

SUPERCLOSENESS OF ORTHOGONAL PROJECTIONS ONTO NEARBY FINITE ELEMENT SPACES

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Abstract. We derive upper bounds on the difference between the orthogonal projections of a smooth function u onto two finite element spaces that are nearby, in the sense that the support of every shape function belonging to one but not both of the spaces is contained in a common region whose measure tends to zero under mesh refinement. The bounds apply, in particular, to the setting in which the two finite element spaces consist of continuous functions that are elementwise polynomials over shape-regular, quasi-uniform meshes that coincide except on a region of measure $O(h^\gamma)$, where γ is a nonnegative scalar and h is the mesh spacing. The projector may be, for example, the orthogonal projector with respect to the L^2 - or H^1 -inner product. In these and other circumstances, the bounds are superconvergent under a few mild regularity assumptions. That is, under mesh refinement, the two projections differ in norm by an amount that decays to zero at a faster rate than the amounts by which each projection differs from u . We present numerical examples to illustrate these superconvergent estimates and verify the necessity of the regularity assumptions on u .

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1. INTRODUCTION

One of the hallmarks of the finite element method is its geometric flexibility: it permits the construction of numerical approximations to solutions of partial differential equations using meshes that are designed according to the practitioner's discretion. When two meshes are used to solve the same problem, the norm of the difference between the corresponding numerical solutions is, of course, no larger than the sum of the norms of the differences between each numerical solution and the exact solution. This paper addresses the question of whether or not a sharper estimate holds in the event that the two meshes coincide over a large fraction of the domain.

Beyond its inherent mathematical appeal, the question raised above has important consequences in the study of numerical solutions to time-dependent PDEs on meshes that change abruptly in time. Notable examples are remeshing during finite element simulations of problems with moving boundaries, and adaptive refinement during finite element simulations of problems on fixed (or moving) domains. The relevance of the aforementioned question in this setting is elucidated in [10], where it is shown that if a parabolic PDE is discretized in space with finite elements and the solution is transferred finitely many times between meshes using a suitable projector,

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then it is possible to derive an upper bound on the error in the numerical solution at a fixed time $T > 0$ that involves the norms of the jumps in $r_h u(t)$ across the remeshing times, where $r_h u(t)$ denotes an elliptic projection of the exact solution $u(t)$ onto the current finite element space. These jumps are precisely the differences between the finite element solutions of an elliptic PDE on two different meshes.

Intuition. It is perhaps not surprising that two finite element solutions associated with nearly identical meshes should differ by an amount that is small relative to their individual differences with the exact solution, under suitable conditions on the finite element spaces and the PDE under consideration. To develop some intuition, it is instructive to first consider the similarity between the *interpolants* of a smooth function u onto two finite element spaces associated with nearby meshes.

To this end, consider two families of shape-regular, quasi-uniform meshes $\{\mathcal{T}_h\}_{h \leq h_0}$ and $\{\mathcal{T}_h^+\}_{h \leq h_0}$ of an open, bounded, Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Assume that the two families are parametrized by a scalar h that equals the maximum diameter of an element among all elements of \mathcal{T}_h and \mathcal{T}_h^+ for every $h \leq h_0$, where h_0 is a positive scalar. Let \mathcal{V}_h and \mathcal{V}_h^+ be finite element spaces consisting of, for definiteness, continuous functions that are elementwise polynomials of degree at most $r - 1$ over \mathcal{T}_h and \mathcal{T}_h^+ , respectively, where $r > 1$ is an integer.

For $s \geq 0$ and $p \in [1, \infty]$, we denote by $W^{s,p}(\Omega)$ the Sobolev space of differentiability s and integrability p , equipped with the norm $\|\cdot\|_{s,p}$ and semi-norm $|\cdot|_{s,p}$. We sometimes write $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$ to emphasize the domain under consideration. We denote $H^s(\Omega) = W^{s,2}(\Omega)$ for every $s \geq 1$ and $L^p(\Omega) = W^{0,p}(\Omega)$ for every $p \in [1, \infty]$.

For finite element spaces of the aforementioned type, the nodal interpolants $i_h u \in \mathcal{V}_h$ and $i_h^+ u \in \mathcal{V}_h^+$ of a function $u \in W^{r,\eta}(\Omega) \cap C^0(\overline{\Omega})$ onto \mathcal{V}_h and \mathcal{V}_h^+ , respectively, satisfy the standard interpolation estimate

$$\|i_h^+ u - u\|_{s,\eta} + \|i_h u - u\|_{s,\eta} \leq Ch^{r-s} |u|_{r,\eta} \quad (1.1)$$

for any $s \in \{0, 1\}$, any $\eta \in [2, \infty]$, and every $h \leq h_0$ [9]. Here and throughout this paper, the letter C denotes a constant that is not necessarily the same at each occurrence and is independent of h .

Using the triangle inequality and (1.1) with $\eta = 2$ gives an immediate upper bound on the L^2 - and H^1 -norms of the difference between $i_h^+ u$ and $i_h u$. Namely,

$$\|i_h^+ u - i_h u\|_{s,2} \leq Ch^{r-s} |u|_{r,2} \quad (1.2)$$

for any $s \in \{0, 1\}$ and every $h \leq h_0$.

Suppose, however, that \mathcal{T}_h and \mathcal{T}_h^+ are nearby in the following sense: the two meshes coincide except on a region of measure $O(h^\gamma)$ for some scalar $\gamma \geq 0$. In this scenario, $i_h u$ and $i_h^+ u$ agree everywhere except in the region over which the meshes differ. Hence, by an application of Holder's inequality (*cf.* Lemma 3.1), the triangle inequality, and (1.1),

$$\begin{aligned} \|i_h^+ u - i_h u\|_{s,2} &\leq Ch^{\gamma(1/2-1/\eta)} \|i_h^+ u - i_h u\|_{s,\eta} \\ &\leq Ch^{\gamma(1/2-1/\eta)} (\|i_h^+ u - u\|_{s,\eta} + \|u - i_h u\|_{s,\eta}) \\ &\leq Ch^{r-s+\gamma(1/2-1/\eta)} |u|_{r,\eta} \end{aligned} \quad (1.3)$$

for any $s \in \{0, 1\}$, any $\eta \in [2, \infty]$, and every $h \leq h_0$.

A comparison of (1.3) with the naive estimate (1.2) reveals that $i_h u$ and $i_h^+ u$ are *superclose* in the L^2 - and H^1 -norms when the corresponding meshes are nearby. The primary goal of this paper is to prove an analogous superconvergent estimate when $i_h u$ and $i_h^+ u$ are replaced by the orthogonal projections $r_h u$ and $r_h^+ u$ of u onto \mathcal{V}_h and \mathcal{V}_h^+ , respectively, with respect to a coercive, continuous bilinear form $a : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$, where $\mathcal{V} \subseteq H^s(\Omega)$ and s is a nonnegative integer. As special cases, our results apply to L^2 -projections (the case $s = 0$) and elliptic projections (the case $s = 1$) onto piecewise polynomial finite element spaces. Another applicable case of interest is that in which the bilinear form a is of the form

$$a(u, w) = \int_{\Omega} \nabla u \cdot \nabla w \, dx - \int_{\Omega} (v \cdot \nabla u) w \, dx + \kappa \int_{\Omega} u w \, dx$$

with a constant $\kappa > 0$ and a vector field $v : \Omega \rightarrow \mathbb{R}^d$. This bilinear form appears in the analysis of finite element methods for the diffusion equation on a moving domain [10], with v playing the role of the velocity of a moving mesh and κ an auxiliary constant introduced to ensure coercivity.

It is not obvious that superconvergent estimates of the form (1.3) should hold in these settings, since the projections of u onto \mathcal{V}_h and \mathcal{V}_h^+ need not agree on the region over which the meshes coincide. Nevertheless, Corollaries 2.3 and 2.5 provide such estimates under suitable assumptions on the finite element spaces \mathcal{V}_h and \mathcal{V}_h^+ and the bilinear form a . The proof uses the observation that, loosely speaking, $a(r_h^+u - r_hu, r_h^+u - r_hu)$ is small if $r_h^+u - r_hu$ is well-approximated by an element of $\mathcal{V}_h^+ \cap \mathcal{V}_h$, since

$$a(r_h^+u - r_hu, w_h) = a(r_h^+u - u, w_h) + a(u - r_hu, w_h) = 0$$

for any $w_h \in \mathcal{V}_h^+ \cap \mathcal{V}_h$. In particular, if $\|r_h^+u - r_hu - w_h\|_{s,2}$ decays to zero more rapidly as $h \rightarrow 0$ than do $\|r_h^+u - u\|_{s,2}$ and $\|r_hu - u\|_{s,2}$, then a superconvergent estimate for $\|r_h^+u - r_hu\|_{s,2}$ follows from the relation

$$a(r_h^+u - r_hu, r_h^+u - r_hu) = a(r_h^+u - r_hu, r_h^+u - r_hu - w_h)$$

together with the coercivity and continuity of a . We in fact prove a more general result that applies to the case in which the projectors r_h and r_h^+ are associated not only with different subspaces \mathcal{V}_h and \mathcal{V}_h^+ , but also with different bilinear forms a_h and a_h^+ that may depend on h .

