$

Proof. The equivalence (4.1) follows from the continuity conditions of the Morley finite element functions (resp. their normal derivatives) at the vertices (resp. midpoints of edges). In order to prove (4.2) consider an arbitrary edge $E\in\mathcal{E}$ and $v_{\mathrm{dG}}\in P_2(\mathcal{T})$. Recall that the edge-patch of an edge $E\in\mathcal{E}$ is denoted by ω_E and $\mathcal{T}(\omega_E)=\{T\in\mathcal{T}\,|\,T\subset\overline{\omega}_E\}$. Define the seminorms ρ_E and θ_E on $P_2(\mathcal{T}(\omega_E))$

$$\rho_E\,(v_{\mathrm{dG}})^2 := \left\|D^2_{\mathrm{NC}}v_{\mathrm{dG}}\right\|^2_{L^2(\omega_E)} + \left(⨍_E [\partial v_{\mathrm{dG}}/\partial\nu_E]_E \,\mathrm{d}s\right)^2 + h_E^{-2}\sum_{z\in\mathcal{N}(E)}[v_{\mathrm{dG}}(z)]_E^2,$$

$$\theta_E\,(v_{\mathrm{dG}})^2 := \left\|D^2_{\mathrm{NC}}v_{\mathrm{dG}}\right\|^2_{L^2(\omega_E)} + h_E^{-1}\left\|\left[\frac{\partial v_{\mathrm{dG}}}{\partial\nu_E}\right]_E\right\|^2_{L^2(E)} + h_E^{-3}\left\|[v_{\mathrm{dG}}]_E\right\|^2_{L^2(E)}.$$

These seminorms on the finite-dimensional space $P_2(\mathcal{T}(\omega_E))$ are equivalent. A scaling argument plus the finite overlap prove (4.2). For an edge $E\in\mathcal{E}$ and $v_{\mathrm{IP}}\in\mathrm{IP}(\mathcal{T})$ define ξ_E by

$$\xi_E(v_{\mathrm{IP}})^2 := \left\|D^2_{\mathrm{NC}}v_{\mathrm{IP}}\right\|^2_{L^2(\omega_E)} + h_E^{-1}\left\|[\partial v_{\mathrm{IP}}/\partial\nu_E]_E\right\|^2_{L^2(\omega_E)}.$$

The seminorms ρ_E and ξ_E are equivalent on $P_2(\mathcal{T}(\omega_E))$. A scaling argument and the finite overlap prove (4.3). $\square$

Remark 4.2. The proof of Theorem 4.1 relies on local equivalence of seminorms. Therefore, the two DG methods of [2,17] are excluded from the analysis of this paper. The DG norm of those papers reads

$$\left(\sum_{T\in\mathcal{T}}\|\Delta v_{\mathrm{dG}}\|^2_{L^2(T)} + \sum_{E\in\mathcal{E}}h_E^{-1}\left\|\left[\frac{\partial v_{\mathrm{dG}}}{\partial\nu_E}\right]_E\right\|^2_{L^2(E)} + \sum_{E\in\mathcal{E}}h_E^{-3}\left\|[v_{\mathrm{dG}}]_E\right\|^2_{L^2(E)}\right)^{1/2}.$$

For an interior edge E, the term

$$\|\Delta_{\mathrm{NC}}v_{\mathrm{dG}}\|^2_{L^2(\omega_E)} + h_E^{-1}\left\|\left[\frac{\partial v_{\mathrm{dG}}}{\partial\nu_E}\right]_E\right\|^2_{L^2(E)} + h_E^{-3}\left\|[v_{\mathrm{dG}}]_E\right\|^2_{L^2(E)}$$

may vanish for any smooth and harmonic v_{dG} on ω_E although $v_{\mathrm{dG}}|_{\omega_E}\not\equiv 0$. Consequently, the arguments that lead to Theorem 4.1 are not applicable for [2,17].

The following quasi-optimality result is proven in [15].

