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A PRIORI ERROR ANALYSIS OF A FULLY-MIXED FINITE ELEMENT METHOD FOR A TWO-DIMENSIONAL FLUID-SOLID INTERACTION PROBLEM *

Carolina Domínguez[1], Gabriel N. Gatica[2], Salim Meddahi[3] and Ricardo Oyarzúa[4]

Abstract. We introduce and analyze a fully-mixed finite element method for a fluid-solid interaction problem in 2D. The model consists of an elastic body which is subject to a given incident wave that travels in the fluid surrounding it. Actually, the fluid is supposed to occupy an annular region, and hence a Robin boundary condition imitating the behavior of the scattered field at infinity is imposed on its exterior boundary, which is located far from the obstacle. The media are governed by the elastodynamic and acoustic equations in time-harmonic regime, respectively, and the transmission conditions are given by the equilibrium of forces and the equality of the corresponding normal displacements. We first apply dual-mixed approaches in both domains, and then employ the governing equations to eliminate the displacement $\mathbf{u}$ of the solid and the pressure p of the fluid. In addition, since both transmission conditions become essential, they are enforced weakly by means of two suitable Lagrange multipliers. As a consequence, the Cauchy stress tensor and the rotation of the solid, together with the gradient of p and the traces of $\mathbf{u}$ and p on the boundary of the fluid, constitute the unknowns of the coupled problem. Next, we show that suitable decompositions of the spaces to which the stress and the gradient of p belong, allow the application of the Babuška–Brezzi theory and the Fredholm alternative for analyzing the solvability of the resulting continuous formulation. The unknowns of the solid and the fluid are then approximated by a conforming Galerkin scheme defined in terms of PEERS elements in the solid, Raviart–Thomas of lowest order in the fluid, and continuous piecewise linear functions on the boundary. Then, the analysis of the discrete method relies on a stable decomposition of the corresponding finite element spaces and also on a classical result on projection methods for Fredholm operators of index zero. Finally, some numerical results illustrating the theory are presented.

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[1] Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile. cdominguez@ing-mat.udec.cl

[2] CI²MA and Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile. ggatica@ing-mat.udec.cl

[3] Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, Calvo Sotelo s/n, Oviedo, España.
salim@uniovi.es

[4] CI²MA (Universidad de Concepción) and Departamento de Matemática, Facultad de Ciencias, Universidad del Bío-Bío, Casilla 3-C, Concepción, Chile.
royarzua@ubiobio.cl

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1. Introduction

In this paper we focus again on the two-dimensional fluid-solid interaction problem studied recently in [7] (see also [9] for a version employing boundary integral equation methods). More precisely, we consider an incident acoustic wave upon a bounded elastic body (obstacle) fully surrounded by a fluid, and are interested in determining both the response of the body and the scattered wave. The obstacle is supposed to be a long cylinder parallel to the x_3-axis whose cross-section is Ω_s. The boundary of Ω_s is denoted by Σ. We assume that the incident wave and the volume force acting on the body exhibit a time-harmonic behaviour with $\mathrm{e}^{-\imath\,\omega\,t}$ ansatz and phasors p_i and $\mathbf{f}$, respectively, so that p_i satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\Omega_s$. Hence, since the phenomenon is supposed to be invariant under a translation in the x_3-direction, we may consider a bidimensional interaction problem posed in the frequency domain. In this way, in what follows we let $\boldsymbol{\sigma}_s : \Omega_s \to \mathbb{C}^{2\times 2}$, $\mathbf{u} : \Omega_s \to \mathbb{C}^2$, and $p : \mathbb{R}^2\backslash\Omega_s \to \mathbb{C}$ be the amplitudes of the Cauchy stress tensor, the displacement field, and the total (incident + scattered) pressure, respectively, where $\mathbb{C}$ stands for the set of complex numbers.

The fluid is assumed to be perfect, compressible, and homogeneous, with density ρ_f and wave number $\kappa_f := \dfrac{\omega}{v_0}$, where v_0 is the speed of sound in the linearized fluid, whereas the solid is supposed to be isotropic and linearly elastic with density ρ_s and Lamé constants μ and λ. The latter means, in particular, that the corresponding constitutive equation is given by Hooke's law, that is

$$\boldsymbol{\sigma}_s \,=\, \lambda\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u})\,\mathbf{I} \,+\, 2\,\mu\,\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in} \quad \Omega_s,$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations, ∇ is the gradient tensor, tr denotes the matrix trace, $^{\mathtt{t}}$ stands for the transpose of a matrix, and $\mathbf{I}$ is the identity matrix of $\mathbb{C}^{2\times 2}$. Consequently, under the hypotheses of small oscillations, both in the solid and the fluid, the unknowns $\boldsymbol{\sigma}_s$, $\mathbf{u}$, and p satisfy the elastodynamic and acoustic equations in time-harmonic regime, that is:

$$\begin{aligned} \mathbf{div}\,\boldsymbol{\sigma}_s \,+\, \kappa_s^2\,\mathbf{u} \,&=\, -\,\mathbf{f} \quad &&\text{in} \quad \Omega_s, \\ \Delta p \,+\, \kappa_f^2\,p \,&=\, 0 &&\text{in} \quad \mathbb{R}^2\backslash\Omega_s, \end{aligned}$$

where κ_s is defined by $\sqrt{\rho_s}\,\omega$, together with the transmission conditions:

$$\begin{aligned} \boldsymbol{\sigma}_s\,\boldsymbol{\nu} \,&=\, -\,p\,\boldsymbol{\nu} \quad &&\text{on} \quad \Sigma, \\ \rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} \,&=\, \frac{\partial p}{\partial\boldsymbol{\nu}} &&\text{on} \quad \Sigma, \end{aligned} \tag{1.1}$$

and the behaviour at infinity given by

$$p - p_i \,=\, O\left(\mathtt{r}^{-1}\right) \tag{1.2}$$

and

$$\frac{\partial(p - p_i)}{\partial\mathtt{r}} \,-\, \imath\,\kappa_f\,(p - p_i) \,=\, o\left(\mathtt{r}^{-1}\right), \tag{1.3}$$

as $\mathtt{r} := \|\mathbf{x}\| \to +\infty$, uniformly for all directions $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}\cdot$ Hereafter, $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, $\|\mathbf{x}\|$ is the euclidean norm of a vector $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \mathbb{R}^2$, and $\boldsymbol{\nu}$ denotes the unit outward normal on Σ, that is pointing toward $\mathbb{R}^2\backslash\Omega_s$. The transmission conditions given in (1.1) constitute the equilibrium of forces and the equality of the normal displacements of the solid and fluid. In other words, the first equation in (1.1) results from the action of pressure forces exerted by the fluid on the solid, and the second one expresses the continuity of the fluid and structural normal displacement components at the interface. In turn, the equation (1.3) is known as the Sommerfeld radiation condition.

Now, it is important to remark that the development of suitable numerical methods for the above described fluid-solid interaction problems has become a subject of increasing interest during the last two decades. Several approaches relying on a primal formulation in the solid, in which the displacement becomes the only unknown in this medium, were originally studied in [5,17–20,23,25]. More recently, and in particular motivated by the need of obtaining direct finite element approximations of the stresses, dual-mixed formulations in the solid have begun to be considered as well (see *e.g.* [7,9]). In fact, the model is first simplified in [7] by assuming that the fluid occupies a bounded annular region Ω_f, whence a Robin boundary condition imitating the behavior of the scattered field at infinity is imposed on the exterior boundary of Ω_f, which is located far from the obstacle. Then, the method in [7] employs a dual-mixed variational formulation for plane elasticity in the solid and keeps the usual primal formulation in the linearized fluid region. In addition, the elastodynamic equation is used to eliminate the displacement unknown from the resulting formulation. Furthermore, since one of the transmission conditions becomes essential, it is enforced weakly by means of a Lagrange multiplier. As a consequence, the stress tensor in the solid and the pressure in the fluid, which solves the Helmholtz equation, constitute the main unknowns. Next, a judicious decomposition of the space of stresses renders suitable the application of the Fredholm alternative and the Babuška–Brezzi theory for the analysis of the whole coupled problem. The corresponding discrete scheme is defined with PEERS elements in the obstacle and the traditional first order Lagrange finite elements in the fluid domain. The stability and convergence of this Galerkin method also relies on a stable decomposition of the finite element space used to approximate the stress variable. On the other hand, the strategy from [7] is modified in [9] in such a way that, instead of introducing a Robin condition on the exterior boundary, a non-local absorbing boundary condition based on boundary integral equations is considered there. Consequently, the exterior boundary can be chosen as any parametrizable smooth closed curve containing the solid, which, in order to minimize the size of the computational domain, is adjusted as sharply as possible to the shape of the obstacle. The rest of the analysis for the corresponding continuous and discrete formulations follows very closely the techniques and arguments developed in [7]. We refer to [9] for further details on this modified approach.

The goal of the present paper is to additionally extend the approach from [7,9] by employing now dual-mixed formulations in both media. The extension concept refers here to the fact that, instead of using a primal approach in the bounded fluid domain, as in [7,9], we now apply in that region the same dual-mixed method that is employed in the solid. In this way, the well-posedness of the formulation that would arise from the additional use of the boundary integral equation method (BIEM) in the unbounded fluid domain, as it was done in [9], will follow straightforwardly from the analyses in that reference and the present paper. By the way, the advantages and disadvantages of using BIEM or not have to do mainly with the computational domain (smaller with BIEM) and the complexity of the resulting Galerkin system (simpler without BIEM). In any case, the above remarks emphasize that, besides $\boldsymbol{\sigma}_s$, from now on we set the additional unknown

$$\boldsymbol{\sigma}_f := \nabla p \quad \text{in} \quad \mathbb{R}^2 \backslash \Omega_s,$$

so that the Helmholtz equation and the second condition in (1.1) are rewritten, respectively, as

$$\operatorname{div} \boldsymbol{\sigma}_f + \kappa_f^2 p = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \Omega_s, \tag{1.4}$$

and

$$\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} = \rho_f \omega^2 \mathbf{u} \cdot \boldsymbol{\nu} \quad \text{on} \quad \Sigma. \tag{1.5}$$

The introduction of $\boldsymbol{\sigma}_f$ and the resulting equation (1.4) is motivated by the eventual need of obtaining direct and more accurate finite element approximations for the pressure gradient $\boldsymbol{\sigma}_f := \nabla p$ (instead of applying numerical differentiation, with the consequent loss of accuracy, to the approximation of p arising from the usual primal formulation). The above is required, for instance, to solve the inverse problem related to the Helmholtz equation, in which the boundary integral representation of the far field pattern, a crucial variable in an associated iterative algorithm, depends on both the trace of p and the normal trace of $\boldsymbol{\sigma}_f$ (see, *e.g.* [6], Chap. 2, Thm. 2.5). To this respect, a $H(\text{div})$-type approximation of $\boldsymbol{\sigma}_f$ is certainly better suited for this purpose. The usefulness of the mixed formulation for the pressure p is also justified by the fact that it is locally mass conservative. Moreover, since both transmission conditions become now essential, they are enforced weakly by using the traces of the displacement and the pressure on the interface as suitable Lagrange multipliers. Hence, the fact that these variables of evident physical interest can also be approximated directly from the associated Galerkin schemes, constitute another important advantage of the fully-mixed approach proposed here. Furthermore, the use of a dual-mixed approach in the solid and the fluid simplify the corresponding computational code since Raviart–Thomas based subspaces can be used in both domains. The rest of this work is organized as follows. In Section 2 we redefine the fluid-solid interaction problem on an annular domain $\Omega_f \subseteq \mathbb{R}^2$ (as in [7, 9]), and derive the associated continuous variational formulation. Then, in Section 3 we utilize the Fredholm and Babuška–Brezzi theories to analyze the resulting saddle point problem and provide sufficient conditions for its well-posedness. The corresponding Galerkin scheme is studied in Section 4. Finally, some numerical experiments illustrating the theoretical results are reported in Section 5.

We end this section with further notations to be used below. Since in the sequel we deal with complex valued functions, we use the symbol $\imath$ for $\sqrt{-1}$, and denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbb{C}$. Also, given $\boldsymbol{\tau}_s := (\tau_{ij})$, $\boldsymbol{\zeta}_s := (\zeta_{ij}) \in \mathbb{C}^{2\times 2}$, we define the deviator tensor $\boldsymbol{\tau}_s^{\mathtt{d}} := \boldsymbol{\tau}_s - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau}_s)\,\mathbf{I}$, the tensor product $\boldsymbol{\tau}_s : \boldsymbol{\zeta}_s := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$, and the conjugate tensor $\overline{\boldsymbol{\tau}_s} := (\overline{\tau}_{ij})$. In turn, in what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, $\mathcal{S}$ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^2.$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^2 and $H^{2\times 2}$, respectively. In addition, we use $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ to denote the usual duality pairings between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and between $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. Furthermore, the Hilbert space

$$\mathbf{H}(\operatorname{div};\mathcal{O}) := \big\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \operatorname{div}\mathbf{w} \in L^2(\mathcal{O}) \big\},$$

is standard in the realm of mixed problems (see [4, 13]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\operatorname{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\operatorname{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\operatorname{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})$. Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. The continuous variational formulation

We first observe, as a consequence of (1.2) and (1.3), that the outgoing waves are absorbed by the far field. According to this fact, and in order to obtain a convenient simplification of our model problem, we now proceed similarly as in [7] and introduce a sufficiently large polyhedral surface Γ approximating a sphere centered at the origin, whose interior contains Ω_s. Then, we define Ω_f as the annular region bounded by Σ and Γ, and consider the Robin boundary condition:

$$\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} - \imath\,\kappa_f\,p = g := \nabla p_i \cdot \boldsymbol{\nu} - \imath\,\kappa_f\,p_i \quad \text{on} \quad \Gamma,$$

where $\boldsymbol{\nu}$ denotes also the unit outward normal on Γ. Therefore, given $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ and $g \in H^{-1/2}(\Gamma)$, we are now interested in the following fluid-solid interaction problem: Find $\boldsymbol{\sigma}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$,

$\boldsymbol{\sigma}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, and $p \in H^1(\Omega_f)$, such that there hold in the distributional sense:

$$\begin{aligned}
\boldsymbol{\sigma}_s \;&=\; \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in} \quad \Omega_s,\\
\mathbf{div}\,\boldsymbol{\sigma}_s \,+\, \kappa_s^2\,\mathbf{u} \;&=\; -\,\mathbf{f} && \text{in} \quad \Omega_s,\\
\boldsymbol{\sigma}_f \;&=\; \nabla p && \text{in} \quad \Omega_f,\\
\mathrm{div}\,\boldsymbol{\sigma}_f \,+\, \kappa_f^2\,p \;&=\; 0 && \text{in} \quad \Omega_f,\\
\boldsymbol{\sigma}_s\,\boldsymbol{\nu} \;&=\; -\,p\,\boldsymbol{\nu} && \text{on} \quad \Sigma,\\
\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu} \;&=\; \rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} && \text{on} \quad \Sigma,\\
\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu} \,-\, \imath\,\kappa_f\,p \;&=\; g && \text{on} \quad \Gamma,
\end{aligned} \tag{2.1}$$

where $\mathcal{C}$ is the elasticity operator given by Hooke's law, that is

$$\mathcal{C}\,\boldsymbol{\zeta}_s \;:=\; \lambda\,\mathrm{tr}(\boldsymbol{\zeta}_s)\,\mathbf{I} \,+\, 2\,\mu\,\boldsymbol{\zeta}_s \qquad \forall\,\boldsymbol{\zeta}_s \in \mathbb{L}^2(\Omega_s). \tag{2.2}$$

Note from (2.1) the full symmetry existing between the dual-mixed formulations in the domains and between the transmission conditions on Σ. This fact motivates later on the use of Raviart–Thomas based subspaces in both domains.

It is clear from (2.2) that $\mathcal{C}$ is bounded and invertible and that the operator $\mathcal{C}^{-1}$ reduces to

$$\mathcal{C}^{-1}\,\boldsymbol{\zeta}_s \;:=\; \frac{1}{2\,\mu}\,\boldsymbol{\zeta}_s \,-\, \frac{\lambda}{4\,\mu\,(\lambda+\mu)}\,\mathrm{tr}(\boldsymbol{\zeta}_s)\,\mathbf{I} \qquad \forall\,\boldsymbol{\zeta}_s \in \mathbb{L}^2(\Omega_s).$$

In addition, the above identity and simple algebraic manipulations yield

$$\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\zeta}_s : \overline{\boldsymbol{\zeta}_s} \;\geq\; \frac{1}{2\mu}\,\|\boldsymbol{\zeta}_s^{\mathtt{d}}\|_{0,\Omega_s}^2 \qquad \forall\,\boldsymbol{\zeta}_s \in \mathbb{L}^2(\Omega_s). \tag{2.3}$$

We now apply dual-mixed approaches in the solid Ω_s and the fluid Ω_f to derive the fully-mixed variational formulation of (2.1). Indeed, following the usual procedure from linear elasticity (see [1,7,27]), we first introduce the rotation

$$\boldsymbol{\gamma} \;:=\; \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}) \;\in\; \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)$$

as a further unknown, where $\mathbb{L}^2_{\mathtt{asym}}(\Omega_s)$ denotes the space of asymmetric tensors with entries in $L^2(\Omega_s)$. According to this, the constitutive equation can be rewritten in the form

$$\mathcal{C}^{-1}\,\boldsymbol{\sigma}_s \;=\; \boldsymbol{\varepsilon}(\mathbf{u}) \;=\; \nabla\mathbf{u} \,-\, \boldsymbol{\gamma},$$

which, multiplying by a function $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$ and integrating by parts, yields

$$\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\sigma}_s : \boldsymbol{\tau}_s \,+\, \int_{\Omega_s} \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \,-\, \langle \boldsymbol{\tau}_s\,\boldsymbol{\nu}, \mathbf{u}\rangle_\Sigma \,+\, \int_{\Omega_s} \boldsymbol{\tau}_s : \boldsymbol{\gamma} \;=\; 0. \tag{2.4}$$

At this point we remark that, given $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, $\boldsymbol{\tau}_s\,\boldsymbol{\nu}|_\Sigma$ is the functional in $\mathbf{H}^{-1/2}(\Sigma)$ defined as

$$\langle \boldsymbol{\tau}_s\,\boldsymbol{\nu}, \boldsymbol{\varphi}\rangle_\Sigma \;:=\; \int_{\Omega_s} \boldsymbol{\tau}_s : \nabla\mathbf{w} \,+\, \int_{\Omega_s} \mathbf{w}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \qquad \forall\,\boldsymbol{\varphi} \in \mathbf{H}^{1/2}(\Sigma),$$

where $\mathbf{w}$ is any function in $\mathbf{H}^1(\Omega_s)$ such that $\mathbf{w} = \boldsymbol{\varphi}$ on Σ and $\mathbf{w} = \mathbf{0}$ on Γ. Then, using the elastodynamic equation (*cf.* second equation of (2.1)) to eliminate $\mathbf{u}$ in Ω_s, and introducing the additional unknown

$$\boldsymbol{\varphi}_s \;:=\; \mathbf{u}|_\Sigma \;\in\; \mathbf{H}^{1/2}(\Sigma), \tag{2.5}$$

we find that (2.4) becomes

$$\int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\sigma}_s : \boldsymbol{\tau}_s \,-\, \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{div}\,\boldsymbol{\sigma}_s\cdot\mathbf{div}\,\boldsymbol{\tau}_s \,-\, \langle \boldsymbol{\tau}_s\boldsymbol{\nu},\boldsymbol{\varphi}_s\rangle_\Sigma \,+\, \int_{\Omega_s}\boldsymbol{\tau}_s:\gamma \,=\, \frac{1}{\kappa_s^2}\int_{\Omega_s}\mathbf{f}\cdot\mathbf{div}\,\boldsymbol{\tau}_s. \tag{2.6}$$

Similarly, multiplying the constitutive equation $\boldsymbol{\sigma}_f = \nabla p$ in Ω_f by $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, integrating by parts, noting that the normal vector points inward Ω_f on Σ, replacing from the Helmholtz equation $p = -\frac{1}{\kappa_f^2}\,\mathrm{div}\,\boldsymbol{\sigma}_f$ in Ω_f, and introducing the auxiliary unknown

$$\boldsymbol{\varphi}_f \,=\, (\varphi_\Sigma,\varphi_\Gamma) \,:=\, (p|_\Sigma, p|_\Gamma) \,\in\, H^{1/2}(\Sigma)\times H^{1/2}(\Gamma), \tag{2.7}$$

we arrive at

$$\int_{\Omega_f}\boldsymbol{\sigma}_f\cdot\boldsymbol{\tau}_f \,-\, \frac{1}{\kappa_f^2}\int_{\Omega_f}\mathrm{div}\,\boldsymbol{\sigma}_f\,\mathrm{div}\,\boldsymbol{\tau}_f \,+\, \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi_\Sigma\rangle_\Sigma \,-\, \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi_\Gamma\rangle_\Gamma \,=\, 0. \tag{2.8}$$

Finally, the symmetry of $\boldsymbol{\sigma}_s$, the transmission conditions on Σ, and the Robin boundary condition on Γ are imposed weakly through the relations:

$$\begin{aligned}
\int_{\Omega_s}\boldsymbol{\sigma}_s:\boldsymbol{\eta} \;&=\; 0 && \forall\,\boldsymbol{\eta}\,\in\,\mathbb{L}^2_{\mathtt{asym}}(\Omega_s),\\
-\,\langle\,\boldsymbol{\sigma}_s\boldsymbol{\nu},\,\boldsymbol{\psi}_s\,\rangle_\Sigma \,-\, \langle\,\varphi_\Sigma\,\boldsymbol{\nu},\,\boldsymbol{\psi}_s\,\rangle_\Sigma \;&=\; 0 && \forall\,\boldsymbol{\psi}_s\,\in\,\mathbf{H}^{1/2}(\Sigma),\\
\langle\,\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu},\,\psi_\Sigma\,\rangle_\Sigma \,-\, \rho_f\,\omega^2\,\langle\,\psi_\Sigma\,\boldsymbol{\nu},\,\boldsymbol{\varphi}_s\,\rangle_\Sigma \;&=\; 0 && \forall\,\psi_\Sigma\,\in\,H^{1/2}(\Sigma),\\
-\,\langle\,\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu},\,\psi_\Gamma\,\rangle_\Gamma \,+\, \imath\,\kappa_f\,\langle\,\varphi_\Gamma,\,\psi_\Gamma\,\rangle_\Gamma \;&=\; -\,\langle\,g,\,\psi_\Gamma\,\rangle_\Gamma && \forall\,\psi_\Gamma\,\in\,H^{1/2}(\Gamma),
\end{aligned} \tag{2.9}$$

where the traces of $\mathbf{u}$ and p have been replaced by the new unknowns introduced in (2.5) and (2.7), the expression $\langle\,\boldsymbol{\varphi}_s\cdot\boldsymbol{\nu},\,\psi_\Sigma\,\rangle_\Sigma$ in the second transmission condition has been rewritten as $\langle\,\psi_\Sigma\,\boldsymbol{\nu},\,\boldsymbol{\varphi}_s\,\rangle_\Sigma$, and the signs of the first transmission condition and the Robin boundary condition have been changed for convenience. Note that $\boldsymbol{\varphi}_s$ and $\boldsymbol{\varphi}_f$ constitute precisely the Lagrange multipliers associated with the transmission and Robin boundary conditions.

