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A PRIORI ERROR ESTIMATES FOR REDUCED ORDER MODELS IN FINANCE *

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Abstract. Mathematical models for option pricing often result in partial differential equations. Recent enhancements are models driven by Lévy processes, which lead to a partial differential equation with an additional integral term. In the context of model calibration, these partial integro differential equations need to be solved quite frequently. To reduce the computational cost the implementation of a reduced order model has shown to be very successful numerically. In this paper we give *a priori* error estimates for the use of the proper orthogonal decomposition technique in the context of option pricing models.

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1. Introduction

Proper Orthogonal Decomposition (POD) has been a successful technique to reduce the order of the complexity of a model described by partial differential equations. There are various examples and a vast literature on both theoretical and numerical aspects of this topic, we mention only [17, 27].

In two areas of applications, modeling options prices in finance [1] and biological models [6], the usual parabolic partial differential equation models of diffusion type are replaced by partial integro-differential equation (PIDE) models. They include a linear spatial integral term as an addition to the traditional PDE formulation. This additional term causes for finite difference and finite element discretization a dense matrix for the resulting system of equations and therefore has to be treated numerically with special care, *cf.* [2, 13, 21, 26] or [12].

The problem of the additional numerical complexity in the solution of PIDEs becomes even more pronounced, if parameters in the models have to be fitted to certain market prices. This results into an optimization problem, a so called calibration problem, formulated as a nonlinear least squares problem, which is usually solved by an optimization algorithm. Such an algorithm requires for each iteration in the optimization a certain number of function evaluations and possibly gradient information of the least squares function and therefore also of the solution of the PIDE. This results in a very high effort from a computational point of view.

Keywords and phrases. Option pricing models, proper orthogonal decomposition, *a priori* error estimate.

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Therefore, the concept of reduced order models carries a lot of potential to be very useful in this context. First results in this direction were obtained in [8, 23] for a reduced basis model, [24, 25] for the calibration of a POD model and [15] for the model reduction of a high dimensional PIDE. The numerical results in [24, 25] show that the use of POD model reduction leads to a substantial saving in computing time for the solution of calibration problems including PIDEs. Another important aspect of the POD model reduction technique is the fact that it preserves the structure of the original model which contains a parabolic and an integral term.

If one replaces the original PIDE by an approximate problem like a POD model, the question of the error for such an approximation arises. For the reduced basis approach, error estimates for parabolic differential equations can be found in [14]. In the context of a PDE approximation *via* POD, papers by [16, 20] show results in this direction. These types of POD error estimates are important also for a calibration process since they can be used to control the improvement of the accuracy of the model in the course of the iteration of the optimization algorithm. However, these estimates do not apply to the problem under consideration since it leads to a formulation with a time-dependent bilinear form not covered by the cited authors.

The goal of this paper is to provide error estimates of the POD approximation for a PIDE as it occurs in the area of mathematical finance. In order to achieve this, we need to develop a weak formulation of the PIDE which also includes time dependent bilinear forms. Since the spatial domain is the whole real line, the weak formulation requires some nonstandard function spaces which have been introduced in this context in [21]. It is a new aspect in this paper that this framework is extended to Dupire PIDEs with time and space dependent coefficients, as they appear in local volatility models. This approach leads to an existence and uniqueness theorem for a weak solution of the PIDEs investigated here. Furthermore, we also derive error estimates for a POD approximation in the framework where the bilinear forms also depend on time. In [4, 5] error estimates for the difference between the finite element solution and the solution of the reduced model are provided for the gradient of an optimization problem driven by the Stokes problem. The reduction is carried out by balanced truncation including techniques from domain decomposition.

The paper is organized as follows. In the following two introductory subsections, the PIDE model for finance is laid out and its weak formulation is motivated and defined. After that we give a brief introduction into the basic concept of POD. We also point out the original projection error introduced by a POD approach using general snapshots.

The second section contains the assumptions on the bilinear form. Furthermore, for the purpose of the proof two projections are defined and a series of Lemmas leads to an error estimate for the error introduced by the POD approximation. Since the original POD error estimate uses an average norm, this also affects the final error estimate, where a similar average norm is estimated.

In the third section we verify all the assumptions for the bilinear form of the PIDE coming from a Lévy model in finance. It proves that this is a viable theoretical framework, since all the assumptions imposed on the example to make the theory applicable are reasonable conditions for the models in finance. The last section contains numerical results for the error of the PIDE model including a comparison of computing time.

1.1. Option pricing models

First we give a brief motivation for the occurrence of partial integro-differential equations in option price modeling. For the definition of an option and the financial background we refer the reader to, *e.g.* [18].

In their seminal paper, Black and Scholes [7] showed that under certain assumptions the price of a call option $C(t, S)$ depending on the current time t and the current underlying price S follows the solution of a partial differential equation. However, if large jumps should be incorporated in this model, a more general model for the development of the stock prices has to be applied. Here the Brownian motion of the Black–Scholes model is replaced by a more general Lévy process such as the addition of a Poisson process to the Brownian motion.

These models are called jump-diffusion models (see [9, 28]) and their call price is given by the solution of the following parabolic integro-differential equation (PIDE):

$$\begin{gathered} C_t(t,S) + \frac{\sigma^2(t,S)}{2} S^2 C_{SS}(t,S) + r(t) S C_S(t,S) - r(t)C(t,S) \\ +\lambda \int_{-\infty}^{+\infty} \Big(C(t, S\mathrm{e}^y) - C(t,S) - S(\mathrm{e}^y - 1) C_S(t,S) \Big) f(y) \mathrm{d}y = 0, \\ (t,x) \in (0,T) \times (0,\infty) \\ C(t,0) = 0, \quad t \in [0,T] \\ C(T,S) = \max\{S - B, 0\}, \quad S \in (0,\infty). \end{gathered} \tag{1.1}$$

Here T represents the maturity, B the strike price of the option, $r(t)$ the risk-free interest rate, $\sigma(t,S)$ the volatility, $\lambda \geq 0$ the frequency of the jumps and $f(y)$ the density function of the distribution of the jump sizes.

Note that the price we are usually interested in is $C(t_0, S_0)$ with $t_0 = 0$, the price today for the current stock price S_0.

If one considers a calibration problem, where for example σ or f should be determined from various market prices of options C_i at time t_0 with stock price S_0 for the same underlying but with various maturities T_i and strike prices B_i, the following optimization problem occurs where we denote the model prices from (1.1) by $C(t_0, S_0; T_i, B_i)$

$$\text{Minimize} \quad \sum_{i=1}^{I} ||C(t_0, S_0; T_i, B_i) - C_i||^2. \tag{1.2}$$

For each function evaluation I partial integro-differential equations would have to be solved, which makes the problem intractable. Instead one can resort to a Dupire-type formulation, first introduced by Dupire [11] for the Black–Scholes model and by Andersen and Andreasen [3] for the PIDE. According to this and the additional variable transformation $x = \ln\left(\frac{B}{S}\right)$ one can compute the prices also by solving the following PIDE, where the current time t and the current stock price S appear in the initial condition, but the prices can be obtained from $D(T,x)$ for all maturities T and strike prices S.

$$\begin{gathered} D_T(T,x) - \frac{\sigma^2(T,x)}{2} D_{xx}(T,x) + \left(r(T) + \frac{\sigma^2(T,x)}{2} - \lambda\zeta \right) D_x(T,x) + \lambda(1+\zeta) D(T,x) \\ -\lambda \int_{-\infty}^{+\infty} D(T, x-y) \mathrm{e}^y f(y) \mathrm{d}y = 0, \quad (T,x) \in (0, T_{\max}) \times (-\infty, \infty) \\ D(0,x) = \max\{S_0 - S_0 \mathrm{e}^x, 0\} =: D_0(x) \end{gathered} \tag{1.3}$$

where we used the abbreviation $\zeta = \int_{I\!R} \mathrm{e}^y f(y)\, \mathrm{d}y - 1$.
Then the calibration problem (1.2) can be rewritten as

$$\text{Minimize} \quad \sum_{i=1}^{I} ||D(T_i, x_i) - C_i||^2, \tag{1.4}$$

where only one PIDE per function evaluation needs to be solved. This could be viewed already as a model reduction compared to the complexity of (1.2). Nevertheless, when an optimization algorithm is applied to solve (1.4), each function evaluation requires the solution of a partial integro-differential equation, where the nonlocal term in the PIDE needs special attention in the discretization, since it leads to dense matrices. The goal of this paper is to replace in (1.4) the map D by a reduced model D^{red} and solve

$$\text{Minimize} \quad \sum_{i=1}^{I} ||D^{\text{red}}(T_i, x_i) - C_i||^2, \tag{1.5}$$

In the sequel we confine ourselves to analyze the choice of $D^{\rm red}$ and to derive error estimates. The analysis for the calibration problem itself will be addressed in a future paper, although first numerical results are promising (see [24]).

Proper orthogonal decomposition (POD) has been a successful model reduction technique in various applications of partial differential equations, like diffusion processes and the Navier-Stokes equation. Since we want to derive error estimates for the reduced models, we require a formulation of the PIDE in a variational setting. Note that, in particular, the integral operator in (1.3) needs special attention. For the case of a constant volatility σ Matache, von Petersdorff and Schwab [21] have derived a variational formulation. Since we want to calibrate σ as function on T and x, we will extend the concept in [21] to suit our model.

The proper variational formulation requires the use of weighted function spaces due to the initial condition, which is not $L_2(\mathbb{R})$-integrable as $D_0(x) \stackrel{x\to-\infty}{\longrightarrow} K_0$.