Theorem 4.3 ([15], p. 2172)**.** *The errors of the Morley FEM, C^0 IP FEM, and DGFEM are quasi-optimal with respect to their norms in the sense that*

$$\|u - u_{\mathrm{M}}\|_{\mathrm{NC}} \lesssim \min_{v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})} \|u - v_{\mathrm{M}}\|_{\mathrm{NC}} + \mathrm{osc}(f, \mathcal{T}),$$

$$\|u - u_{\mathrm{IP}}\|_{\mathrm{IP}} \lesssim \min_{v_{\mathrm{IP}} \in \mathrm{IP}(\mathcal{T})} \|u - v_{\mathrm{IP}}\|_{\mathrm{IP}} + \mathrm{osc}(f, \mathcal{T}),$$

$$\|u - u_{\mathrm{dG}}\|_{\mathrm{dG}} \lesssim \min_{v_{\mathrm{dG}} \in P_2(\mathcal{T})} \|u - v_{\mathrm{dG}}\|_{\mathrm{dG}} + \mathrm{osc}(f, \mathcal{T}).$$

Proof. The first two estimates are derived explicitly in [15]. For the proof of the third inequality, the abstract framework of [15] shows that it suffices to verify the existence of an enriching operator $E_{\mathrm{dG}} : P_2(\mathcal{T}) \to V_C$, for some conforming finite element space V_C, such that for all $v_{\mathrm{dG}} \in P_2(\mathcal{T})$

$$\left\| D^2 E_{\mathrm{dG}} v_{\mathrm{dG}} \right\|_{L^2(\Omega)} \lesssim \|v_{\mathrm{dG}}\|_{\mathrm{dG}}$$

$$\text{and} \qquad \sup_{w_{\mathrm{dG}} \in P_2(\mathcal{T}) \setminus \{0\}} \frac{\int_\Omega f\,(w_{\mathrm{dG}} - E_{\mathrm{dG}} w_{\mathrm{dG}})\ \mathrm{d}x - a_{\mathrm{dG}}(v_{\mathrm{dG}}, w_{\mathrm{dG}} - E_{\mathrm{dG}} w_{\mathrm{dG}})}{\|w_{\mathrm{dG}}\|_{\mathrm{dG}}}$$
$$\lesssim \left(\|u - v_{\mathrm{dG}}\|_{\mathrm{dG}} + \mathrm{osc}(f, \mathcal{T}) \right).$$

The design of such an operator can be found in [13]. □

Proof of Theorem 2.1. Theorem 4.1 allows to replace the upper bounds in Theorem 4.3 by the respective best-approximations with respect to the norm $\|\cdot\|_h$. Theorem 3.1 establishes the equivalence

$$\|u - u_{\mathrm{M}}\|_h \approx \|u - u_{\mathrm{IP}}\|_h \approx \|u - u_{\mathrm{dG}}\|_h \lesssim \left\| (1 - \Pi_0) D^2 u \right\|_{L^2(\Omega)}$$

up to oscillations $\mathrm{osc}(f, \mathcal{T})$. □

Remark 4.4. For the C^0 IP and DGFEM, the multiplicative constants of the second and third inequality of Theorem 4.3 depend on the penalisation parameters σ_{IP}, σ_{dG} and deteriorate for large penalisation parameters. Therefore, the best-approximation constants in the estimates $\|u - u_{\mathrm{IP}}\|_{\mathrm{IP}} \lesssim \left\| (1 - \Pi_0) D^2 u \right\|_{L^2(\Omega)}$ and $\|u - u_{\mathrm{dG}}\|_{\mathrm{dG}} \lesssim \left\| (1 - \Pi_0) D^2 u \right\|_{L^2(\Omega)}$ of Theorem 2.1 depend critically on the fixed choice of the penalisation parameters.

Remark 4.5. The assumption $f \in L^2$ is essential in the analysis because the results of [15] are based on the efficiency of volume parts of a residual-based error estimators.

Remark 4.6 (Higher-order FEMs)**.** The analysis of this paper focuses on the lowest-order case of piecewise quadratic FEM nonconforming approximations which are attractive for a convenient implementation. A comparison with conforming schemes, for example, with the quintic Argyris triangle based on piecewise P_5 polynomials will in general result in a superior convergence rate of the Argyris FEM. The arguments of [15] may be applied for a comparison of the DGFEM or the C^0 IP FEM based on $P_5(\mathcal{T})$ polynomials with the Argyris FEM and yield best-approximation of those discontinuous Galerkin schemes in their respective norms whereas the classical Galerkin orthogonality for the Argyris FEM only gives the best-approximation in the subspace of $P_5(\mathcal{T})$ with C^1 continuity constraints. It is expected that this best-approximation is comparable with the best-approximation in $P_5(\mathcal{T})$, but the proof remains as an open question. A corresponding statement for the approximation by piecewise gradients was proven in [19].