Throughout the rest of the paper we make the identification $H^t(\partial\Omega_f)\,\equiv\,H^t(\Sigma)\times H^t(\Gamma)$ for each $t\in\mathbb{R}$, with the norm $\|\boldsymbol{\psi}_f\|_{t,\partial\Omega_f} := \|\psi_\Sigma\|_{t,\Sigma} + \|\psi_\Gamma\|_{t,\Gamma}$ for each $\boldsymbol{\psi}_f := (\psi_\Sigma,\psi_\Gamma)\in H^t(\partial\Omega_f)$.

Therefore, adding (2.6)–(2.9), and defining the spaces

$$\mathbf{H} \,:=\, \mathbb{H}(\mathbf{div};\Omega_s)\times\mathbf{H}(\mathrm{div};\Omega_f) \quad\text{and}\quad \mathbf{Q} \,:=\, \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\times\mathbf{H}^{1/2}(\Sigma)\times H^{1/2}(\partial\Omega_f),$$

we arrive at the following fully-mixed variational formulation of (2.1): Find $\widehat{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_s,\boldsymbol{\sigma}_f)\in\mathbf{H}$ and $\widehat{\boldsymbol{\gamma}} := (\boldsymbol{\gamma},\boldsymbol{\varphi}_s,\boldsymbol{\varphi}_f)\in\mathbf{Q}$ such that

$$\begin{aligned}
A(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\tau}}) \,+\, B(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\gamma}}) &= F(\widehat{\boldsymbol{\tau}}) && \forall\,\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}_s,\boldsymbol{\tau}_f)\in\mathbf{H},\\
B(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\eta}}) \,+\, K(\widehat{\boldsymbol{\gamma}},\widehat{\boldsymbol{\eta}}) &= G(\widehat{\boldsymbol{\eta}}) && \forall\,\widehat{\boldsymbol{\eta}} := (\boldsymbol{\eta},\boldsymbol{\psi}_s,\boldsymbol{\psi}_f)\in\mathbf{Q},
\end{aligned} \tag{2.10}$$

where $F:\mathbf{H}\to\mathbb{C}$ and $G:\mathbf{Q}\to\mathbb{C}$ are the lineal functionals

$$F(\widehat{\boldsymbol{\tau}}) \,:=\, \frac{1}{\kappa_s^2}\int_{\Omega_s}\mathbf{f}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \quad \forall\,\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}_s,\boldsymbol{\tau}_f)\in\mathbf{H},$$

$$G(\widehat{\boldsymbol{\eta}}) \,:=\, -\,\langle\,g,\,\psi_\Gamma\,\rangle_\Gamma \quad \forall\,\widehat{\boldsymbol{\eta}} := (\boldsymbol{\eta},\boldsymbol{\psi}_s,\boldsymbol{\psi}_f) := (\boldsymbol{\eta},\boldsymbol{\psi}_s,(\psi_\Sigma,\psi_\Gamma))\in\mathbf{Q},$$

and $A:\mathbf{H}\times\mathbf{H}\to\mathbb{C}$, $B:\mathbf{H}\times\mathbf{Q}\to\mathbb{C}$, and $K:\mathbf{Q}\times\mathbf{Q}\to\mathbb{C}$ are the bilinear forms defined by

$$\begin{gathered}
A(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) \,:=\, \int_{\Omega_s}\mathcal{C}^{-1}\boldsymbol{\zeta}_s:\boldsymbol{\tau}_s \,-\, \frac{1}{\kappa_s^2}\int_{\Omega_s}\mathbf{div}\,\boldsymbol{\zeta}_s\cdot\mathbf{div}\,\boldsymbol{\tau}_s \,+\, \int_{\Omega_f}\boldsymbol{\zeta}_f\cdot\boldsymbol{\tau}_f - \frac{1}{\kappa_f^2}\int_{\Omega_f}\mathrm{div}\,\boldsymbol{\zeta}_f\,\mathrm{div}\,\boldsymbol{\tau}_f\\
\forall\,(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) := ((\boldsymbol{\zeta}_s,\boldsymbol{\zeta}_f),(\boldsymbol{\tau}_s,\boldsymbol{\tau}_f))\in\mathbf{H}\times\mathbf{H},
\end{gathered} \tag{2.11}$$

$$B(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) \,:=\, B_s(\boldsymbol{\tau}_s,(\boldsymbol{\eta},\boldsymbol{\psi}_s)) \,+\, B_f(\boldsymbol{\tau}_f,\boldsymbol{\psi}_f) \qquad \forall\,(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) := ((\boldsymbol{\tau}_s,\boldsymbol{\tau}_f),(\boldsymbol{\eta},\boldsymbol{\psi}_s,\boldsymbol{\psi}_f))\in\mathbf{H}\times\mathbf{Q}, \tag{2.12}$$

with

$$B_s(\boldsymbol{\tau}_s, (\boldsymbol{\eta}, \boldsymbol{\psi}_s)) := \int_{\Omega_s} \boldsymbol{\tau}_s : \boldsymbol{\eta} - \langle \boldsymbol{\tau}_s \boldsymbol{\nu}, \boldsymbol{\psi}_s \rangle_\Sigma, \tag{2.13}$$

$$B_f(\boldsymbol{\tau}_f, \boldsymbol{\psi}_f) := \langle \boldsymbol{\tau}_f \cdot \boldsymbol{\nu}, \psi_\Sigma \rangle_\Sigma - \langle \boldsymbol{\tau}_f \cdot \boldsymbol{\nu}, \psi_\Gamma \rangle_\Gamma, \tag{2.14}$$

and

$$\begin{gathered} K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\eta}}) := -\langle \xi_\Sigma \boldsymbol{\nu}, \boldsymbol{\psi}_s \rangle_\Sigma - \rho_f \omega^2 \langle \psi_\Sigma \boldsymbol{\nu}, \boldsymbol{\xi}_s \rangle_\Sigma + \imath \kappa_f \langle \xi_\Gamma, \psi_\Gamma \rangle_\Gamma \\ \forall \widehat{\boldsymbol{\chi}} := (\boldsymbol{\chi}, \boldsymbol{\xi}_s, \boldsymbol{\xi}_f) := (\boldsymbol{\chi}, \boldsymbol{\xi}_s, (\xi_\Sigma, \xi_\Gamma)) \in \mathbf{Q}, \\ \forall \widehat{\boldsymbol{\eta}} := (\boldsymbol{\eta}, \boldsymbol{\psi}_s, \boldsymbol{\psi}_f) := (\boldsymbol{\eta}, \boldsymbol{\psi}_s, (\psi_\Sigma, \psi_\Gamma)) \in \mathbf{Q}. \end{gathered} \tag{2.15}$$

It is straightforward to see, applying the Cauchy–Schwarz inequality, the duality pairings $\langle \cdot, \cdot \rangle_\Sigma$ and $\langle \cdot, \cdot \rangle_\Gamma$, and the usual trace theorems in $\mathbb{H}(\mathbf{div}; \Omega_s)$ and $\mathbf{H}(\mathrm{div}; \Omega_f)$, that F, G, A, B, B_s, B_f, and K are all bounded with constants depending on κ_s, μ, κ_f, ρ_f, and ω.

3. Analysis of the continuous variational formulation

In this section we proceed analogously to [7] and employ suitable decompositions of $\mathbb{H}(\mathbf{div}; \Omega_s)$ and $\mathbf{H}(\mathrm{div}; \Omega_f)$ to show that (2.10) becomes a compact perturbation of a well-posed problem. To this end, we now need to introduce two projectors defined in terms of auxiliary Neumann boundary value problems posed in Ω_s and Ω_f, respectively.

3.1. The associated projectors

We begin by recalling from the analysis in [7], Section 4.1, the definition of the projector in Ω_s. In fact, let us first denote by $\mathbb{RM}(\Omega_s)$ the space of rigid body motions in Ω_s, that is

$$\mathbb{RM}(\Omega_s) := \left\{ \mathbf{v} : \Omega_s \to \mathbb{C}^2 : \quad \mathbf{v}(\mathbf{x}) = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \quad \forall\, \mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \Omega_s,\ a, b, c \in \mathbb{C} \right\},$$

and let $\mathbf{M} : \mathbf{L}^2(\Omega_s) \to \mathbb{RM}(\Omega_s)$ be the associated orthogonal projector. Then, given $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div}; \Omega_s)$, we consider the boundary value problem

$$\begin{gathered} \tilde{\boldsymbol{\sigma}}_s = \mathcal{C}\, \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) \quad \text{in} \quad \Omega_s, \quad \mathbf{div}\, \tilde{\boldsymbol{\sigma}}_s = (\mathbf{I} - \mathbf{M})\big(\mathbf{div}\, \boldsymbol{\tau}_s\big) \quad \text{in} \quad \Omega_s, \\ \tilde{\boldsymbol{\sigma}}_s \boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Sigma, \quad \tilde{\mathbf{u}} \in (\mathbf{I} - \mathbf{M})(\mathbf{L}^2(\Omega_s)), \end{gathered} \tag{3.1}$$

where $\mathcal{C}\, \boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ is defined according to (2.2). Hereafter, $\mathbf{I}$ denotes also a generic identity operator. Note that the application of the operator $\mathbf{I} - \mathbf{M}$ on the right hand side of the equilibrium equation is needed to guarantee the usual compatibility condition for the Neumann problem (3.1) (*cf.* [3], Thm. 9.2.30), and that the orthogonality condition on $\tilde{\mathbf{u}}$ is required for uniqueness. Indeed, it is well known (see, *e.g.* [8], Sect. 3, Thm. 3.1) that (3.1) is well-posed. In addition, owing to the regularity result for the elasticity problem with Neumann boundary conditions (see, *e.g.* [14, 15]), we know that $(\tilde{\boldsymbol{\sigma}}_s, \tilde{\mathbf{u}}) \in \mathbb{H}^\epsilon(\Omega_s) \times \mathbf{H}^{1+\epsilon}(\Omega_s)$, for some $\epsilon > 0$, and there holds

$$\|\tilde{\boldsymbol{\sigma}}_s\|_{\epsilon, \Omega_s} + \|\tilde{\mathbf{u}}\|_{1+\epsilon, \Omega_s} \le C\, \|\mathbf{div}\, \boldsymbol{\tau}_s\|_{0, \Omega_s}. \tag{3.2}$$

We now introduce the linear operator $\mathbf{P}_s : \mathbb{H}(\mathbf{div}; \Omega_s) \to \mathbb{H}(\mathbf{div}; \Omega_s)$ defined by

$$\mathbf{P}_s(\boldsymbol{\tau}_s) := \tilde{\boldsymbol{\sigma}}_s \qquad \forall\, \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div}; \Omega_s), \tag{3.3}$$

where $\tilde{\boldsymbol{\sigma}}_s := \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ and $\tilde{\mathbf{u}}$ is the unique solution of (3.1). It is clear from (3.1) that

$$\mathbf{P}_s(\boldsymbol{\tau}_s)^{\mathtt{t}} \;=\; \mathbf{P}_s(\boldsymbol{\tau}_s) \quad \text{in} \quad \Omega_s, \quad \mathbf{div}\,\mathbf{P}_s(\boldsymbol{\tau}_s) \;=\; (\mathbf{I}-\mathbf{M})\big(\mathbf{div}\,\boldsymbol{\tau}_s\big) \quad \text{in} \quad \Omega_s, \tag{3.4}$$

and

$$\mathbf{P}_s(\boldsymbol{\tau}_s)\,\boldsymbol{\nu} \;=\; \mathbf{0} \quad \text{on} \quad \Sigma. \tag{3.5}$$

Then, the continuous dependence result for (3.1) gives

$$\|\mathbf{P}_s(\boldsymbol{\tau}_s)\|_{\mathbf{div};\Omega_s} \;\le\; C\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s} \qquad \forall\,\boldsymbol{\tau}_s \,\in\, \mathbb{H}(\mathbf{div};\Omega_s),$$

which shows that $\mathbf{P}_s$ is bounded. Moreover, it is easy to see from (3.1)–(3.5) that $\mathbf{P}_s$ is actually a projector, and hence there holds

$$\mathbb{H}(\mathbf{div};\Omega_s) \;=\; \mathbf{P}_s(\mathbb{H}(\mathbf{div};\Omega_s)) \,\oplus\, (\mathbf{I}-\mathbf{P}_s)(\mathbb{H}(\mathbf{div};\Omega_s)). \tag{3.6}$$

Finally, it is clear from (3.2) that $\mathbf{P}_s(\boldsymbol{\tau}_s) \in \mathbb{H}^{\epsilon}(\Omega_s)$ and

$$\|\mathbf{P}_s(\boldsymbol{\tau}_s)\|_{\epsilon,\Omega_s} \;\le\; C\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s} \qquad \forall\,\boldsymbol{\tau}_s \,\in\, \mathbb{H}(\mathbf{div};\Omega_s). \tag{3.7}$$

We proceed analogously for the domain Ω_f. In fact, let $P_0(\Omega_f)$ be the space of constant polynomials on Ω_f, and let $\mathbf{J} : L^2(\Omega_f) \to P_0(\Omega_f)$ be the corresponding orthogonal projector. Then, given $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, we consider the Neumann boundary value problem

$$\begin{gathered} \tilde{\boldsymbol{\sigma}}_f \;=\; \nabla\tilde{p} \quad \text{in} \quad \Omega_f, \quad \mathrm{div}\,\tilde{\boldsymbol{\sigma}}_f \;=\; (\mathbf{I}-\mathbf{J})\big(\mathrm{div}\,\boldsymbol{\tau}_f\big) \quad \text{in} \quad \Omega_f, \\ \tilde{\boldsymbol{\sigma}}_f\cdot\boldsymbol{\nu} \;=\; 0 \quad \text{on} \quad \Sigma\cup\Gamma, \quad \tilde{p} \,\in\, (\mathbf{I}-\mathbf{J})(L^2(\Omega_f)). \end{gathered} \tag{3.8}$$

Analogue remarks to those given for the compatibility condition and uniqueness of solution of (3.1) are valid here with $\mathbf{J}$ instead of $\mathbf{M}$. In addition, it is not difficult to see that (3.8) is well-posed as well. Furthermore, the classical regularity result for the Poisson problem with Neumann boundary conditions (see, *e.g.* [14,15]) implies that $(\tilde{\boldsymbol{\sigma}}_f,\tilde{p}) \in \mathbf{H}^{\epsilon}(\Omega_f)\times H^{1+\epsilon}(\Omega_f)$, for some $\epsilon > 0$ (parameter that can be assumed, from now on, to be the same of (3.2)), and that

$$\|\tilde{\boldsymbol{\sigma}}_f\|_{\epsilon,\Omega_f} \,+\, \|\tilde{p}\|_{1+\epsilon,\Omega_f} \;\le\; C\,\|\mathrm{div}\,\boldsymbol{\tau}_f\|_{0,\Omega_f}. \tag{3.9}$$

We now define the linear operator $\mathbf{P}_f : \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}(\mathrm{div};\Omega_f)$ by

$$\mathbf{P}_f(\boldsymbol{\tau}_f) \;:=\; \tilde{\boldsymbol{\sigma}}_f \qquad \forall\,\boldsymbol{\tau}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f), \tag{3.10}$$

where $\tilde{\boldsymbol{\sigma}}_f := \nabla\tilde{p}$ and $\tilde{p}$ is the unique solution of (3.8). It follows that

$$\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_f) \;=\; (\mathbf{I}-\mathbf{J})\big(\mathrm{div}\,\boldsymbol{\tau}_f\big) \quad \text{in} \quad \Omega_f \quad \text{and} \quad \mathbf{P}_f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} \;=\; 0 \quad \text{on} \quad \Sigma\cup\Gamma. \tag{3.11}$$

In addition, thanks to the continuous dependence result for (3.8), there holds

$$\|\mathbf{P}_f(\boldsymbol{\tau}_f)\|_{\mathrm{div};\Omega_f} \;\le\; C\,\|\mathrm{div}\,\boldsymbol{\tau}_f\|_{0,\Omega_f} \qquad \forall\,\boldsymbol{\tau}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f),$$

which shows that $\mathbf{P}_f$ is bounded. Furthermore, it is straightforward from (3.8)–(3.11) that $\mathbf{P}_f$ is a projector, and therefore

$$\mathbf{H}(\mathrm{div};\Omega_f) \;=\; \mathbf{P}_f\left(\mathbf{H}(\mathrm{div};\Omega_f)\right) \,\oplus\, (\mathbf{I}-\mathbf{P}_f)\left(\mathbf{H}(\mathrm{div};\Omega_f)\right). \tag{3.12}$$

Also, it is clear from (3.9) that $\mathbf{P}_f(\boldsymbol{\tau}_f) \in \mathbf{H}^{\epsilon}(\Omega_f)$ and

$$\|\mathbf{P}_f(\boldsymbol{\tau}_f)\|_{\epsilon,\Omega_f} \;\le\; C\,\|\mathrm{div}\,\boldsymbol{\tau}_f\|_{0,\Omega_f} \qquad \forall\,\boldsymbol{\tau}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f). \tag{3.13}$$

3.2. Decomposition of the bilinear form A

We begin the analysis by introducing the bilinear forms $A_s^+ : \mathbb{H}(\mathbf{div};\Omega_s)\times\mathbb{H}(\mathbf{div};\Omega_s)\to\mathbb{C}$ and $A_f^+ : \mathbf{H}(\mathrm{div};\Omega_f)\times\mathbf{H}(\mathrm{div};\Omega_f)\to\mathbb{C}$ given by

$$A_s^+(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s) := \int_{\Omega_s}\mathcal{C}^{-1}\boldsymbol{\zeta}_s:\boldsymbol{\tau}_s + \frac{1}{\kappa_s^2}\int_{\Omega_s}\mathbf{div}\,\boldsymbol{\zeta}_s\cdot\mathbf{div}\,\boldsymbol{\tau}_s \qquad \forall\,\boldsymbol{\zeta}_s,\,\boldsymbol{\tau}_s\in\mathbb{H}(\mathbf{div};\Omega_s), \tag{3.14}$$

and

$$A_f^+(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f) := \int_{\Omega_f}\boldsymbol{\zeta}_f\cdot\boldsymbol{\tau}_f + \frac{1}{\kappa_f^2}\int_{\Omega_f}\mathrm{div}\,\boldsymbol{\zeta}_f\mathrm{div}\,\boldsymbol{\tau}_f \qquad \forall\,\boldsymbol{\zeta}_f,\,\boldsymbol{\tau}_f\in\mathbf{H}(\mathrm{div};\Omega_f), \tag{3.15}$$

which are clearly bounded, symmetric, and positive semi-definite. Actually, it is straightforward to see from (3.15) that A_f^+ is $\mathbf{H}(\mathrm{div};\Omega_f)$-elliptic, that is there exists $\alpha_f^+ := \min\left\{1,\frac{1}{\kappa_f^2}\right\} > 0$ such that

$$A_f^+(\boldsymbol{\tau}_f,\overline{\boldsymbol{\tau}}_f) \ \geq\ \alpha_f^+\,\|\boldsymbol{\tau}_f\|^2_{\mathrm{div};\Omega_f} \qquad \forall\,\boldsymbol{\tau}_f\in\mathbf{H}(\mathrm{div};\Omega_f), \tag{3.16}$$

and we show below in Section 3.3 that A_s^+ is also elliptic but on a subspace of $\mathbb{H}(\mathbf{div};\Omega_s)$.