Organization. This paper is organized as follows. In Section 2, we summarize our main results. We begin with an abstract estimate (Thm. 2.1) for the H^s -norm of $r_h^+u - r_hu$. We then apply Theorem 2.1 to the setting of finite element spaces with nontrivial intersection in Theorem 2.2. Under some additional assumptions on the finite element spaces, the bilinear forms, and the regularity of u , we deduce in Corollary 2.3 a superconvergent estimate for $\|r_h^+u - r_hu\|_{s,2}$ that parallels (1.3). Next, we specialize to the case in which $s = 1$ and a_h and a_h^+ are bilinear forms associated with elliptic operators that possess smoothing properties. We use a duality argument to prove a superconvergent estimate (Thm. 2.4 and Cor. 2.5) for the L^2 -norm of $r_h^+u - r_hu$ that is up to one order higher than the corresponding estimate in the H^1 -norm given by Corollary 2.3.

In Section 3, we present proofs of the preceding results and provide a few remarks along the way.

In Section 4, we demonstrate the necessity of the regularity assumptions on u that are imposed in the theorems by exhibiting an example of a pair of projectors r_h and r_h^+ and a function u whose insufficient regularity leads to a reduction in the rates of convergence of $\|r_h^+u - r_hu\|_{1,2}$ and $\|r_h^+u - r_hu\|_{0,2}$.

Finally, we give numerical examples to illustrate our positive theoretical results in Section 5.

Related work. The results presented in this paper bear resemblance to the well-studied phenomenon of superconvergence in finite element theory, where the functions under comparison are typically the solution to a PDE and the numerical solution to a finite element discretization of the same problem. The phenomenon often manifests itself as an exceptional rate of convergence of the finite element solution to the exact solution at isolated points in the domain, as in [3, 5, 11, 15, 19, 20]. Related results involve exceptional rates of convergence of the finite element solution to a discrete representative of the exact solution, such as its interpolant [1, 4, 6, 13, 16–18]. Finally, post-processing techniques can lead to modifications of a finite element solution that converge more rapidly to the exact solution than the unprocessed finite element solution [2, 4, 7, 12, 14, 15, 21]. To our knowledge, however, little attention has been paid to the supercloseness of finite element solutions associated with differing meshes.

2. STATEMENT OF RESULTS

Notation. Fixing a nonnegative integer s and an open, bounded, Lipschitz domain $\Omega \subset \mathbb{R}^d$, let \mathcal{V} be a closed subspace of $H^s(\Omega)$. Let $a_h : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ and $a_h^+ : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ be bilinear forms that may depend on a parameter $h \leq h_0$, where h_0 is a positive scalar. We assume that a_h and a_h^+ are continuous and coercive uniformly in h . In

other words, for every $h \leq h_0$ and every $u, w \in \mathcal{V}$, the inequalities

$$\begin{aligned} a_h(u, u) &\geq \alpha \|u\|_{s,2}^2, \\ a_h(u, w) &\leq M \|u\|_{s,2} \|w\|_{s,2} \end{aligned}$$

hold with constants α and M independent of h , and similarly for a_h^+ (with the same constants α and M).

Let $\{\mathcal{V}_h\}_{0 < h \leq h_0}$ and $\{\mathcal{V}_h^+\}_{0 < h \leq h_0}$ be two families of finite element subspaces of \mathcal{V} . It is a consequence of the Lax–Milgram theorem that the maps $r_h : \mathcal{V} \rightarrow \mathcal{V}_h$ and $r_h^+ : \mathcal{V} \rightarrow \mathcal{V}_h^+$ defined by the relations

$$a_h(r_h u - u, w_h) = 0 \quad \forall w_h \in \mathcal{V}_h$$

and

$$a_h^+(r_h^+ u - u, w_h^+) = 0 \quad \forall w_h^+ \in \mathcal{V}_h^+,$$

respectively, are well-defined linear projectors.

For intuition, it is useful to think of \mathcal{V}_h and \mathcal{V}_h^+ as finite element spaces associated with a pair of meshes \mathcal{T}_h and \mathcal{T}_h^+ of Ω , with the parameter h denoting the maximum diameter of an element among all elements of \mathcal{T}_h and \mathcal{T}_h^+ . This level of concreteness, however, is not needed for a presentation of the results that follow.

Abstract estimate. Our first result is an abstract estimate for the H^s -norm of $r_h^+ u - r_h u$. It provides an alternative to the obvious upper bound

$$\|r_h^+ u - r_h u\|_{s,2} \leq \|r_h^+ u - u\|_{s,2} + \|u - r_h u\|_{s,2}$$

that one obtains from the triangle inequality. Its utility will be made apparent shortly.

Theorem 2.1. *Let a_h^+ and a_h be uniformly coercive and continuous bilinear forms on $\mathcal{V} \times \mathcal{V}$. Then for every $u \in \mathcal{V}$ and every $h \leq h_0$,*

$$\begin{aligned} \|r_h^+ u - r_h u\|_{s,2} \leq & \inf_{\substack{e_h \in \mathcal{V}_h \\ e_h^+ \in \mathcal{V}_h^+}} \left[\frac{M}{\alpha} \|r_h^+ u - r_h u - (e_h + e_h^+)\|_{s,2} + \frac{1}{\sqrt{\alpha}} \left(|a_h^+(r_h^+ u - u, e_h)|^{1/2} \right. \right. \\ & \left. \left. + |a_h(r_h u - u, e_h^+)|^{1/2} + |a_h^+(r_h u - u, e_h + e_h^+) - a_h(r_h u - u, e_h + e_h^+)|^{1/2} \right) \right]. \end{aligned} \quad (2.1)$$

The preceding theorem provides a heuristic for estimating the H^s -norm of $r_h^+ u - r_h u$. Namely, one seeks functions $e_h \in \mathcal{V}_h$ and $e_h^+ \in \mathcal{V}_h^+$ that are nearly (right-) orthogonal to $r_h^+ u - u$ and $r_h u - u$ with respect to $a_h^+(\cdot, \cdot)$ and $a_h(\cdot, \cdot)$, respectively, but whose sum is close to $r_h^+ u - r_h u$. In general, near orthogonality and closeness to $r_h^+ u - r_h u$ are competing interests. Exact orthogonality holds for $e_h, e_h^+ \in \mathcal{V}_h^+ \cap \mathcal{V}_h$, whereas $e_h + e_h^+$ can be made equal to $r_h^+ u - r_h u$ by choosing, for instance, $e_h^+ = r_h^+ u$ and $e_h = -r_h u$. If a suitable choice of e_h and e_h^+ leads to adequate satisfaction of both interests simultaneously, and if a_h^+ is close to a_h (in the sense that the last term in (2.1) is small), then the prospects of producing a superconvergent bound on $\|r_h^+ u - r_h u\|_{s,2}$ are favorable.

Finite element spaces with nontrivial intersection. We now apply Theorem 2.1 to the case in which the finite element spaces \mathcal{V}_h^+ and \mathcal{V}_h intersect nontrivially. The setting that we have in mind is that in which \mathcal{V}_h and \mathcal{V}_h^+ consist of continuous functions that are elementwise polynomials over shape-regular, quasi-uniform meshes of Ω that coincide except on a region of measure $O(h^\gamma)$ for some constant $\gamma \geq 0$. To allow for more generality, we state the assumptions on \mathcal{V}_h^+ and \mathcal{V}_h abstractly, and we refer the reader to Appendix A for a proof of their satisfaction in the aforementioned setting.

In particular, we assume the existence of a constant $\eta \in [2, \infty]$ such that the following properties hold:

(2.2.i) For every $h \leq h_0$, $\mathcal{V}_h, \mathcal{V}_h^+ \subset W^{s,\eta}(\Omega) \cap \mathcal{V}$.

(2.2.ii) There exists $C > 0$ independent of h such that the inverse estimate

$$\|w_h\|_{m,\eta} \leq Ch^{-m}\|w_h\|_{0,\eta}$$

holds for every $m = 0, 1, \dots, s$, every $w_h \in \mathcal{V}_h^+ \cap \mathcal{V}_h$, and every $h \leq h_0$.

(2.2.iii) There exist constants $\gamma \geq 0$ and $C > 0$ independent of h and a map $\pi_h : \mathcal{V}_h^+ + \mathcal{V}_h \rightarrow \mathcal{V}_h^+ \cap \mathcal{V}_h$ such that

$$\|\pi_h w_h\|_{0,\eta} \leq C\|w_h\|_{0,\eta}$$

and

$$|\text{supp}(\pi_h w_h - w_h)| \leq Ch^\gamma$$

for every $w_h \in \mathcal{V}_h^+ + \mathcal{V}_h$ and every $h \leq h_0$.