Definition 1.1. For $\mu > 0$ we define

$$L^2_{-\mu}(\mathbb{R}) := \left\{ v \in L^1_{\rm loc}(\mathbb{R}) : v(\cdot)\mathrm{e}^{-\mu|\cdot|} \in L^2(\mathbb{R}) \right\}$$

with inner product $\langle v, w\rangle_{L^2_{-\mu}} := \int_{\mathbb{R}} v(x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x$ and the weighted Sobolev space

$$H^1_{-\mu}(\mathbb{R}) := \left\{ v \in L^1_{\rm loc}(\mathbb{R}) : v(\cdot)\mathrm{e}^{-\mu|\cdot|}, v'(\cdot)\mathrm{e}^{-\mu|\cdot|} \in L^2(\mathbb{R}) \right\}$$

with inner product $\langle v, w\rangle_{H^1_{-\mu}} := \langle v, w\rangle_{L^2_{-\mu}} + \langle v', w'\rangle_{L^2_{-\mu}}$.

Remark 1.2. $L^2_{-\mu}(\mathbb{R})$ as well as $H^1_{-\mu}(\mathbb{R})$ with the above defined inner products are Hilbert spaces.

It is clear that $D_0(\cdot) \in H^1_{-\mu}(\mathbb{R})$ for all $\mu > 0$.

We motivate the variational formulation of (1.3) in the following lines. First, we multiply the PIDE (1.3) by $w(x)\mathrm{e}^{-2\mu|x|}$ and integrate over $\mathbb{R}$ where $\mu > 0$ and w is an arbitrary function in $C_0^\infty(\mathbb{R})$.

$$\begin{aligned}
&\int_{\mathbb{R}} D_T(T,x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x = \int_{\mathbb{R}} \frac{\sigma^2(T,x)}{2} D_{xx}(T,x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x \\
&- \int_{\mathbb{R}} \left(r(T) + \frac{\sigma^2(T,x)}{2} - \lambda\zeta \right) D_x(T,x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x \\
&- \int_{\mathbb{R}} \lambda(1+\zeta)D(T,x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x + \lambda \int_{\mathbb{R}}\int_{\mathbb{R}} D(T,x-y)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{e}^y f(y)\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

If the first term on the right hand side of this equation is integrated by parts we obtain the following equation

$$\int_{\mathbb{R}} D_T(T,x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x = -a^{-\mu}(T; D(T,\cdot), w(\cdot))$$

where the bilinear form $a^{-\mu}$ is defined as:

Definition 1.3. Let $r(T), \lambda, \zeta$ be given constants and assume that $\sigma(T,\cdot), \sigma(T,\cdot)_x$ are continuous and bounded functions on $\mathbb{R}$. For each constant $\mu > 0$ and $T > 0$ the bilinear form

$$a^{-\mu}(T;\cdot,\cdot) : H^1_{-\mu}(\mathbb{R}) \times H^1_{-\mu}(\mathbb{R}) \to \mathbb{R}$$

is defined as

$$a^{-\mu}(T;\ v,w) := \int_{\mathbb{R}} \frac{\sigma^2(T,x)}{2} v'(x)w'(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x \tag{1.6}$$
$$+ \int_{\mathbb{R}} \left(r(T) + \frac{\sigma^2(T,x)}{2} - \lambda\zeta + \frac{(\sigma^2(T,x))_x}{2} + \sigma^2(T,x)\mu\ \mathrm{sign}(x) \right) v'(x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x$$
$$+ \int_{\mathbb{R}} \lambda(1+\zeta)v(x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x - \ \lambda \int_{\mathbb{R}}\int_{\mathbb{R}} v(x-y)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{e}^{y}f(y)\mathrm{d}y\ \mathrm{d}x.$$

We set

$$W([a,b],V) := \left\{ u : u \in L^2((a,b),V),\quad u' \in L^2((a,b),V') \right\} \quad a,b \in \mathbb{R},\quad V \text{ Hilbert space.}$$

Hence the variational formulation of the Dupire PIDE (1.3) can be expressed in the following form:

Definition 1.4. The variational formulation of the PIDE (1.3) consists of finding $D \in W([0,T_{\max}], H^1_{-\mu}(\mathbb{R}))$ such that for all $T \in (0,T_{\max})$

$$\frac{\mathrm{d}}{\mathrm{d}T}\langle D(T,\cdot), w(\cdot)\rangle_{L^2_{-\mu}} + a^{-\mu}(T; D(T,\cdot), w(\cdot)) = 0 \quad \forall\ w \in H^1_{-\mu}(\mathbb{R}) \tag{1.7}$$

holds with initial condition

$$\langle D(0,\cdot), w(\cdot)\rangle_{L^2_{-\mu}} = \langle D_0(\cdot), w(\cdot)\rangle_{L^2_{-\mu}}\ \forall\ w \in H^1_{-\mu}(\mathbb{R}). \tag{1.8}$$

The existence of a solution is guaranteed, if the bilinear form $a^{-\mu}(T;\ v,w)$ meets some certain demands. This will be discussed in the following sections.

For the numerical solution of the PIDE above the spatial variable is discretized *via* a finite element approach, *i.e.* after the variational formulation and some transformations we approximate the solution by the linear combination of spline basis functions. The time variable is discretized by applying either the implicit Euler or the Crank–Nicolson method as some special cases of the θ-method.

Using a Galerkin approximation of the Hilbert space $H^1_{-\mu}$ with n basis functions (*e.g.* linear splines) and discretizing the time variable with m time steps leads to m linear systems of equations of size $n \times n$. In order to obtain an approximate solution of the PIDE one has to solve a linear system of equations in each time step. Since there is a double-integral term in the bilinear form, the stiffness matrix is dense even if we take basis functions with compact support, hence the cost for matrix vector multiplications can be of the order of $\mathcal{O}(n^2)$.

This illustrated the potential benefits of a reduced order approach, especially in a calibration process, where the problem has to be solved several times.

1.2. Proper orthogonal decomposition

We now give a brief introduction to proper orthogonal decomposition (POD), a technique to obtain a problem of much smaller order than the original discretized version.

Let u_i, $i = 1,\ldots,n$, be elements of a real separable Hilbert space H which, for example, approximate the solution $u(t_i)$ of a differential equation at various time instances t_i. Those elements u_i are sometimes called "snapshots". The space spanned by the snapshots has dimension $r \geq 1$, *i.e.* $dim(\mathrm{span}(u_1,\ldots,u_n)) = r$. Thus, at least one snapshot is assumed to be nonzero. Proper orthogonal decomposition consists of first finding elements $\Psi_j \in H$, $j = 1,\ldots,r$, that build an orthonormal basis of $\mathrm{span}(u_1,\ldots,u_n)$ and have the following additional property: Considering the partial basis $\Psi_1,\ldots,\Psi_p$ for an arbitrary $p \in \{1,\ldots,r\}$, there are no other orthonormal basis functions $\Phi_1,\ldots,\Phi_p$, which approximate an "average" element of $\mathrm{span}(u_1,\ldots,u_n)$ in a better way. The projection of a $v \in \text{span}(u_1, \dots, u_n)$ on the space spanned by arbitrary orthonormal functions $\{\Psi_j\}_{j=1}^p$ can be computed from its Fourier expansion:

$$\tilde{v} = \sum_{j=1}^{p} \langle v, \Psi_j \rangle_H \Psi_j.$$

The mathematical formulation for the POD basis functions is formulated as follows:

Definition 1.5. Given vectors $u_1, ..., u_n \in H$, find orthonormal vectors $\Psi_1, \dots, \Psi_r \in \text{span}(u_1, \dots, u_n)$ by solving the minimization problem:

$$\min_{\Psi_1,\dots,\Psi_p} \sum_{i=1}^{n} \gamma_i \left\| u_i - \sum_{j=1}^{p} \langle u_i, \Psi_j \rangle_H \Psi_j \right\|_H^2 \qquad (1.9)$$
$$s.t. \quad \langle \Psi_k, \Psi_l \rangle_H = \delta_{kl} \quad \forall k, l = 1, \dots, p$$

for all $p \in \{1, \dots, r\}$ with weights $\gamma_i > 0$, $i = 1, \dots, n$. The first p vectors $\Psi_1, ..., \Psi_p$ are called a POD basis of rank p. The spanning subspace is denoted by $\mathcal{V}^p = \text{span}(\Psi_1, ..., \Psi_p)$.

We shortly review how to calculate these POD basis functions. For this purpose we introduce the matrix $\mathcal{K} \in \mathbb{R}^{n \times n}$ with

$$\mathcal{K}_{ij} := \sqrt{\gamma_i \gamma_j} \langle u_j, u_i \rangle_H \quad \forall\, i, j = 1, \dots, n.$$

Solving the eigenvalue problem

$$\mathcal{K} v^k = \lambda_k v^k \quad k = 1, \dots, r$$

[30] where $v^k \in \mathbb{R}^n$, the POD basis functions Ψ_k for a basis of rank p $(\leq r)$ are given by

$$\Psi_k = \frac{1}{\sqrt{\lambda_k}} \sum_{i=1}^{n} \sqrt{\gamma_i} v_i^k u_i$$

where u_i, $i = 1, \dots, n$ are the snapshots from above.

Considering only the first $p < r$ POD basis functions for a representation of the u_i, we have to deal with an approximation, the projection of u_i on the space spanned by $\{\Psi_k\}_{k=1}^p$. The error resulting from dropping the information stored in $\Psi_{p+1}, \dots, \Psi_r$, *i.e.* the function value of the minimization problem above is estimated in the following theorem. See, *e.g.* [29], for a proof.

Theorem 1.6. *Let $\Psi_1, \dots, \Psi_p$ be a solution to the minimization problem (1.9). Then it holds*

$$\sum_{i=1}^{n} \gamma_i \left\| u_i - \sum_{j=1}^{p} \langle u_i, \Psi_j \rangle_H \Psi_j \right\|_H^2 = \sum_{k=p+1}^{r} \lambda_k. \qquad (1.10)$$

If this technique is applied to the problem outlined above, one has to specify the snapshots. Here we choose the approximation of the solution of the problem at fixed time steps $t_1, \dots, t_n$, *i.e.* $D(\cdot, t_1), \dots, D(\cdot, t_n)$ by the finite element method. We obtain some orthonormal basis functions containing specific information about the solution of the PIDE. Approximating the PIDE problem *via* a POD approach means that we replace the finite element basis functions by the POD basis function calculated from a given solution to the problem. Since we only need a few basis functions – numerical tests show that 10 is already a sufficient quantity – compared to, *e.g.*, 1000 finite element basis functions, the systems of equations that have to be solved are quite smaller.