In what follows, we employ the decompositions (3.6) and (3.12) to reformulate (2.10) in a more suitable form. More precisely, the unknown $\widehat{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_s,\boldsymbol{\sigma}_f)$ and the corresponding test function $\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}_s,\boldsymbol{\tau}_f)$, both in $\mathbf{H}$, are replaced, respectively, by the expressions

$$\boldsymbol{\sigma}_s = \mathbf{P}_s(\boldsymbol{\sigma}_s) + (\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\sigma}_s), \quad \boldsymbol{\sigma}_f = \mathbf{P}_f(\boldsymbol{\sigma}_f) + (\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\sigma}_f) \tag{3.17}$$

and

$$\boldsymbol{\tau}_s = \mathbf{P}_s(\boldsymbol{\tau}_s) + (\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\tau}_s), \quad \boldsymbol{\tau}_f = \mathbf{P}_f(\boldsymbol{\tau}_f) + (\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\tau}_f). \tag{3.18}$$

To this respect, we observe, according to (3.4), (3.5), and the fact that $\nabla\mathbf{v}\in\mathbb{L}^2_{\mathtt{asym}}(\Omega_s)$ for all $\mathbf{v}\in\mathbb{RM}(\Omega_s)$, that for all $\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s\in\mathbb{H}(\mathbf{div};\Omega_s)$, there holds

$$\begin{aligned}\int_{\Omega_s}\mathbf{div}(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s)\cdot\mathbf{div}\,\mathbf{P}_s(\boldsymbol{\tau}_s) \ &=\ \int_{\Omega_s}\mathbf{M}(\mathbf{div}\,\boldsymbol{\zeta}_s)\cdot\mathbf{div}\,\mathbf{P}_s(\boldsymbol{\tau}_s)\\ =\ -\int_{\Omega_s}\nabla\mathbf{M}(\mathbf{div}\,\boldsymbol{\zeta}_s):\mathbf{P}_s(\boldsymbol{\tau}_s) \ &+\ \langle\,\mathbf{P}_s(\boldsymbol{\tau}_s)\,\boldsymbol{\nu},\,\mathbf{M}(\mathbf{div}\,\boldsymbol{\zeta}_s)\,\rangle_\Sigma = 0.\end{aligned} \tag{3.19}$$

Analogously, according to (3.11), we deduce that for all $\boldsymbol{\zeta}_f,\,\boldsymbol{\tau}_f\in\mathbf{H}(\mathrm{div};\Omega_f)$, there holds

$$\begin{aligned}\int_{\Omega_f}\mathrm{div}\big(\mathbf{I}-\mathbf{P}_f\big)(\boldsymbol{\zeta}_f)\,\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_f) \ &=\ \mathbf{J}(\mathrm{div}\,\boldsymbol{\zeta}_f)\int_{\Omega_f}\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_f)\\ &=\ \mathbf{J}(\mathrm{div}\,\boldsymbol{\zeta}_f)\Big\{\langle\mathbf{P}_f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu},1\rangle_\Gamma - \langle\mathbf{P}_f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu},1\rangle_\Sigma\Big\} \ =\ 0.\end{aligned} \tag{3.20}$$

Hence, using the decompositions (3.6) and (3.12), and the identities (3.19) and (3.20), and adding and substracting suitable terms, we find that A (*cf.* (2.11)) can be decomposed as

$$A(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) = A_0(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) + K_0(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) \qquad \forall\,(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) := ((\boldsymbol{\zeta}_s,\boldsymbol{\zeta}_f),(\boldsymbol{\tau}_s,\boldsymbol{\tau}_f))\in\mathbf{H}\times\mathbf{H},$$

where $A_0:\mathbf{H}\times\mathbf{H}\to\mathbb{C}$ and $K_0:\mathbf{H}\times\mathbf{H}\to\mathbb{C}$ are given by

$$A_0(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) = A_s(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s) + A_f(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f), \tag{3.21}$$

and

$$K_0(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) = K_s(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s) + K_f(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f), \tag{3.22}$$

with the bilinear forms $A_s : \mathbb{H}(\mathbf{div};\Omega_s)\times\mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{C}$, $A_f : \mathbf{H}(\mathrm{div};\Omega_f)\times\mathbf{H}(\mathrm{div};\Omega_f) \to \mathbb{C}$, $K_s : \mathbb{H}(\mathbf{div};\Omega_s)\times\mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{C}$, and $K_f : \mathbf{H}(\mathrm{div};\Omega_f) \times \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbb{C}$ defined by

$$A_s\,(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s)) \;:=\; -\,A_s^{+}(\mathbf{P}_s(\boldsymbol{\zeta}_s),\mathbf{P}_s(\boldsymbol{\tau}_s)) \;+\; A_s^{+}((\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s),(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\tau}_s)), \tag{3.23}$$

$$A_f\,\big(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f\big) \;:=\; -\,A_f^{+}(\mathbf{P}_f(\boldsymbol{\zeta}_f),\mathbf{P}_f(\boldsymbol{\tau}_f)) \;+\; A_f^{+}((\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\zeta}_f),(\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\tau}_f)), \tag{3.24}$$

$$\begin{aligned} K_s\,(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s)) \;:=\; & 2\int_{\Omega_s} \mathcal{C}^{-1}\mathbf{P}_s(\boldsymbol{\zeta}_s):\mathbf{P}_s(\boldsymbol{\tau}_s) \;+\; \int_{\Omega_s} \mathcal{C}^{-1}\mathbf{P}_s(\boldsymbol{\zeta}_s):(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\tau}_s) \\ & + \int_{\Omega_s} \mathcal{C}^{-1}(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s):\mathbf{P}_s(\boldsymbol{\tau}_s) \;-\Big(1+\frac{1}{\kappa_s^2}\Big)\int_{\Omega_s} \mathbf{div}(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s)\cdot\mathbf{div}(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\tau}_s)\;, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} K_f\,\big(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f\big) \;:=\; & 2\int_{\Omega_f} \mathbf{P}_f(\boldsymbol{\zeta}_f)\cdot\mathbf{P}_f(\boldsymbol{\tau}_f) + \int_{\Omega_f} \mathbf{P}_f(\boldsymbol{\zeta}_f)\cdot(\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\tau}_f) \\ & + \int_{\Omega_f} (\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\zeta}_f)\cdot\mathbf{P}_f(\boldsymbol{\tau}_f) \;-\;\Big(1+\frac{1}{\kappa_f^2}\Big)\int_{\Omega_f} \mathrm{div}(\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\zeta}_f)\cdot\mathrm{div}(\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\tau}_f). \end{aligned} \tag{3.26}$$

Next, we let $\mathbf{A}_0 : \mathbf{H} \to \mathbf{H}$, $\mathbf{K}_0 : \mathbf{H} \to \mathbf{H}$, $\mathbf{B} : \mathbf{H} \to \mathbf{Q}$ and $\mathbf{K} : \mathbf{Q} \to \mathbf{Q}$ be the linear and bounded operators induced by the bilinear forms (3.21)–(2.15), respectively. In addition, we let $\mathbf{B}^* : \mathbf{Q} \to \mathbf{H}$ be the adjoint of $\mathbf{B}$, and denote by $\mathbf{F}$ and $\mathbf{G}$ the Riesz representants of the functionals F and G. Hence, using these notations and taking into account the decompositions (3.17) and (3.18), the fully-mixed variational formulation (2.10) can be rewritten as the following operator equation: Find $(\widehat{\boldsymbol{\sigma}},\widehat{\gamma}) \in \mathbf{H}\times\mathbf{Q}$ such that

$$\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \widehat{\gamma} \end{pmatrix} \;+\; \begin{pmatrix} \mathbf{K}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \widehat{\gamma} \end{pmatrix} \;=\; \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}. \tag{3.27}$$

Moreover, it is quite straightforward from the definitions of A_0 (*cf.* (3.21)) and B (*cf.* (2.12)) that (up to a permutation of rows) there holds

$$\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \widehat{\gamma} \end{pmatrix} = \left(\begin{array}{cc|cc} \mathbf{A}_s & \mathbf{B}_s^* & \multicolumn{2}{c}{\mathbf{0}} \\ \mathbf{B}_s & \mathbf{0} & & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{A}_f & \mathbf{B}_f^* \\ & & \mathbf{B}_f & \mathbf{0} \end{array}\right) \left(\begin{array}{c} \boldsymbol{\sigma}_s \\ (\gamma,\boldsymbol{\varphi}_s) \\ \hline \boldsymbol{\sigma}_f \\ \boldsymbol{\varphi}_f \end{array}\right), \tag{3.28}$$

where $\mathbf{A}_s : \mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{H}(\mathbf{div};\Omega_s)$, $\mathbf{B}_s : \mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\times\mathbf{H}^{1/2}(\Sigma)$, $\mathbf{A}_f : \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}(\mathrm{div};\Omega_f)$, and $\mathbf{B}_f : \mathbf{H}(\mathrm{div};\Omega_f) \to H^{1/2}(\partial\Omega_f)$ are the bounded linear operators induced by A_s, B_s, A_f, and B_f, respectively.

In the following section we show that the matrix operators on the left hand side of (3.27) become bijective and compact, respectively. In particular, concerning the bijectivity issue, and because of the block-diagonal saddle point structure shown by the right-hand side of (3.28), it suffices to apply the well known Babuška–Brezzi theory independently to each one of the two blocks arising there.

3.3. Application of the Babuška–Brezzi and Fredholm theories

We begin with the continuous inf-sup conditions for the bilinear forms B_s and B_f, which are equivalent to the surjectivity of $\mathbf{B}_s$ and $\mathbf{B}_f$, respectively. For this purpose, we first notice from (2.13) and (2.14) that these operators are given by

$$\mathbf{B}_s(\boldsymbol{\tau}_s) \;:=\; \Big(\frac{1}{2}\big(\boldsymbol{\tau}_s-\boldsymbol{\tau}_s^{\mathtt{t}}\big),-\mathcal{R}_s(\boldsymbol{\tau}_s\,\boldsymbol{\nu})\Big) \qquad \forall\,\boldsymbol{\tau}_s \,\in\, \mathbb{H}(\mathbf{div};\Omega_s), \tag{3.29}$$

and

$$\mathbf{B}_f(\boldsymbol{\tau}_f) := (\mathcal{R}_{\Sigma}(\boldsymbol{\tau}_f\cdot\boldsymbol{\nu}), -\mathcal{R}_{\Gamma}(\boldsymbol{\tau}_f\cdot\boldsymbol{\nu})) \qquad \forall\,\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f), \tag{3.30}$$

where $\mathcal{R}_s : \mathbf{H}^{-1/2}(\Sigma) \to \mathbf{H}^{1/2}(\Sigma)$, $\mathcal{R}_{\Sigma} : H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$, and $\mathcal{R}_{\Gamma} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, are the respective Riesz operators. Hence, we have the following lemmas.

Lemma 3.1. *There exists $\beta_s > 0$ such that*

$$\sup_{\boldsymbol{\tau}_s\in\mathbb{H}(\mathbf{div};\Omega_s)\setminus\{\mathbf{0}\}} \frac{|\,B_s(\boldsymbol{\tau}_s,(\boldsymbol{\eta},\boldsymbol{\psi}_s)\,)\,|}{\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}} \,\geq\, \beta_s\,\|(\boldsymbol{\eta},\boldsymbol{\psi}_s)\| \qquad \forall\,(\boldsymbol{\eta},\boldsymbol{\psi}_s) \in \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\times\mathbf{H}^{1/2}(\Sigma).$$

Proof. We proceed as in the proof of [10], Lemma 4.1. Given $(\boldsymbol{\eta},\boldsymbol{\psi}_s) \in \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\times\mathbf{H}^{1/2}(\Sigma)$ we let $\mathbf{z} \in \mathbf{H}^1(\Omega_s)$ be the unique (up to a rigid motion) solution of the variational formulation

$$\int_{\Omega_s}\boldsymbol{\varepsilon}(\mathbf{z}):\boldsymbol{\varepsilon}(\mathbf{w}) \,=\, -\int_{\Omega_s}\mathbf{r}(\boldsymbol{\eta},\boldsymbol{\psi}_s)\cdot\mathbf{w} \,-\, \int_{\Omega_s}\boldsymbol{\eta}:\nabla\mathbf{w} \,+\, \langle\mathcal{R}_s^{-1}(\boldsymbol{\psi}_s),\mathbf{w}\rangle_{\Sigma} \qquad \forall\,\mathbf{w}\in\mathbf{H}^1(\Omega_s), \tag{3.31}$$

where $\mathbf{r}(\boldsymbol{\eta},\boldsymbol{\psi}_s) \in \mathbb{RM}(\Omega_s)$ is characterized by

$$\int_{\Omega_s}\mathbf{r}(\boldsymbol{\eta},\boldsymbol{\psi}_s)\cdot\mathbf{w} \,=\, -\int_{\Omega_s}\boldsymbol{\eta}:\nabla\mathbf{w} \,+\, \langle\mathcal{R}_s^{-1}(\boldsymbol{\psi}_s),\mathbf{w}\rangle_{\Sigma} \qquad \forall\,\mathbf{w} \in \mathbb{RM}(\Omega_s).$$

Then, defining $\boldsymbol{\zeta}_s := \boldsymbol{\varepsilon}(\mathbf{z})+\boldsymbol{\eta}$, we find from (3.31) that $\mathbf{div}\,\boldsymbol{\zeta}_s = \mathbf{r}(\boldsymbol{\eta},\boldsymbol{\psi}_s)$ in Ω_s, whence $\boldsymbol{\zeta}_s\in\mathbb{H}(\mathbf{div};\Omega_s)$, and thus $\boldsymbol{\zeta}_s\boldsymbol{\nu} = -\mathcal{R}_s^{-1}(\boldsymbol{\psi}_s)$ on Σ. It follows that $\mathbf{B}_s(\boldsymbol{\zeta}_s) = (\boldsymbol{\eta},\boldsymbol{\psi}_s)$, which proves the surjectivity of $\mathbf{B}_s$. □

Lemma 3.2. *There exists $\beta_f > 0$ such that*

$$\sup_{\boldsymbol{\tau}_f\in\mathbf{H}(\mathrm{div};\Omega_f)\setminus\{\mathbf{0}\}} \frac{|\,B_f(\boldsymbol{\tau}_f,\boldsymbol{\psi}_f)\,|}{\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}} \,\geq\, \beta_f\,\|\boldsymbol{\psi}_f\|_{1/2,\partial\Omega_f} \qquad \forall\,\boldsymbol{\psi}_f := (\psi_{\Sigma},\psi_{\Gamma}) \in H^{1/2}(\partial\Omega_f).$$

Proof. Given $\boldsymbol{\psi}_f := (\psi_{\Sigma},\psi_{\Gamma}) \in H^{1/2}(\partial\Omega_f)$, we let $z \in H^1(\Omega_f)$ be the unique solution (up to a constant) of the Neumann boundary value problem

$$\begin{aligned} \Delta z \,&=\, -\frac{1}{|\Omega_f|}\Big\{\langle\mathcal{R}_{\Sigma}^{-1}(\psi_{\Sigma}),1\rangle_{\Sigma} \,+\, \langle\mathcal{R}_{\Gamma}^{-1}(\psi_{\Gamma}),1\rangle_{\Gamma}\Big\} \quad \text{in} \quad \Omega_f,\\ \nabla z\cdot\boldsymbol{\nu} \,&=\, \mathcal{R}_{\Sigma}^{-1}(\psi_{\Sigma}) \quad \text{on} \quad \Sigma, \quad \nabla z\cdot\boldsymbol{\nu} \,=\, -\mathcal{R}_{\Gamma}^{-1}(\psi_{\Gamma}) \quad \text{on} \quad \Gamma. \end{aligned} \tag{3.32}$$

Then, defining $\boldsymbol{\zeta}_f := \nabla z \quad \text{in} \quad \Omega_f$, we easily see that

$$\mathbf{B}_f(\boldsymbol{\zeta}_f) := \big(\mathcal{R}_{\Sigma}(\boldsymbol{\zeta}_f\cdot\boldsymbol{\nu}), -\mathcal{R}_{\Gamma}(\boldsymbol{\zeta}_f\cdot\boldsymbol{\nu})\big) \,=\, (\psi_{\Sigma},\psi_{\Gamma}),$$

which shows that $\mathbf{B}_f$ is surjective. □

We now let $\mathbf{V}_s$ and $\mathbf{V}_f$ be the kernels of $\mathbf{B}_s$ and $\mathbf{B}_f$, respectively, that is, according to (3.29) and (3.30),

$$\mathbf{V}_s := \Big\{\boldsymbol{\tau}_s\in\mathbb{H}(\mathbf{div};\Omega_s): \quad \boldsymbol{\tau}_s \,=\, \boldsymbol{\tau}_s^{\mathtt{t}} \quad \text{in} \quad \Omega_s, \quad \boldsymbol{\tau}_s\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Sigma\Big\}, \tag{3.33}$$

$$\mathbf{V}_f := \Big\{\boldsymbol{\tau}_f\in\mathbf{H}(\mathrm{div};\Omega_f): \quad \boldsymbol{\tau}_f\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Sigma, \quad \boldsymbol{\tau}_f\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma\Big\}, \tag{3.34}$$

and aim to prove that $A_s|_{\mathbf{V}_s\times\mathbf{V}_s}$ and $A_f|_{\mathbf{V}_f\times\mathbf{V}_f}$ induce bijective operators. In particular, for A_s we proceed as in [7], Section 4.2 and make use of the decomposition

$$\mathbb{H}(\mathbf{div};\Omega_s) \,=\, \mathbb{H}_0(\mathbf{div};\Omega_s) \,\oplus\, \mathbb{C}\,\mathbf{I},$$

with

$$\mathbb{H}_0(\mathbf{div};\Omega_s) := \left\{ \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \int_{\Omega_s} \operatorname{tr}\boldsymbol{\tau}_s = 0 \right\}, \tag{3.35}$$

and the inequalities

$$\|\boldsymbol{\tau}_s^{\mathtt{d}}\|_{0,\Omega_s}^2 + \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}^2 \geq c_1\,\|\boldsymbol{\tau}_{s,0}\|_{0,\Omega_s}^2 \qquad \forall\,\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s) \tag{3.36}$$

(*cf.* [4], Prop. 3.1, Chap. IV), and

$$\|\boldsymbol{\tau}_{s,0}\|_{\mathbf{div};\Omega_s}^2 \geq c_2\,\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}^2 \qquad \forall\,\boldsymbol{\tau}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s) \tag{3.37}$$

(*cf.* [7], Lem. 4.5), with

$$\tilde{\mathbb{H}}(\mathbf{div};\Omega_s) := \left\{ \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \boldsymbol{\tau}_s\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Sigma \right\}, \tag{3.38}$$

where each $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$ is written as $\boldsymbol{\tau}_s = \boldsymbol{\tau}_{s,0} + d\,\mathbf{I}$, with $\boldsymbol{\tau}_{s,0} \in \mathbb{H}_0(\mathbf{div};\Omega_s)$ and $d \in \mathbb{C}$.

The following lemma establishes the $\tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$-ellipticity of A_s^+.

Lemma 3.3. *There exists $\alpha_s^+ > 0$, depending on μ, κ_s, c_1, and c_2, such that*

$$A_s^+(\boldsymbol{\tau}_s,\overline{\boldsymbol{\tau}}_s) \geq \alpha_s^+\,\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}^2 \qquad \forall\,\boldsymbol{\tau}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s). \tag{3.39}$$

Proof. According to the definition of A_s^+ (*cf.* (3.14)), and using the inequalities (2.3)–(3.37), we find that for each $\boldsymbol{\tau}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ there holds

$$\begin{aligned}
A_s^+(\boldsymbol{\tau}_s,\overline{\boldsymbol{\tau}}_s) &\geq \frac{1}{2\mu}\,\|\boldsymbol{\tau}_s^{\mathtt{d}}\|_{0,\Omega_s}^2 + \frac{1}{\kappa_s^2}\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}^2 \\
&\geq \min\left\{\frac{1}{2\mu},\frac{1}{2\kappa_s^2}\right\}\left\{\|\boldsymbol{\tau}_s^{\mathtt{d}}\|_{0,\Omega_s}^2 + \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}^2\right\} + \frac{1}{2\kappa_s^2}\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}^2 \\
&\geq \quad \tilde{c}_1\,\|\boldsymbol{\tau}_{s,0}\|_{0,\Omega_s}^2 + \frac{1}{2\kappa_s^2}\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}^2 \\
&\geq \min\left\{\tilde{c}_1,\frac{1}{2\kappa_s^2}\right\}\|\boldsymbol{\tau}_{s,0}\|_{\mathbf{div};\Omega_s}^2 \geq \alpha_s^+\,\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}^2,
\end{aligned}$$

with $\tilde{c}_1 := c_1 \min\left\{\frac{1}{2\mu},\frac{1}{2\kappa_s^2}\right\}$ and $\alpha_s^+ := c_2 \min\left\{\tilde{c}_1,\frac{1}{2\kappa_s^2}\right\}$, which completes the proof. □

We are now in a position to prove that A_s and A_f satisfy the continuous inf-sup conditions required by the Babuška–Brezzi theory. To this end, we need to introduce the operators

$$\Xi_s := (\mathbf{I} - 2\,\mathbf{P}_s) : \mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{H}(\mathbf{div};\Omega_s) \tag{3.40}$$

and

$$\Xi_f := (\mathbf{I} - 2\,\mathbf{P}_f) : \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}(\mathrm{div};\Omega_f), \tag{3.41}$$

which, recalling that $\mathbf{P}_s$ and $\mathbf{P}_f$ are projectors, are certainly bounded and satisfy

$$\mathbf{P}_s\,\Xi_s = -\mathbf{P}_s, \qquad (\mathbf{I} - \mathbf{P}_s)\,\Xi_s = \mathbf{I} - \mathbf{P}_s, \tag{3.42}$$

$$\mathbf{P}_f\,\Xi_f = -\mathbf{P}_f, \quad \text{and} \quad (\mathbf{I} - \mathbf{P}_f)\,\Xi_f = \mathbf{I} - \mathbf{P}_f. \tag{3.43}$$

Then, we can establish the following lemmas.

Lemma 3.4. *There exist* $\alpha_s,\ C_s > 0$ *such that*

$$A_s\big(\boldsymbol{\zeta}_s, \varXi_s(\overline{\boldsymbol{\zeta}}_s)\big) \,\geq\, \alpha_s \,\|\boldsymbol{\zeta}_s\|^2_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\zeta}_s \,\in\, \tilde{\mathbb{H}}(\mathbf{div};\Omega_s), \tag{3.44}$$

and

$$\sup_{\boldsymbol{\tau}_s\in\mathbf{V}_s\setminus\{\mathbf{0}\}} \frac{|A_s(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s)|}{\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}} \,\geq\, C_s\,\|\boldsymbol{\zeta}_s\|_{\mathbf{div};\Omega_s} \qquad \forall\,\boldsymbol{\zeta}_s \,\in\, \mathbf{V}_s. \tag{3.45}$$

In addition, there holds

$$\sup_{\boldsymbol{\zeta}_s\in\mathbf{V}_s\setminus\{\mathbf{0}\}} |A_s(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s)| \,>\, 0 \qquad \forall\,\boldsymbol{\tau}_s \,\in\, \mathbf{V}_s, \quad \boldsymbol{\tau}_s \,\neq\, \mathbf{0}. \tag{3.46}$$

Proof. We first observe, thanks to the definitions of $\mathbf{V}_s$ and $\tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ (*cf.* (3.33), (3.38)), and the properties of $\mathbf{P}_s$ (*cf.* (3.4), (3.5)), that $\mathbf{V}_s \subseteq \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ and $\mathbf{P}_s(\boldsymbol{\zeta}_s) \in \mathbf{V}_s$ for each $\boldsymbol{\zeta}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, and hence, in particular both $\mathbf{P}_s(\boldsymbol{\zeta}_s)$ and $(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s)$ belong to $\tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ for each $\boldsymbol{\zeta}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$. It follows, according to the definition of A_s (*cf.* (3.23)), the properties of $\varXi_s$ (*cf.* (3.42)), and the ellipticity of A_s^+ (*cf.* (3.39)), that for each $\boldsymbol{\zeta}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ there holds

$$\begin{aligned}
A_s\big(\boldsymbol{\zeta}_s, \varXi_s(\overline{\boldsymbol{\zeta}}_s)\big) \;&=\; A_s^+(\mathbf{P}_s(\boldsymbol{\zeta}_s), \mathbf{P}_s(\overline{\boldsymbol{\zeta}}_s)) \,+\, A_s^+((\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s), (\mathbf{I}-\mathbf{P}_s)(\overline{\boldsymbol{\zeta}}_s)) \\
&\geq\;\; \alpha_s^+ \left\{ \|\mathbf{P}_s(\boldsymbol{\zeta}_s)\|^2_{\mathbf{div};\Omega_s} \,+\, \|(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s)\|^2_{\mathbf{div};\Omega_s} \right\} \\
&\geq\;\; \frac{\alpha_s^+}{2}\,\|\boldsymbol{\zeta}_s\|^2_{\mathbf{div};\Omega_s},
\end{aligned}$$

which shows (3.44) with $\alpha_s := \alpha_s^+/2$. Next, given $\boldsymbol{\zeta}_s \in \mathbf{V}_s\setminus\{\mathbf{0}\}$, it is clear from the above analysis that $\varXi_s(\overline{\boldsymbol{\zeta}}_s) \in \mathbf{V}_s\setminus\mathbf{0}$, and therefore, applying (3.44), we deduce that

$$\sup_{\boldsymbol{\tau}_s\in\mathbf{V}_s\setminus\{\mathbf{0}\}} \frac{|A_s(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s)|}{\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}} \,\geq\, \frac{\big|A_s(\boldsymbol{\zeta}_s,\varXi_s(\overline{\boldsymbol{\zeta}}_s))\big|}{\|\varXi_s(\overline{\boldsymbol{\zeta}}_s)\|_{\mathbf{div};\Omega_s}} \,\geq\, \alpha_s\,\frac{\|\boldsymbol{\zeta}_s\|^2_{\mathbf{div};\Omega_s}}{\|\varXi_s(\overline{\boldsymbol{\zeta}}_s)\|_{\mathbf{div};\Omega_s}},$$

which yields (3.45) with $C_s := \alpha_s/\|\varXi_s\|$. Finally, (3.46) is a straightforward consequence of (3.45) and the symmetry of A_s. $\square$