In the context of finite element spaces consisting of continuous functions that are elementwise polynomials over shape-regular, quasi-uniform meshes of Ω , a befitting choice for π_h in (2.2.iii) is the nodal interpolant onto $\mathcal{V}_h^+ \cap \mathcal{V}_h$; see Appendix B. In that setting, the constant γ appearing in (2.2.iii) may take on any real value between 0 and d , unless the two meshes coincide entirely (in which case γ may be chosen arbitrarily large). To realize a pair of meshes \mathcal{T}_h and \mathcal{T}_h^+ fulfilling (2.2.iii) with $\gamma \in [0, d]$, one may, for instance, consider a shape-regular, quasi-uniform mesh \mathcal{T}_h of Ω and perturb the positions of $O(h^{-d+\gamma})$ of its nodes by a sufficiently small amount to define \mathcal{T}_h^+ .

The following theorem results from applying Theorem 2.1 to the setting delineated in conditions (2.2.i–2.2.iii), with the choice $e_h = \pi_h(r_h^+ u - r_h u)$ and $e_h^+ = 0$ in (2.1).

Theorem 2.2. *Suppose the conditions of Theorem 2.1 hold and the finite element spaces \mathcal{V}_h^+ and \mathcal{V}_h satisfy conditions (2.2.i–2.2.iii). Suppose further that there exist constants $C_1 > 0$, $\delta \geq 0$, $1 \leq q \leq \eta$, and $\mu, \nu \in \{0, 1, \dots, s\}$ independent of h such that*

$$|a_h^+(v, w) - a_h(v, w)| \leq C_1 h^\delta \|v\|_{\mu,\eta} \|w\|_{\nu,q} \tag{2.2}$$

for every $v, w \in W^{s,\eta}(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$. Then there exists $C > 0$ independent of h such that for any $h \leq h_0$ and any $u \in W^{s,\eta}(\Omega) \cap \mathcal{V}$,

$$\begin{aligned} \|r_h^+ u - r_h u\|_{s,2} &\leq Ch^{\sigma-s} \left[h^s \|r_h^+ u - u\|_{s,\eta} + h^s \|r_h u - u\|_{s,\eta} + \|r_h^+ u - u\|_{0,\eta} + \|r_h u - u\|_{0,\eta} \right. \\ &\quad \left. + (h^\mu \|r_h u - u\|_{\mu,\eta})^{1/2} (\|r_h^+ u - u\|_{0,\eta} + \|r_h u - u\|_{0,\eta})^{1/2} \right] \end{aligned}$$

with

$$\sigma = \min \left\{ \gamma \left(\frac{1}{2} - \frac{1}{\eta} \right), \frac{\delta + 2s - \mu - \nu}{2} \right\}. \tag{2.3}$$

The meaning of Theorem 2.2 is clearest when the quantities $h^m \|r_h u - u\|_{m,p}$ and $h^m \|r_h^+ u - u\|_{m,p}$, $m = 0, 1, \dots, s$, $p = 2, \eta$, all decay at the same rate with respect to h as $h \rightarrow 0$. In such a setting, the theorem states that $\|r_h^+ u - r_h u\|_{s,2}$ tends to zero faster than $\|r_h u - u\|_{s,2} + \|r_h^+ u - u\|_{s,2}$ by a factor $O(h^\sigma)$, where the order of superconvergence σ depends primarily upon two features: (1) the extent to which the finite element spaces \mathcal{V}_h and \mathcal{V}_h^+ coincide, as measured by the constant γ in (2.2.iii), and (2) the difference between the bilinear forms a_h and a_h^+ , as measured by the constants δ , μ , and ν in (2.2). The regularity of u also plays a role in the estimate via the constant η , which is in the best case equal to ∞ .

To be more concrete, let us point out that in many contexts (which we detail in Appendix B), the quantities $r_h u - u$ and $r_h^+ u - u$ satisfy estimates of the form

$$\|r_h u - u\|_{0,\eta} + \|r_h^+ u - u\|_{0,\eta} \leq C\ell(h)h^r |u|_{r,\eta}, \tag{2.4}$$

$$\|r_h u - u\|_{m,\eta} + \|r_h^+ u - u\|_{m,\eta} \leq Ch^{r-m} |u|_{r,\eta}, \quad m = 1, 2, \dots, s, \tag{2.5}$$

for every $u \in W^{r,\eta}(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$, where $r > s$ is an integer and $\ell(h)$ is either identically unity or equal to $\log(h^{-1})$. Note that (2.5) is vacuous when $s = 0$. When such estimates hold, the following corollary to Theorem 2.2 is immediate.

Corollary 2.3. *Suppose that the conditions of Theorem 2.2 are satisfied and that both r_h and r_h^+ satisfy estimates of the form (2.4) and (2.5) for an integer $r > s$. Then there exists $C > 0$ independent of h such that*

$$\|r_h^+ u - r_h u\|_{s,2} \leq C \ell(h) h^{r-s+\sigma} |u|_{r,\eta}$$

for every $u \in W^{r,\eta}(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$, with σ given by (2.3).

In particular, if $a_h = a_h^+$, then

$$\|r_h^+ u - r_h u\|_{s,2} \leq C \ell(h) h^{r-s+\gamma(1/2-1/\eta)} |u|_{r,\eta}$$

for every $u \in W^{r,\eta}(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$.

Note that to deduce the preceding corollary, the case $a_h = a_h^+$ is handled by taking $\delta = \infty$ and choosing any admissible μ, ν and q in (2.2).

L^2 estimates for elliptic projections. Finally, we restrict our attention to the case $s = 1$ with $\mathcal{V} = H_0^1(\Omega)$, so that a_h and a_h^+ are coercive, continuous bilinear forms on $H_0^1(\Omega) \times H_0^1(\Omega)$, uniformly in h . Here, $H_0^1(\Omega)$ denotes the space of functions in $H^1(\Omega)$ with vanishing trace on $\partial\Omega$. Our aim is to provide an estimate for the L^2 -norm of $r_h^+ u - r_h u$ that parallels the estimate in the H^1 -norm provided by Corollary 2.3 but is of a higher order by up to one power of h .

In addition to the assumptions stated in Theorem 2.2, we make the following assumptions on the bilinear forms a_h and a_h^+ .

- (2.4.i) The bilinear forms a_h and a_h^+ are associated with elliptic operators whose adjoints possess *smoothing properties* (cf. [9], Def. 3.14), uniformly in h . Precisely, let $f \in L^2(\Omega)$ and consider the following problem: Find $w \in \mathcal{V}$ such that

$$a_h(y, w) = (f, y) \quad \forall y \in \mathcal{V}, \tag{2.6}$$

where $(f, y) := \int_{\Omega} f y$. Then a_h is said to have smoothing properties (uniformly in h) if there exists a constant $C > 0$ independent of h such that for every $f \in L^2(\Omega)$ and every $h \leq h_0$, there exists a unique solution w to (2.6) satisfying the elliptic regularity estimate

$$\|w\|_{2,2} \leq C \|f\|_{0,2}.$$

- (2.4.ii) There exists $C > 0$ such that for any $h \leq h_0$, any subdomain $R \subseteq \Omega$, and any $v, w \in \mathcal{V}$ with $\text{supp}(w) \subseteq R$,

$$|a_h(v, w)| \leq C \|v\|_{1,2,R} \|w\|_{1,2,R},$$

where the constant C is independent of h and R , and similarly for a_h^+ .

- (2.4.iii) The constant q appearing in the bound (2.2) satisfies the additional restriction

$$\begin{cases} q < \infty & \text{if } d = 4 - 2\nu, \\ q \leq \frac{2d}{d-4+2\nu} & \text{if } d > 4 - 2\nu. \end{cases}$$

Condition (2.4.iii) guarantees the validity of the Sobolev emdedding $H^2(\Omega) \subset W^{\nu,q}(\Omega)$. Note that it places no additional restriction on q if $d < 4 - 2\nu$.

Furthermore, we assume the existence of interpolation operators $i_h : \bar{\mathcal{V}} \rightarrow \mathcal{V}_h$ and $i_h^+ : \bar{\mathcal{V}} \rightarrow \mathcal{V}_h^+$ defined on a space $H^2(\Omega) \cap \mathcal{V} \subseteq \bar{\mathcal{V}} \subseteq \mathcal{V}$ that satisfy the following properties.

(2.4.iv) There exists $C > 0$ independent of h such that

$$\|i_h w\|_{\nu,q} + \|i_h^+ w\|_{\nu,q} \leq C \|w\|_{\nu,q}$$

for every $w \in H^2(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$.