2. *A priori* error estimates

Since the original problem, the PIDE, is replaced by a smaller one, the POD approximation, we want to estimate the error involved in this process. We make the following assumptions on the bilinear form following [10], p. 509 ff.:

Assumption 2.1.

a.) *Let V and H be two real, separable Hilbert spaces with the inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_H$ and the induced norms $||\cdot||_V$ and $||\cdot||_H$, respectively. With the dual spaces V^* and H^* they form a Gelfand triple:*

$$V \hookrightarrow H = H^* \hookrightarrow V^* \tag{2.1}$$

with dense embeddings. Furthermore, assume an $\alpha > 0$ with $||v||_H^2 \leq \alpha ||v||_V^2$ for all $v \in V$.

b.) *Let $a : [0,T] \times (V \times V) \to I\!R$ for all $t \in [0,T]$ be a uniformly continuous and coercive bilinear form,* i.e. *there exist constants $\beta, \kappa > 0$ independently of t with*

$$|a(t;\ v,w)| \leq \beta ||v||_V ||w||_V \qquad \forall\, v,w \in V\ \forall\, t \in [0,T], \tag{2.2}$$

$$a(t;\ v,v) \geq \kappa ||v||_V^2 \qquad \forall\, v \in V\ \forall\, t \in [0,T]. \tag{2.3}$$

In addition let $a(\cdot;\cdot,\cdot)$ be Lipschitz-continuous with respect to t, i.e.

$$|a(t_1;\ v,w) - a(t_2;\ v,w)| \leq c_{\text{lip}} |t_1 - t_2|\ ||v||_V ||w||_V \qquad \forall\, v,w \in V. \tag{2.4}$$

c.) *Let $L :\ [0,T] \times V \to I\!R$ be a linear form with*

$$|L(t;\ v)| \leq c_L\ ||v||_V \quad \forall\, t \in [0,T],\ v \in V \tag{2.5}$$

With the notation fixed and the assumptions stated, we can formulate the weak form of an abstract parabolic initial value problem.

Problem 2.2. For given initial value $w_0 \in H$ find a solution $u \in W([0,T],V)$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t),v)_H + a(t;\ u(t),v) = L(t;\ v) \quad \forall\, v \in V,\ t \in (0,T) \tag{2.6}$$

and initial condition

$$(u(0),v)_H = (w_0,v)_H \quad \forall\, v \in V. \tag{2.7}$$

As an abbreviation we set for the finite difference quotient

$$\bar{\partial} u_i := \frac{u_i - u_{i-1}}{\Delta t}\cdot$$

The discretized version in time of Problem 2.2 on the subspace $\mathcal{V}^p$ of V with equidistant time steps $t_1, \ldots, t_m$ (*i.e.* $\Delta t = t_i - t_{i-1}\ \forall i = 1, \ldots, m$) looks as follows:

Problem 2.3. For given initial value $w_0 \in H$ and some $\theta \in [0,1]$ find$\{u_i^{\text{POD}}\}_{i=0}^m \subset \mathcal{V}^p$ with

$$\left(\bar{\partial} u_i^{\text{POD}}, v\right)_H + \theta \cdot a\Big(t_i;\ u_i^{\text{POD}}, v\Big) + (1-\theta)\cdot a\Big(t_{i-1};\ u_{i-1}^{\text{POD}}, v\Big) = \theta \cdot L(t_i;\ v) + (1-\theta)\cdot L(t_{i-1}, v) \qquad \forall\, v \in \mathcal{V}^p,\ i = 1, \ldots, m \tag{2.8}$$

and initial condition

$$\left(u_0^{\text{POD}}, v\right)_H = (w_0,v)_H \quad \forall\, v \in \mathcal{V}^p. \tag{2.9}$$

Using the stated assumptions we can invoke an existence and uniqueness theorem in [10], p. 512 ff., to conclude that there exists a unique solution for both problems.

Since there are different possibilities to create a POD basis we want to clarify which ones we use and which errors we address.

Error 1. Average error between the solution $u(t)$ of Problem 2.2 and the solution on the POD subspace, discretized in time *via* the θ-method (this is Problem 2.3). The POD basis functions are calculated from the snapshots of the solution $u(t)$ and the corresponding difference quotients, *i.e.* the snapshots are $y_i = u(t_{i-1})$, $i = 1, \ldots, n+1$ and $y_{i+n+1} = \frac{u(t_i)-u(t_{i-1})}{\Delta t}$, $i = 1, \ldots, n$. To avoid confusion we call the POD solution $u^{\mathrm{POD},1}$

$$ERR_1 = \frac{1}{n}\sum_{i=1}^{n} \left|\left|u_i^{\mathrm{POD},1} - u(t_i)\right|\right|_H^2 \tag{2.10}$$

Error 2. Average error between the finite element approximation u^{FEM} discretized in time and space (this is the solution of Problem 2.3 in which we replace the POD space $\mathcal{V}^p$ by the finite element subspace $\mathcal{H}^m$) and the POD approximation discretized in time (Problem 2.3), whereas the POD basis functions are calculated from the snapshots of the finite element solution and the corresponding difference quotients ($u^{\mathrm{POD},2}$)

$$ERR_2 = \frac{1}{n}\sum_{i=1}^{n} \left|\left|u_i^{\mathrm{POD},2} - u_i^{\mathrm{FEM}}\right|\right|_H^2 \tag{2.11}$$

The fact that the difference quotients are included in the calculation of the POD basis is often reported to yield numerically better approximation results. Here we note, that it also facilitates the proof for the error estimates.

In the proof we use two different projections and in the next lemma we show some characteristic properties of these projections.

Definition 2.4. Let $\mathcal{V}^p$ be a subspace of V. We define the H-projection Π_H^p

$$\Pi_H^p : V \to \mathcal{V}^p \quad \Leftrightarrow \quad \langle \Pi_H^p u - u, v\rangle_H = 0 \;\forall\, v \in \mathcal{V}^p$$

and the Ritz-projection $\Pi_{a,t}^p$

$$\Pi_{a,t}^p : V \to \mathcal{V}^p \quad \Leftrightarrow \quad a(t;\; \Pi_{a,t}^p u - u, v) = 0 \;\forall\, v \in \mathcal{V}^p, t \in [0,T].$$

We show a relationship of the Ritz-projection to the H-projection and the Lipschitz-continuity for the Ritz-projection with respect to the time variable.

Lemma 2.5. *For the projections we have*

$$||\Pi_{a,t}^p u - u||_V^2 \le \frac{\beta}{\kappa}||v-u||_V^2 \quad v \in \mathcal{V}^p, \quad \textit{in particular} \quad v = \Pi_H^p u \tag{2.12}$$

$$||(\Pi_{a,t}^p - \Pi_{a,s}^p)u||_V \le |t-s|\frac{c_{\mathrm{lip}}}{\kappa}||\Pi_{a,s}^p v - v||_V. \tag{2.13}$$

Proof. Using (2.2) and (2.3) one easily verifies the following inequalities:

$$\kappa||\Pi_{a,t}^p u - u||_V^2 \le a(t;\; \Pi_{a,t}^p u - u, \Pi_{a,t}^p u - u) \le a(t;\; v-u, v-u) \le \beta||v-u||_V^2$$

for all $t \in [0,T]$ and $v \in \mathcal{V}^p$. The coercivity (2.3) yields

$$\kappa||(\Pi^p_{a,t} - \Pi^p_{a,s})u||^2_V \leq a(t;\ (\Pi^p_{a,t} - \Pi^p_{a,s})u, (\Pi^p_{a,t} - \Pi^p_{a,s})u)$$

and using the Ritz-projection property as well as the Lipschitz continuity (2.4) we get

$$\begin{aligned} a(t;\ (\Pi^p_{a,t} - \Pi^p_{a,s})u, (\Pi^p_{a,t} - \Pi^p_{a,s})u) &= a(s;\ \Pi^p_{a,s}u - u, (\Pi^p_{a,t} - \Pi^p_{a,s})u) - a(t;\ \Pi^p_{a,s}u - u, (\Pi^p_{a,t} - \Pi^p_{a,s})u) \\ &\leq c_{\text{lip}}|t-s|\ ||\Pi^p_{a,s}u - u||_V\ ||(\Pi^p_{a,t} - \Pi^p_{a,s})u||_V. \end{aligned}$$

Combining these two results yields the second statement. □

By Assumption 2.1 we have $||v||^2_H \leq \alpha||v||^2_V$ for all $v \in V$. A reverse inequality holds, if we consider the finite-dimensional subspace $\mathcal{V}^r$

$$||u||_V \leq \sqrt{||S||_2}||u||_H \quad \forall u \in \mathcal{V}^r \quad \text{with} \quad S \in I\!R^{r\times r}, \quad S_{ij} = \langle \Psi_j, \Psi_i\rangle_V \tag{2.14}$$

see *e.g.* [20], Lemma 2.

For the POD error compared to the Ritz-projection we have the following error estimate.