Lemma 3.5. *There exist* $\alpha_f,\ C_f > 0$ *such that*

$$A_f\big(\boldsymbol{\zeta}_f, \varXi_f(\overline{\boldsymbol{\zeta}}_f)\big) \,\geq\, \alpha_f \,\|\boldsymbol{\zeta}_f\|^2_{\mathrm{div};\Omega_f} \qquad \forall\, \boldsymbol{\zeta}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f), \tag{3.47}$$

and

$$\sup_{\boldsymbol{\tau}_f\in\mathbf{V}_f\setminus\{\mathbf{0}\}} \frac{\big|A_f(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f)\big|}{\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}} \,\geq\, C_f\,\|\boldsymbol{\zeta}_f\|_{\mathrm{div};\Omega_f} \qquad \forall\,\boldsymbol{\zeta}_f \,\in\, \mathbf{V}_f. \tag{3.48}$$

In addition, there holds

$$\sup_{\boldsymbol{\zeta}_f\in\mathbf{V}_f\setminus\{\mathbf{0}\}} \big|A_f(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f)\big| \,>\, 0 \qquad \forall\,\boldsymbol{\tau}_f \,\in\, \mathbf{V}_f, \quad \boldsymbol{\tau}_f \,\neq\, \mathbf{0}. \tag{3.49}$$

Proof. We proceed analogously to the proof of the previous lemma. In fact, according to the definition of A_f (*cf.* (3.24)) and the properties of $\varXi_f$ (*cf.* (3.43)), and applying the ellipticity of A_f^+ (*cf.* (3.16)), we find that for each $\boldsymbol{\zeta}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$ there holds

$$\begin{aligned} A_f\big(\boldsymbol{\zeta}_f, \Xi_f(\overline{\boldsymbol{\zeta}}_f)\big) &= A_f^+(\mathbf{P}_f(\boldsymbol{\zeta}_f), \mathbf{P}_f(\overline{\boldsymbol{\zeta}}_f)) + A_f^+((\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\zeta}_f), (\mathbf{I}-\mathbf{P}_f)(\overline{\boldsymbol{\zeta}}_f)) \\ &\geq \alpha_f^+ \Big\{ \|\mathbf{P}_f(\boldsymbol{\zeta}_f)\|^2_{\mathrm{div};\Omega_f} + \|(\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\zeta}_f)\|^2_{\mathrm{div};\Omega_f} \Big\} \\ &\geq \frac{\alpha_f^+}{2} \|\boldsymbol{\zeta}_f\|^2_{\mathrm{div};\Omega_f}, \end{aligned}$$

which proves (3.47) with $\alpha_f := \alpha_f^+/2$. Next, it is clear from (3.47) that $\Xi_f(\overline{\boldsymbol{\zeta}}_f) \neq \mathbf{0}$ for each $\boldsymbol{\zeta}_f \in \mathbf{H}(\mathrm{div};\Omega_f) \setminus \{\mathbf{0}\}$. In addition, thanks to the properties of $\mathbf{P}_f$ (*cf.* (3.11)) and the definition of $\mathbf{V}_f$ (*cf.* (3.34)), we deduce that $\Xi_f(\overline{\boldsymbol{\zeta}}_f)$ belong to $\mathbf{V}_f \setminus \{\mathbf{0}\}$ for each $\boldsymbol{\zeta}_f \in \mathbf{V}_f \setminus \{\mathbf{0}\}$, and hence

$$\sup_{\boldsymbol{\tau}_f \in \mathbf{V}_f \setminus \{\mathbf{0}\}} \frac{\big|A_f(\boldsymbol{\zeta}_f, \boldsymbol{\tau}_f)\big|}{\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}} \geq \frac{\big|A_f(\boldsymbol{\zeta}_f, \Xi_f(\overline{\boldsymbol{\zeta}}_f))\big|}{\|\Xi_f(\overline{\boldsymbol{\zeta}}_f)\|_{\mathrm{div};\Omega_f}} \geq \alpha_f \frac{\|\boldsymbol{\zeta}_f\|^2_{\mathrm{div};\Omega_f}}{\|\Xi_f(\overline{\boldsymbol{\zeta}}_f)\|_{\mathrm{div};\Omega_f}},$$

which implies (3.48) with $C_f := \alpha_f/\|\Xi_f\|$. Finally, the inequality (3.49) follows directly from (3.48) and the symmetry of A_f. □

As a consequence of Lemmas 3.1–3.4, and 3.5, and having in mind the identity (3.28) and the classical Babuška–Brezzi theory (*cf.* [4], Thm. 1.1, Chap. II), we conclude that the matrix operator $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} : \mathbf{H}\times\mathbf{Q} \to \mathbf{H}\times\mathbf{Q}$ is an isomorphism. In turn, the compactness of $\begin{pmatrix} \mathbf{K}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix} : \mathbf{H}\times\mathbf{Q} \to \mathbf{H}\times\mathbf{Q}$ is proved by the following lemma.

Lemma 3.6. *The operators* $\mathbf{K}_0 : \mathbf{H} \to \mathbf{H}$ *and* $\mathbf{K} : \mathbf{Q} \to \mathbf{Q}$ *are compact.*

Proof. We first recall from Section 3.1 (*cf.* (3.7) and (3.13)) that there exists $\epsilon > 0$ such that $\mathbf{P}_s(\boldsymbol{\tau}_s) \in \mathbb{H}^\epsilon(\Omega_s)$ for each $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, and $\mathbf{P}_f(\boldsymbol{\tau}_f) \in \mathbf{H}^\epsilon(\Omega_f)$ for each $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, which, thanks to the compact imbeddings $\mathbb{H}^\epsilon(\Omega_s) \hookrightarrow \mathbb{L}^2(\Omega_s)$ and $\mathbf{H}^\epsilon(\Omega_f) \hookrightarrow \mathbf{L}^2(\Omega_f)$, imply the compactness of $\mathbf{P}_s : \mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{L}^2(\Omega_s)$ and $\mathbf{P}_f : \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{L}^2(\Omega_f)$. It follows that the adjoints $\mathbf{P}_s^* : \mathbb{L}^2(\Omega_s) \to \mathbb{H}(\mathbf{div};\Omega_s)$ and $\mathbf{P}_f^* : \mathbf{L}^2(\Omega_f) \to \mathbf{H}(\mathrm{div};\Omega_f)$, and hence the operators $\mathbf{P}_s^*\,\mathcal{C}^{-1}\,\mathbf{P}_s$, $(\mathbf{I}-\mathbf{P}_s)^*\,\mathcal{C}^{-1}\,\mathbf{P}_s$, $\mathbf{P}_s^*\,\mathcal{C}^{-1}\,(\mathbf{I}-\mathbf{P}_s)$, $\mathbf{P}_f^*\,\mathbf{P}_f$, $(\mathbf{I}-\mathbf{P}_f)^*\,\mathbf{P}_f$, and $\mathbf{P}_f^*\,(\mathbf{I}-\mathbf{P}_f)$ are all compact. This shows that the first three terms defining the bilinear forms K_s (*cf.* (3.25)) and K_f (*cf.* (3.26)) induce compact operators. In addition, it is clear from the second identity in (3.4) and the first identity in (3.11) that the fourth terms of K_s and K_f yield finite rank operators, and therefore $\mathbf{K}_0 : \mathbf{H} \to \mathbf{H}$ becomes compact.

Furthermore, the three terms defining $\mathbf{K}$ (*cf.* (2.15)), that is $\langle \xi_\Sigma\, \boldsymbol{\nu}, \boldsymbol{\psi}_s\rangle_\Sigma$, $\rho_f\,\omega^2\,\langle \psi_\Sigma\,\boldsymbol{\nu}, \boldsymbol{\xi}_s\rangle_\Sigma$, and $\imath\,\kappa_f\,\langle \xi_\Gamma, \psi_\Gamma\rangle_\Gamma$ also yield compact operators because of the compactness of the composition defined by the following diagram

$$\begin{array}{ccccccc} H^{1/2}(\Sigma) & \stackrel{\text{compact}}{\longrightarrow} & L^2(\Sigma) & \stackrel{\text{continuous}}{\longrightarrow} & \mathbf{L}^2(\Sigma) & \stackrel{\text{compact}}{\longrightarrow} & \mathbf{H}^{-1/2}(\Sigma) \\ \psi_\Sigma & \longrightarrow & \psi_\Sigma & \longrightarrow & \psi_\Sigma\,\boldsymbol{\nu} & \longrightarrow & \psi_\Sigma\,\boldsymbol{\nu}, \end{array}$$

and thanks to the compact imbedding $H^{1/2}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$. This completes the proof. □

We are able now to provide the main result of this section.

Theorem 3.7. *Assume that the homogeneous problem associated to* (2.10) *has only the trivial solution. Then, given* $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ *and* $g \in H^{-1/2}(\Gamma)$, *there exists a unique solution* $(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) \in \mathbf{H}\times\mathbf{Q}$ *to* (2.10) (*equivalently* (3.27)). *In addition, there exists* $C > 0$ *such that*

$$\|(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma})\|_{\mathbf{H}\times\mathbf{Q}} \leq C \Big\{ \|\mathbf{f}\|_{0,\Omega_s} + \|g\|_{-1/2,\Gamma} \Big\}.$$

Proof. It suffices to notice, according to our previous analysis, that the left hand side of (3.27) constitutes a Fredholm operator of index zero. □

We end this section by remarking that the extension of the previous continuous analysis to the 3D version of our interaction problem is quite straightforward. However, this is not exactly the case when trying to extend to 3D the Galerkin analysis shown below in Section 4. In particular, the proofs of the discrete inf-sup conditions involving boundary or interface terms are rather technical and they require additional hypotheses on the triangulations of both domains. In order to circumvent these difficulties, in the recent works [10, 11] we have developed a new approach which incorporates the exact satisfaction of the transmission conditions into the definitions of the continuous and discrete spaces.

4. Analysis of the Galerkin scheme

In this section we introduce a Galerkin approximation of (2.10) and show, under the same assumption of Theorem 3.7, that it is well-posed. The corresponding result is given by Theorem 4.11, whose proof is obtained as a consequence of the analysis in the following sections. In fact, we first define in Section 4.1 the main finite element subspaces to be employed in the definition of the Galerkin scheme (*cf.* (4.7)) and provide their approximation properties in Section 4.2. Then, in Section 4.3 we prove the existence of stable discrete liftings of the normal traces on Σ and Γ of the finite element subspaces approximating the stresses. These lifting operators allow us to establish certain equivalence results (*cf.* Lems. 4.3 and 4.4), which later on simplify the proofs of the discrete inf-sup conditions for the bilinear forms B_f and B_s (*cf.* Lems. 4.7 and 4.8). Next, in Section 4.4 we introduce uniformly bounded discrete operators $\mathbf{P}_{f,h}$ and $\mathbf{P}_{s,h}$ approximating $\mathbf{P}_f$ and $\mathbf{P}_s$, respectively. Recall that the latter operators were utilized in Section 3.3 to prove the continuous inf-sup conditions for the bilinear forms A_s and A_f (*cf.* Lems. 3.4 and 3.5). Hence, the key results in Section 4.4 refer to the upper estimates for the errors $\|\mathbf{P}_s - \mathbf{P}_{s,h}\|$ and $\|\mathbf{P}_f - \mathbf{P}_{f,h}\|$ (*cf.* Lems. 4.5 and 4.6), which are utilized in Lemma 4.10 to prove the discrete inf-sup conditions for A_s and A_f. Finally, after establishing all the above mentioned discrete inf-sup conditions in Section 4.5, the well-posedness of the Galerkin scheme, which follows at once, is summarized in Theorem 4.11.

4.1. Preliminaries

We first let $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$ be triangulations, belonging to shape-regular families, of the polygonal regions $\bar{\Omega}_s$ and $\bar{\Omega}_f$, respectively, by triangles T of diameter h_T, with global mesh size

$$h := \max\Big\{ \max\big\{ h_T : \quad T \in \mathcal{T}_h^s \big\} ; \max\big\{ h_T : \quad T \in \mathcal{T}_h^f \big\} \Big\},$$

and such that the vertices of $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$ coincide on Σ. In what follows, given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^2$, $P_\ell(S)$ denotes the space of polynomials defined in S of total degree $\leq \ell$. In addition, following the same terminology described at the end of the introduction, we denote $\mathbf{P}_\ell(S) := [P_\ell(S)]^2$. Furthermore, given $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $\mathbf{x} := (x_1, x_2)^{\mathtt{t}}$ a generic vector of $\mathbb{R}^2$, we let $\mathrm{RT}_0(T) := \operatorname{span}\Big\{(1,0), (0,1), (x_1, x_2)\Big\}$ be the local Raviart–Thomas space of order 0 (*cf.* [4, 26]), and set $\mathbf{curl}^{\mathtt{t}}\, b_T := \left(\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1}\right)$, where b_T is the usual cubic bubble function on T. Then we define

$$\mathbf{H}_h^s := \Big\{ \mathbf{v}_{s,h} \in \mathbf{H}(\mathrm{div}; \Omega_s) : \quad \mathbf{v}_{s,h}|_T \in \mathrm{RT}_0(T) \oplus P_0(T)\, \mathbf{curl}^{\mathtt{t}}\, b_T \quad \forall\, T \in \mathcal{T}_h^s \Big\},$$

$$\mathbb{H}_h^s := \Big\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}(\mathbf{div}; \Omega_s) : \quad \mathbf{c}^{\mathtt{t}}\, \boldsymbol{\tau}_{s,h} \in \mathbf{H}_h^s \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \Big\}, \tag{4.1}$$

$$\mathbf{H}_h^f := \Big\{ \boldsymbol{\tau}_{f,h} \in \mathbf{H}(\mathrm{div}; \Omega_f) : \quad \boldsymbol{\tau}_{f,h}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^f \Big\}, \tag{4.2}$$

$$\mathbb{Q}_h^s := \left\{ \boldsymbol{\eta}_h := \begin{pmatrix} 0 & \eta_h \\ -\eta_h & 0 \end{pmatrix} : \quad \eta_h \in C(\bar{\Omega}_s), \quad \eta_h|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h^s \right\}, \tag{4.3}$$

$$\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma), \tag{4.4}$$

$$\mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma), \tag{4.5}$$

where $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$ are generic finite dimensional subspaces (to be specified later on) of $H^{1/2}(\Sigma)$ and $H^{1/2}(\Gamma)$, respectively, and introduce the finite element subspaces $\mathbf{H}_h \subseteq \mathbf{H}$ and $\mathbf{Q}_h \subseteq \mathbf{Q}$, given by

$$\mathbf{H}_h := \mathbb{H}_h^s \times \mathbf{H}_h^f \qquad \text{and} \qquad \mathbf{Q}_h := \mathbb{Q}_h^s \times \mathbf{Q}_h^s \times \mathbf{Q}_h^f. \tag{4.6}$$

Note that the associated generic subspaces $\mathbf{Q}_h^f$ and $\mathbf{Q}_h^s$ are employed below (*cf.* Lemmas 4.3 and 4.4) to establish preliminary equivalence results concerning the discrete inf-sup conditions for B_f and B_s. Explicit definitions of $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$, and hence of $\mathbf{Q}_h^f$ and $\mathbf{Q}_h^s$, are given later on in Section 4.5 (*cf.* (4.48)–(4.51)) to finally guarantee the ocurrence of the discrete inf-sup conditions for those bilinear forms (*cf.* Lems. 4.7 and 4.8).

In addition, our analysis below will also require the subspaces

$$\tilde{\mathbf{H}}_h^s := \left\{ \mathbf{v}_{s,h} \in \mathbf{H}(\mathrm{div}; \Omega_s) : \quad \mathbf{v}_{s,h}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^s \right\},$$

$$\tilde{\mathbb{H}}_h^s := \left\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}(\mathbf{div}; \Omega_s) : \quad \mathbf{c}^{\mathrm{t}}\, \boldsymbol{\tau}_{s,h} \in \tilde{\mathbf{H}}_h^s \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \right\},$$

$$\mathbf{U}_h^s := \left\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega_s) : \quad \mathbf{v}_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^s \right\}$$

and

$$U_h^f := \left\{ v_h \in L^2(\Omega_f) : \quad v_h|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^f \right\}.$$

We recall here that $\mathbb{H}_h^s \times \mathbf{U}_h^s \times \mathbb{Q}_h^s$ constitutes the well known PEERS space introduced in [1] for a mixed finite element aproximation of the linear elasticity problem in the plane. In turn, $\mathbf{H}_h^f \times U_h^f$ is the lowest order Raviart–Thomas mixed finite element approximation of the Poisson problem for the Laplace equation (see [4, 26]). Also, it is important to notice, which will be used below, that $\tilde{\mathbf{H}}_h^s \subseteq \mathbf{H}_h^s$ and hence $\tilde{\mathbb{H}}_h^s \subseteq \mathbb{H}_h^s$.

The Galerkin scheme associated to our continuous problem (2.10) is then defined as follows: Find $\widehat{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}) \in \mathbf{H}_h$ and $\widehat{\boldsymbol{\gamma}}_h := (\boldsymbol{\gamma}_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h}) \in \mathbf{Q}_h$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\tau}}_h) + B(\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\gamma}}_h) &= F(\widehat{\boldsymbol{\tau}}_h) \qquad \forall\, \widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h, \\ B(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\eta}}_h) + K(\widehat{\boldsymbol{\gamma}}_h, \widehat{\boldsymbol{\eta}}_h) &= G(\widehat{\boldsymbol{\eta}}_h) \qquad \forall\, \widehat{\boldsymbol{\eta}}_h := (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}, \boldsymbol{\psi}_{f,h}) \in \mathbf{Q}_h, \end{aligned} \tag{4.7}$$

We collect next the approximation properties of the finite element subspaces introduced above.

4.2. Approximation properties of the subspaces

We begin with the subspaces $\mathbb{H}_h^s$ and $\mathbf{H}_h^f$. Indeed, given $\delta \in (0,1]$, we let

$$\mathcal{E}_h^s : \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div}; \Omega_s) \to \tilde{\mathbb{H}}_h^s \subseteq \mathbb{H}_h^s \qquad \text{and} \qquad \mathcal{E}_h^f : \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\mathrm{div}; \Omega_f) \to \mathbf{H}_h^f$$

be the usual Raviart–Thomas interpolation operators (see [4, 26]), which, given $\boldsymbol{\tau}_s \in \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div}; \Omega_s)$ and $\boldsymbol{\tau}_f \in \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\mathrm{div}; \Omega_f)$, are characterized by the identities

$$\int_e \mathcal{E}_h^s(\boldsymbol{\tau}_s)\, \boldsymbol{\nu} \cdot \mathbf{q} = \int_e \boldsymbol{\tau}_s\, \boldsymbol{\nu} \cdot \mathbf{q} \quad \forall\, \mathbf{q} \in \mathbf{P}_0(e), \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h^s, \tag{4.8}$$

and

$$\int_e \mathcal{E}_h^f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu}\, q \;=\; \int_e \boldsymbol{\tau}_f\cdot\boldsymbol{\nu}\, q \quad \forall\, q\,\in\, P_0(e), \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h^f. \tag{4.9}$$

In addition, the corresponding conmuting diagram properties yield

$$\mathbf{div}(\mathcal{E}_h^s(\boldsymbol{\tau}_s)) \;=\; \mathcal{P}_h^s(\mathbf{div}\,\boldsymbol{\tau}_s) \qquad \forall\,\boldsymbol{\tau}_s\,\in\,\mathbb{H}^\delta(\Omega_s)\cap\mathbb{H}(\mathbf{div};\Omega_s), \tag{4.10}$$

and

$$\mathrm{div}(\mathcal{E}_h^f(\boldsymbol{\tau}_f)) \;=\; \mathcal{P}_h^f(\mathrm{div}\,\boldsymbol{\tau}_f) \qquad \forall\,\boldsymbol{\tau}_f\,\in\,\mathbf{H}^\delta(\Omega_f)\cap\mathbf{H}(\mathrm{div};\Omega_f), \tag{4.11}$$

where $\mathcal{P}_h^s:\mathbf{L}^2(\Omega_s)\to\mathbf{U}_h^s$ and $\mathcal{P}_h^f: L^2(\Omega_f)\to U_h^f$ are the corresponding orthogonal projectors, which satisfy the following error estimates (see, *e.g.* [4])

(AP_h^s) For each $t\in(0,1]$ and for each $\mathbf{v}\,\in\,\mathbf{H}^t(\Omega_s)$, there holds

$$\|\mathbf{v}-\mathcal{P}_h^s(\mathbf{v})\|_{0,\Omega_s}\;\le\; C\,h^t\,\|\mathbf{v}\|_{t,\Omega_s}.$$

(AP_h^f) For each $t\in(0,1]$ and for each $v\,\in\,H^t(\Omega_f)$, there holds

$$\|v-\mathcal{P}_h^f(v)\|_{0,\Omega_f}\;\le\; C\,h^t\,\|v\|_{t,\Omega_f}.$$

Furthermore, it is easy to show, using the well-known Bramble–Hilbert Lemma and the boundedness of the local interpolation operators on the reference element $\widehat{T}$ (see, *e.g.* [16], Eq. (3.39)), that there exist $\widehat{C}_s$, $\widehat{C}_f>0$, independent of h, such that for each $\boldsymbol{\tau}_s\in\mathbb{H}^\delta(\Omega_s)\cap\mathbb{H}(\mathbf{div};\Omega_s)$ and for each $\boldsymbol{\tau}_f\in\mathbf{H}^\delta(\Omega_f)\cap\mathbf{H}(\mathrm{div};\Omega_f)$, there hold

$$\|\boldsymbol{\tau}_s-\mathcal{E}_h^s(\boldsymbol{\tau}_s)\|_{0,T}\;\le\;\widehat{C}_s\,h_T^\delta\left\{|\boldsymbol{\tau}_s|_{\delta,T}\,+\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,T}\right\}\quad\forall\,T\in\mathcal{T}_h^s, \tag{4.12}$$

and

$$\|\boldsymbol{\tau}_f-\mathcal{E}_h^f(\boldsymbol{\tau}_f)\|_{0,T}\;\le\;\widehat{C}_f\,h_T^\delta\left\{|\boldsymbol{\tau}_f|_{\delta,T}\,+\,\|\mathrm{div}\,\boldsymbol{\tau}_f\|_{0,T}\right\}\quad\forall\,T\in\mathcal{T}_h^f. \tag{4.13}$$

Hence, as a consequence of (4.10), (4.12), and (AP_h^s) (respectively, (4.11), (4.13), and (AP_h^f)), one can derive the following two statements

($\mathrm{AP}_h^{\boldsymbol{\sigma}_s}$) For each $\delta\in(0,1]$ and for each $\boldsymbol{\tau}_s\in\mathbb{H}^\delta(\Omega_s)$, with $\mathbf{div}\,\boldsymbol{\tau}_s\,\in\,\mathbf{H}^\delta(\Omega_s)$, there holds

$$\|\boldsymbol{\tau}_s-\mathcal{E}_h^s(\boldsymbol{\tau}_s)\|_{\mathbf{div};\Omega_s}\;\le\; C\,h^\delta\left\{\|\boldsymbol{\tau}_s\|_{\delta,\Omega_s}\;+\;\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{\delta,\Omega_s}\right\}.$$

($\mathrm{AP}_h^{\boldsymbol{\sigma}_f}$) For each $\delta\in(0,1]$ and for each $\boldsymbol{\tau}_f\in\mathbf{H}^\delta(\Omega_f)$, with $\mathrm{div}\,\boldsymbol{\tau}_f\,\in\,H^\delta(\Omega_f)$, there holds

$$\|\boldsymbol{\tau}_f-\mathcal{E}_h^f(\boldsymbol{\tau}_f)\|_{\mathrm{div};\Omega_f}\;\le\; C\,h^\delta\left\{\|\boldsymbol{\tau}_f\|_{\delta,\Omega_f}\;+\;\|\mathrm{div}\,\boldsymbol{\tau}_f\|_{\delta,\Omega_f}\right\}.$$

Finally, the orthogonal projector $\mathcal{R}_h:\mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\to\mathbb{Q}_h^s$ satisfies the following property (see [4])

($\mathrm{AP}_h^{\boldsymbol{\gamma}}$) For each $t\in(0,1]$ and for each $\boldsymbol{\eta}\,\in\,\mathbb{H}^t(\Omega_s)\,\cap\,\mathbb{L}^2_{\mathtt{asym}}(\Omega_s)$, there holds

$$\|\boldsymbol{\eta}-\mathcal{R}_h(\boldsymbol{\eta})\|_{0,\Omega_s}\;\le\; C\,h^t\,\|\boldsymbol{\eta}\|_{t,\Omega_s}.$$

The approximation properties of $\mathbf{Q}_h^s$ and $\mathbf{Q}_h^f$ will be provided once we introduce the specific finite element subspaces $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$. In fact, as already mentioned, the choice of these discrete spaces will be indicated throughout the analysis of well-posedness of our Galerkin scheme (4.7) (see Sect. 4.5 below), particularly when proving the discrete inf-sup conditions for B_s and B_f. We previously need to define in Section 4.3 stable discrete liftings towards Ω_s and Ω_f of normal traces on Σ and Γ and establish its connection with those stability conditions for B_s and B_f. Then in Section 4.4 we introduce suitable discrete approximations of the operators $\mathbf{P}_s|_{\mathbb{H}_h^s}$ and $\mathbf{P}_f|_{\mathbf{H}_s^f}$, which will be employed in Section 4.5 to show the discrete inf-sup conditions for A_s and A_f.