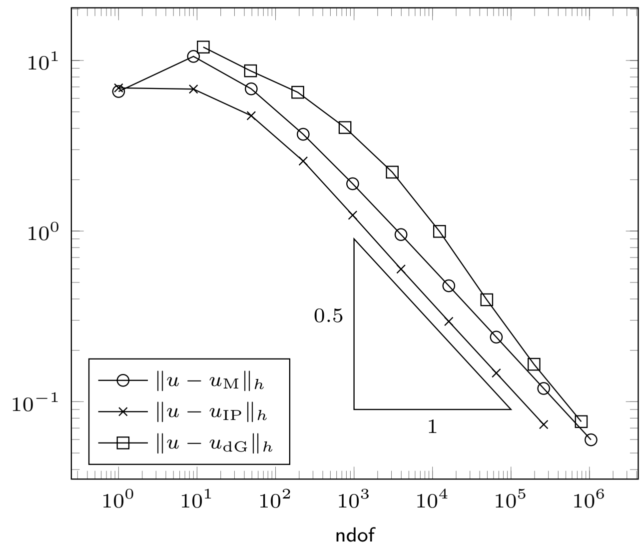

FIGURE 3. Convergence history for the analytic example.

5. Numerical experiments

Two numerical examples illustrate the result that the Morley FEM, C^0 IP and DGFEM are equivalent. In all the experiments, the penalty parameters are $\sigma_{\mathrm{IP}} = \sigma_{\mathrm{dG}} = 15$. The schemes were implemented in Matlab in the framework of [1]; *cf.* [8] for an outline of an implementation for the Morley FEM.

5.1. Analytic solution

Let $u = -(x^4 - 2x^2 + 1)(y^4 - 2y^2 + 1)$ be the solution with respect to the right-hand side $f := \Delta^2 u$ on the unit square. The initial triangulation consists of a criss grid of $(-1,1)^2$ into two triangles. Figure 3 shows the convergence history of the exact error in the energy norm $\|\cdot\|_h$ on uniform red-refined meshes. In all the experiments, the equivalence constants range between 2 and 6.

5.2. L-shaped domain with uniform mesh refinement

Consider the L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1] \times [-1,0])$ with $\omega := 3\pi/2$ and $\alpha := 0.5444837$ as a noncharacteristic root of $\sin^2(\alpha\omega) = \alpha^2 \sin^2(\omega)$. The exact singular solution from ([14], p. 107) [6] reads in polar coordinates

$$u(r,\theta) = \left(r^2\cos^2\theta - 1\right)^2 \left(r^2\sin^2\theta - 1\right)^2 r^{1+\alpha}\, g(\theta) \quad \text{for}$$

$$\begin{aligned} g(\theta) = &\left(\frac{1}{\alpha-1}\sin((\alpha-1)\omega) - \frac{1}{\alpha+1}\sin((\alpha+1)\omega)\right)(\cos((\alpha-1)\theta) - \cos((\alpha+1)\theta)) \\ &- \left(\frac{1}{\alpha-1}\sin((\alpha-1)\theta) - \frac{1}{\alpha+1}\sin((\alpha+1)\theta)\right)(\cos((\alpha-1)\omega) - \cos((\alpha+1)\omega)). \end{aligned}$$

Figure 4 displays the convergence history of the exact error on uniform red-refined meshes.

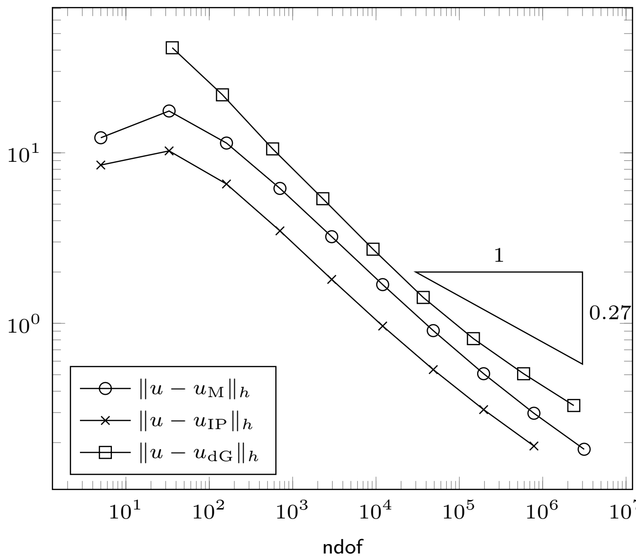

FIGURE 4. Convergence history of the error for Grisvard's example.

Algorithm 1. Adaptive Morley finite element method.