Lemma 2.6. *For the implicit Euler method ($\theta = 1$) we have*

$$\left|\left|u_i^{\text{POD},1} - \Pi^p_{a,t_i}u(t_i)\right|\right|_H \leq \left|\left|u_{i-1}^{\text{POD},1} - \Pi^p_{a,t_{i-1}}u(t_{i-1})\right|\right|_H + \Delta t||v_i||_H \tag{2.15}$$

and for the Crank–Nicolson scheme ($\theta = 1/2$)

$$\left|\left|u_i^{\text{POD},1} - \Pi^p_{a,t_i}u(t_i)\right|\right|_H \leq (1 + 4\xi\Delta t^3)\left|\left|u_{i-1}^{\text{POD},1} - \Pi^p_{a,t_{i-1}}u(t_{i-1})\right|\right|_H + 2\Delta t||v_i||_H \tag{2.16}$$

provided $\xi\Delta t^3 < 1/2$ with $\xi = c^2_{\text{lip}}\alpha||S||^2_2/(32\kappa)$. Here v_i is defined as

$$v_i = \theta\cdot u_t(t_i) + (1-\theta)\cdot u_t(t_{i-1}) - \bar{\partial}\Pi^p_{a,t_i}u(t_i). \tag{2.17}$$

Proof. Set

$$w_i = u_i^{\text{POD},1} - \Pi^p_{a,t_i}u(t_i).$$

In the equalities below, for the first equation we use the definition of w_i, for the second equation recall (2.8) for the first part and Definition 2.4b for the second part (note that $w_i \in \mathcal{V}^p$). Thus we have for an arbitrary $\Psi \in \mathcal{V}^p$:

$$\begin{aligned} &\langle\bar{\partial}w_i, \Psi\rangle_H + \theta\cdot a(t_i;\ w_i, \Psi) + (1-\theta)\cdot a(t_{i-1};\ w_{i-1}, \Psi) \\ &= \langle\bar{\partial}u_i^{\text{POD},1}, \Psi\rangle_H + \theta\cdot a\left(t_i;\ u_i^{\text{POD},1}, \Psi\right) + (1-\theta)\cdot a\left(t_{i-1};\ u_{i-1}^{\text{POD},1}, \Psi\right) \\ &\quad - \langle\bar{\partial}\Pi^p_{a,t_i}u(t_i), \Psi\rangle_H - \theta\cdot a(t_i;\ \Pi^p_{a,t_i}u(t_i), \Psi) - (1-\theta)\cdot a(t_{i-1};\ \Pi^p_{a,t_i}u(t_{i-1}), \Psi) \\ &= \theta\cdot L(t_i;\ \Psi) + (1-\theta)\cdot L(t_{i-1};\ \Psi) \\ &\quad - \langle\bar{\partial}\Pi^p_{a,t_i}u(t_i), \Psi\rangle_H - \theta\cdot a(t_i;\ u(t_i), \Psi) - (1-\theta)\cdot a(t_{i-1};\ u(t_{i-1}), \Psi). \end{aligned}$$

Since $u(t)$ is the solution of (2.6) we obtain

$$\begin{aligned} &\langle\bar{\partial}w_i, \Psi\rangle_H + \theta\cdot a(t_i;\ w_i, \Psi) + (1-\theta)\cdot a(t_{i-1};\ w_{i-1}, \Psi) \\ &= \theta\cdot\langle u_t(t_i), \Psi\rangle_H + (1-\theta)\cdot\langle u_t(t_{i-1}), \Psi\rangle_H - \langle\bar{\partial}\Pi^p_{a,t_i}u(t_i), \Psi\rangle_H = \langle v_i, \Psi\rangle_H \end{aligned} \tag{2.18}$$

with v_i defined in (2.17).

If we set $\Psi = w_i$ we obtain for the implicit Euler method ($\theta = 1$)

$$\begin{aligned}||w_i||_H^2 &= \langle w_i, w_{i-1}\rangle_H - \Delta t\; a(t_i;\; w_i, w_i) + \Delta t\langle v_i, w_i\rangle_H \\ &\le ||w_i||_H||w_{i-1}||_H - \Delta t\frac{\kappa}{\alpha}||w_i||_H^2 + \Delta t||v_i||_H||w_i||_H\end{aligned}$$

and hence

$$||w_i||_H \le \frac{1}{1+\Delta t\frac{\kappa}{\alpha}}(||w_{i-1}||_H + \Delta t||v_i||_H) \le ||w_{i-1}||_H + \Delta t||v_i||_H. \tag{2.19}$$

Before we derive the estimate for the Crank–Nicolson scheme we show that

$$a(t_i;\; w_i, w_i + w_{i-1}) + a(t_{i-1};\; w_{i-1}, w_i + w_{i-1}) \ge -\frac{c_{\text{lip}}^2\alpha||S||_2^2}{16\kappa}\Delta t^2(||w_i||_H + ||w_{i-1}||_H)^2.$$

We use the assumptions on the bilinear form to derive

$$\begin{aligned}&a(t_i;\; w_i, w_i + w_{i-1}) + a(t_{i-1};\; w_{i-1}, w_i + w_{i-1}) \\ &= a(t_i;\; w_i + w_{i-1}, w_i + w_{i-1}) + (a(t_{i-1};\; w_{i-1}, w_i + w_{i-1}) - a(t_i;\; w_{i-1}, w_i + w_{i-1})) \\ &\ge \kappa||w_i + w_{i-1}||_V^2 - c_{\text{lip}}|t_i - t_{i-1}|\; ||w_{i-1}||_V||w_i + w_{i-1}||_V\end{aligned}$$

and similarly

$$a(t_i;\; w_i, w_i + w_{i-1}) + a(t_{i-1};\; w_{i-1}, w_i + w_{i-1}) \ge \kappa||w_i + w_{i-1}||_V^2 - c_{\text{lip}}|t_i - t_{i-1}|\; ||w_i||_V||w_i + w_{i-1}||_V.$$

First, we add the last two inequalities and divide by 2, then use $w_i \in \mathcal{V}^p$ to estimate with (2.14), and finally complete the squares to obtain

$$\begin{aligned}&a(t_i;\; w_i, w_i + w_{i-1}) + a(t_{i-1};\; w_{i-1}, w_i + w_{i-1}) \\ &\ge \kappa||w_i + w_{i-1}||_V^2 - \frac{1}{2}c_{\text{lip}}\Delta t(||w_i||_V + ||w_{i-1}||_V)||w_i + w_{i-1}||_V \\ &\ge \frac{\kappa}{\alpha}\Big(||w_i + w_{i-1}||_H^2 - \frac{c_{\text{lip}}\alpha||S||_2}{2\kappa}\Delta t(||w_i||_H + ||w_{i-1}||_H)||w_i + w_{i-1}||_H\Big) \\ &= \frac{\kappa}{\alpha}\Big(\big(||w_i + w_{i-1}||_H - \frac{c_{\text{lip}}\alpha||S||_2}{4\kappa}\Delta t(||w_i||_H + ||w_{i-1}||_H)\big)^2 \\ &\quad - \frac{c_{\text{lip}}^2\alpha^2||S||_2^2}{16\kappa^2}\Delta t^2(||w_i||_H + ||w_{i-1}||_H)^2\Big) \ge -2\xi\Delta t^2(||w_i||_H + ||w_{i-1}||_H)^2\end{aligned}$$

with $\xi := c_{\text{lip}}^2\alpha||S||_2^2/(32\kappa)$, which was claimed to be shown above.

We return to formula (2.18) and use $\Psi = w_i + w_{i-1} \in \mathcal{V}^p$ in the Crank–Nicolson case ($\theta = 1/2$)

$$\begin{aligned}||w_i||_H^2 &= ||w_{i-1}||_H^2 - \frac{\Delta t}{2}(a(t_i;\; w_i, w_i + w_{i-1}) + a(t_{i-1};\; w_{i-1}, w_i + w_{i-1})) \\ &\quad + \Delta t\langle v_i, w_i + w_{i-1}\rangle_H \\ &\le ||w_{i-1}||_H^2 + \xi\Delta t^3(||w_i||_H + ||w_{i-1}||_H)^2 + \Delta t||v_i||_H(||w_i||_H + ||w_{i-1}||_H)\end{aligned}$$

and therefore

$$||w_i||_H - ||w_{i-1}||_H = \frac{||w_i||_H^2 - ||w_{i-1}||_H^2}{||w_i||_H + ||w_{i-1}||_H} \le \xi\Delta t^3(||w_i||_H + ||w_{i-1}||_H) + \Delta t||v_i||_H.$$

If we assume that Δt is chosen so small that $\xi\Delta t^3 < 1/2$ we obtain

$$||w_i||_H \le \frac{1+\xi\Delta t^3}{1-\xi\Delta t^3}||w_{i-1}||_H + \frac{1}{1-\xi\Delta t^3}\Delta t||v_i||_H \le (1+4\xi\Delta t^3)||w_{i-1}||_H + 2\Delta t||v_i||_H. \tag{2.20}$$

□

After proving these lemmas, we can state and show the main error estimate. First, we consider Error 1 as defined in (2.10).

Theorem 2.7. *Let $u(t)$ be the solution of Problem 2.2, $\{u_i^{\mathrm{POD},1}\}_{i=0}^n$ the solution of Problem 2.3. Then with appropriate constants C_i $(i = 0, 1, 2)$, independent of n, we have*

$$\frac{1}{n}\sum_{i=1}^{n}\left|\left|u(t_i)-u_i^{\mathrm{POD},1}\right|\right|_H^2 \le C_0||u(t_0)-\varPi_H^p u(t_0)||_H^2 + C_1\varDelta t^j + C_2||S||_2\sum_{j=p+1}^{r}\lambda_j$$

with $j = 2$ for the implicit Euler method assuming $u_{tt} \in L_2([0,T];H)$
and $j = 4$ for the Crank–Nicolson method, assuming $u_{ttt} \in L_2([0,T];H)$ and $\varDelta t$ sufficiently small.
Furthermore, for some constant C we have

$$||u(t_0)-\varPi_H^p u(t_0)||_H^2 \le n\ C\sum_{j=p+1}^{r}\lambda_j.$$

Proof. Define the snapshots y_i:

$$y_i = u(t_{i-1}) \qquad i = 1,\dots,n+1$$

$$y_{i+n+1} = \bar{\partial}u(t_i) = \frac{u(t_i)-u(t_{i-1})}{\varDelta t} \qquad i = 1,\dots,n$$