4.3. Stable discrete liftings of normal traces on Σ and Γ

In what follows we proceed as in [12], Sections 4.3 and 5.2 and assume from now on that $\{\mathcal{T}_h^s\}_{h>0}$ and $\{\mathcal{T}_h^f\}_{h>0}$ are quasi-uniform around Σ and Γ. This means that there exist Lipschitz-continuous neighborhoods Ω_Σ and Ω_Γ of Σ and Γ, respectively, such that the elements of $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$ intersecting those regions are more or less of the same size. Equivalently, we define

$$\mathcal{T}_{\Sigma,h} := \Big\{ T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f : \quad T \cap \Omega_\Sigma \neq \emptyset \Big\}, \tag{4.14}$$

$$\mathcal{T}_{\Gamma,h} := \Big\{ T \in \mathcal{T}_h^f : \quad T \cap \Omega_\Gamma \neq \emptyset \Big\}, \tag{4.15}$$

and assume that there exist $c > 0$, independent of h, such that

$$\max \left\{ \max_{T\in\mathcal{T}_{\Sigma,h}} h_T \,;\, \max_{T\in\mathcal{T}_{\Gamma,h}} h_T \right\} \leq c \min \left\{ \min_{T\in\mathcal{T}_{\Sigma,h}} h_T \,;\, \min_{T\in\mathcal{T}_{\Gamma,h}} h_T \right\} \qquad \forall\, h > 0. \tag{4.16}$$

Note that the above assumption and the shape-regularity property of the meshes imply that Σ_h, the partition on Σ inherited from $\mathcal{T}_h^s$ (or from $\mathcal{T}_h^f$), and Γ_h, the partition on Γ inherited from $\mathcal{T}_h^f$, are also quasi-uniform, which means that there exist C_Σ, $C_\Gamma > 0$, independent of h, such that

$$h_\Sigma := \max \Big\{ |e| : \ e \text{ edge of } \Sigma_h \Big\} \leq C_\Sigma \min \Big\{ |e| : \ e \text{ edge of } \Sigma_h \Big\}$$

and

$$h_\Gamma := \max \Big\{ |e| : \ e \text{ edge of } \Gamma_h \Big\} \leq C_\Gamma \min \Big\{ |e| : \ e \text{ edge of } \Gamma_h \Big\}.$$

Also, it is easy to see that there exist c, $C > 0$, independent of h, such that

$$c\, h_\Sigma \leq h_\Gamma \leq C\, h_\Sigma. \tag{4.17}$$

In addition, the quasi-uniformity of Σ_h and Γ_h guarantees the inverse inequality on the spaces

$$\Phi_h(\Sigma) := \Big\{ \phi_h \in L^2(\Sigma) : \quad \phi_h|_e \in P_0(e) \quad \forall\, e \text{ edge of } \Sigma_h \Big\}$$

and

$$\Phi_h(\Gamma) := \Big\{ \phi_h \in L^2(\Gamma) : \quad \phi_h|_e \in P_0(e) \quad \forall\, e \text{ edge of } \Gamma_h \Big\},$$

which means that

$$\|\phi_h\|_{-1/2+\delta,\Sigma} \leq C\, h_\Sigma^{-\delta}\, \|\phi_h\|_{-1/2,\Sigma} \quad \forall\, \phi_h \in \Phi_h(\Sigma), \quad \forall\, \delta \in [0, 1/2] \tag{4.18}$$

and

$$\|\phi_h\|_{-1/2+\delta,\Gamma} \leq C\, h_\Gamma^{-\delta}\, \|\phi_h\|_{-1/2,\Gamma} \quad \forall\, \phi_h \in \Phi_h(\Gamma), \quad \forall\, \delta \in [0, 1/2]. \tag{4.19}$$

The following two lemmas establish our results on the existence of stable discrete liftings. These lifting operators will then be employed to prove the equivalence results given by Lemmas 4.3 and 4.4, which later on simplify the proofs of the discrete inf-sup conditions for B_f and B_s.

Lemma 4.1. *There exist uniformly bounded linear operators* $\mathcal{L}_h^f : \Phi_h(\Sigma) \times \Phi_h(\Gamma) \to \mathbf{H}_h^f$ *such that*

$$\mathcal{L}_h^f(\boldsymbol{\phi}_h) \cdot \boldsymbol{\nu} = \phi_{h,\Sigma} \text{ on } \Sigma \qquad \text{and} \qquad \mathcal{L}_h^f(\boldsymbol{\phi}_h) \cdot \boldsymbol{\nu} = -\,\phi_{h,\Gamma} \text{ on } \Gamma \tag{4.20}$$

for each $\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \in \Phi_h(\Sigma) \times \Phi_h(\Gamma)$.

Proof. Given $\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \in \Phi_h(\Sigma) \times \Phi_h(\Gamma)$, we let $z \in H^1(\Omega_f)$ be the unique solution (up to a constant) of the Neumann boundary value problem

$$\begin{aligned} \Delta z \,&=\, -\frac{1}{|\Omega_f|}\Big\{\langle \phi_{h,\Sigma}, 1\rangle_\Sigma \,+\, \langle \phi_{h,\Gamma}, 1\rangle_\Gamma\Big\} \quad \text{in} \quad \Omega_f, \\ \nabla z\cdot\boldsymbol{\nu} \,&=\, \phi_{h,\Sigma} \quad \text{on} \quad \Sigma, \quad \nabla z\cdot\boldsymbol{\nu} \,=\, -\,\phi_{h,\Gamma} \quad \text{on} \quad \Gamma, \end{aligned} \tag{4.21}$$

which can be seen as a discrete version of (3.32), and whose corresponding continuous dependence result says that

$$\|z\|_{1,\Omega_f} \,\le\, C\,\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f} \,:=\, C\Big\{\|\phi_{h,\Sigma}\|_{-1/2,\Sigma} \,+\, \|\phi_{h,\Gamma}\|_{-1/2,\Gamma}\Big\}. \tag{4.22}$$

Furthermore, since the Neumann datum $\boldsymbol{\phi}_h$ belongs to $H^\delta(\Sigma) \times H^\delta(\Gamma)$ for any $\delta \in [-1/2, 1/2)$, the classical regularity result for mixed boundary value problems on polygonal domains (see, *e.g.* [15]) implies that $z \in H^{5/4}(\Omega_f)$ and

$$\|z\|_{5/4,\Omega_f} \,\le\, C\,\|\boldsymbol{\phi}_h\|_{-1/4,\partial\Omega_f} \,:=\, C\Big\{\|\phi_{h,\Sigma}\|_{-1/4,\Sigma} \,+\, \|\phi_{h,\Gamma}\|_{-1/4,\Gamma}\Big\}. \tag{4.23}$$

In addition, since $\Omega_f^{\texttt{int}} \,:=\, \Omega_f\backslash\big(\Omega_\Sigma \cup \Omega_\Gamma\big)$ is strictly contained in Ω_f, the interior elliptic regularity estimate (see, *e.g.* [24], Thm. 4.16) yields

$$\|z\|_{2,\Omega_f^{\texttt{int}}} \,\le\, C\,\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}. \tag{4.24}$$

According to the above, we now let $\boldsymbol{\zeta}_f \,:=\, \nabla z$ in Ω_f, whence $\boldsymbol{\zeta}_f$ belongs to $\mathbf{H}^{1/4}(\Omega_f)$, and notice from the first equation in (4.21) that

$$\operatorname{div}\boldsymbol{\zeta}_f \,=\, -\frac{1}{|\Omega_f|}\Big\{\langle \phi_{h,\Sigma}, 1\rangle_\Sigma \,+\, \langle \phi_{h,\Gamma}, 1\rangle_\Gamma\Big\} \quad \text{in} \quad \Omega_f, \tag{4.25}$$

thus showing that $\boldsymbol{\zeta}_f \,\in\, \mathbf{H}(\operatorname{div};\Omega_f)$. Then we can define

$$\mathcal{L}_h^f(\boldsymbol{\phi}_h) \,:=\, \mathcal{E}_h^f(\boldsymbol{\zeta}_f) \,\in\, \mathbf{H}_h^f,$$

which, in virtue of the conmuting diagram property (4.11) and the characterization (4.9), and having in mind (4.25) and the boundary conditions in (4.21), clearly satisfies

$$\operatorname{div}\mathcal{L}_h^f(\boldsymbol{\phi}_h) \,=\, -\frac{1}{|\Omega_f|}\Big\{\langle \phi_{h,\Sigma}, 1\rangle_\Sigma \,+\, \langle \phi_{h,\Gamma}, 1\rangle_\Gamma\Big\} \quad \text{in} \quad \Omega_f, \tag{4.26}$$

and the identities required by (4.20).

It remains to show that $\mathcal{L}_h^f$ is uniformly bounded. We first deduce, using (4.26), that there exists $C > 0$, independent of h, such that

$$\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{\operatorname{div};\Omega_f} \,\le\, C\Big\{\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f} \,+\, \|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{0,\Omega_f}\Big\}. \tag{4.27}$$

Next, in order to estimate $\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{0,\Omega_f}$, we divide Ω_f into three regions by defining (*cf.* (4.14), (4.15))

$$\begin{aligned} \Omega_{\Sigma,h}^f \,&:=\, \cup\Big\{T: \quad T \,\in\, \mathcal{T}_h^f \cap \mathcal{T}_{\Sigma,h}\Big\}, \\ \Omega_{\Gamma,h} \,&:=\, \cup\Big\{T: \quad T \in \mathcal{T}_{\Gamma,h}\Big\}, \end{aligned}$$

and

$$\Omega_{f,h}^{\texttt{int}} \,:=\, \Omega_f \setminus \Big(\Omega_{\Sigma,h}^f \,\cup\, \Omega_{\Gamma,h}\Big).$$

It follows, using the stability of $\mathcal{E}_h^f$ in $\mathbf{H}^1(\Omega_{f,h}^{\mathtt{int}})$, the fact that $\boldsymbol{\zeta}_f|_{\Omega_{f,h}^{\mathtt{int}}} \in \mathbf{H}^1(\Omega_{f,h}^{\mathtt{int}})$, the inclusion $\Omega_{f,h}^{\mathtt{int}} \subseteq \Omega_f^{\mathtt{int}}$, and the estimate (4.24), that

$$\begin{aligned}\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{0,\Omega_f} &= \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_f} \leq \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{f,h}^{\mathtt{int}}} + \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f} + \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Gamma,h}} \\ &\leq C\,\|z\|_{2,\Omega_f^{\mathtt{int}}} + \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f} + \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Gamma,h}} \\ &\leq C\,\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f} + \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f} + \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Gamma,h}}. \end{aligned} \tag{4.28}$$

Now, adding and substracting $\boldsymbol{\zeta}_f = \nabla z$ in $\Omega_{\Sigma,h}^f \subseteq \Omega_f$, noting that $\|\boldsymbol{\zeta}_f\|_{0,\Omega_{\Sigma,h}^f} \leq \|z\|_{1,\Omega_f}$, and employing the estimates (4.22), (4.13) (with $\delta = 1/4$) and (4.23), together with the identity (4.26), the quasi-uniformity bound (4.16), the inverse inequalities (4.18) and (4.19), and the equivalence between h_Σ and h_Γ (*cf.* (4.17)), we arrive at

$$\begin{aligned}\|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f}^2 &\leq C\left\{\|\boldsymbol{\zeta}_f - \mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f}^2 + \|\boldsymbol{\zeta}_f\|_{0,\Omega_{\Sigma,h}^f}^2\right\} \\ &\leq C\left\{\sum_{T\in\mathcal{T}_{\Sigma,h}^f} h_T^{1/2}\,\|z\|_{5/4,T}^2 + \|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}^2\right\} \\ &\leq C\left\{h_\Sigma^{1/2}\,\|\boldsymbol{\phi}_h\|_{-1/4,\partial\Omega_f}^2 + \|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}^2\right\} \\ &\leq C\,\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}^2. \end{aligned} \tag{4.29}$$

The estimate for $\|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Gamma,h}}^2$ proceeds similarly and yields the same upper bound. In this way, (4.27)–(4.29) provide the uniform boundedness of $\mathcal{L}_h^f$, which completes the proof. □

Lemma 4.2. *There exist uniformly bounded linear operators $\mathcal{L}_h^s : \Phi_h(\Sigma)\times\Phi_h(\Sigma) \to \mathbb{H}_h^s$ such that*

$$\mathcal{L}_h^s(\boldsymbol{\phi}_h)\,\boldsymbol{\nu} = \boldsymbol{\phi}_h \text{ on } \Sigma \qquad \forall\,\boldsymbol{\phi}_h \in \Phi_h(\Sigma)\times\Phi_h(\Sigma). \tag{4.30}$$

Proof. Given $\boldsymbol{\phi}_h \in \Phi_h(\Sigma)\times\Phi_h(\Sigma)$ we let $\mathbf{z}\in\mathbf{H}^1(\Omega_s)$ be the unique solution (up to a constant vector) of the Neumann boundary value problem (in vectorial form)

$$\Delta\mathbf{z} = \frac{1}{|\Omega_s|}\int_\Sigma \boldsymbol{\phi}_h \quad \text{in} \quad \Omega_s, \qquad \nabla\mathbf{z}\,\boldsymbol{\nu} = \boldsymbol{\phi}_h \quad \text{on} \quad \Sigma,$$

whose corresponding continuous dependence result states that

$$\|\mathbf{z}\|_{1,\Omega_s} \leq C\,\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}.$$

Since the Neumann datum $\boldsymbol{\phi}_h$ belongs to $\mathbf{H}^\delta(\Sigma)$ for any $\delta \in [0,1/2)$, we know that we have at least $\mathbf{H}^{3/2}(\Omega_s)$-regularity for $\mathbf{z}$ and

$$\|\mathbf{z}\|_{3/2,\Omega_s} \leq C\,\|\boldsymbol{\phi}_h\|_{0,\Sigma}.$$

In addition, noting that $\Omega_s^{\mathtt{int}} := \Omega_s\setminus\Omega_\Sigma$ is an interior region of Ω_s, the interior elliptic regularity estimate again (see, *e.g.* [24], Thm. 4.16) yields

$$\|\mathbf{z}\|_{2,\Omega_s^{\mathtt{int}}} \leq C\,\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}.$$

Next, we set $\boldsymbol{\zeta}_s := \nabla\mathbf{z}$ in Ω_s, which belongs to $\mathbb{H}^{1/2}(\Omega_s)\cap\mathbb{H}(\mathbf{div};\Omega_s)$, define $\mathcal{L}_h^s(\boldsymbol{\phi}_h) := \mathcal{E}_h^s(\boldsymbol{\zeta}_s)$, and proceed analogously to the proof of the previous lemma, by using now the conmuting diagram property (4.10), the characterization (4.8), the error estimate (4.12), the quasi-uniformity bound (4.16), and the inverse inequality (4.18). We omit further details. □

As a first consequence of Lemmas 4.1 and 4.2, and noting from the definitions of $\mathbf{H}_h^f$ (*cf.* (4.2)) and $\mathbb{H}_h^s$ (*cf.* (4.1)) that

$$\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}|_{\partial\Omega_f} \equiv \big(\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}|_{\Sigma}, \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}|_{\Gamma}\big) \in \Phi_h(\Sigma) \times \Phi_h(\Gamma) \qquad \forall\, \boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f,$$

and

$$\boldsymbol{\tau}_{s,h}\, \boldsymbol{\nu}|_{\Sigma} \in \Phi_h(\Sigma) \times \Phi_h(\Sigma) \qquad \forall\, \boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s,$$

we deduce that actually there hold

$$\Phi_h(\Sigma) \times \Phi_h(\Gamma) = \Big\{ \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}|_{\partial\Omega_f} : \quad \boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f \Big\}, \tag{4.31}$$

and

$$\Phi_h(\Sigma) \times \Phi_h(\Sigma) = \Big\{ \boldsymbol{\tau}_{s,h}\, \boldsymbol{\nu}|_{\Sigma} : \quad \boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s \Big\}. \tag{4.32}$$

Hence, the stable discrete liftings $\mathcal{L}_h^f$ and $\mathcal{L}_h^s$, and the identities (4.31) and (4.32) allow to show equivalence results concerning the discrete inf-sup conditions for B_f (*cf.* (2.14)) and for the second term defining B_s (*cf.* (2.13)). More precisely, we have the following lemmas.

Lemma 4.3. *Let us define, for each* $\boldsymbol{\psi}_{f,h} := (\psi_{h,\Sigma}, \psi_{h,\Gamma}) \in \mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$,

$$S(\boldsymbol{\psi}_{f,h}) := \sup_{\boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f \setminus \{\mathbf{0}\}} \frac{|\, B_f(\boldsymbol{\tau}_{f,h}, \boldsymbol{\psi}_{f,h})\,|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}}$$

and

$$\widetilde{S}(\boldsymbol{\psi}_{f,h}) := \sup_{\substack{\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \\ \in \Phi_h(\Sigma) \times \Phi_h(\Gamma) \setminus \{\mathbf{0}\}}} \frac{|\, \langle \phi_{h,\Sigma}, \psi_{h,\Sigma} \rangle_\Sigma + \langle \phi_{h,\Gamma}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}}.$$

Then there exist C_1, $C_2 > 0$, independent of h, such that

$$C_1\, \widetilde{S}(\boldsymbol{\psi}_{f,h}) \le S(\boldsymbol{\psi}_{f,h}) \le C_2\, \widetilde{S}(\boldsymbol{\psi}_{f,h}) \qquad \forall\, \boldsymbol{\psi}_{f,h} \in \mathbf{Q}_h^f. \tag{4.33}$$

Proof. Let $c_f > 0$, independent of h, whose existence is provided by Lemma 4.1, such that

$$\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{\mathrm{div};\Omega_f} \le c_f\, \|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f} \qquad \forall\, \boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \in \Phi_h(\Sigma) \times \Phi_h(\Gamma).$$

Then, for each $\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \in \Phi_h(\Sigma) \times \Phi_h(\Gamma) \setminus \{\mathbf{0}\}$ there holds, using (4.20),

$$\begin{aligned}
\frac{|\, \langle \phi_{h,\Sigma}, \psi_{h,\Sigma} \rangle_\Sigma + \langle \phi_{h,\Gamma}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}} &\le c_f\, \frac{|\, \langle \phi_{h,\Sigma}, \psi_{h,\Sigma} \rangle_\Sigma + \langle \phi_{h,\Gamma}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{\mathrm{div};\Omega_f}} \\
&= c_f\, \frac{|\, \langle \mathcal{L}_h^f(\boldsymbol{\phi}_h) \cdot \boldsymbol{\nu}, \psi_{h,\Sigma} \rangle_\Sigma - \langle \mathcal{L}_h^f(\boldsymbol{\phi}_h) \cdot \boldsymbol{\nu}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{\mathrm{div};\Omega_f}} \le c_f\, S(\boldsymbol{\psi}_{f,h}),
\end{aligned}$$

which implies the left-hand side of (4.33) with $C_1 = c_f^{-1}$. Similarly, for each $\boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f$ we find, using that $\|\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}\|_{-1/2,\partial\Omega_f} := \|\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}\|_{-1/2,\Sigma} + \|\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}\|_{-1/2,\Gamma} \le C\, \|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}$ and (4.31), that

$$\begin{aligned}
\frac{|\, B_f(\boldsymbol{\tau}_{f,h}, \boldsymbol{\psi}_{f,h})\,|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}} &= \frac{|\, \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Sigma} \rangle_\Sigma - \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}} \\
&\le C\, \frac{|\, \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Sigma} \rangle_\Sigma - \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}\|_{-1/2,\partial\Omega_f}} \le C\, \widetilde{S}(\boldsymbol{\psi}_{f,h}),
\end{aligned}$$

which yields the right-hand side of (4.33) with $C_2 = C$. □

Lemma 4.4. *Let us define for each* $\boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma)$

$$T(\boldsymbol{\psi}_{s,h}) := \sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s \backslash \{\mathbf{0}\}} \frac{|\langle \boldsymbol{\tau}_{s,h} \boldsymbol{\nu}, \boldsymbol{\psi}_{s,h} \rangle_\Sigma|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}}$$

and

$$\widetilde{T}(\boldsymbol{\psi}_{s,h}) := \sup_{\substack{\boldsymbol{\phi}_h \in \Phi_h(\Sigma) \times \Phi_h(\Sigma) \\ \boldsymbol{\phi}_h \neq \mathbf{0}}} \frac{|\langle \boldsymbol{\phi}_h, \boldsymbol{\psi}_{s,h} \rangle_\Sigma|}{\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}}.$$

Then there exist C_3, $C_4 > 0$, *independent of* h, *such that*

$$C_3 \, \widetilde{T}(\boldsymbol{\psi}_{s,h}) \leq T(\boldsymbol{\psi}_{s,h}) \leq C_4 \, \widetilde{T}(\boldsymbol{\psi}_{s,h}) \qquad \forall \, \boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s. \tag{4.34}$$

Proof. It follows analogously to the proof of Lemma 4.3 by using now, thanks to Lemma 4.2, that there exists $c_s > 0$, independent of h, such that $\|\mathcal{L}_h^s(\boldsymbol{\phi}_h)\|_{\mathbf{div};\Omega_s} \leq c_s \|\boldsymbol{\phi}_h\|_{-1/2,\Sigma} \quad \forall \, \boldsymbol{\phi}_h \in \Phi_h(\Sigma) \times \Phi_h(\Sigma)$, and noting that $\|\boldsymbol{\tau}_{s,h} \boldsymbol{\nu}\|_{-1/2,\Sigma} \leq C \|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}$. We omit further details. □

The previous two lemmas, more precisely the left-hand sides of the equivalences (4.33) and (4.34), will be employed below in Section 4.5 to show that the bilinear forms B_f and B_s satisfy the discrete inf-sup conditions on the corresponding finite element subspaces.