(2.4.v) There exists $C > 0$ independent of h such that

$$\|i_h w - w\|_{1,2} + \|i_h^+ w - w\|_{1,2} \leq Ch |w|_{2,2}$$

for every $w \in H^2(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$.

(2.4.vi) For every $w \in H^2(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$,

$$\text{supp}(i_h^+ w - i_h w) \subseteq \mathcal{R}_h,$$

where

$$\mathcal{R}_h := \bigcup_{w_h \in \mathcal{V}_h + \mathcal{V}_h^+} \text{supp}(w_h - \pi_h w_h)$$

and π_h is the map introduced in (2.2.iii).

Our estimate for the L^2 -norm of $r_h^+ u - r_h u$, whose proof employs a duality argument, is as follows.

Theorem 2.4. *Suppose the conditions of Theorem 2.2 hold with $s = 1$. Assume further that conditions (2.4.i–2.4.vi) hold. Then there exists $C > 0$ independent of h such that for every $u \in W^{1,\eta}(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$,*

$$\begin{aligned} \|r_h^+ u - r_h u\|_{0,2} &\leq Ch^{\sigma'} \left[h \|r_h^+ u - u\|_{1,\eta} + h \|r_h u - u\|_{1,\eta} + \|r_h^+ u - u\|_{0,\eta} + \|r_h u - u\|_{0,\eta} \right. \\ &\quad \left. + (h^\mu \|r_h u - u\|_{\mu,\eta})^{1/2} (\|r_h^+ u - u\|_{0,\eta} + \|r_h u - u\|_{0,\eta})^{1/2} \right. \\ &\quad \left. + h^\mu \|r_h u - u\|_{\mu,\eta} \right], \end{aligned}$$

with

$$\sigma' = \min \left\{ \gamma \left(\frac{1}{2} - \frac{1}{\eta} \right), \frac{\delta + 2 - \mu - \nu}{2}, \delta - \mu \right\}. \tag{2.7}$$

Just as in Theorem 2.2, the meaning of Theorem 2.4 is clearest when the quantities $h^m \|r_h u - u\|_{m,p}$ and $h^m \|r_h^+ u - u\|_{m,p}$, $m = 0, 1, \dots, s$, $p = 2, \eta$, all decay at the same rate with respect to h as $h \rightarrow 0$. In such a setting, Theorem 2.4 states that $\|r_h^+ u - r_h u\|_{0,2}$ tends to zero faster than $\|r_h u - u\|_{0,2} + \|r_h^+ u - u\|_{0,2}$ by a factor $O(h^{\sigma'})$, where the order of superconvergence σ' is given by (2.7). Note that $\sigma' \leq \sigma$, where σ is the order of superconvergence of the H^1 -norm of $r_h^+ u - r_h u$ that was provided in Theorem 2.2.

Concretely, when estimates of the form (2.4) and (2.5) hold for $u \in W^{r,\eta}(\Omega) \cap \mathcal{V}$ with an integer $r > 1$, we arrive immediately at the following corollary to Theorem 2.4.

Corollary 2.5. *Suppose that the conditions of Theorem 2.2 are satisfied and that both r_h and r_h^+ satisfy estimates of the form (2.4) and (2.5) for an integer $r > 1$. Then there exists $C > 0$ independent of h such that*

$$\|r_h^+ u - r_h u\|_{0,2} \leq Cl(h) h^{r+\sigma'} |u|_{r,\eta}$$

for every $u \in W^{r,\eta}(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$, with σ' given by (2.7).

In particular, if $a_h = a_h^+$, then

$$\|r_h^+ u - r_h u\|_{0,2} \leq Cl(h) h^{r+\gamma(1/2-1/\eta)} |u|_{r,\eta}$$

for every $u \in W^{r,\eta}(\Omega) \cap \mathcal{V}$ and every $h \leq h_0$.

Note that to deduce the preceding corollary, the case $a_h = a_h^+$ is again handled by taking $\delta = \infty$ and choosing any admissible μ, ν and q in (2.2).

3. PROOFS

This section presents proofs of Theorems 2.1, 2.2, and 2.4.

Proof of Theorem 2.1. Let $e_h \in \mathcal{V}_h$ and $e_h^+ \in \mathcal{V}_h^+$, and write

$$a_h^+(r_h^+u - r_hu, r_h^+u - r_hu) = a_h^+(r_h^+u - r_hu, r_h^+u - r_hu - (e_h + e_h^+)) + a_h^+(r_h^+u - r_hu, e_h + e_h^+).$$

The uniform coercivity and continuity of a_h^+ imply

$$\|r_h^+u - r_hu\|_{s,2}^2 \leq \frac{1}{\alpha} (M\|r_h^+u - r_hu\|_{s,2}\|r_h^+u - r_hu - (e_h + e_h^+)\|_{s,2} + |a_h^+(r_h^+u - r_hu, e_h + e_h^+)|).$$

Using the fact that for real numbers $x, a, b \geq 0$,

$$x^2 \leq ax + b \implies x \leq a + \sqrt{b},$$

we deduce that

$$\|r_h^+u - r_hu\|_{s,2} \leq \frac{M}{\alpha}\|r_h^+u - r_hu - (e_h + e_h^+)\|_{s,2} + \frac{1}{\sqrt{\alpha}}|a_h^+(r_h^+u - r_hu, e_h + e_h^+)|^{1/2}$$

The result will then follow from the identity

$$a_h^+(r_h^+u - r_hu, e_h + e_h^+) = a_h^+(r_h^+u - u, e_h) + a_h(u - r_hu, e_h^+) + a_h^+(u - r_hu, e_h + e_h^+) - a_h(u - r_hu, e_h + e_h^+) \tag{3.1}$$

together with the subadditivity of the square root operator.

To prove (3.1), use the decomposition $r_h^+u - r_hu = (r_h^+u - u) + (u - r_hu)$ to write

$$a_h^+(r_h^+u - r_hu, e_h + e_h^+) = a_h^+(r_h^+u - u, e_h + e_h^+) + a_h^+(u - r_hu, e_h + e_h^+).$$

Now add and subtract $a_h(u - r_hu, e_h + e_h^+)$ to obtain

$$a_h^+(r_h^+u - r_hu, e_h + e_h^+) = a_h^+(r_h^+u - u, e_h + e_h^+) + a_h(u - r_hu, e_h + e_h^+) + a_h^+(u - r_hu, e_h + e_h^+) - a_h(u - r_hu, e_h + e_h^+).$$

Finally, use the definitions of r_h^+ and r_h to simplify the first two terms, giving (3.1). □

We remark that while the estimate (2.1) is not symmetric in the “+” variables and their unadorned counterparts, it can easily be made symmetric by exchanging the roles of r_h^+ and a_h^+ with r_h and a_h , respectively, and averaging the resulting estimates. The same holds true for the estimates in Theorems 2.2 and 2.4.

We now turn to the proof of Theorem 2.2. We begin with a lemma concerning the relationship between a function’s support and its Sobolev norms.

Lemma 3.1. *Let $f \in W^{k,p}(\Omega)$, $k \geq 0$, $p \in [1, \infty]$. Then for any $1 \leq t \leq p$,*

$$\|f\|_{k,t} \leq |\text{supp}(f)|^{1/t-1/p} \|f\|_{k,p}.$$

Proof. Let $\chi : \Omega \rightarrow \{0, 1\}$ denote the indicator function for $\text{supp}(f)$. We have

$$\begin{aligned} \|f\|_{k,t} &= \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{0,t} \\ &= \sum_{|\alpha| \leq k} \|\chi \partial^\alpha f\|_{0,t}. \end{aligned}$$

Now let $\tilde{p} \in [1, \infty]$ be such that $\frac{1}{\tilde{p}} + \frac{1}{p} = \frac{1}{t}$. By Holder's inequality,

$$\begin{aligned} \|f\|_{k,t} &\leq \sum_{|\alpha| \leq k} \|\chi\|_{0,\tilde{p}} \|\partial^\alpha f\|_{0,p} \\ &= |\text{supp}(f)|^{1/\tilde{p}} \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{0,p} \\ &= |\text{supp}(f)|^{1/t-1/p} \|f\|_{k,p}. \end{aligned}$$

□

The Proof of Theorem 2.2 is as follows.