Let $dim(\mathrm{span}(y_1,\dots,y_{2n+1})) = r$. We compute the POD basis $\varPsi_1,\dots,\varPsi_r$ with the corresponding eigenvalues $\lambda_1,\dots,\lambda_r$ using the norm $||\cdot||_H$. For simplicity, the weighting factors are set constant, *i.e.* $\gamma_i = \frac{1}{2n+1}$ $\forall$ i. However, a different choice with $\gamma_i \ne \gamma_j$, for $i \ne j$, would only cause slight modifications. Denote by $\mathcal{V}^p$ the space spanned by $\{\varPsi_i\}_{i=1}^p$. Then (1.10) yields:

$$\frac{1}{2n+1}\sum_{i=0}^{n}\left|\left|u(t_i)-\varPi_H^p u(t_i)\right|\right|_H^2 + \frac{1}{2n+1}\sum_{i=1}^{n}\left|\left|\bar{\partial}u(t_i)-\varPi_H^p\bar{\partial}u(t_i)\right|\right|_H^2 = \sum_{k=p+1}^{r}\lambda_k. \tag{2.21}$$

Let us define

$$w_i^1 = u_i^{\mathrm{POD},1} - \varPi_{a,t_i}^p u(t_i) \quad \text{and} \quad w_i^2 = \varPi_{a,t_i}^p u(t_i) - u(t_i)$$

so that the triangle inequality yields:

$$\frac{1}{n}\sum_{i=1}^{n}\left|\left|u_i^{\mathrm{POD},1}-u(t_i)\right|\right|_H^2 \le \frac{2}{n}\sum_{i=1}^{n}\left|\left|w_i^1\right|\right|_H^2 + \frac{2}{n}\sum_{i=1}^{n}\left|\left|w_i^2\right|\right|_H^2. \tag{2.22}$$

Let us first give an estimate for w_i^2. Using the assumption on the norms of the Hilbert spaces in Assumption 2.1a, Lemma 2.5 (2.12)–(2.21):

$$\frac{1}{n}\sum_{i=1}^{n}||w_i^2||_H^2 \le \frac{1}{n}\frac{\alpha\beta||S||_2}{\kappa}\sum_{i=1}^{n}\left|\left|u(t_i)-\varPi_H^p u(t_i)\right|\right|_H^2 \le \frac{3\alpha\beta||S||_2}{\kappa}\sum_{j=p+1}^{r}\lambda_j. \tag{2.23}$$

Since we included the difference quotients in the set of snapshots we obtain analogously:

$$\frac{1}{n}\sum_{i=1}^{n}||\bar{\partial}u(t_i)-\varPi_{a,t_i}^p\bar{\partial}u(t_i)||_H^2 \le \frac{3\alpha\beta||S||_2}{\kappa}\sum_{j=p+1}^{r}\lambda_j \tag{2.24}$$

a result, which will be needed later.

Estimates for w_i^1 are provided in Lemma 2.6: for the implicit Euler we have

$$||w_i^1||_H \le ||w_{i-1}^1||_H + \Delta t||v_i||_H \tag{2.25}$$

and for Crank–Nicolson

$$||w_i^1||_H \le (1 + 4\xi \Delta t^3)||w_{i-1}^1||_H + 2\Delta t||v_i||_H \tag{2.26}$$

with $v_i = r_i + z_i$ from (2.17), where

$$r_i := \theta \cdot u_t(t_i) + (1-\theta) \cdot u_t(t_{i-1}) - \bar{\partial} u(t_i) \quad \text{and} \quad z_i := \bar{\partial} u(t_i) - \bar{\partial} \Pi_{a,t_i}^p u(t_i).$$

If we apply Lemma 2.8 formulated below to (2.25) and (2.26) this leads to

$$\begin{aligned}
\theta = 1: \quad & \frac{1}{n}\sum_{i=1}^n ||w_i^1||_H^2 \le \max_{1\le i\le n} ||w_i^1||_H^2 \le 2\left|\left|w_0^1\right|\right|_H^2 + 2n\sum_{k=1}^n \Delta t^2 ||v_k||_H^2 \\
& \le 2\left|\left|w_0^1\right|\right|_H^2 + 4T\Delta t \sum_{k=1}^n (||r_k||_H^2 + ||z_k||_H^2) && (2.27)\\
\theta = \frac{1}{2}: \quad & \frac{1}{n}\sum_{i=1}^n \left|\left|w_i^1\right|\right|_H^2 \le 2\mathrm{e}^{8\Delta t^3 \xi n}\left(\left|\left|w_0^1\right|\right|_H^2 + \frac{1-\mathrm{e}^{-8\Delta t^3\xi n}}{8\Delta t^3 \xi}\sum_{k=1}^n 4\Delta t^2 ||v_k||_H^2 \right) \\
& = 2\mathrm{e}^{8\Delta t^2 \xi T}\left(\left|\left|w_0^1\right|\right|_H^2 + \frac{1-\mathrm{e}^{-8\Delta t^2 \xi T}}{2\Delta t \xi}\sum_{k=1}^n ||v_k||_H^2 \right) \\
& \le \tilde{C}_{CN}||w_0^1||_H^2 + C_{CN}\Delta t \sum_{k=1}^n (||r_k||_H^2 + ||z_k||_H^2). && (2.28)
\end{aligned}$$

We split z_i as follows and use Lemma 2.5 (2.13)

$$\begin{aligned}
||z_i||_H^2 &\le 2||\bar{\partial} u(t_i) - \Pi_{a,t_i}^p \bar{\partial} u(t_i))||_H^2 + 2||\Pi_{a,t_i}^p \bar{\partial} u(t_i) - \bar{\partial} \Pi_{a,t_i}^p u(t_i)||_H^2 \\
&= 2||\bar{\partial} u(t_i) - \Pi_{a,t_i}^p \bar{\partial} u(t_i)||_H^2 + \frac{2}{\Delta t}||\Pi_{a,t_i}^p u(t_{i-1}) - \Pi_{a,t_{i-1}}^p u(t_{i-1})||_H^2 \\
&\le 2||\bar{\partial} u(t_i) - \Pi_{a,t_i}^p \bar{\partial} u(t_i)||_H^2 + \frac{c_{\text{lip}}}{\kappa}||w_{i-1}^2||_H^2.
\end{aligned}$$

We use (2.24) for the first part we apply Lemma 2.5 (2.13) and (2.23) for the second to get

$$\frac{1}{n}\sum_{i=1}^n ||z_i||_H^2 \le \frac{6\alpha\beta||S||_2}{\kappa}\sum_{j=p+1}^r \lambda_j + \frac{c_{\text{lip}}^2}{\kappa^2}\frac{6\alpha\beta||S||_2}{\kappa}\sum_{j=p+1}^r \lambda_j \tag{2.29}$$

With regard to r_i one can easily show the following results

$$\theta = 1: \quad \sum_{i=1}^n ||u_t(t_i) - \bar{\partial} u(t_i)||_H^2 \le \Delta t \int_0^T ||u_{tt}(s)||_H^2 \,\mathrm{d}s = \bar{C}\Delta t \tag{2.30}$$

$$\theta = \frac{1}{2}: \quad \sum_{i=1}^n \left|\left|\frac{1}{2}u_t(t_i) + \frac{1}{2}u_t(t_{i-1}) - \bar{\partial} u(t_i)\right|\right|_H^2 \le \frac{\Delta t^3}{16}\int_0^T ||u_{ttt}(s)||_H^2 \,\mathrm{d}s = \tilde{C}\Delta t^3 \tag{2.31}$$

under the assumption that $u_{tt} \in L^2([0,T];H)$ for $\theta = 1$ and $u_{ttt}(t) \in L^2([0,T];H)$ for $\theta = \frac{1}{2}$.

Altogether, we obtain for $ERR_1 = \frac{1}{n}\sum\limits_{i=1}^{n} ||u_i^{\mathrm{POD},1} - u(t_i)||_H^2$ combining (2.22) and (2.23)

$$ERR_1 \le \frac{2}{n}\sum_{i=1}^{n}\left|\left|w_i^1\right|\right|_H^2 + \frac{2}{n}\sum_{i=1}^{n}\left|\left|w_i^2\right|\right|_H^2 \le \frac{2}{n}\sum_{i=1}^{n}\left|\left|w_i^1\right|\right|_H^2 + \frac{6\alpha\beta||S||_2}{\kappa}\sum_{j=p+1}^{r}\lambda_j.$$

With appropriate constants $d_1, ..., d_4$ we estimate further using (2.27)–(2.30) or (2.28)–(2.31) and $j = 2$ for implicit Euler and $j = 4$ for Crank–Nicolson

$$\begin{aligned} ERR_1 &\le d_1\left|\left|w_0^1\right|\right|_H^2 + d_2\Delta t\sum_{k=1}^{n}(||r_k||_H^2 + ||z_k||_H^2) + \frac{6\alpha\beta||S||_2}{\kappa}\sum_{j=p+1}^{r}\lambda_j \\ &\le d_1\left|\left|w_0^1\right|\right|_H^2 + d_3\Delta t^j + d_4\sum_{j=p+1}^{r}\lambda_j \end{aligned}$$

what yields to the proposition. □

The following lemma gives a useful estimate which is being used in the proof of the previous theorem.