Input initial triangulation $\mathcal{T}_0$, bulk parameter $0 < \theta \leq 1$.
for $\ell = 0, 1, 2, \ldots$ **do**
Solve. Compute the solution of Morley FEM $u_\ell := u_{\mathrm{M}}$ with respect to $\mathcal{T}_\ell$ and `ndof` degrees of freedom.
Estimate. Compute local contributions of the error estimator

$$\eta^2(T) := \left\| h_{\mathcal{T}}^2 f \right\|_{L^2(T)}^2 + \sum_{E \in \mathcal{E}(T)} h_E \left\| \left[D_{\mathrm{NC}}^2 u_\ell \right]_E \tau_E \right\|_{L^2(E)}^2 \quad \text{for all } T \in \mathcal{T}_\ell.$$

Mark. The Dörfler marking chooses a minimal subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ such that

$$\theta \sum_{T \in \mathcal{T}_\ell} \eta^2(T) \leq \sum_{T \in \mathcal{M}_\ell} \eta^2(T).$$

Refine. Compute the closure of $\mathcal{M}_\ell$ and generate a new triangulation $\mathcal{T}_{\ell+1}$ using newest vertex bisection [3,18]. **od**
Output Sequence of triangulations $(\mathcal{T}_\ell)_\ell$ and discrete solutions$(u_\ell)_\ell$.

5.3. L-Shaped domain with adaptive mesh refinement

In this experiment, the methods are compared on a sequence of adaptively refined meshes. The meshes are generated by the AFEM loop for the Morley FEM from [16], see Algorithm 1 below.

The triangulations only depend on the Morley FEM solution and the error estimator. The C^0 IP and DGFEM show the same convergence rate as the Morley FEM on these meshes for $\theta = 0.5$, see Figure 5. An adaptive mesh is shown in Figure 6.

6. EQUIVALENCE OF APPROXIMATION CLASSES

The comparison of various nonconforming schemes in this paper is for a fixed triangulation and states equivalence of the errors of the Morley FEM, C^0 IP and DGFEM with the best approximation. Some more far-reaching question is the comparison of the three schemes and their optimal mesh-design.

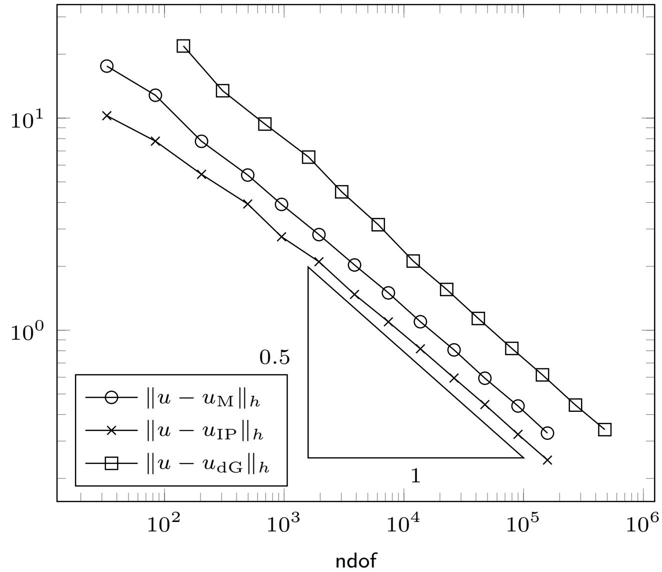

FIGURE 5. Convergence history of the error for Grisvard's example.