4.4. Discrete approximations of $\mathbf{P}_s|_{\mathbb{H}_h^s}$ and $\mathbf{P}_f|_{\mathbf{H}_h^f}$

In what follows we introduce uniformly bounded linear operators $\mathbf{P}_{s,h} : \mathbb{H}_h^s \to \mathbb{H}_h^s$ and $\mathbf{P}_{f,h} : \mathbf{H}_h^f \to \mathbf{H}_h^f$ approximating $\mathbf{P}_s|_{\mathbb{H}_h^s} : \mathbb{H}_h^s \to \mathbb{H}(\mathbf{div};\Omega_s)$ and $\mathbf{P}_f|_{\mathbf{H}_h^f} : \mathbf{H}_h^f \to \mathbf{H}(\mathrm{div};\Omega_f)$, respectively, and derive upper bounds for the associated errors given by $\|\mathbf{P}_s(\boldsymbol{\tau}_{s,h}) - \mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})\|_{\mathbf{div};\Omega_s}$ (*cf.* Lema 4.5) and $\|\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) - \mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\|_{\mathrm{div};\Omega_f}$ (*cf.* Lem. 4.6) for each $(\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h := \mathbb{H}_h^s \times \mathbf{H}_h^f$. These are the key estimates utilized below in Section 4.5 to prove the discrete inf-sup conditions for the bilinear forms A_s and A_f (*cf.* Lem. 4.10).

Indeed, given $(\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h$, we first recall from (3.3) and (3.1) that $\mathbf{P}_s(\boldsymbol{\tau}_{s,h}) := \tilde{\boldsymbol{\sigma}}_s$, where $\tilde{\boldsymbol{\sigma}}_s = \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ and $\tilde{\mathbf{u}}$ is the unique solution of

$$\begin{aligned} \tilde{\boldsymbol{\sigma}}_s &= \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) \quad \text{in} \quad \Omega_s, \quad \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_s = (\mathbf{I} - \mathbf{M})\big(\mathbf{div}\,\boldsymbol{\tau}_{s,h}\big) \quad \text{in} \quad \Omega_s, \\ \tilde{\boldsymbol{\sigma}}_s \boldsymbol{\nu} &= \mathbf{0} \quad \text{on} \quad \Sigma, \quad \tilde{\mathbf{u}} \in (\mathbf{I} - \mathbf{M})(\mathbf{L}^2(\Omega_s)), \end{aligned} \tag{4.35}$$

In turn, we know from (3.10) and (3.8) that $\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) := \tilde{\boldsymbol{\sigma}}_f$, where $\tilde{\boldsymbol{\sigma}}_f := \nabla \tilde{p}$ and $\tilde{p}$ is the unique solution of

$$\begin{aligned} \tilde{\boldsymbol{\sigma}}_f &= \nabla \tilde{p} \quad \text{in} \quad \Omega_f, \quad \mathrm{div}\,\tilde{\boldsymbol{\sigma}}_f = (\mathbf{I} - \mathbf{J})\big(\mathrm{div}\,\boldsymbol{\tau}_{f,h}\big) \quad \text{in} \quad \Omega_f, \\ \tilde{\boldsymbol{\sigma}}_f \cdot \boldsymbol{\nu} &= 0 \quad \text{on} \quad \Sigma \cup \Gamma, \quad \tilde{p} \in (\mathbf{I} - \mathbf{J})(L^2(\Omega_f)). \end{aligned} \tag{4.36}$$

We now let $(\tilde{\boldsymbol{\sigma}}_{s,h}, \tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\gamma}}_h) \in \mathbb{H}_h^s \times (\mathbf{I} - \mathbf{M})(\mathbf{U}_h^s) \times \mathbb{Q}_h^s$ be the mixed finite element approximation of (4.35), which was introduced and analyzed in [7], Section 5.2, and define

$$\mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h}) := \tilde{\boldsymbol{\sigma}}_{s,h}. \tag{4.37}$$

Hence, we know from [7], Section 5.2 that there hold

$$\|\mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})\|_{\mathbf{div};\Omega_s} \leq C \|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}, \tag{4.38}$$

$$\mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})\,\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Sigma \qquad \text{and} \qquad \int_{\Omega_s} \mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h}) : \tilde{\boldsymbol{\eta}}_h = 0 \quad \forall \, \tilde{\boldsymbol{\eta}}_h \in \mathbb{Q}_h^s. \tag{4.39}$$

The uniform boundedness of $\mathbf{P}_{s,h}$ is obvious from (4.38), whereas the first equation of (4.39) says that $\mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})$ belongs to $\tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ (*cf.* (3.38)). Furthermore, in virtue of [7], Lemma 5.4, whose proof makes use of the definition (4.37), the conmuting diagram identity (4.10), the approximation properties (4.12), (AP^s_h), and (AP^{γ}_h), and the regularity estimate for (4.35) (*cf.* (3.2), (3.7)), we have the following error estimate.

Lemma 4.5. *Let $\epsilon > 0$ be the parameter defining the regularity of the solution of* (4.35)*. Then, there exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau}_{s,h} \in \mathbb{H}^s_h$ there holds*

$$\|\mathbf{P}_s(\boldsymbol{\tau}_{s,h}) - \mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})\|_{\mathbf{div};\Omega_s} \le C\, h^{\epsilon}\, \|\mathbf{div}\,\boldsymbol{\tau}_{s,h}\|_{0,\Omega_s}. \tag{4.40}$$

We now turn to the definition and properties of $\mathbf{P}_{f,h}$. According to the regularity estimates given by (3.9) and (3.13), we know that $\mathbf{P}_f(\boldsymbol{\tau}_{f,h})$ belongs to $\mathbf{H}^{\epsilon}(\Omega_f)$ and

$$\|\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\|_{\epsilon,\Omega_f} \le C\, \|\mathrm{div}\,\boldsymbol{\tau}_{f,h}\|_{0,\Omega_f}, \tag{4.41}$$

which suggests to consider the Raviart–Thomas interpolation operator $\mathcal{E}^f_h$ and define

$$\mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h}) := \mathcal{E}^f_h\big(\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\big). \tag{4.42}$$

It follows, employing the conmuting diagram property (4.11), the second equation in (4.36) (which says that $\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) = \big(\mathbf{I}-\mathbf{J}\big)(\mathrm{div}\,\boldsymbol{\tau}_{f,h}))$, and the fact that $\mathrm{div}\,\boldsymbol{\tau}_{f,h}$ is piecewise constant, that

$$\mathrm{div}\,\mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h}) = \mathcal{P}^f_h\big(\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\big) = \mathcal{P}^f_h\big((\mathbf{I}-\mathbf{J})(\mathrm{div}\,\boldsymbol{\tau}_{f,h})\big) = \mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_{f,h}). \tag{4.43}$$

Also, it is easy to see that the uniform boundedness of $\mathcal{E}^f_h : \mathbf{H}^{\epsilon}(\Omega_f) \cap \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}^f_h$ (which follows from (4.13) and (4.11)), together with the estimate (4.41) and the identity (4.43), imply that $\mathbf{P}_{f,h}$ is uniformly bounded as well. In addition, using the characterization property (4.9) and the third equation in (4.36) (which says that $\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Sigma\cup\Gamma$), we easily deduce that

$$\mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Sigma\cup\Gamma. \tag{4.44}$$

We are now in a position to establish our second error estimate.

Lemma 4.6. *Let $\epsilon > 0$ be the parameter defining the regularity of the solution of* (4.36)*. Then, there exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau}_{f,h} \in \mathbf{H}^f_h$ there holds*

$$\|\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) - \mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\|_{\mathrm{div};\Omega_f} \le C\, h^{\epsilon}\, \|\mathrm{div}\,\boldsymbol{\tau}_{f,h}\|_{0,\Omega_f}. \tag{4.45}$$

Proof. We proceed as in the proof of [7], Lemma 5.4, though the present one becomes simpler. Let us first notice, in virtue of (4.42) and (4.43), that

$$\|\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) - \mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\|_{\mathrm{div};\Omega_f} = \|\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) - \mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\|_{0,\Omega_f} = \|(\mathbf{I}-\mathcal{E}^f_h)\big(\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\big)\|_{0,\Omega_f}.$$

Hence, applying the approximation property (4.13) and the identity (4.43), we find that

$$\begin{aligned}
\|(\mathbf{I}-\mathcal{E}^f_h)\big(\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\big)\|^2_{0,\Omega_f} &= \sum_{T\in\mathcal{T}^f_h} \|(\mathbf{I}-\mathcal{E}^f_h)\big(\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\big)\|^2_{0,T} \\
&\le C \sum_{T\in\mathcal{T}^f_h} h_T^{2\epsilon}\Big\{|\mathbf{P}_f(\boldsymbol{\tau}_{f,h})|^2_{\epsilon,T} + \|\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\|^2_{0,T}\Big\} \\
&\le C\, h^{2\epsilon}\Big\{\|\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\|^2_{\epsilon,\Omega_f} + \|\big(\mathbf{I}-\mathbf{J}\big)(\mathrm{div}\,\boldsymbol{\tau}_{f,h})\|^2_{0,\Omega_f}\Big\},
\end{aligned}$$

which, together with the estimate (4.41) and the fact that $\|\mathbf{I}-\mathbf{J}\| \le 1$, completes the proof. $\square$

4.5. Well-posedness of the Galerkin scheme

We now aim to show the well-posedness of the mixed finite element scheme (4.7). For this purpose, as established by a classical result on projection methods for Fredholm operators of index zero (see, *e.g.* [21], Thm. 13.7), one just needs to prove that the Galerkin scheme associated to the isomorphism $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is well-posed. Equivalently, in virtue of the identity (3.28), it suffices to apply the discrete Babuška–Brezzi theory to each one of the blocks $\begin{pmatrix} \mathbf{A}_s & \mathbf{B}_s^* \\ \mathbf{B}_s & \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{A}_f & \mathbf{B}_f^* \\ \mathbf{B}_f & \mathbf{0} \end{pmatrix}$. According to the above, in what follows we show that the bilinear forms A_s, B_s, A_f, and B_f (not necessarily in this order) satisfy the discrete inf-sup conditions on the corresponding finite element subspaces.

We begin our analysis with the derivation of the discrete inf-sup condition for B_f. To this end, and in order to apply Lemma 4.3, we first notice that for each $\boldsymbol{\psi}_{f,h} := (\psi_{h,\Sigma}, \psi_{h,\Gamma}) \in \mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$ there holds

$$
\begin{aligned}
\widetilde{S}(\boldsymbol{\psi}_h) &:= \sup_{\substack{\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \\ \in \Phi_h(\Sigma) \times \Phi_h(\Gamma) \setminus \{\mathbf{0}\}}} \frac{|\langle \phi_{h,\Sigma}, \psi_{h,\Sigma} \rangle_\Sigma + \langle \phi_{h,\Gamma}, \psi_{h,\Gamma} \rangle_\Gamma|}{\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}} \\
&\geq \frac{1}{2} \left\{ \sup_{\phi_{h,\Sigma} \in \Phi_h(\Sigma) \setminus \{\mathbf{0}\}} \frac{|\langle \phi_{h,\Sigma}, \psi_{h,\Sigma} \rangle_\Sigma|}{\|\phi_{h,\Sigma}\|_{-1/2,\Sigma}} + \sup_{\phi_{h,\Gamma} \in \Phi_h(\Gamma) \setminus \{\mathbf{0}\}} \frac{|\langle \phi_{h,\Gamma}, \psi_{h,\Gamma} \rangle_\Gamma|}{\|\phi_{h,\Gamma}\|_{-1/2,\Gamma}} \right\}.
\end{aligned}
$$

It follows, in virtue also of the left-hand side of (4.33), that a sufficient condition for the required inequality concerning B_f is the existence of $\tilde{\beta}_{f,\Sigma}$, $\tilde{\beta}_{f,\Gamma} > 0$, independent of h, such that

$$
\sup_{\phi_{h,\Sigma} \in \Phi_h(\Sigma) \setminus \{\mathbf{0}\}} \frac{|\langle \phi_{h,\Sigma}, \psi_{h,\Sigma} \rangle_\Sigma|}{\|\phi_{h,\Sigma}\|_{-1/2,\Sigma}} \geq \tilde{\beta}_{f,\Sigma} \|\psi_{h,\Sigma}\|_{1/2,\Sigma} \qquad \forall\, \psi_{h,\Sigma} \in \Lambda_h(\Sigma), \tag{4.46}
$$

and

$$
\sup_{\phi_{h,\Gamma} \in \Phi_h(\Gamma) \setminus \{\mathbf{0}\}} \frac{|\langle \phi_{h,\Gamma}, \psi_{h,\Gamma} \rangle_\Gamma|}{\|\phi_{h,\Gamma}\|_{-1/2,\Gamma}} \geq \tilde{\beta}_{f,\Gamma} \|\psi_{h,\Gamma}\|_{1/2,\Gamma} \qquad \forall\, \psi_{h,\Gamma} \in \Lambda_h(\Gamma). \tag{4.47}
$$

Note that (4.46) and (4.47) constitute two independent discrete inf-sup conditions holding between subspaces living in Σ and Γ, respectively. Then, we recall from [12], Lemma 5.2, that a suitable choice of the subspaces $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$ guarantees the ocurrence of the above. More precisely, let us assume, without loss of generality, that the number of edges of Σ_h and Γ_h are even numbers. Then, we let Σ_{2h} (resp. Γ_{2h}) be the partition of Σ (resp. Γ) arising by joining pairs of adjacent elements, and define

$$
\Lambda_h(\Sigma) := \Big\{ \psi_h \in C(\Sigma) : \quad \psi_h|_e \in P_1(e) \quad \forall\, e \text{ edge of } \Sigma_{2h} \Big\}, \tag{4.48}
$$

$$
\Lambda_h(\Gamma) := \Big\{ \psi_h \in C(\Gamma) : \quad \psi_h|_e \in P_1(e) \quad \forall\, e \text{ edge of } \Gamma_{2h} \Big\}, \tag{4.49}
$$

$$
\mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma). \tag{4.50}
$$

and

$$
\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma). \tag{4.51}
$$

In this way, we are in a position to establish the following result.

Lemma 4.7. *Let $\mathbf{Q}_h^f$ be given by (4.50). Then there exists $\tilde{\beta}_f > 0$, independent of h, such that*

$$
\sup_{\boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f \setminus \{\mathbf{0}\}} \frac{|B_f(\boldsymbol{\tau}_{f,h}, \boldsymbol{\psi}_{f,h})|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}} \geq \tilde{\beta}_f \|\boldsymbol{\psi}_{f,h}\|_{1/2,\partial\Omega_f} \qquad \forall\, \boldsymbol{\psi}_{f,h} \in \mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma).
$$

Proof. A straightforward application of [12], Lemma 5.2, to the pairs of subspaces $(\Phi_h(\Sigma), \Lambda_h(\Sigma))$ and $(\Phi_h(\Gamma), \Lambda_h(\Gamma))$ imply (4.46) and (4.47), and hence the previous discussion completes the proof with the constant $\tilde{\beta}_f = \frac{C_1}{2} \min\left\{ \tilde{\beta}_{f,\Sigma}, \tilde{\beta}_{f,\Gamma} \right\}$. □

Before continuing the analysis, we let $\Pi_{\Sigma} : H^{1/2}(\Sigma) \to \Lambda_h(\Sigma)$ and $\Pi_{\Gamma} : H^{1/2}(\Gamma) \to \Lambda_h(\Gamma)$ be the orthogonal projectors, and recall from [2] that the approximation properties of $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$ are given as follows:

($\mathrm{AP}_{\Sigma,h}$) For each $\delta \in (0,1]$ and for each $\psi \in H^{1/2+\delta}(\Sigma)$, there holds

$$\|\psi - \Pi_{\Sigma}(\psi)\|_{1/2,\Sigma} \le C\, h_{\Sigma}^{\delta}\, \|\psi\|_{1/2+\delta,\Sigma}.$$

($\mathrm{AP}_{\Gamma,h}$) For each $\delta \in (0,1]$ and for each $\psi \in H^{1/2+\delta}(\Gamma)$, there holds

$$\|\psi - \Pi_{\Gamma}(\psi)\|_{1/2,\Gamma} \le C\, h_{\Gamma}^{\delta}\, \|\psi\|_{1/2+\delta,\Gamma}.$$

Note that ($\mathrm{AP}_{\Sigma,h}$) and ($\mathrm{AP}_{\Gamma,h}$) yield the approximation properties of $\mathbf{Q}_h^s$ and $\mathbf{Q}_h^f$ (*cf.* (4.4), (4.5)).

We now turn to the connection between Lemma 4.4 and the discrete inf-sup condition for the bilinear form B_s (*cf.* (2.13)) with $\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma)$ and $\Lambda_h(\Sigma)$ given by (4.48). We first notice that for each $\boldsymbol{\psi}_{s,h} := (\psi_{h,\Sigma}, \widetilde{\psi}_{h,\Sigma}) \in \mathbf{Q}_h^s$ there holds, denoting $\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \widetilde{\phi}_{h,\Sigma}) \in \Phi_h(\Sigma) \times \Phi_h(\Sigma)$,

$$\begin{aligned}\widetilde{T}(\boldsymbol{\psi}_{s,h}) &:= \sup_{\substack{\boldsymbol{\phi}_h \in \Phi_h(\Sigma)\times\Phi_h(\Sigma)\\ \boldsymbol{\phi}_h \neq \mathbf{0}}} \frac{|\langle \boldsymbol{\phi}_h, \boldsymbol{\psi}_{s,h}\rangle_{\Sigma}|}{\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}}\\ &\ge \frac{1}{2}\left\{ \sup_{\phi_{h,\Sigma} \in \Phi_h(\Sigma)\setminus\{\mathbf{0}\}} \frac{|\langle \phi_{h,\Sigma}, \psi_{h,\Sigma}\rangle_{\Sigma}|}{\|\phi_{h,\Sigma}\|_{-1/2,\Sigma}} + \sup_{\widetilde{\phi}_{h,\Sigma} \in \Phi_h(\Sigma)\setminus\{\mathbf{0}\}} \frac{|\langle \widetilde{\phi}_{h,\Sigma}, \widetilde{\psi}_{h,\Sigma}\rangle_{\Sigma}|}{\|\widetilde{\phi}_{h,\Sigma}\|_{-1/2,\Sigma}} \right\}.\end{aligned}$$

Hence, since [12], Lemma 5.2, guarantees (4.46), we deduce from the above inequality that

$$\widetilde{T}(\boldsymbol{\psi}_{s,h}) \ge \tilde{\beta}_{f,\Sigma} \left\{ \|\psi_{h,\Sigma}\|_{1/2,\Sigma} + \|\widetilde{\psi}_{h,\Sigma}\|_{1/2,\Sigma} \right\} \qquad \forall\, \boldsymbol{\psi}_{s,h} := (\psi_{h,\Sigma}, \widetilde{\psi}_{h,\Sigma}) \in \mathbf{Q}_h^s,$$

which, combined with the left-hand side of (4.34), yields

$$T(\boldsymbol{\psi}_{s,h}) := \sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\setminus\{\mathbf{0}\}} \frac{|\langle \boldsymbol{\tau}_{s,h}\boldsymbol{\nu}, \boldsymbol{\psi}_{s,h}\rangle_{\Sigma}|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \ge C_3\, \tilde{\beta}_{f,\Sigma}\, \|\boldsymbol{\psi}_{s,h}\|_{1/2,\Sigma} \qquad \forall\, \boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s. \tag{4.52}$$

Consequently, we are now able to prove the following lemma.