Lemma 2.8. *Assume that $r_i \le (1+\delta)r_{i-1} + b_i$, $i = 1, \ldots, n$, holds for some given sequence b_i and some r_0. Then*

$$\max_{1\le i\le n}|r_i|^2 \le 2\mathrm{e}^{2\delta n}\left(r_0^2 + \frac{1-\mathrm{e}^{-2\delta n}}{2\delta}\sum_{k=1}^{n} b_k^2\right) \qquad \text{if} \quad \delta > 0 \tag{2.32}$$

$$\max_{1\le i\le n}|r_i|^2 \le 2r_0^2 + 2n\sum_{k=1}^{n} b_k^2 \qquad \text{if} \quad \delta = 0. \tag{2.33}$$

Proof. We only prove the proposition for $\delta > 0$ since the special case $\delta = 0$ can easily be obtained from this. From the assumption we infer that $r_i \le \mathrm{e}^{\delta i}r_0 + \sum_{k=1}^{i}\mathrm{e}^{\delta(i-k)}b_k$. Since $\delta > 0$ an application of the binomial formula as well as the Cauchy–Schwarz inequality and a geometric series argument we obtain that

$$\begin{aligned} \max_{1\le i\le n}|r_i|^2 &\le 2\mathrm{e}^{2\delta n}r_0^2 + 2\left(\sum_{k=1}^{n}\mathrm{e}^{\delta(n-k)}b_k\right)^2 \le 2\mathrm{e}^{2\delta n}r_0^2 + 2\left(\sum_{k=1}^{n}\mathrm{e}^{2\delta(n-k)}\sum_{k=1}^{n}b_k^2\right) \\ &\le 2\mathrm{e}^{2\delta n}\left(r_0^2 + 2\frac{1-\mathrm{e}^{-2\delta n}}{\mathrm{e}^{2\delta}-1}\sum_{k=1}^{n}b_k^2\right) \le 2\mathrm{e}^{2\delta n}\left(r_0^2 + 2\frac{1-\mathrm{e}^{-2\delta n}}{2\delta}\sum_{k=1}^{n}b_k^2\right). \end{aligned}$$

□

In the next theorem, we take a look at Error 2. Here we estimate the difference between the POD solution compared to the discretized FEM solution as defined in (2.11).

Theorem 2.9. *Let $\left\{u_i^{\mathrm{FEM}}\right\}_{i=0}^{n}$ be the finite element solution using the finite element space $\mathcal{H}^m$ in the Galerkin approximation. Let $\left\{u_i^{\mathrm{POD},2}\right\}_{i=0}^{n}$ be the solution of Problem 2.3 based on the FEM snapshots.*
Then with appropriate constants $\tilde{C}_0, \tilde{C}_1$ independent of n we have for the implicit Euler method and, for sufficiently small Δt, also for the Crank–Nicolson method

$$\frac{1}{n}\sum_{i=1}^{n}\left|\left|u_i^{\mathrm{FEM}} - u_i^{\mathrm{POD},2}\right|\right|_H^2 \le \tilde{C}_0\ ||u_0^{\mathrm{FEM}} - \Pi_H^p u_0^{\mathrm{FEM}}||_H^2 + \tilde{C}_1||S||_2\sum_{j=p+1}^{r}\lambda_j$$

where $||u_0^{\mathrm{FEM}} - \Pi_H^p u_0^{\mathrm{FEM}}||_H^2 \le 3n\sum_{j=p+1}^{r}\lambda_j$.

Proof. The proof is analogous to Theorem 2.7. Instead of $u(t_i)$ we use the snapshots $u_i^{\rm FEM}$. Defining $w_i^1 = u_i^{\rm POD,2} - P_{t_i}^p u_i^{\rm FEM}$ the main difference is:

$$\begin{aligned}
&\langle \bar\partial w_i^1, \Psi\rangle_H + \theta\cdot a(t_i;\ w_i^1,\Psi) + (1-\theta)\cdot a(t_{i-1};\ w_{i-1}^1,\Psi)\\
&= \langle \bar\partial u_i^{\rm POD,2}, \Psi\rangle_H + \theta\cdot a(t_i;\ u_i^{\rm POD,2},\Psi) + (1-\theta)\cdot a(t_{i-1};\ u_{i-1}^{\rm POD,2},\Psi)\\
&\quad - \langle \bar\partial \Pi_{a,t_i}^p u_i^{\rm FEM}, \Psi\rangle_H - \theta\cdot a(t_i;\ \Pi_{a,t_i}^p u_i^{\rm FEM},\Psi) - (1-\theta)\cdot a(t_{i-1};\ \Pi_{a,t_i}^p u_{i-1}^{\rm FEM},\Psi)\\
&= \theta\cdot L(t_i;\ \Psi) + (1-\theta)\cdot L(t_{i-1};\ \Psi)\\
&\quad - \theta\cdot a(t_i;\ u_i^{\rm FEM},\Psi) - (1-\theta)\cdot a(t_{i-1};\ u_{i-1}^{\rm FEM},\Psi) - \langle \bar\partial P_{t_i}^p u_i^{\rm FEM}, \Psi\rangle_H\\
&= \langle \bar\partial u_i^{\rm FEM} - \bar\partial \Pi_{a,t_i}^p u_i^{\rm FEM}, \Psi\rangle_H =: \langle v_i, \Psi\rangle_H.
\end{aligned}$$

Compared to Theorem 2.7 the r_i's drop out, which leads immediately to the statement of the theorem. □

If we use the maximal number of POD basis functions, the whole error is equal to zero, because the error resulting from the time discretization is present in both solutions.

Note that the norm $||S||_2$, which occurs in the estimates in Theorem 2.7 and 2.9, in general depends on n. This can be avoided by using the stronger topology V in (1.9) (*cf.* [20]).

Error 2 seems to be more interesting because in practice we do not have the exact solution $u(t)$ for our PIDE, but only an approximation, *e.g.* from a finite element method, available.

3. Example PIDE

In the previous section we derived several the error estimates for the abstract model. As mentioned in the two first sections, we are interested in reduced order models for a partial integro-differential equation which is one of the fundamental differential equations in mathematical finance. We showed in the second section, how this PIDE can be cast into a weak formulation using time-dependent variational inequalities.

The error estimates in the previous section for the reduced order models were obtained under certain conditions on the underlying problem, in particular its bilinear form defining the variational form of the PIDE. The objective in this section is to provide the groundwork for verifying the assumptions of the theory of the previous sections. In particular, we will address all the assumptions to be met for the bilinear form in Definition 1.3.

The Gelfand triple we use in our special case is given by the Hilbert spaces

$$H = L^2_{-\mu}(\mathbb{R}), \qquad V = H^1_{-\mu}(\mathbb{R}).$$

The next task is to check if the requirements concerning the bilinear form $a^{-\mu}(T;\ v,w)$ are met. Recall that it is defined by

$$\begin{aligned}
a^{-\mu}(T;\ v,w) &:= \int_{\mathbb{R}} \frac{\sigma^2(T,x)}{2} v'(x)w'(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x\\
&+ \int_{\mathbb{R}} \left(r(T) + \frac{\sigma^2(T,x)}{2} - \lambda\zeta + \frac{(\sigma^2(T,x))_x}{2} + \sigma^2(T,x)\mu\ \mathrm{sign}(x)\right) v'(x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x\\
&+ \int_{\mathbb{R}} \lambda(1+\zeta)v(x)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{d}x - \lambda\int_{\mathbb{R}}\int_{\mathbb{R}} v(x-y)w(x)\mathrm{e}^{-2\mu|x|}\mathrm{e}^{y}f(y)\mathrm{d}y\ \mathrm{d}x.
\end{aligned}$$

We impose the following assumptions on the functions r and σ, which are not restrictive for the application in finance.

Assumption 3.1. For each $T \in [0, T_{\max}]$, let $\sigma(T, \cdot)$ be continuously differentiable on $\mathbb{R}$. Furthermore, let $r(\cdot)$, $\sigma(\cdot, x)$ and $\sigma_x(\cdot, x)$ be uniformly Lipschitz-continuous functions in the variable T with Lipschitz constants $r_{\text{lip}}, \sigma_{\text{lip}}, \sigma_{x,lip}$. We assume that there are constants $r_{\max}, \sigma_{\min}, \sigma_{\max}, \sigma_{\text{der}}$ which satisfy

$$\begin{aligned} 0 \le r(T) \le r_{\max} \quad & \forall\, T \in [0, T_{\max}], \\ 0 < \sigma_{\min} \le \sigma(T, x) \le \sigma_{\max} \quad & (T, x) \in [0, T_{\max}] \times \mathbb{R}, \\ |\sigma_x(T, x)| \le \sigma_{\text{der}} \quad & (T, x) \in [0, T_{\max}] \times \mathbb{R}. \end{aligned}$$

In the following theorem we prove that the bilinear form defined in Definition 1.3 is bounded and that Garding's inequality holds. For this to hold, we need an assumption on the asymptotic decay of the function f.

Assumption 3.2. For some $\mu > 0$ assume that $\int_{\mathbb{R}} \mathrm{e}^{y+\mu|y|} y f(y)\ \mathrm{d}y < \infty$.

We will show at the end of this section that this condition is usually satisfied for common choices of f in finance.

Theorem 3.3. *If Assumptions 3.1 and 3.2 hold, then there exist constants $c_1, c_2 > 0$ and $c_3 \in \mathbb{R}$ independent of $T \in [0, T_{\max}]$, such that the following inequalities hold for the bilinear form (1.3) of the PIDE (1.7):*

$$|a^{-\mu}(T;\ v, w)| \le c_1\ ||v||_{H^1_{-\mu}} ||w||_{H^1_{-\mu}} \qquad \forall\, T \in [0, T_{\max}] \tag{3.1}$$

$$a^{-\mu}(T;\ v, v) + c_3\ ||v||^2_{L^2_{-\mu}} \ge c_2\ ||v||^2_{H^1_{-\mu}} \qquad \forall\, T \in [0, T_{\max}] \tag{3.2}$$

$$\begin{aligned} |a^{-\mu}(T_1;\ v, w) - a^{-\mu}(T_2;\ v, w)| \le c_{\text{lip}}\ |T_1 - T_2|\ ||v||_{H^1_{-\mu}} ||w||_{H^1_{-\mu}} \\ \forall\, T_1, T_2 \in [0, T_{\max}],\ v, w \in H^1_{-\mu} \end{aligned} \tag{3.3}$$

Proof. In order to prove the continuity of the bilinear form, we estimate the terms in $a^{-\mu}$ separately: first

$$\left| \int_{\mathbb{R}} \frac{\sigma^2(T, x)}{2} v'(x) w'(x) \mathrm{e}^{-2\mu|x|}\ \mathrm{d}x \right| \le \frac{\sigma^2_{\max}}{2} ||v||_{H^1_{-\mu}} ||w||_{H^1_{-\mu}}. \tag{3.4}$$