Proof of Theorem 2.2. Choose $e_h^+ = 0$ and $e_h = \pi_h(r_h^+ u - r_h u)$ in (2.1). By the stability assumption in (2.2.iii),

$$\begin{aligned} \|e_h\|_{0,\eta} &\leq C \|r_h^+ u - r_h u\|_{0,\eta} \\ &\leq C (\|r_h^+ u - u\|_{0,\eta} + \|u - r_h u\|_{0,\eta}). \end{aligned}$$

Thus, for any $m = 0, 1, \dots, s$,

$$\|e_h\|_{m,\eta} \leq Ch^{-m} (\|r_h^+ u - u\|_{0,\eta} + \|u - r_h u\|_{0,\eta}) \tag{3.2}$$

by (2.2.ii). It follows that

$$\begin{aligned} \|r_h^+ u - r_h u - (e_h + e_h^+)\|_{s,\eta} &\leq \|r_h^+ u - u\|_{s,\eta} + \|u - r_h u\|_{s,\eta} + \|e_h\|_{s,\eta} + \|e_h^+\|_{s,\eta} \\ &\leq C (\|r_h^+ u - u\|_{s,\eta} + \|u - r_h u\|_{s,\eta} \\ &\quad + h^{-s} \|r_h^+ u - u\|_{0,\eta} + h^{-s} \|u - r_h u\|_{0,\eta}). \end{aligned}$$

Now note that $r_h^+ u - r_h u - (e_h + e_h^+)$ has support of measure $O(h^\gamma)$ by (2.2.iii). Consequently, by Lemma 3.1,

$$\begin{aligned} \|r_h^+ u - r_h u - (e_h + e_h^+)\|_{s,2} &\leq Ch^{\gamma(1/2-1/\eta)} \|r_h^+ u - r_h u - (e_h + e_h^+)\|_{s,\eta} \\ &\leq Ch^{\gamma(1/2-1/\eta)} (\|r_h^+ u - u\|_{s,\eta} + \|r_h u - u\|_{s,\eta} \\ &\quad + h^{-s} \|r_h^+ u - u\|_{0,\eta} + h^{-s} \|r_h u - u\|_{0,\eta}). \end{aligned} \tag{3.3}$$

To estimate the remaining terms that appear in (2.1), note that

$$a_h^+(r_h^+ u - u, e_h) = 0$$

since $e_h \in \mathcal{V}_h^+ \cap \mathcal{V}_h \subseteq \mathcal{V}_h^+$, and

$$a_h(r_h u - u, e_h^+) = 0$$

since $e_h^+ = 0$. Finally, using (3.2) with $m = \nu$ together with (2.2) shows that

$$\begin{aligned} &|a_h^+(r_h u - u, e_h + e_h^+) - a_h(r_h u - u, e_h + e_h^+)| \\ &\leq Ch^\delta \|r_h u - u\|_{\mu,\eta} \|e_h\|_{\nu,q} \\ &\leq Ch^\delta \|r_h u - u\|_{\mu,\eta} \|e_h\|_{\nu,\eta} \\ &\leq Ch^{\delta-\nu} \|r_h u - u\|_{\mu,\eta} (\|r_h^+ u - u\|_{0,\eta} + \|u - r_h u\|_{0,\eta}). \end{aligned}$$

Taking the square root and adding (3.3) proves the claim. □

Note that the preceding proof treats the estimate (2.2) wastefully when $q < \eta$, in the sense that the ultimate bound on $\|r_h^+ u - r_h u\|_{s,2}$ is unchanged if q is replaced by η . The importance of considering scenarios in which q may be chosen less than η is made apparent in Theorem 2.4, where the restriction (2.4.iii) is enforced.

With this in mind, we now prove Theorem 2.4.

Proof of Theorem 2.4. Define $w \in \mathcal{V}$ as the solution to the dual problem

$$a_h^+(y, w) = (r_h^+ u - r_h u, y) \quad \forall y \in \mathcal{V}. \quad (3.4)$$

Note that $w \in H^2(\Omega) \cap \mathcal{V}$ by (2.4.i).

For any $w_h^+ \in \mathcal{V}_h^+$, $w_h \in \mathcal{V}_h$, we have

$$\begin{aligned} \|r_h^+ u - r_h u\|_{0,2}^2 &= a_h^+(r_h^+ u - r_h u, w) \\ &= a_h^+(r_h^+ u - r_h u, w - w_h^+) + a_h^+(r_h^+ u - r_h u, w_h^+) \\ &= a_h^+(r_h^+ u - r_h u, w - w_h^+) + a_h^+(u - r_h u, w_h^+) \\ &= a_h^+(r_h^+ u - r_h u, w - w_h^+) + a_h^+(u - r_h u, w_h^+ - w_h) \\ &\quad + a_h^+(u - r_h u, w_h) - a_h(u - r_h u, w_h) \\ &=: T_1 + T_2 + T_3, \end{aligned}$$

where

$$\begin{aligned} T_1 &= a_h^+(r_h^+ u - r_h u, w - w_h^+), \\ T_2 &= a_h^+(u - r_h u, w_h^+ - w_h), \\ T_3 &= a_h^+(u - r_h u, w_h) - a_h(u - r_h u, w_h). \end{aligned}$$

Now choose $w_h^+ = i_h^+ w$ and $w_h = i_h w$ and bound each term separately. By the continuity of a_h^+ and (2.4.v),

$$\begin{aligned} |T_1| &\leq C \|r_h^+ u - r_h u\|_{1,2} \|w - w_h^+\|_{1,2} \\ &\leq Ch \|r_h^+ u - r_h u\|_{1,2} |w|_{2,2}. \end{aligned}$$

To bound T_2 , note that $\text{supp}(w_h^+ - w_h) \subseteq \mathcal{R}_h$ has measure $O(h^\gamma)$ by (2.4.vi) and (2.2.iii). Thus,

$$\begin{aligned} |T_2| &\leq C \|u - r_h u\|_{1,2,\mathcal{R}_h} \|w_h^+ - w_h\|_{1,2,\mathcal{R}_h} \\ &\leq Ch^{\gamma(1/2-1/\eta)} \|u - r_h u\|_{1,\eta} (\|w_h^+ - w\|_{1,2,\mathcal{R}_h} + \|w - w_h\|_{1,2,\mathcal{R}_h}) \\ &\leq Ch^{\gamma(1/2-1/\eta)+1} \|u - r_h u\|_{1,\eta} |w|_{2,2} \end{aligned}$$

by (2.4.ii), Lemma 3.1, and (2.4.v). For T_3 , we have by (2.2) that

$$|T_3| \leq Ch^\delta \|u - r_h u\|_{\mu,\eta} \|w_h\|_{\nu,q}.$$

Using (2.4.iv) together with the Sobolev embedding $H^2(\Omega) \subset W^{\nu,q}(\Omega)$ ensured by (2.4.iii) gives

$$|T_3| \leq Ch^\delta \|u - r_h u\|_{\mu,\eta} \|w\|_{2,2}.$$

Combining results and invoking the regularity estimate (i) leads to

$$\|r_h^+ u - r_h u\|_{0,2} \leq C \left[h \|r_h^+ u - r_h u\|_{1,2} + h^{\min\{\gamma(1/2-1/\eta), \delta-\mu\}} (h \|u - r_h u\|_{1,\eta} + h^\mu \|u - r_h u\|_{\mu,\eta}) \right].$$

Conclude using Theorem 2.2. □

4. THE NEED FOR REGULARITY

When $a_h^+ = a_h$ and γ is fixed, the estimates of Corollaries 2.3 and 2.5 are of the highest order in h when $\eta = \infty$, but in this case they demand that $u \in W^{r,\infty}(\Omega) \cap \mathcal{V}$. If the regularity requirement $u \in W^{r,\infty}(\Omega) \cap \mathcal{V}$ is relaxed, the rates of convergence of $\|r_h^+ u - r_h u\|_{0,2}$ and $\|r_h^+ u - r_h u\|_{1,2}$ as $h \rightarrow 0$ may deteriorate.

Indeed, consider the case in which \mathcal{V}_h is the space of continuous, piecewise affine functions on a grid $(0, h, 2h, 3h, \dots, 1)$ of the unit interval in one dimension that vanish at 0 and 1. Let \mathcal{V}_h^+ be the space of continuous, piecewise affine functions on the nearby grid $(0, 3h/2, 2h, 3h, \dots, 1)$ that vanish at 0 and 1. Let

$$a_h^+(u, w) = a_h(u, w) = \int_0^1 \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} dx,$$

so that the projectors r_h and r_h^+ coincide with the nodal interpolants onto \mathcal{V}_h and \mathcal{V}_h^+ , respectively ([9], Rem. 3.25(i)). In this setting, the conditions of Corollaries 2.3 and 2.5 hold with $\eta = \infty$, $\gamma = 1$, $r = 2$, and $\ell(h) \equiv 1$, leading to the estimates

$$\begin{aligned} \|r_h^+ u - r_h u\|_{0,2} &\leq Ch^{5/2} |u|_{r,\infty}, \\ \|r_h^+ u - r_h u\|_{1,2} &\leq Ch^{3/2} |u|_{r,\infty} \end{aligned}$$

for $u \in W^{2,\infty}(0, 1) \cap H_0^1(0, 1)$.