Lemma 4.8. *Let $\mathbf{Q}_h^s$ be given by (4.51). Then there exists $\tilde{\beta}_s > 0$, independent of h, such that*

$$\sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\setminus\{\mathbf{0}\}} \frac{|B_s(\boldsymbol{\tau}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}))|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \ge \tilde{\beta}_s\, \|(\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h})\| \qquad \forall\, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}) \in \mathbb{Q}_h^s \times \mathbf{Q}_h^s.$$

Proof. Given $(\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}) \in \mathbb{Q}_h^s \times \mathbf{Q}_h^s$ we have, according to the definition of B_s (*cf.* (2.13)), that

$$\sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\setminus\{\mathbf{0}\}} \frac{|B_s(\boldsymbol{\tau}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}))|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \ge \sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\setminus\{\mathbf{0}\}} \frac{|\langle \boldsymbol{\tau}_{s,h}\boldsymbol{\nu}, \boldsymbol{\psi}_{s,h}\rangle_{\Sigma}|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} - \|\boldsymbol{\eta}_h\|_{0,\Omega_s},$$

which, thanks to (4.52), implies that

$$\sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\setminus\{\mathbf{0}\}} \frac{|B_s(\boldsymbol{\tau}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}))|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \ge C_3\, \tilde{\beta}_{f,\Sigma}\, \|\boldsymbol{\psi}_{s,h}\|_{1/2,\Sigma} - \|\boldsymbol{\eta}_h\|_{0,\Omega_s}. \tag{4.53}$$

Furthermore, we know from [22], Theorem 4.5, (see also [1], Lem. 4.4) that there exists $\boldsymbol{\zeta}_{s,h} \in \mathbb{H}_h^s$ such that $\boldsymbol{\zeta}_{s,h}\,\boldsymbol{\nu} = \mathbf{0}$ on Σ, $\mathbf{div}\,\boldsymbol{\zeta}_{s,h} = \mathbf{0}$ in Ω_s, and

$$| \, B_s(\boldsymbol{\zeta}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h})) \, | \;\geq\; C \, \|\boldsymbol{\zeta}_{s,h}\|_{0,\Omega_s} \, \|\boldsymbol{\eta}\|_{0,\Omega_s} \;=\; C \, \|\boldsymbol{\zeta}_{s,h}\|_{\mathbf{div};\Omega_s} \, \|\boldsymbol{\eta}\|_{0,\Omega_s},$$

which yields

$$\sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s \setminus \{\mathbf{0}\}} \frac{| \, B_s(\boldsymbol{\tau}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h})) \, |}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \;\geq\; C \, \|\boldsymbol{\eta}_h\|_{0,\Omega_s}. \tag{4.54}$$

Finally, a suitable linear combination of (4.53) and (4.54) gives the required inequality. □

We now let $\mathbf{V}_{s,h}$ and $\mathbf{V}_{f,h}$ be the discrete kernels of B_s (*cf.* (2.13)) and B_f (*cf.* (2.14)), that is,

$$\mathbf{V}_{s,h} := \Big\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s : \quad \int_{\Omega_s} \boldsymbol{\tau}_{s,h} : \boldsymbol{\eta}_h = 0 \quad \forall\, \boldsymbol{\eta}_h \in \mathbb{Q}_h^s, \quad \langle \boldsymbol{\tau}_{s,h}\,\boldsymbol{\nu}, \boldsymbol{\psi}_{s,h} \rangle_\Sigma = 0 \quad \forall\, \boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s \Big\}, \tag{4.55}$$

$$\mathbf{V}_{f,h} \;:=\; \Big\{ \boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f : \quad \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Sigma} \rangle_\Sigma \;=\; \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Gamma} \rangle_\Gamma = 0 \quad \forall\, (\psi_{h,\Sigma}, \psi_{h,\Gamma}) \in \mathbf{Q}_h^f \Big\}, \tag{4.56}$$

and aim to prove that the bilinear forms A_s and A_f satisfy the discrete inf-sup conditions on $\mathbf{V}_{s,h} \times \mathbf{V}_{s,h}$ and $\mathbf{V}_{f,h} \times \mathbf{V}_{f,h}$, respectively.

We begin by observing that $\mathbf{V}_{s,h}$ is certainly contained in

$$\tilde{\mathbf{V}}_{s,h} \;:=\; \Big\{ \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \langle \boldsymbol{\tau}_s\,\boldsymbol{\nu}, \boldsymbol{\psi}_{s,h} \rangle_\Sigma = 0 \quad \forall\, \boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s \Big\},$$

which is not a subspace of $\tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ (*cf.* (3.38)) but on the contrary contains it. While this latter fact prevent us of applying directly (3.37) (and hence the ellipticity estimates (3.39) and (3.44)) to the whole $\tilde{\mathbf{V}}_{s,h}$, we show next that actually (3.37) does also hold in this bigger space. In fact, let us first pick one corner point of Σ and define a function v that is continuous, linear on each side of Σ, equal to one in the chosen vertex and zero on all other ones. Then, it is easy to check that, if $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ are the normal vectors on the two sides of Σ that meet at the corner point, the function $\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Sigma)$ given by $\boldsymbol{\psi} := v\,(\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2)$ belongs to $\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma)$ for each $h > 0$, and satisfies

$$\langle \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma \neq 0.$$

This function $\boldsymbol{\psi}$ in $\mathbf{Q}_h^s$ is employed next to prove the validity of (3.37) in $\tilde{\mathbf{V}}_{s,h}$.

Lemma 4.9. *There exists $\tilde{c}_2 > 0$, independent of h, such that*

$$\|\boldsymbol{\tau}_{s,0}\|^2_{\mathbf{div};\Omega_s} \;\geq\; \tilde{c}_2 \, \|\boldsymbol{\tau}_s\|^2_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\tau}_s \;\in\; \tilde{\mathbf{V}}_{s,h}, \tag{4.57}$$

where $\boldsymbol{\tau}_s \;=\; \boldsymbol{\tau}_{s,0} \,+\, d\,\mathbf{I}$, with $\boldsymbol{\tau}_{s,0} \in \mathbb{H}_0(\mathbf{div};\Omega_s)$ (cf. (3.35)) and $d \in \mathbb{C}$.

Proof. Given $\boldsymbol{\tau}_s \in \tilde{\mathbf{V}}_{s,h}$ we clearly have, using that $\boldsymbol{\psi} \in \mathbf{Q}_h^s$ for each $h > 0$, that

$$0 \;=\; \langle \boldsymbol{\tau}_s\,\boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma \;=\; \langle \boldsymbol{\tau}_{s,0}\,\boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma \,+\, d\,\langle \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma,$$

which gives

$$d \;=\; -\,\frac{\langle \boldsymbol{\tau}_{s,0}\,\boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma}{\langle \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma},$$

and hence

$$|d| \;\leq\; C\,\frac{\|\boldsymbol{\psi}\|_{1/2,\Sigma}}{|\,\langle \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma\,|}\; \|\boldsymbol{\tau}_{s,0}\|_{\mathbf{div};\Omega_s}.$$

This inequality and the fact that $\|\boldsymbol{\tau}_s\|^2_{\mathbf{div};\Omega_s} \;=\; \|\boldsymbol{\tau}_{s,0}\|^2_{\mathbf{div};\Omega_s} \,+\, 2\,d^2\,|\Omega_s|$ imply (4.57). □

As a consequence of Lemma 4.9, and following basically the same arguments employed in the proofs of Lemmas 3.3 and 3.4, we deduce that the inequalities (3.39) and (3.44) also hold in $\tilde{\mathbf{V}}_{s,h}$. In particular, the latter says that there exists $\tilde{\alpha}_s > 0$, independent of h, such that

$$A_s\big(\boldsymbol{\tau}_s, \Xi_s(\overline{\boldsymbol{\tau}}_s)\big) \,\geq\, \tilde{\alpha}_s \, \|\boldsymbol{\tau}_s\|^2_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\tau}_s \,\in\, \tilde{\mathbf{V}}_{s,h}. \tag{4.58}$$

We are now ready to prove the discrete analogues of (3.45) (*cf.* Lem. 3.4) and (3.48) (*cf.* Lem. 3.5), which constitute the required discrete inf-sup conditions for A_s and A_f.

Lemma 4.10. *There exist $\widetilde{C}_s$, $\widetilde{C}_f$, $h_0 > 0$, independent of h, such that for each $h \leq h_0$ there holds*

$$\sup_{\boldsymbol{\tau}_{s,h}\in\mathbf{V}_{s,h}\backslash\{\mathbf{0}\}} \frac{\big|A_s(\boldsymbol{\zeta}_{s,h},\boldsymbol{\tau}_{s,h})\big|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,\geq\, \widetilde{C}_s \, \|\boldsymbol{\zeta}_{s,h}\|_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\zeta}_{s,h} \,\in\, \mathbf{V}_{s,h}. \tag{4.59}$$

and

$$\sup_{\boldsymbol{\tau}_{f,h}\in\mathbf{V}_{f,h}\backslash\{\mathbf{0}\}} \frac{\big|A_f(\boldsymbol{\zeta}_{f,h},\boldsymbol{\tau}_{f,h})\big|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}} \,\geq\, \widetilde{C}_f \, \|\boldsymbol{\zeta}_{f,h}\|_{\mathrm{div};\Omega_f} \qquad \forall\, \boldsymbol{\zeta}_{f,h} \,\in\, \mathbf{V}_{f,h}. \tag{4.60}$$

Proof. In order to prove (4.59) we introduce the natural discrete approximation of the operator Ξ_s (*cf.* (3.40)) given by $\Xi_{s,h} := \big(\mathbf{I} - 2\,\mathbf{P}_{s,h}\big) : \mathbb{H}^s_h \to \mathbb{H}^s_h$, with $\mathbf{P}_{s,h}$ defined by (4.37). In this way, it follows directly from (4.40) (*cf.* Lem. 4.5) that

$$\|\Xi_s(\boldsymbol{\zeta}_{s,h}) \,-\, \Xi_{s,h}(\boldsymbol{\zeta}_{s,h})\|_{\mathbf{div};\Omega_s} \,\leq\, C\, h^{\epsilon}\, \|\boldsymbol{\zeta}_{s,h}\|_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\zeta}_{s,h} \in \mathbb{H}^s_h.$$

Hence, taking in particular $\boldsymbol{\zeta}_{s,h} \in \mathbf{V}_{s,h}$, adding and substracting $\Xi_s\big(\overline{\boldsymbol{\zeta}}_{s,h}\big)$, using the boundedness of A_s, and applying the inequality (4.58) (having in mind that $\mathbf{V}_{s,h} \subseteq \tilde{\mathbf{V}}_{s,h}$), we find that

$$\big|\, A_s(\boldsymbol{\zeta}_{s,h}, \Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h}))\,\big| \,\geq\, \big|\, A_s(\boldsymbol{\zeta}_{s,h}, \Xi_s(\overline{\boldsymbol{\zeta}}_{s,h}))\,\big| \,-\, \tilde{C}\, h^{\epsilon}\, \|\boldsymbol{\zeta}_{s,h}\|^2_{\mathbf{div};\Omega_s} \,\geq\, \Big\{\tilde{\alpha}_s \,-\, \tilde{C}\, h^{\epsilon}\Big\}\, \|\boldsymbol{\zeta}_{s,h}\|^2_{\mathbf{div};\Omega_s},$$

from which we deduce the existence of c, $h_0 > 0$, independent of h, such that

$$\big|\, A_s(\boldsymbol{\zeta}_{s,h}, \Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h}))\,\big| \,\geq\, c\, \|\boldsymbol{\zeta}_{s,h}\|^2_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\zeta}_{s,h} \in \mathbf{V}_{s,h}, \quad \forall\, h \leq h_0. \tag{4.61}$$

Note from this inequality that $\Xi_{s,h}(\boldsymbol{\zeta}_{s,h}) \neq \mathbf{0}$ for each $\boldsymbol{\zeta}_{s,h} \neq \mathbf{0}$. Also, it is clear from (4.39) and the characterization of $\mathbf{V}_{s,h}$ (*cf.* (4.55)) that $\mathbf{P}_{s,h}(\boldsymbol{\zeta}_{s,h})$, and hence $\Xi_{s,h}(\boldsymbol{\zeta}_{s,h})$, belong to $\mathbf{V}_{s,h}$ for each $\boldsymbol{\zeta}_{s,h} \in \mathbf{V}_{s,h}$. Consequently, we employ (4.61) to bound the supremum on $\mathbf{V}_{s,h}\backslash\{\mathbf{0}\}$ as follows

$$\sup_{\boldsymbol{\tau}_{s,h}\in\mathbf{V}_{s,h}\backslash\{\mathbf{0}\}} \frac{\big|A_s(\boldsymbol{\zeta}_{s,h},\boldsymbol{\tau}_{s,h})\big|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,\geq\, \frac{\big|A_s(\boldsymbol{\zeta}_{s,h},\Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h}))\big|}{\|\Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h})\|_{\mathbf{div};\Omega_s}} \,\geq\, c\, \frac{\|\boldsymbol{\zeta}_{s,h}\|^2_{\mathbf{div};\Omega_s}}{\|\Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h})\|_{\mathbf{div};\Omega_s}}$$

for each $\boldsymbol{\zeta}_{s,h} \in \mathbf{V}_{s,h}$ and for each $h \leq h_0$, which, thanks to the uniform boundedness of $\|\Xi_{s,h}\|$, say by a constant $\bar{C} > 0$, imply (4.59) with $\widetilde{C}_s \,=\, c/\bar{C}$.

The proof of (4.60) proceeds analogously by considering now $\Xi_{f,h} := \big(\mathbf{I} - 2\,\mathbf{P}_{f,h}\big) : \mathbf{H}^f_h \to \mathbf{H}^f_h$, with $\mathbf{P}_{f,h}$ defined by (4.42), applying the inequality (3.47) (*cf.* Lem. 3.5), using, thanks to (4.45) (*cf.* Lem. 4.6), that

$$\|\Xi_f(\boldsymbol{\zeta}_{f,h}) \,-\, \Xi_{f,h}(\boldsymbol{\zeta}_{f,h})\|_{\mathrm{div};\Omega_f} \,\leq\, C\, h^{\epsilon}\, \|\boldsymbol{\zeta}_{f,h}\|_{\mathrm{div};\Omega_f} \qquad \forall\, \boldsymbol{\zeta}_{f,h} \in \mathbf{H}^f_h,$$

and noting, in virtue of (4.44), that $\Xi_{f,h}(\boldsymbol{\zeta}_{f,h}) \in \mathbf{V}_{f,h}$ (*cf.* (4.56)) for each $\boldsymbol{\zeta}_{f,h} \in \mathbf{V}_{f,h}$. □

The following theorem establishes the well-posedness and convergence of the discrete scheme (4.7) with the finite element subspaces $\mathbb{H}^s_h$, $\mathbf{H}^f_h$, $\mathbb{Q}^s_h$, $\mathbf{Q}^s_h$, $\mathbf{Q}^f_h$, $\Lambda_h(\Sigma)$, and $\Lambda_h(\Gamma)$, given, respectively, by (4.1)–(4.49).

Theorem 4.11. *Assume that the homogeneous problem associated to* (2.10) *has only the trivial solution, and let $h_0 > 0$ be the constant provided by Lemma 4.10. Then there exists $h_1 \in (0, h_0]$ such that for each $h \in (0, h_1]$, the fully-mixed finite element scheme* (4.7) *has a unique solution $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\gamma}}_h) := ((\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}), (\boldsymbol{\gamma}_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h})) \in \mathbf{H}_h \times \mathbf{Q}_h$. In addition, there exist C_1, $C_2 > 0$, independent of h, such that for each $h \in (0, h_1]$ there hold*

$$\|(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\gamma}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_1 \left\{ \sup_{\widehat{\boldsymbol{\tau}}_h \in \mathbf{H}_h \setminus \{\mathbf{0}\}} \frac{|F(\widehat{\boldsymbol{\tau}}_h)|}{\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} + \sup_{\widehat{\boldsymbol{\eta}}_h \in \mathbf{Q}_h \setminus \{\mathbf{0}\}} \frac{|G(\widehat{\boldsymbol{\eta}}_h)|}{\|\widehat{\boldsymbol{\eta}}_h\|_{\mathbf{Q}}} \right\} \le C_1 \left\{ \|\mathbf{f}\|_{0,\Omega_s} + \|g\|_{-1/2,\Gamma} \right\}$$

and

$$\|(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\gamma}}) - (\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\gamma}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_2 \inf_{(\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\eta}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h} \|(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\gamma}}) - (\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\eta}}_h)\|_{\mathbf{H}\times\mathbf{Q}}, \tag{4.62}$$

where $(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\gamma}}) := ((\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f), (\boldsymbol{\gamma}, \boldsymbol{\varphi}_s, \boldsymbol{\varphi}_f)) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution of (2.10)*. Furthermore, if there exists $\delta \in (0, 1]$ such that $\boldsymbol{\sigma}_s \in \mathbb{H}^\delta(\Omega_s)$, $\mathbf{div}\, \boldsymbol{\sigma}_s \in \mathbf{H}^\delta(\Omega_s)$, $\boldsymbol{\sigma}_f \in \mathbf{H}^\delta(\Omega_f)$, $\mathrm{div}\, \boldsymbol{\sigma}_f \in H^\delta(\Omega_f)$, $\boldsymbol{\gamma} \in \mathbb{H}^\delta(\Omega_s)$, $\boldsymbol{\varphi}_s \in \mathbf{H}^{1/2+\delta}(\Sigma)$, and $\boldsymbol{\varphi}_f \in H^{1/2+\delta}(\partial\Omega_f)$, then for each $h \in (0, h_1]$ there holds*

$$\begin{aligned} \|(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\gamma}}) - (\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\gamma}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_3\, h^\delta \Big\{ & \|\boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|\mathbf{div}\, \boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|\boldsymbol{\sigma}_f\|_{\delta,\Omega_f} \\ + \;\; & \|\mathrm{div}\, \boldsymbol{\sigma}_f\|_{\delta,\Omega_f} + \|\boldsymbol{\gamma}\|_{\delta,\Omega_s} + \|\boldsymbol{\varphi}_s\|_{1/2+\delta,\Sigma} + \|\boldsymbol{\varphi}_f\|_{1/2+\delta,\partial\Omega_f} \Big\}, \end{aligned}$$

with a constant $C_3 > 0$, independent of h.

Proof. Because of Lemmas 4.7–4.10, the proof of the first part is a straightforward application of [21], Theorem 13.7. In turn, the rate of convergence follows directly from the Cea estimate (4.62) and the approximation properties of the finite element subspaces involved (see $(\mathrm{AP}_h^{\boldsymbol{\sigma}_s})$, $(\mathrm{AP}_h^{\boldsymbol{\sigma}_f})$, $(\mathrm{AP}_h^{\boldsymbol{\gamma}})$ in Section 4.2, and $(\mathrm{AP}_{\Sigma,h})$ and $(\mathrm{AP}_{\Gamma,h})$ above in the present section). □

5. Numerical results

In this section we present three examples showing the performance of our fully-mixed finite element scheme (4.7). Examples 1 and 2 consider smooth exact solutions, whereas Example 3, whose exact solution is singular, is utilized to illustrate the regularity dependence of the rate of convergence (*cf.* Thm. 4.11). We begin by introducing additional notations. The variable N stands for the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$ (*cf.* (4.6)), and the individual errors are denoted by

$$\mathsf{e}(\boldsymbol{\sigma}_s) := \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{\mathbf{div};\Omega_s}, \quad \mathsf{e}(\boldsymbol{\sigma}_f) := \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{\mathrm{div};\Omega_f}, \quad \mathsf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega_s},$$

$$\mathsf{e}(\boldsymbol{\varphi}_s) := \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{1/2,\Sigma}, \quad \mathsf{e}(\varphi_\Sigma) := \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{1/2,\Sigma} \quad \text{and} \quad \mathsf{e}(\varphi_\Gamma) := \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{1/2,\Gamma},$$

where $\boldsymbol{\varphi}_f := (\varphi_\Sigma, \varphi_\Gamma) \in H^{1/2}(\Sigma) \times H^{1/2}(\Gamma)$ and $\boldsymbol{\varphi}_{f,h} := (\varphi_{\Sigma,h}, \varphi_{\Gamma,h}) \in \mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$. Also, we let $r(\boldsymbol{\sigma}_s)$, $r(\boldsymbol{\sigma}_f)$, $r(\boldsymbol{\gamma})$, $r(\boldsymbol{\varphi}_s)$, $r(\varphi_\Sigma)$ and $r(\varphi_\Gamma)$ be the experimental rates of convergence given by

$$r(\boldsymbol{\sigma}_s) := \frac{\log\big(\mathsf{e}(\boldsymbol{\sigma}_s)/\mathsf{e}'(\boldsymbol{\sigma}_s)\big)}{\log(h/h')}, \quad r(\boldsymbol{\sigma}_f) := \frac{\log\big((\mathsf{e}(\boldsymbol{\sigma}_f)/\mathsf{e}'(\boldsymbol{\sigma}_f)\big)}{\log(h/h')},$$

$$r(\boldsymbol{\gamma}) := \frac{\log\big(\mathsf{e}(\boldsymbol{\gamma})/\mathsf{e}'(\boldsymbol{\gamma})\big)}{\log(h/h')}, \quad r(\boldsymbol{\varphi}_s) := \frac{\log\big(\mathsf{e}(\boldsymbol{\varphi}_s)/\mathsf{e}'(\boldsymbol{\varphi}_s)\big)}{\log(h/h')},$$

$$r(\varphi_\Sigma) := \frac{\log\big(\mathsf{e}(\varphi_\Sigma)/\mathsf{e}'(\varphi_\Sigma)\big)}{\log(h/h')} \quad \text{and} \quad r(\varphi_\Gamma) := \frac{\log\big(\mathsf{e}(\varphi_\Gamma)/\mathsf{e}'(\varphi_\Gamma)\big)}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with corresponding errors e and e'.

TABLE 1. Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (Ex. 1).

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	1117	6.150E−02	–	8.865E−01	–	6.642E−03	–
$2\pi/96$	2090	4.264E−02	0.903	5.996E−01	0.964	3.975E−03	1.266
$2\pi/128$	3686	3.112E−02	1.095	4.414E−01	1.065	2.570E−03	1.516
$2\pi/192$	7869	2.107E−02	0.962	3.044E−01	0.917	1.530E−03	1.279
$2\pi/256$	13666	1.586E−02	0.987	2.249E−01	1.053	1.018E−03	1.415
$2\pi/384$	31282	1.038E−02	1.046	1.489E−01	1.017	6.623E−04	1.061
$2\pi/512$	55438	7.784E−03	1.000	1.106E−01	1.035	4.324E−04	1.482
$2\pi/768$	125069	5.152E−03	1.017	7.397E−02	0.991	2.745E−04	1.121
$2\pi/1024$	221848	3.871E−03	0.994	5.540E−02	1.005	2.034E−04	1.041
$2\pi/1536$	498545	2.579E−03	1.001	3.670E−02	1.016	1.298E−04	1.109
$2\pi/2048$	887629	1.927E−03	1.014	2.770E−02	0.978	9.678E−05	1.019

We first consider $\Omega_s := (-0.2, 0.2) \times (-0.4, 0.4)$ and let the artificial boundary Γ be the ellipse centered at the origin with minor and major semiaxis given by 0.4 and 0.6, respectively, that is $\Omega_f := \left\{(x_1, x_2)^{\mathtt{t}} \in \mathbb{R}^2 : \frac{x_1^2}{0.4^2} + \frac{x_2^2}{0.6^2} < 1\right\} \setminus \overline{\Omega}_s$. We take $\rho_s = \rho_f = \lambda = \mu = 1$, and the rest of parameters are given by the sets

$$\Big\{ v_0 = 1;\ \omega = 5;\ \kappa_s = 5;\ \kappa_f = 5 \Big\} \quad \text{and} \quad \Big\{ v_0 = 0.7;\ \omega = 7;\ \kappa_s = 7;\ \kappa_f = 10 \Big\},$$

which define Examples 1 and 2, respectively. Furthermore, let K_0, K_1 and K_2 be the modified Bessel functions of the second kind and order 0, 1, and 2, respectively, and let $H_0^{(1)}$ be the Hankel function of the first kind and order zero. Then, we choose the data in such a way that the exact solution of (2.1) (or (2.10)) is determined by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \dfrac{1}{2\pi}\,\psi(\mathbf{x}) - \dfrac{(x_1-1)^2}{r_1^2}\,\chi(\mathbf{x}) \\ -\dfrac{(x_1-1)\,x_2}{r_1^2}\,\chi(\mathbf{x}) \end{pmatrix} \ \forall\, \mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega_s, \ \text{and} \ p(\mathbf{x}) = H_0^{(1)}(\omega\,|\mathbf{x}|) \ \ \forall\, \mathbf{x} \in \Omega_f,$$

where $r_1 := \sqrt{(x_1-1)^2 + x_2^2}$, $\psi(\mathbf{x}) := K_0(\imath\,\omega\,r_1) + \frac{1}{\imath\,\omega\,r_1}\left\{K_1(\imath\,\omega\,r_1) - \frac{1}{\sqrt{3}}\,K_1\left(\frac{\imath\,\omega\,r_1}{\sqrt{3}}\right)\right\}$, and $\chi(\mathbf{x}) := K_2(\imath\,\omega\,r_1) - \frac{1}{3}\,K_2\left(\frac{\imath\,\omega\,r_1}{\sqrt{3}}\right)$. Actually, $\mathbf{u}$ is the fundamental solution, centered at $(1,0)^{\mathtt{t}}$, of the elastodynamic equation, which yields $\mathbf{f} = \mathbf{0}$ in Ω_s, and p is the fundamental solution, centered at the origin, of the Helmholtz equation in Ω_f.