If we set $k_1 = |r_{\max} + \frac{\sigma^2_{\max}}{2} + \lambda\zeta + \sigma_{\max}\sigma_{\text{der}} + \mu\sigma^2_{\max}|$ we obtain for the next term of the bilinear form:

$$\begin{aligned} &\left| \int_{\mathbb{R}} \left(r(T) + \frac{\sigma^2(T, x)}{2} - \lambda\zeta + \left(\frac{\sigma(T, x)}{2} \right)_x + \sigma^2(T, x)\mu\ \mathrm{sign}(x) \right) v'(x) w(x) \mathrm{e}^{-2\mu|x|}\ \mathrm{d}x \right| \\ &\le k_1 \langle v'(x), w(x) \rangle_{L^2_{-\mu}} \le k_1 \cdot ||v'||_{L^2_{-\mu}} ||w||_{L^2_{-\mu}} \le k_1 \cdot ||v||_{H^1_{-\mu}} ||w||_{H^1_{-\mu}}. \end{aligned} \tag{3.5}$$

The two remaining terms are merged using $1 + \zeta = \int_{\mathbb{R}} \mathrm{e}^y f(y) \mathrm{d}y$ (see (1.3))

$$\begin{aligned} g(v, w) :=& \lambda \left| \int_{\mathbb{R}} \left((1 + \zeta) v(x) w(x)\ - \int_{\mathbb{R}} v(x - y) \mathrm{e}^y f(y)\ \mathrm{d}y\ w(x) \right) \mathrm{e}^{-2\mu|x|} \mathrm{d}x \right| \\ =& \lambda \left| \int_{\mathbb{R}} \int_{\mathbb{R}} (v(x) - v(x - y)) \mathrm{e}^y f(y)\ \mathrm{d}y\ w(x) \mathrm{e}^{-2\mu|x|}\ \mathrm{d}x \right| \\ =& \lambda \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^1 v'(x - \xi y) \cdot y\ \mathrm{d}\xi\ \mathrm{e}^y f(y)\ \mathrm{d}y\ w(x) \mathrm{e}^{-2\mu|x|}\ \mathrm{d}x \right|. \end{aligned}$$

We rearrange the order of integration, use the Cauchy–Schwarz inequality and a variable transformation by introducing $z = x - \xi y$

$$\begin{aligned} g(v,w) &= \lambda \left| \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} v'(x-\xi y) w(x) \mathrm{e}^{-2\mu|x|} \; \mathrm{d}x \; y \cdot \mathrm{e}^y f(y) \; \mathrm{d}y \; \mathrm{d}\xi \right| \\ &\le \lambda \left| \int_0^1 \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} v'^2(x-\xi y) \mathrm{e}^{-2\mu|x|} \; \mathrm{d}x \Big)^{1/2} ||w||_{L^2_{-\mu}} \; y \cdot \mathrm{e}^y f(y) \; \mathrm{d}y \; \mathrm{d}\xi \right| \\ &\le \lambda \left| \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} v'^2(z) \mathrm{e}^{-2\mu|z|} \mathrm{e}^{2\mu|y|} \; dz \Big)^{1/2} ||w||_{L^2_{-\mu}} \; y \cdot \mathrm{e}^y f(y) \; \mathrm{d}y \right| \\ &\le \lambda \left| \int_{\mathbb{R}} y \cdot \mathrm{e}^{\mu|y|+y} f(y) \; \mathrm{d}y ||v'||_{L^2_{-\mu}} ||w||_{L^2_{-\mu}} \right|. \end{aligned} \tag{3.6}$$

By Assumption 3.2 we can define a constant $k_2 = \int_{\mathbb{R}} \mathrm{e}^{y+\mu|y|} y f(y) \; \mathrm{d}y$ and finally obtain

$$g(v,w) \le \lambda \cdot k_2 \cdot ||v'||_{L^2_{-\mu}} ||w||_{L^2_{-\mu}} \le \lambda \cdot k_2 \cdot ||v||_{H^1_{-\mu}} ||w||_{H^1_{-\mu}}. \tag{3.7}$$

Collecting the estimates from (3.4)–(3.7) the continuity of the bilinear form is proven

$$|a^{-\mu}(T;\; v,w)| \le \left(\frac{\sigma^2_{\max}}{2} + k_1 + \lambda \cdot k_2 \right) \cdot ||v||_{H^1_{-\mu}} ||w||_{H^1_{-\mu}}.$$

Next we prove Garding's inequality for the bilinear form. We estimate the first term by

$$\begin{aligned} \int_{\mathbb{R}} \frac{\sigma^2(T,x)}{2} v'^2(x) \mathrm{e}^{-2\mu|x|} \; \mathrm{d}x &\ge \frac{\sigma^2_{\min}}{2} \int_{\mathbb{R}} v'^2(x) \mathrm{e}^{-2\mu|x|} \; \mathrm{d}x \\ &= \frac{\sigma^2_{\min}}{2} ||v'||^2_{L^2_{-\mu}} = \frac{\sigma^2_{\min}}{2} \cdot ||v||^2_{H^1_{-\mu}} - \frac{\sigma^2_{\min}}{2} ||v||^2_{L^2_{-\mu}}. \end{aligned} \tag{3.8}$$

Due to (3.5), (3.6) we obtain for the remaining terms

$$\begin{aligned} &\int_{\mathbb{R}} \left(r(T) + \frac{\sigma^2(T,x)}{2} - \lambda\zeta + \left(\frac{\sigma^2(T,x)}{2} \right)_x + \sigma^2(T,x)\mu \; \mathrm{sign}(x) \right) v'(x) v(x) \mathrm{e}^{-2\mu|x|} \; \mathrm{d}x \\ &\quad + \lambda \int_{\mathbb{R}} \left((1+\zeta) v(x) w(x) \; - \int_{\mathbb{R}} v(x-y) \mathrm{e}^y f(y) \; \mathrm{d}y \; v(x) \right) \mathrm{e}^{-2\mu|x|} \mathrm{d}x \\ &\ge -(k_1 + k_2 \cdot \lambda) \cdot ||v'||_{L^2_{-\mu}} ||v||_{L^2_{-\mu}} \ge -\frac{k^2_{arb}}{4} ||v'||^2_{L^2_{-\mu}} - \frac{(k_1 + k_2 \cdot \lambda)^2}{k^2_{arb}} ||v||^2_{L^2_{-\mu}} \\ &\ge -\frac{k^2_{arb}}{4} ||v||^2_{H^1_{-\mu}} - \frac{(k_1 + k_2 \cdot \lambda)^2}{k^2_{arb}} ||v||^2_{L^2_{-\mu}} \end{aligned} \tag{3.9}$$

for any (arbitrary) constant $k_{arb} > 0$. If we chose $k_{arb} = \sigma_{\min}$, then the estimates (3.8) and (3.9) lead to

$$a^{-\mu}(T;\; v,v) \ge \frac{\sigma^2_{\min}}{4} ||v||^2_{H^1_{-\mu}} - \left(\frac{(k_1 + k_2 \cdot \lambda)^2}{\sigma^2_{\min}} + \frac{\sigma^2_{\min}}{2} \right) ||v||^2_{L^2_{-\mu}}.$$

Finally, we address the Lipschitz-continuity (3.3) of the bilinear form. Similar to the proof of the continuity we divide the bilinear form into three different parts. Since the term $g(v,w)$ is independent of time, it can be ignored. For the next term we obtain

$$\left|\int_{I\!R} \frac{\sigma^2(T_1,x)-\sigma^2(T_2,x)}{2} v'(x)w'(x)\mathrm{e}^{-2\mu|x|}\;\mathrm{d}x\right| \leq |T_1-T_2|\frac{\sigma_{\mathrm{lip}}\sigma_{\max}}{2}||v||_{H^1_{-\mu}}||w||_{H^1_{-\mu}} \tag{3.10}$$

where σ_{lip} is the Lipschitz-constant of $\sigma(\cdot,x)$. The remaining term can be treated analogously

$$\begin{aligned}&\left|\int_{I\!R}\left(\left(r(T_1)+\frac{\sigma^2(T_1,x)}{2}-\lambda\zeta+\left(\frac{\sigma^2(T_1,x)}{2}\right)_x+\sigma^2(T_1,x)\mu\;\mathrm{sign}(x)\right)\right.\right.\\&\left.\left.\quad-\left(r(T_2)+\frac{\sigma^2(T_2,x)}{2}-\lambda\zeta+\left(\frac{\sigma^2(T_2,x)}{2}\right)_x+\sigma^2(T_2,x)\mu\;\mathrm{sign}(x)\right)\right)v'(x)w(x)\mathrm{e}^{-2\mu|x|}\;\mathrm{d}x\right|\\&\leq|T_1-T_2|(r_{\mathrm{lip}}+\sigma_{\mathrm{lip}}\sigma_{\max}+\sigma_{\mathrm{lip}}\sigma_{\mathrm{der}}+\sigma_{x,lip}\sigma_{\max}+2\sigma_{\mathrm{lip}}\sigma_{\max}\mu)\;||v||_{H^1_{-\mu}}||w||_{H^1_{-\mu}}\end{aligned} \tag{3.11}$$

which completes the proof. □

It is well-known, that the boundedness of $a^{-\mu}$ together with the weak coercivity (3.2) yields the existence of a unique solution of the variational equality.

Theorem 3.4. *If Assumptions 3.1 and 3.2 hold, then there exists a unique solution $D \in W([0,T_{\max}], H^1_{-\mu}(I\!R))$ of the PIDE (1.7) with initial condition (1.8).*

Proof. Under the given assumptions, we can define a new bilinear form on $L^2_{-\mu}$

$$\tilde{a}^{-\mu}(T;\;v,w)=a^{-\mu}(T;\;v,w)+c_3\langle v,w\rangle,$$

which satisfies all the assumptions on the boundedness, the coercivity and Lipschitz continuity as spelled out previously.