However, consider the function

$$u(x) = x^{2-1/p} - x$$

with $2 < p < \infty$, so that $u \in W^{2,p-\varepsilon}(0, 1) \cap H_0^1(0, 1)$ for any $\varepsilon > 0$. Then a direct calculation renders that

$$\begin{aligned} \|r_h^+ u - r_h u\|_{0,2} &\geq Ch^{5/2-1/p}, \\ \|r_h^+ u - r_h u\|_{1,2} &\geq Ch^{3/2-1/p}, \end{aligned}$$

which are of a lower order than the rates $h^{5/2}$ and $h^{3/2}$, respectively, obtainable for a function in $W^{2,\infty}(0, 1) \cap H_0^1(0, 1)$. In fact, by letting $p \rightarrow 2$, these rates can be made arbitrarily close to the quadratic and linear rates that hold in the L^2 - and H^1 -norms, respectively, on a pair of unrelated meshes.

5. NUMERICAL EXAMPLES

In this section, we numerically illustrate the superconvergent estimates of Corollaries 2.3 and 2.5 on test cases in one and two dimensions.

One dimension. Consider the case in which \mathcal{V}_h is the space of continuous, piecewise polynomial functions of degree at most $r - 1$ on a grid $(0, h, 2h, 3h, \dots, 1)$ of the unit interval in one dimension that vanish at 0 and 1. Let \mathcal{V}_h^+ be the space of continuous, piecewise polynomial functions of the same degree that vanish at 0 and 1, on the same grid but with the node nearest to $x = 1/4$ perturbed by $h/4$ in the positive direction. In this scenario, assumption (2.2.iii) is satisfied with $\gamma = 1$. Let $u(x) = \sin(\pi x)$ and let

$$a_h^+(u, w) = a_h(u, w) = \int_0^1 u w dx,$$

so that r_h and r_h^+ are the L^2 -projectors onto \mathcal{V}_h and \mathcal{V}_h^+ , respectively.

Table 1 shows the L^2 -norm of the difference $r_h^+ u - r_h u$ for several values of h , beginning with $h = 1/8 =: h_0$. The table illustrates the predictions of Corollary 2.3, namely

$$\|r_h^+ u - r_h u\|_{0,2} \leq \begin{cases} Ch^{5/2} |u|_{2,\infty} & \text{if } r = 2, \\ Ch^{7/2} |u|_{3,\infty} & \text{if } r = 3. \end{cases}$$

TABLE 1. L^2 -supercloseness of L^2 -projections onto piecewise affine ($r = 2$) and piecewise quadratic ($r = 3$) finite element spaces over nearby meshes ($\gamma = 1$) in one dimension.

| h_0/h | Affine ($r = 2$) | | Quadratic ($r = 3$) | |
|---------|-----------------------------|--------|-----------------------------|--------|
| | $\ r_h^+ u - r_h u\ _{0,2}$ | Order | $\ r_h^+ u - r_h u\ _{0,2}$ | Order |
| 1 | 3.2150e-03 | – | 1.2843e-04 | – |
| 2 | 5.6505e-04 | 2.5084 | 1.0676e-05 | 3.5886 |
| 4 | 9.9837e-05 | 2.5007 | 9.1277e-07 | 3.5480 |
| 8 | 1.7645e-05 | 2.5003 | 7.9301e-08 | 3.5248 |
| 16 | 3.1189e-06 | 2.5002 | 6.9484e-09 | 3.5126 |
| 32 | 5.5132e-07 | 2.5001 | 6.1146e-10 | 3.5063 |

TABLE 2. H^1 -supercloseness of elliptic projections onto piecewise affine ($r = 2$) and piecewise quadratic ($r = 3$) finite element spaces over nearby meshes ($\gamma = 1$) in one dimension.

| h_0/h | Affine ($r = 2$) | | Quadratic ($r = 3$) | |
|---------|-----------------------------|--------|-----------------------------|--------|
| | $\ r_h^+ u - r_h u\ _{1,2}$ | Order | $\ r_h^+ u - r_h u\ _{1,2}$ | Order |
| 1 | 1.4451e-01 | – | 7.4390e-03 | – |
| 2 | 5.1203e-02 | 1.4968 | 1.2835e-03 | 2.5351 |
| 4 | 1.8081e-02 | 1.5017 | 2.2408e-04 | 2.5180 |
| 8 | 6.3851e-03 | 1.5017 | 3.9364e-05 | 2.5090 |
| 16 | 2.2558e-03 | 1.5011 | 6.9369e-06 | 2.5045 |
| 32 | 7.9723e-04 | 1.5006 | 1.2243e-06 | 2.5023 |

Next, consider the same setup as above, but with

$$a_h^+(u, w) = a_h(u, w) = \int_0^1 \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} dx,$$

so that r_h and r_h^+ are the standard elliptic projectors onto \mathcal{V}_h and \mathcal{V}_h^+ , respectively. Table 2 shows the H^1 -norm of the difference $r_h^+ u - r_h u$ for the sequence of grids described above. The table illustrates the predictions of Corollary 2.3, namely

$$\|r_h^+ u - r_h u\|_{1,2} \leq \begin{cases} Ch^{3/2} \log(h^{-1}) |u|_{2,\infty} & \text{if } r = 2, \\ Ch^{5/2} |u|_{3,\infty} & \text{if } r = 3. \end{cases}$$

Table 3 shows the L^2 -norm of the difference $r_h^+ u - r_h u$ for the same sequence of grids. The table illustrates the predictions of Corollary 2.5, namely

$$\|r_h^+ u - r_h u\|_{0,2} \leq \begin{cases} Ch^{5/2} \log(h^{-1}) |u|_{2,\infty} & \text{if } r = 2, \\ Ch^{7/2} |u|_{3,\infty} & \text{if } r = 3. \end{cases}$$

Note that we have not attempted to detect the presence of the factor $\log(h^{-1})$ in these numerical experiments.

Two dimensions. Consider now the case in which $\mathcal{V}_h \subset H_0^1((0, 1) \times (0, 1))$ is the space of continuous, piecewise affine functions on a mesh of the unit square in two dimensions consisting of equally sized isosceles right triangles, as in Figure 1a. Let $\mathcal{V}_h^+ \subset H_0^1((0, 1) \times (0, 1))$ be the space of continuous, piecewise affine functions on the same mesh, but with the node nearest to $(x, y) = (1/4, 1/4)$ perturbed by $h/4$ in the positive x direction, as in Figure 1b. In this scenario, assumption (2.2.iii) is satisfied with $\gamma = 2$. Let $u(x) = \sin(\pi x) \sin(\pi y)$ and let

$$a_h^+(u, w) = a_h(u, w) = \int_0^1 \int_0^1 uw \, dx dy,$$

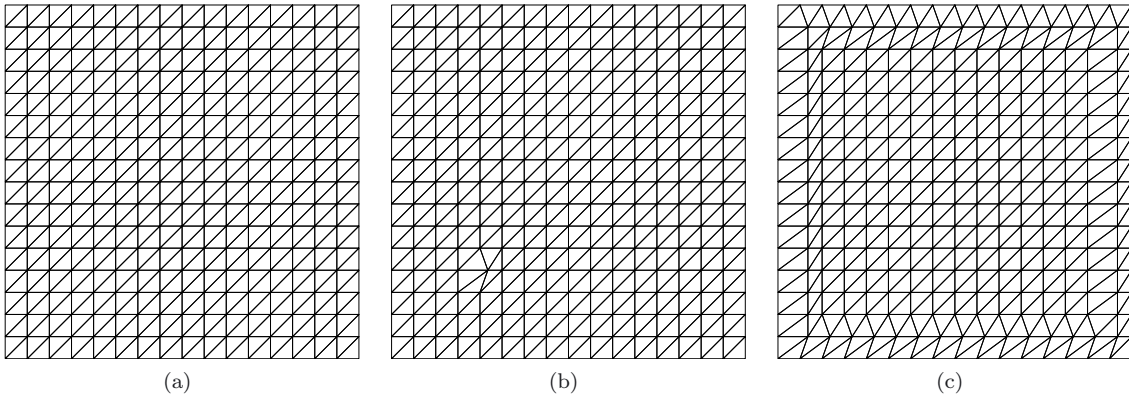


FIGURE 1. (a) Mesh of the unit square consisting of equally sized isosceles right triangles. (b) Identical mesh, but with the node at $(x, y) = (1/4, 1/4)$ perturbed by $h/4$ in the positive x direction. (c) Identical mesh, but with all nodes having distance $h/\sqrt{2}$ from the boundary perturbed by $h/4$ in the positive x direction.