Then, for Example 3 we let Ω_s be the L-shaped domain $(-0.3, 0.3)^2 \setminus (0, 0.3)^2$ and consider Γ as the boundary of the unit circle $B(\mathbf{0}, 1)$. In addition, we take $\rho_s = \rho_f = \lambda = \mu = 1$, $v_0 = 6/11$, and $\omega = 6$, so that $\kappa_s = 6$ and $\kappa_f = 11$. Then, we choose the data in such a way that the exact solution of (2.1) (or (2.10)) is given by

$$\mathbf{u}(\mathbf{r}, \theta) \ := \ \mathbf{r}^{5/3}\,\sin\big((2\,\theta - \pi)/3\big) \begin{pmatrix} 1+\imath \\ 1+\imath \end{pmatrix} \qquad \forall\,(\mathbf{r}, \theta) \in \Omega_s,$$

and

$$p(\mathbf{x}) = H_0^{(1)}(\omega\,|\mathbf{x} + (0.15, 0)|) \qquad \forall\, \mathbf{x} \in \Omega_f,$$

Note that $\mathbf{u}$ becomes singular at the origin, the corner of the L. More precisely, it is not difficult to see that around this singularity $\mathbf{div}\,\boldsymbol{\sigma}_s$ behaves of order $\mathbf{r}^{-1/3}$. It follows that $\mathbf{div}\,\boldsymbol{\sigma}_s$ belongs to $\mathbf{H}^{2/3-\epsilon}(\Omega_s)$ for each $\epsilon > 0$, and hence, according to Theorem 4.11, we expect experimental rates of convergence, particularly $r(\boldsymbol{\sigma}_s)$, close to 2/3.

In Tables 1 to 4 we present the convergence history of Examples 1 and 2 for finite sequences of quasi-uniform triangulations of the computational domain $\overline{\Omega}_s \cup \overline{\Omega}_f$. We remark that the rate of convergence $O(h)$ predicted

TABLE 2. Convergence history for $\boldsymbol{\varphi}_s$, φ_Σ, and φ_Γ (Ex. 1).

h	N	$\mathsf{e}(\boldsymbol{\varphi}_s)$	$r(\boldsymbol{\varphi}_s)$	$\mathsf{e}(\varphi_\Sigma)$	$r(\varphi_\Sigma)$	$\mathsf{e}(\varphi_\Gamma)$	$r(\varphi_\Gamma)$
$2\pi/64$	1117	9.684E−03	–	1.689E−01	–	4.819E−02	–
$2\pi/96$	2090	4.899E−03	1.681	7.439E−02	2.022	2.030E−02	2.133
$2\pi/128$	3686	2.727E−03	2.037	4.415E−02	1.813	1.226E−02	1.752
$2\pi/192$	7869	1.427E−03	1.598	2.362E−02	1.542	5.610E−03	1.928
$2\pi/256$	13666	8.446E−04	1.822	1.348E−02	1.951	3.850E−03	1.308
$2\pi/384$	31282	4.023E−04	1.829	6.741E−03	1.708	1.834E−03	1.830
$2\pi/512$	55438	2.521E−04	1.625	3.849E−03	1.948	1.187E−03	1.511
$2\pi/768$	125069	1.266E−04	1.699	1.896E−03	1.746	6.280E−04	1.571
$2\pi/1024$	221848	8.236E−05	1.494	1.290E−03	1.339	4.437E−04	1.208
$2\pi/1536$	498545	4.112E−05	1.713	6.765E−04	1.592	2.231E−04	1.695
$2\pi/2048$	887629	2.633E−05	1.550	4.455E−04	1.452	1.533E−04	1.305

TABLE 3. Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (Ex. 2).

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	1117	1.260E−01	–	9.166E−01	–	1.166E−02	–
$2\pi/96$	2090	7.827E−02	1.174	6.046E−01	1.026	5.671E−03	1.777
$2\pi/128$	3686	5.687E−02	1.111	4.434E−01	1.077	3.591E−03	1.588
$2\pi/192$	7869	3.851E−02	0.962	3.052E−01	0.921	2.119E−03	1.301
$2\pi/256$	13666	2.880E−02	1.009	2.252E−01	1.057	1.414E−03	1.406
$2\pi/384$	31282	1.880E−02	1.052	1.490E−01	1.019	8.978E−04	1.121
$2\pi/512$	55438	1.410E−02	1.001	1.106E−01	1.036	5.736E−04	1.557
$2\pi/768$	125069	9.319E−03	1.021	7.398E−02	0.992	3.624E−04	1.133
$2\pi/1024$	221848	6.999E−03	0.995	5.541E−02	1.005	2.665E−04	1.069
$2\pi/1536$	498545	4.662E−03	1.002	3.670E−02	1.016	1.682E−04	1.135
$2\pi/2048$	887629	3.485E−03	1.012	2.770E−02	0.978	1.247E−04	1.040

TABLE 4. Convergence history for $\boldsymbol{\varphi}_s$, φ_Σ, and φ_Γ (Ex. 2).

h	N	$\mathsf{e}(\boldsymbol{\varphi}_s)$	$r(\boldsymbol{\varphi}_s)$	$\mathsf{e}(\varphi_\Sigma)$	$r(\varphi_\Sigma)$	$\mathsf{e}(\varphi_\Gamma)$	$r(\varphi_\Gamma)$
$2\pi/64$	1117	2.051E−02	–	2.498E−01	–	7.683E−02	–
$2\pi/96$	2090	8.132E−03	2.281	9.442E−02	2.399	2.670E−02	2.607
$2\pi/128$	3686	4.515E−03	2.045	5.483E−02	1.890	1.581E−02	1.820
$2\pi/192$	7869	2.478E−03	1.480	2.897E−02	1.573	7.554E−03	1.822
$2\pi/256$	13666	1.438E−03	1.892	1.611E−02	2.041	4.685E−03	1.660
$2\pi/384$	31282	7.075E−04	1.749	7.925E−03	1.749	2.200E−03	1.865
$2\pi/512$	55438	4.504E−04	1.570	4.488E−03	1.976	1.393E−03	1.587
$2\pi/768$	125069	2.114E−04	1.865	2.162E−03	1.802	7.204E−04	1.627
$2\pi/1024$	221848	1.435E−04	1.346	1.448E−03	1.393	5.041E−04	1.241
$2\pi/1536$	498545	7.019E−05	1.764	7.478E−04	1.629	2.517E−04	1.713
$2\pi/2048$	887629	4.461E−05	1.575	4.897E−04	1.472	1.728E−04	1.307

by Theorem 4.11 (when $\delta = 1$) is attained for all the unknowns in both cases. In particular, we observe that the errors $\mathsf{e}(\boldsymbol{\varphi}_s)$, $\mathsf{e}(\varphi_\Sigma)$, and $\mathsf{e}(\varphi_\Gamma)$ converge a bit faster than expected. On the other hand, in Table 5 we display the convergence history of some unknowns of Example 3 for finite sequences of quasi-uniform triangulations of the computational domain $\overline{\Omega}_s \cup \overline{\Omega}_f$. We notice here, as already announced, that $r(\boldsymbol{\sigma}_s)$ oscillates in fact around $2/3$. However, the other rates of convergence shown there are not affected by the lack of regularity of $\boldsymbol{\sigma}_s$. Finally, in

TABLE 5. Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (Ex. 3).

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	2215	9.938E−01	–	1.375E+01	–	1.115E−01	–
$2\pi/96$	4767	6.768E−01	0.947	8.337E−00	1.235	2.291E−02	3.903
$2\pi/128$	8495	5.373E−01	0.802	5.973E−00	1.159	1.020E−02	2.814
$2\pi/192$	19067	4.468E−01	0.455	3.971E−00	1.007	5.789E−03	1.396
$2\pi/256$	33331	3.899E−01	0.474	3.001E−00	0.974	3.776E−03	1.485
$2\pi/384$	75077	2.800E−01	0.817	1.973E−00	1.034	1.680E−03	1.998
$2\pi/512$	133497	2.351E−01	0.607	1.488E−00	0.981	1.154E−03	1.303
$2\pi/768$	299000	1.883E−01	0.547	9.898E−01	1.006	6.706E−04	1.340
$2\pi/1024$	534105	1.493E−01	0.807	7.408E−01	1.007	4.519E−04	1.372
$2\pi/1536$	1199275	1.109E−01	0.735	4.947E−01	0.996	2.701E−04	1.270

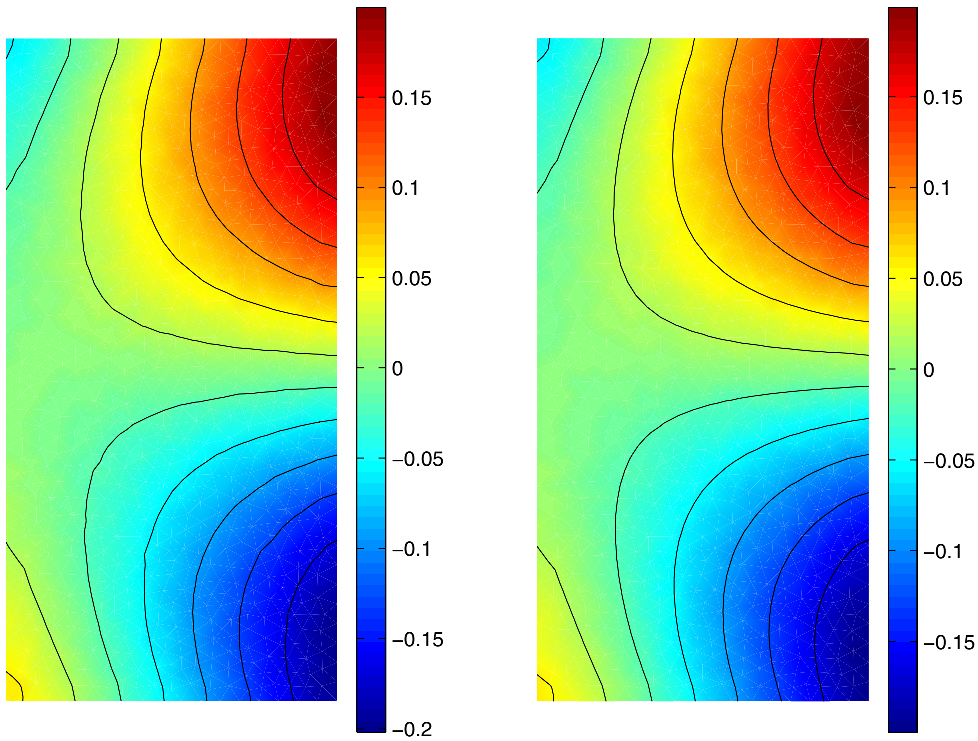


FIGURE 1. Approximate and exact imaginary part of $\boldsymbol{\sigma}_{s,12}$ (Ex. 1).

Figures 1 to 8 we display real and imaginary parts of some components of the approximate and exact solutions of Examples 1 and 2 for $N = 13666$. The fact that they do not distinguish from each other illustrates the accurateness of the proposed fully-mixed method. Note that in the case of the unknowns on the boundaries, they are depicted along straight lines beginning at the points $(0.2, 0.4)$ and $(0.4, 0.0)$ for Σ and Γ, respectively, and then continuing counterclockwise.

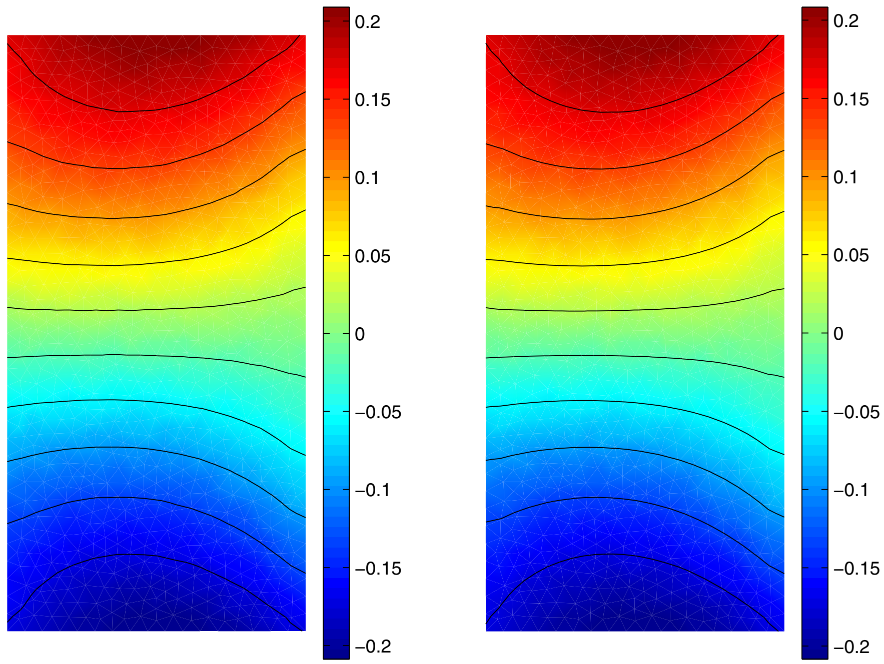


FIGURE 2. Approximate and exact real part of $\boldsymbol{\sigma}_{s,21}$ (Ex. 1).

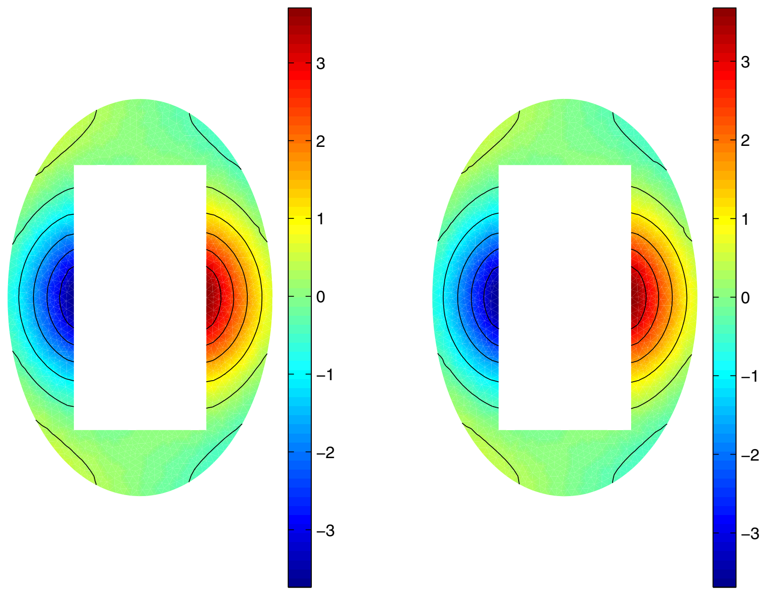


FIGURE 3. Approximate and exact imaginary part of $\boldsymbol{\sigma}_{f,1}$ (Ex. 1).

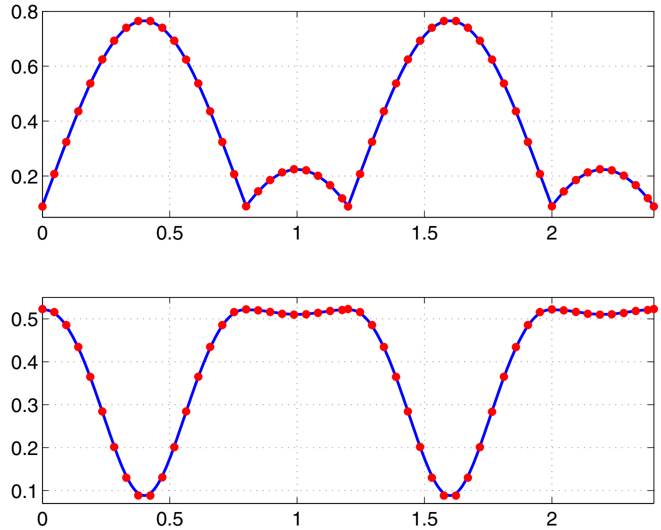


FIGURE 4. Approximate (red) and exact (blue) real and imaginary parts of φ_{Σ} (Ex. 1).

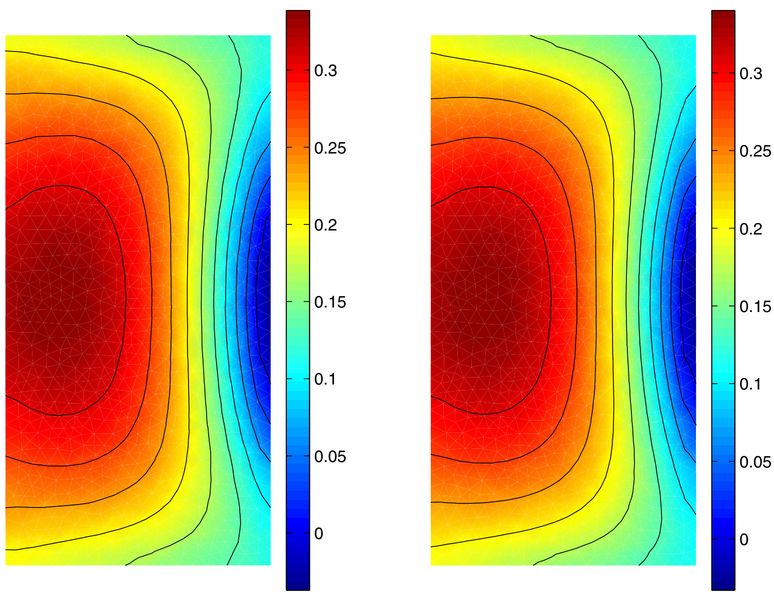


FIGURE 5. Approximate and exact imaginary part of $\boldsymbol{\sigma}_{s,11}$ (Ex. 2).

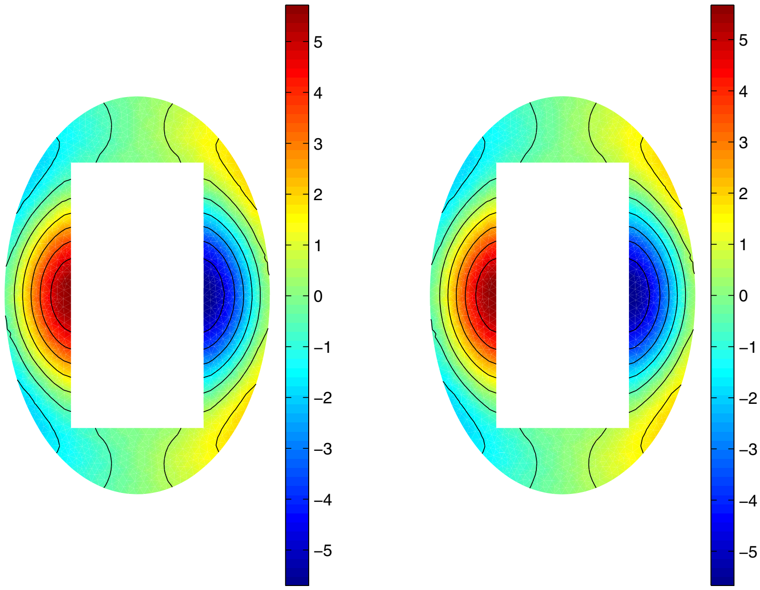


FIGURE 6. Approximate and exact real part of $\boldsymbol{\sigma}_{f,1}$ (Ex. 2).

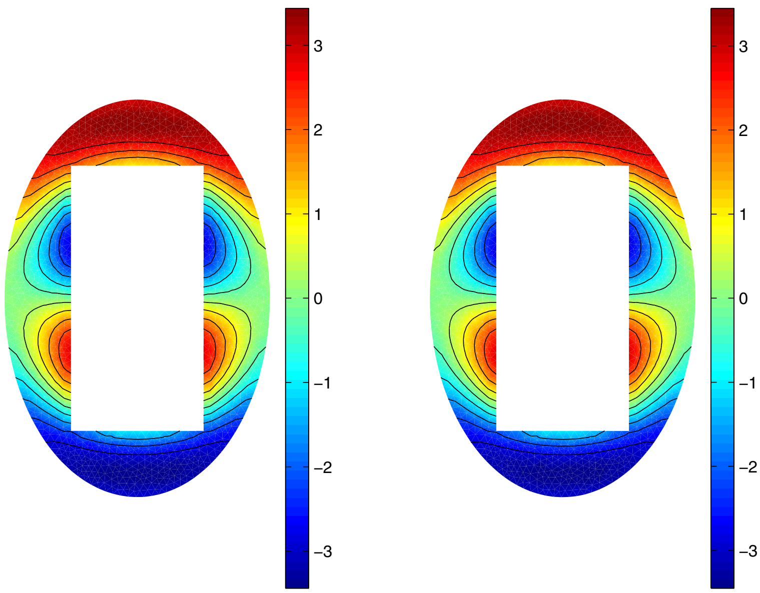
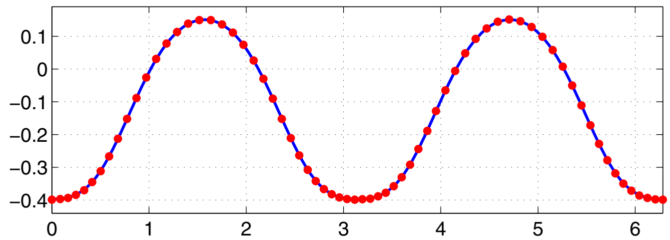


FIGURE 7. Approximate and exact real part of $\boldsymbol{\sigma}_{f,2}$ (Ex. 2).



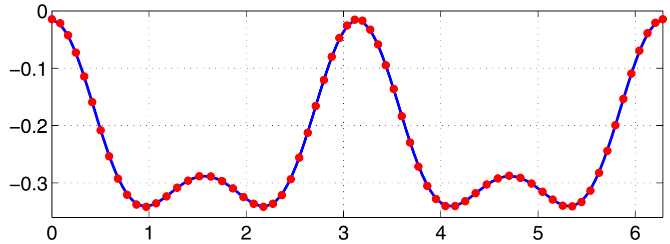


FIGURE 8. Approximate (red) and exact (blue) real and imaginary parts of φ_Γ (Ex. 2).

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