According to Theorem 3.3 this bilinear form is uniformly bounded and coercive on the Hilbert space $H^1_{-\mu}$. By [10], p. 509 ff., there exists a unique solution $\tilde{D} \in W([0,T_{\max}], H^1_{-\mu}(I\!R))$ of the variational equality for all $T \in (0,T_{\max})$

$$\frac{\mathrm{d}}{\mathrm{d}T}\langle\tilde{D}(T,\cdot),w(\cdot)\rangle_{L^2_{-\mu}}+\tilde{a}^{-\mu}(T;\tilde{D}(T,\cdot),w(\cdot))=0\quad\forall\;w\in H^1_{-\mu}(I\!R) \tag{3.12}$$

with initial condition

$$\langle\tilde{D}(0,\cdot),w(\cdot)\rangle_{L^2_{-\mu}}=\langle D_0(\cdot),w(\cdot)\rangle_{L^2_{-\mu}}\;\forall\;w\in H^1_{-\mu}(I\!R). \tag{3.13}$$

Then it is easy to verify that

$$D(T,x)=\mathrm{e}^{-c_3T}\tilde{D}(T,x)$$

satisfies the desired variational equality (1.7) with initial condition (1.8). □

In financial applications the two following distribution functions f play an important role in the formulation of PIDEs. For those, we check the assumption

$$\int_{I\!R}\mathrm{e}^{y+\mu|y|}yf(y)\;\mathrm{d}y<\infty.$$

Remark 3.5. The following models specified by the functions $f(y)$ defined below satisfy the requirement $\int_{\mathbb{R}} \mathrm{e}^{y+\mu|y|} y f(y)\ \mathrm{d}y < \infty$.

- Merton model [22]: $f(y) = \dfrac{1}{\sqrt{2\pi}\sigma_M} \exp\left\{-\dfrac{(y-\mu_M)^2}{2\sigma_M^2}\right\}$
- Kou model [19]: $f(y) = p \cdot \eta_1 \cdot \mathrm{e}^{-\eta_1 y} \cdot 1_{\{y \geq 0\}} + (1-p) \cdot \eta_2 \cdot \mathrm{e}^{\eta_2 y} \cdot 1_{\{y<0\}}$ with $\eta_1 > 1$ and $\eta_2 > 0$.

Proof. Regarding the Merton model, the finiteness of the integral term is clear, since we have a (shifted) $-y^2$-term in the density function.

For the proof in the case of the Kou model we divide the integral into two terms. First, for a given $\eta_1 > 1$ there is a $\mu > 0$ sufficiently small, such that $-\eta_1 + \mu + 1 < 0$. Hence, we obtain

$$\int_0^{+\infty} p \cdot \eta_1 \cdot \mathrm{e}^{-\eta_1 y} \mathrm{e}^{\mu y + y} y\ \mathrm{d}y \int_0^{+\infty} p \cdot \eta_1 \mathrm{e}^{(-\eta_1+\mu+1)y} y\ \mathrm{d}y < \infty.$$

For the other half of the integration interval, let $\mu > 0$ be small enough such that $\eta_2 - \mu + 1 > 0$, which can be achieved, since $\eta_2 > 0$ by assumption. Then

$$\int_{-\infty}^0 (1-p) \cdot \eta_2 \cdot \mathrm{e}^{\eta_2 y} \mathrm{e}^{-\mu_2 y + y} y\ \mathrm{d}y = \int_{-\infty}^0 (1-p) \cdot \eta_2 \mathrm{e}^{(\eta_2 - \mu_2 + 1)y} y\ \mathrm{d}y < \infty \qquad \square$$

Note that infinite activity models like Variance Gamma or CGMY with $\sigma = 0$ do not fit the theory presented here and need to be analyzed differently.

4. Numerical results

We now give some numerical results concerning the accuracy of a solution which is calculated *via* the POD approach. As our test problem we use the PIDE problem developed by Merton (see Rem. 3.5), where the parameters are set as follows:

$$r(T) \equiv 3\%, \quad \sigma(x,T) \equiv 30\%, \quad S_0 = 1, \quad \lambda = 50\%, \quad \mu_M = 0\%, \quad \sigma_M = 50\%.$$

Fortunately, at least for constant volatility there exists a closed form solution for the Merton model in terms of an infinite series of Black–Scholes prices. So we can show results concerning the POD error compared to this closed form solution (as in (2.10)) and compared to the finite element solution (as in (2.11)). Thus, the snapshots that are used to compute the POD basis functions are taken from the closed form solution and the finite element solution, respectively. Note that the snapshots of the closed form solution are calculated for a discretization grid with 4000 x-steps and 400 T-steps.

In Table 1 we see in column two the error as defined in (2.10) between the closed-form solution and the corresponding POD solution, *i.e.* $ERR_1 = \sum_{i=1}^n ||u(t_i) - u_i^{\mathrm{POD},1}||^2_{H(-R,R)}$, where we choose $R = 5$.

We compare this error with the sum over the remaining eigenvalues in column three depending on the number p of POD basis functions that are used. We include two pairs of tables, one including difference quotients and the other without. At the beginning, the reduction of the true errors occurs in a similar way the reduction in the sum of the remaining eigenvalues. For $p > 10$ (incl diff. quot.) and $p > 8$ (excl. diff. quot.), respectively, we see no further decrease because here the error is dominated by the time discretization error of the POD solution.

As expected this effect is not observable in Table 2, where we compare a finite element solution with the corresponding POD solution, analogously. The discretization of the FEM solution consists of 4000 x-steps and 400 T-steps. Thus, the time discretization error occurs in both methods.

After having complemented the error estimation theory above with numerical results, we now turn to some results concerning the time. The solution of the original system, *i.e.* the generation of the snapshots was achieved

TABLE 1. Compare between the error as defined in (2.10) and the sum of the remaining eigenvalues for Merton's model.

	Incl. difference quotients		Excl. difference quotients	
p	ERR_1	$\sum_{k=p+1}^{r} \lambda_k$	ERR_1	$\sum_{k=p+1}^{r} \lambda_k$
5	1.03e-003	5.93e-003	5.13e-005	4.01e-005
6	2.31e-004	1.55e-003	1.09e-005	7.22e-006
7	5.00e-005	3.85e-004	3.62e-006	1.57e-006
8	1.08e-005	9.22e-005	2.52e-006	3.61e-007
9	3.10e-006	2.14e-005	3.52e-006	8.03e-008
10	2.55e-006	4.81e-006	5.61e-006	1.71e-008
11	4.04e-006	1.06e-006	8.41e-006	3.51e-009
12	6.57e-006	2.28e-007	1.13e-005	7.03e-010
13	9.66e-006	4.78e-008	1.36e-005	1.38e-010
14	1.25e-005	9.84e-009	1.47e-005	2.64e-011
15	1.44e-005	1.98e-009	1.45e-005	5.07e-012

TABLE 2. Compare between the error as defined in (2.11) and the sum of the remaining eigenvalues for Merton's model.

	Incl. difference quotients		Excl. difference quotients	
p	ERR_2	$\sum_{k=p+1}^{r} \lambda_k$	ERR_2	$\sum_{k=p+1}^{r} \lambda_k$
5	1.87e-003	1.16e-002	5.13e-005	4.04e-005
6	7.81e-004	5.24e-003	1.07e-005	7.56e-006
7	3.44e-004	2.48e-003	3.03e-006	1.90e-006
8	1.54e-004	1.19e-003	1.13e-006	6.31e-007
9	6.88e-005	5.66e-004	5.21e-007	2.57e-007
10	3.05e-005	2.67e-004	2.47e-007	1.15e-007
11	1.34e-005	1.25e-004	1.11e-007	5.26e-008
12	5.85e-006	5.81e-005	5.08e-008	2.40e-008
13	2.52e-006	2.68e-005	2.31e-008	1.09e-008
14	1.08e-006	1.23e-005	1.03e-008	4.90e-009
15	4.55e-007	5.54e-006	4.61e-009	2.19e-009

by the use of an iterative solver combined with a fast Fourier transformation to solve the dense systems of equations in each time step in an efficient way. Indeed, in Table 3 using a Crank–Nicolson scheme we can observe an almost linear relation of problem size (n_T and n_x represent the number of discretization steps in time and space, respectively) and computing time (see column three; FEM-LSE is the time for solving the Linear Systems of Equations in the finite element approach) proving the efficiency of this solution approach. We have already noted that $p = 10$ seems to provide sufficient accuracy for the POD system. This was the size of the POD system which one has to solve, *i.e.* a linear system of equations of size 10×10 per time step. The computing for this step is listed in column four and is almost negligible. More expensive is the calculation of the POD basis function, *i.e.* the solution of the eigenvalue problem (see column five). The timings show clearly that once the POD model has been set up, it is almost free to use and its computational cost is a small fraction of the cost for generating the snapshots.

5. CONCLUSION AND OUTLOOK

In this paper we have shown that POD is a successful model reduction technique for PIDEs. Using weighted function spaces and time-dependent bilinear forms, we provide a rigorous framework for the existence of solutions

Table 3. Computing times of FE- and POD-method for different mesh sizes: Merton's model

Discretization		Computing times (s)		
n_T	$\mathrm{n_x}$	FEM − LSE	POD − LSE	POD − Basis
200	4000	2.96	0.02	0.05
	8000	6.15	0.03	0.14
400	4000	5.76	0.04	0.30
	8000	12.18	0.03	0.56

of PIDEs and their discretizations. Furthermore, this gives the basis for a proof of *a priori* error estimates for the reduced order model through POD. It can be shown, that this theory is applicable to realistic models in option pricing using Lévy processes. The numerical results confirm the theoretical findings. Techniques from a posteriori estimates could lead to improved information for more refined statements on the optimal allocation of snapshots and optimal number of POD basis functions.

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