TABLE 3. L^2 -supercloseness of elliptic projections onto piecewise affine ($r = 2$) and piecewise quadratic ($r = 3$) finite element spaces over nearby meshes ($\gamma = 1$) in one dimension.

| h_0/h | Affine ($r = 2$) | | Quadratic ($r = 3$) | |
|---------|-----------------------------|--------|-----------------------------|--------|
| | $\ r_h^+ u - r_h u\ _{0,2}$ | Order | $\ r_h^+ u - r_h u\ _{0,2}$ | Order |
| 1 | 3.4546e-03 | – | 1.7770e-04 | – |
| 2 | 6.1937e-04 | 2.4796 | 1.5493e-05 | 3.5198 |
| 4 | 1.1019e-04 | 2.4908 | 1.3576e-06 | 3.5124 |
| 8 | 1.9537e-05 | 2.4957 | 1.1943e-07 | 3.5069 |
| 16 | 3.4587e-06 | 2.4979 | 1.0530e-08 | 3.5036 |
| 32 | 6.1186e-07 | 2.4990 | 9.2955e-10 | 3.5018 |

TABLE 4. L^2 -supercloseness of L^2 -projections onto piecewise affine ($r = 2$) finite element spaces over nearby meshes ($\gamma = 2$; see Figs. 1a and 1b) in two dimensions.

| Affine ($r = 2$) | | |
|--------------------|-----------------------------|--------|
| h_0/h | $\ r_h^+ u - r_h u\ _{0,2}$ | Order |
| 1 | 6.3533e-03 | – |
| 2 | 7.5614e-04 | 3.0708 |
| 4 | 8.8718e-05 | 3.0914 |
| 8 | 1.1020e-05 | 3.0091 |
| 16 | 1.3781e-06 | 2.9993 |

so that r_h and r_h^+ are the L^2 -projectors onto \mathcal{V}_h and \mathcal{V}_h^+ , respectively.

Table 4 shows the L^2 -norm of the difference $r_h^+ u - r_h u$ for several values of h , beginning with $h = \sqrt{2}/4 =: h_0$. The table illustrates the predictions of Corollary 2.3, namely

$$\|r_h^+ u - r_h u\|_{0,2} \leq Ch^3 |u|_{2,\infty}. \tag{5.1}$$

TABLE 5. H^1 - and L^2 -supercloseness of elliptic projections onto piecewise affine ($r = 2$) finite element spaces over nearby meshes ($\gamma = 2$; see Figs. 1a and 1b) in two dimensions.

| Affine ($r = 2$) | | | | |
|--------------------|-----------------------------|--------|-----------------------------|--------|
| h_0/h | $\ r_h^+ u - r_h u\ _{1,2}$ | Order | $\ r_h^+ u - r_h u\ _{0,2}$ | Order |
| 1 | 2.1441e-01 | – | 6.6386e-03 | – |
| 2 | 4.7374e-02 | 2.1782 | 7.8678e-04 | 3.0768 |
| 4 | 1.1359e-02 | 2.0603 | 9.6370e-05 | 3.0293 |
| 8 | 2.8114e-03 | 2.0144 | 1.2033e-05 | 3.0016 |
| 16 | 7.0176e-04 | 2.0023 | 1.5106e-06 | 2.9937 |

Next, consider the same setup as above, but with

$$a_h^+(u, w) = a_h(u, w) = \int_0^1 \int_0^1 \left(\frac{\partial u}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial w}{\partial y} \right) dx dy,$$

so that r_h and r_h^+ are the elliptic projectors onto \mathcal{V}_h and \mathcal{V}_h^+ , respectively. Table 5 shows the H^1 - and L^2 -norms of the difference $r_h^+ u - r_h u$ for the sequence of meshes described above. The table illustrates the predictions of Corollaries 2.3 and 2.5, namely

$$\|r_h^+ u - r_h u\|_{m,2} \leq \begin{cases} Ch^2 \log(h^{-1})|u|_{2,\infty} & \text{if } m = 0, \\ Ch^3 \log(h^{-1})|u|_{2,\infty} & \text{if } m = 1. \end{cases} \quad (5.2)$$

Again, we have not attempted to detect the presence of the factor $\log(h^{-1})$.

More substantial mesh perturbation in two dimensions. Finally, consider the same two-dimensional tests as above, but with the mesh of Figure 1b replaced by a different perturbation of the uniform mesh. Namely, consider perturbing all nodes whose distance from the boundary of the unit square is equal to $h/\sqrt{2}$ (the length of the shortest edge of each triangle) *via* a translation by $h/4$ in the positive x direction, as in Figure 1c.

In this scenario, assumption (2.2.iii) is satisfied with $\gamma = 1$, so that the estimates (5.1) and (5.2) no longer apply. Their analogues in this case read

$$\|r_h^+ u - r_h u\|_{0,2} \leq Ch^{5/2}|u|_{2,\infty}.$$

and

$$\|r_h^+ u - r_h u\|_{m,2} \leq \begin{cases} Ch^{3/2} \log(h^{-1})|u|_{2,\infty} & \text{if } m = 0, \\ Ch^{5/2} \log(h^{-1})|u|_{2,\infty} & \text{if } m = 1, \end{cases}$$

respectively. Tables 6 and 7 illustrate these predictions. Again, we have not attempted to detect the presence of the factor $\log(h^{-1})$.

6. SUMMARY

We have derived estimates for the difference between the orthogonal projections $r_h u$ and $r_h^+ u$ of a smooth function u onto nearby finite element spaces \mathcal{V}_h and \mathcal{V}_h^+ , respectively, with respect to bilinear forms $a_h, a_h^+ : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$, respectively, where \mathcal{V} is a closed subspace of $H^s(\Omega)$. When $s \in \{0, 1\}$ and \mathcal{V}_h and \mathcal{V}_h^+ consist of continuous functions that are elementwise polynomials over shape-regular, quasi-uniform meshes that coincide except on a region of measure $O(h^\gamma)$ for a constant $\gamma \geq 0$, the estimates for $\|r_h^+ u - r_h u\|_{s,2}$ are superconvergent by $O(h^{\gamma/2})$, provided that $u \in W^{s,\infty}(\Omega)$ and a_h and a_h^+ are sufficiently close. In addition, when $s = 1$ and a few more mild assumptions (namely (2.4.i)–(2.4.vi)) are satisfied, an $O(h^{\gamma/2})$ -superconvergent estimate for $\|r_h^+ u - r_h u\|_{0,2}$ holds. Numerical experiments illustrated these estimates and verified the necessity of the regularity assumptions on u .

TABLE 6. L^2 -supercloseness of L^2 -projections onto piecewise affine ($r = 2$) finite element spaces over nearby meshes ($\gamma = 1$; see Figs. 1a and 1c) in two dimensions. Relative to Table 4, a lower order of superconvergence is observed due to the larger fraction of perturbed elements present in the perturbed mesh.

| Affine ($r = 2$) | | |
|--------------------|-----------------------------|--------|
| h_0/h | $\ r_h^+ u - r_h u\ _{0,2}$ | Order |
| 1 | 2.2504e-02 | – |
| 2 | 4.8445e-03 | 2.2158 |
| 4 | 1.0019e-03 | 2.2736 |
| 8 | 1.9159e-04 | 2.3866 |
| 16 | 3.5132e-05 | 2.4472 |
| 32 | 6.3195e-06 | 2.4749 |

TABLE 7. H^1 - and L^2 -supercloseness of elliptic projections onto piecewise affine ($r = 2$) finite element spaces over nearby meshes ($\gamma = 1$; see Figs. 1a and 1c) in two dimensions. Relative to Table 5, lower orders of superconvergence are observed due to the larger fraction of perturbed elements present in the perturbed mesh.

| Affine ($r = 2$) | | | | |
|--------------------|-----------------------------|--------|-----------------------------|--------|
| h_0/h | $\ r_h^+ u - r_h u\ _{1,2}$ | Order | $\ r_h^+ u - r_h u\ _{0,2}$ | Order |
| 1 | 5.4318e-01 | – | 1.9864e-02 | – |
| 2 | 2.8504e-01 | 0.9303 | 4.8794e-03 | 2.0254 |
| 4 | 1.2522e-01 | 1.1867 | 1.0528e-03 | 2.2125 |
| 8 | 4.8674e-02 | 1.3632 | 1.9842e-04 | 2.4075 |
| 16 | 1.7931e-02 | 1.4407 | 3.5671e-05 | 2.4758 |
| 32 | 6.4595e-03 | 1.4730 | 6.3290e-06 | 2.4947 |

APPENDIX A. PROPERTIES OF PIECEWISE POLYNOMIAL FINITE ELEMENT SPACES

In this section, we verify conditions (2.2.i)–(2.2.iii) for piecewise polynomial finite element spaces on nearby meshes for the cases $s = 0$ and $s = 1$.

As in Section 1, consider two families of shape-regular, quasi-uniform meshes $\{\mathcal{T}_h\}_{h \leq h_0}$ and $\{\mathcal{T}_h^+\}_{h \leq h_0}$ of an open, bounded, Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Assume that the two families are parametrized by a scalar h that equals the maximum diameter of an element among all elements of \mathcal{T}_h and \mathcal{T}_h^+ for every $h \leq h_0$. Let \mathcal{V}_h and \mathcal{V}_h^+ be finite element spaces consisting of continuous functions that are elementwise polynomials of degree at most $r - 1$ over \mathcal{T}_h and \mathcal{T}_h^+ , respectively, where $r > 1$ is an integer.