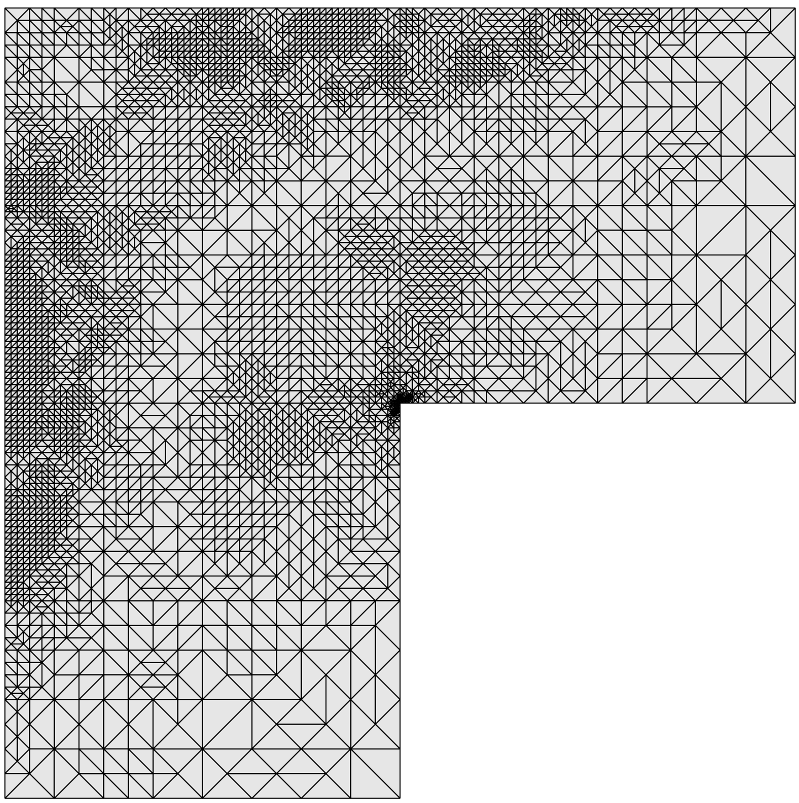

FIGURE 6. Adaptive mesh, 6977 triangles.

Given the initial mesh $\mathcal{T}_0$, let $\mathbb{T}$ be the set of all admissible refinements of $\mathcal{T}_0$ created by newest-vertex bisection [3,18], and (recall that $|\mathcal{T}|$ abbreviates the number of triangles in $\mathcal{T}$) let

$$\mathbb{T}(N) := \left\{\mathcal{T} \in \mathbb{T} \,\middle|\, |\mathcal{T}| - |\mathcal{T}_0| \leq N\right\} \quad \text{for } N \in \mathbb{N}.$$

Define the seminorms for $(u,f) \in V \times L^2(\Omega)$

$$|(u,f)|_{s,h} := \sup_{N\in\mathbb{N}} N^s \inf_{\mathcal{T}\in\mathbb{T}(N)} \inf_{v_2\in P_2(\mathcal{T})} \left(\|u - v_2\|_h^2 + \operatorname{osc}^2(f,\mathcal{T})\right)^{1/2}$$

$$|(u,f)|_{s,\mathrm{M}} := \sup_{N\in\mathbb{N}} N^s \inf_{\mathcal{T}\in\mathbb{T}(N)} \left(|\!|\!|u - u_{\mathrm{M}}(\mathcal{T})|\!|\!|_{\mathrm{NC}}^2 + \operatorname{osc}^2(f,\mathcal{T})\right)^{1/2}$$

$$|(u,f)|_{s,\mathrm{IP}} := \sup_{N\in\mathbb{N}} N^s \inf_{\mathcal{T}\in\mathbb{T}(N)} \left(\|u - u_{\mathrm{IP}}(\mathcal{T})\|_{\mathrm{IP}}^2 + \operatorname{osc}^2(f,\mathcal{T})\right)^{1/2}$$

$$|(u,f)|_{s,\mathrm{dG}} := \sup_{N\in\mathbb{N}} N^s \inf_{\mathcal{T}\in\mathbb{T}(N)} \left(\|u - u_{\mathrm{dG}}(\mathcal{T})\|_{\mathrm{dG}}^2 + \operatorname{osc}^2(f,\mathcal{T})\right)^{1/2}.$$

Here $u_{\mathrm{M}}(\mathcal{T})$, $u_{\mathrm{IP}}(\mathcal{T})$ and $u_{\mathrm{dG}}(\mathcal{T})$ denote the solutions of the Morley FEM, C^0 IP and DGFEM with respect to the triangulation $\mathcal{T}$. The comparison results of this paper imply that the four approximation classes for $j \in \{h, \mathrm{M}, \mathrm{IP}, \mathrm{dG}\}$

$$\mathcal{A}_{s,j} := \left\{(u,f) \in V \times L^2(\Omega) \,\middle|\, u \text{ solves (1.2) with right-hand side } f \text{ and } |(u,f)|_{s,j} < \infty\right\}$$

are equal, *i.e.*,

$$\mathcal{A}_{s,h} = \mathcal{A}_{s,\mathrm{M}} = \mathcal{A}_{s,\mathrm{IP}} = \mathcal{A}_{s,\mathrm{dG}} \quad \text{with equivalent norms.}$$

The equivalence constants depend on $\mathcal{T}_0$ and the stabilisation parameters σ_{IP} and σ_{dG} (through the quasi-optimal constants by [15]) but not on $0 < s < \infty$.

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