In this setting, condition (2.2.i) is automatic for any $\eta \in [2, \infty]$, $s \in \{0, 1\}$. Condition (2.2.ii) is trivial for $s = 0$ and is satisfied for $s = 1$ and any $\eta \in [2, \infty]$ [9].

Condition (2.2.iii) holds for any $\eta \in [2, \infty]$ when \mathcal{T}_h and \mathcal{T}_h^+ coincide except on a region of measure $O(h^\gamma)$. To prove this, let $\{N_a\}_{a=1}^A \subset \mathcal{V}_h$ and $\{N_a^+\}_{a=1}^{A^+} \subset \mathcal{V}_h^+$ be the standard Lagrange shape functions that form bases for \mathcal{V}_h and \mathcal{V}_h^+ , respectively. Our assumptions on \mathcal{T}_h and \mathcal{T}_h^+ imply the existence of an integer I such that $N_a = N_a^+$ for every $1 \leq a \leq I$ and such that

$$\left| \left(\bigcup_{a=I+1}^A \text{supp}(N_a) \right) \cup \left(\bigcup_{a=I+1}^{A^+} \text{supp}(N_a^+) \right) \right| \leq Ch^\gamma \tag{A.1}$$

for every $h \leq h_0$.

Define $\pi_h : \mathcal{V}_h^+ + \mathcal{V}_h \rightarrow \mathcal{V}_h^+ \cap \mathcal{V}_h$ as follows: For any

$$w_h = \sum_{a=1}^I c_a N_a + \sum_{a=I+1}^A c_a N_a + \sum_{a=I+1}^{A^+} c_a^+ N_a^+ \tag{A.2}$$

belonging to $\mathcal{V}_h^+ + \mathcal{V}_h$, set

$$\pi_h w_h := \sum_{a=1}^I c_a N_a. \tag{A.3}$$

Clearly,

$$|\text{supp}(\pi_h w_h - w_h)| \leq Ch^\gamma$$

for every $w_h \in \mathcal{V}_h^+ + \mathcal{V}_h$ and every $h \leq h_0$. To prove that

$$\|\pi_h w_h\|_{0,\eta} \leq C \|w_h\|_{0,\eta} \tag{A.4}$$

for every $w_h \in \mathcal{V}_h^+ + \mathcal{V}_h$ and every $h \leq h_0$, there are two cases to consider: $\eta = \infty$ and $2 \leq \eta < \infty$.

For $\eta = \infty$, it is enough to note that for each of the two finite element spaces, every shape function is bounded uniformly in h in the maximum norm, the number of shape functions whose support intersects any given element is bounded uniformly in h , and the coefficients c_a , $1 \leq a \leq I$, in the expansion (A.2) of w_h are bounded by $\|w_h\|_{0,\infty}$. Indeed, the standard degrees of freedom σ_a , $1 \leq a \leq I$, for the Lagrange shape functions $N_a (= N_a^+)$, $1 \leq a \leq I$, satisfy

$$\sigma_a(N_b) = \delta_{ab}, \quad 1 \leq b \leq A$$

and

$$\sigma_a(N_b^+) = \delta_{ab}, \quad 1 \leq b \leq A^+,$$

where δ_{ab} denotes the Kronecker delta. Hence, for any $1 \leq a \leq I$,

$$|c_a| = |\sigma_a(w_h)| \leq \|w_h\|_{0,\infty}.$$

For $2 \leq \eta < \infty$, the proof of (A.4) relies on the following lemma.

Lemma A.1. *Let $\{\mathcal{T}_h\}_{h \leq h_0}$ be a shape-regular, quasi-uniform family of meshes of an open, bounded, Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with h denoting the maximum diameter of an element $K \in \mathcal{T}_h$. Let $r > 1$ be an integer. For any $K \in \mathcal{T}_h$, let $\theta_1, \theta_2, \dots, \theta_{n_{sh}}$ denote the local shape functions for the Lagrange finite element of degree at most $r - 1$ on K . Then for any $2 \leq \eta < \infty$, there exist $C_1, C_2 > 0$ independent of h such that for every $h \leq h_0$, every $K \in \mathcal{T}_h$, and every $v = \sum_{i=1}^{n_{sh}} d_i \theta_i$,*

$$C_1 h^d \sum_{i=1}^{n_{sh}} |d_i|^\eta \leq \|v\|_{0,\eta,K}^\eta \leq C_2 h^d \sum_{i=1}^{n_{sh}} |d_i|^\eta.$$

Proof. A proof of this fact when $\eta = 2$ is given in ([9], Lem. 9.7). The case $2 < \eta < \infty$ is a trivial modification thereof. \square

Now let w_h and $\pi_h w_h$ be as in (A.2) and (A.3), respectively. Note that the support of $\pi_h w_h$ is contained within the region $Q_h \subseteq \Omega$ over which \mathcal{T}_h and \mathcal{T}_h^+ coincide. On any $K \in \mathcal{T}_h$ with $K \subseteq Q_h$, we can write

$$w_h|_K = \sum_{i=1}^{n_{sh}} d_i \theta_i$$

and

$$\pi_h w_h|_K = \sum_{i=1}^{n_{sh}} \bar{d}_i \theta_i,$$

with scalars $d_i \in \mathbb{R}$ and $\bar{d}_i \in \{0, d_i\}$ for every i . By Lemma A.1,

$$\begin{aligned} \|\pi_h w_h\|_{0,\eta,K}^\eta &\leq C_2 h^d \sum_{i=1}^{n_{sh}} |\bar{d}_i|^\eta \\ &\leq C_2 h^d \sum_{i=1}^{n_{sh}} |d_i|^\eta \\ &\leq C_2 C_1^{-1} \|w_h\|_{0,\eta,K}^\eta \end{aligned}$$

on every such K . Summing over all $K \in \mathcal{T}_h$ with $K \subseteq Q_h$ proves (A.4) for $2 \leq \eta < \infty$.

APPENDIX B. ESTIMATES FOR THE L^2 -PROJECTION AND ELLIPTIC PROJECTIONS

Two exemplary cases in which estimates of the form (2.4) and (2.5) are known to hold are the following. Suppose that $\mathcal{V} = H^s(\Omega) \cap H_0^1(\Omega)$ and \mathcal{V}_h is the space of continuous functions in \mathcal{V} that are elementwise polynomials of degree at most $r-1$ on a shape-regular, quasi-uniform family of meshes $\{\mathcal{T}_h\}_{h \leq h_0}$ whose maximum element diameter is h . Then:

- (i) If $s = 0$, $d \in \{1, 2\}$, and

$$a_h(u, w) = \int_\Omega u w \, dx$$

so that r_h is the L^2 -projector onto \mathcal{V}_h , then (2.4) holds with $\ell(h) \equiv 1$ for any $\eta \in [2, \infty]$ [8]. Note that the estimate (2.5) is vacuous in this case, since $s = 0$.

- (ii) If $s = 1$, $d \in \{2, 3\}$, and

$$a_h(u, w) = \int_\Omega \left(\sum_{i,j=1}^d a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} + \sum_{j=1}^d b_j(x) \frac{\partial u}{\partial x_j} w + b_0(x) u w \right) dx$$

with h -independent coefficients a_{ij} , $i, j = 1, 2, \dots, d$ and b_j , $j = 0, 1, \dots, d$, then (2.4) and (2.5) hold [9] with $\ell(h) \equiv 1$ for any $2 \leq \eta < \infty$ (if $r = 2$) and any $\eta \in [2, \infty]$ (if $r > 2$), provided that

- The coefficients satisfy $b_j \in L^\infty(\Omega)$, $j = 0, 1, \dots, d$, and $a_{ij} \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, $i, j = 1, 2, \dots, d$, with $p > 2$ if $d = 2$ and $p \geq 12/15$ if $d = 3$.
- The coefficients a_{ij} are coercive pointwise, *i.e.* there exists $c > 0$ independent of x such that

$$\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq c |\xi|^2 \tag{B.1}$$

for every $0 \neq \xi \in \mathbb{R}^d$ and a.e. $x \in \Omega$.

- There exists $C > 0$, $q_0 > d$ such that the continuous Dirichlet problem

$$a_h(u, w) = \int_\Omega f w \, dx \quad \forall w \in \mathcal{V}$$

has a unique solution satisfying

$$\|u\|_{2,q} \leq C \|f\|_{0,q} \tag{B.2}$$

for every $f \in L^p(\Omega)$ and every $1 < q < q_0$.

Under the same conditions as above but with $r = 2$ and $\eta = \infty$, the estimates (2.4) and (2.5) hold with $\ell(h) = \log(h^{-1})$ in dimension $d = 2$ [9].

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